Chapter III
Fundamental Models on the Lattice

Here we shall give a complete list of results pertaining to the Toda model, a fundamental model on the lattice. We will show that the r-matrix approach applies to this case and may be used to prove the complete integrability of the model in the quasi-periodic case. For the rapidly decreasing boundary conditions we will analyze the mapping $\mathscr{F}$ from the initial data of the auxiliary linear problem to the transition coefficients and outline a method for solving the inverse problem, i. e. for constructing $\mathscr{F}^{-1}$. On the basis of the r-matrix approach it will be shown that $\mathscr{F}$ is a canonical transformation to action-angle type variables establishing the complete integrability of the Toda model in the rapidly decreasing case. We shall also define a lattice version of the LL model, the most general integrable lattice system with two-dimensional auxiliary space.

§ 1. Complete Integrability of the Toda Model in the Quasi-Periodic Case

The equations of motion for the model are

$$\frac{d^2 q_n}{dt^2} = e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}}, \qquad n=1,\dots,N, \tag{1.1}$$

where

$$q_{N+n} = q_n + c. \tag{1.2}$$

This is a Hamiltonian system on the phase space $\mathscr{M} = \mathbb{R}^{2N}$ with coordinates $(p_1, \dots, p_N, q_1, \dots, q_N)$, endowed with the standard Poisson structure

$$\{p_i, p_j\} = \{q_i, q_j\} = 0, \qquad \{p_i, q_j\} = \delta_{ij}, \qquad i,j = 1, \dots, N. \tag{1.3}$$

L. D. Faddeev et al., *Hamiltonian Methods in the Theory of Solitons*

The Hamiltonian is

$$H=\sum_{n=1}^{N}\left(\frac{1}{2}p_n^2+e^{q_{n+1}-q_n}\right) \tag{1.4}$$

(see § I.2).

We will show that *the model is completely integrable* in the sense of classical mechanics with finitely many degrees of freedom. By the Liouville-Arnold theorem, one need only to produce a set of N involutive integrals of the motion I_n

$$\{H, I_n\}=0, \quad \{I_n, I_m\}=0, \quad n,m=1,\ldots,N, \tag{1.5}$$

that are functionally independent,

$$\operatorname{rank}\left(\frac{\partial I_m}{\partial p_n}, \frac{\partial I_m}{\partial q_n}\right)=N \tag{1.6}$$

on a dense subset of $\mathscr{M}$. The left hand side of (1.6) is an $N\times 2N$ matrix composed of the first derivatives of the I_m.

For the proof we consider the auxiliary linear problem for the Toda model,

$$F_{n+1}=L_n(\lambda)F_n, \tag{1.7}$$

where

$$L_n(\lambda)=\begin{pmatrix} p_n+\lambda & e^{q_n} \\ -e^{-q_n} & 0 \end{pmatrix} \tag{1.8}$$

(see § I.2) and apply the r-matrix approach. *A natural analogue of the fundamental Poisson brackets of Chapter II is given by*

$$\{L_n(\lambda) \underset{,}{\otimes} L_m(\mu)\}=[r(\lambda,\mu), L_n(\lambda)\otimes L_n(\mu)]\,\delta_{nm}. \tag{1.9}$$

Indeed, $L_n(\lambda)$ may be regarded as a transition matrix (over one lattice step), and so its Poisson brackets should be modelled on the corresponding expressions for $T(x,y,\lambda)$.

To calculate the r-matrix we express $L_n(\lambda)$ as

$$L_n(\lambda)=(p_n+\lambda)\sigma+\operatorname{sh}q_n\,\sigma_1+i\operatorname{ch}q_n\,\sigma_2, \tag{1.10}$$

with

$$\sigma=\frac{I+\sigma_3}{2}. \tag{1.11}$$

From

$$\{p_n, \operatorname{sh} q_m\} = \operatorname{ch} q_n \delta_{nm}, \qquad \{p_n, \operatorname{ch} q_m\} = \operatorname{sh} q_n \delta_{nm} \tag{1.12}$$

we find

$$\begin{aligned}\{L_n(\lambda) \underset{,}{\otimes} L_m(\mu)\} = (i \operatorname{sh} q_n(\sigma \otimes \sigma_2 - \sigma_2 \otimes \sigma) \\ + \operatorname{ch} q_n(\sigma \otimes \sigma_1 - \sigma_1 \otimes \sigma))\delta_{nm},\end{aligned} \tag{1.13}$$

so that the left hand side of (1.9) is linear in $\operatorname{sh} q_n$ and $\operatorname{ch} q_n$ and does not depend on λ, μ or p_n. In the product $L_n(\lambda) \otimes L_n(\mu)$ the terms linear in $\operatorname{sh} q_n$ and $\operatorname{ch} q_n$ have the form

$$\operatorname{sh} q_n(\lambda \sigma \otimes \sigma_1 + \mu \sigma_1 \otimes \sigma) + i \operatorname{ch} q_n(\lambda \sigma \otimes \sigma_2 + \mu \sigma_2 \otimes \sigma)$$

and the remaining terms commute with the permutation matrix P. We shall therefore look for an r-matrix of the form

$$r(\lambda, \mu) = f(\lambda, \mu) P, \tag{1.14}$$

where $f(\lambda, \mu)$ is an unknown function. We have

$$\begin{aligned}[P, \lambda \sigma \otimes \sigma_1 + \mu \sigma_1 \otimes \sigma] &= (\lambda - \mu) P(\sigma \otimes \sigma_1 - \sigma_1 \otimes \sigma) \\ &= i(\lambda - \mu)(\sigma \otimes \sigma_2 - \sigma_2 \otimes \sigma),\end{aligned} \tag{1.15}$$

$$\begin{aligned}[P, \lambda \sigma \otimes \sigma_2 + \mu \sigma_2 \otimes \sigma] &= (\lambda - \mu) P(\sigma \otimes \sigma_2 - \sigma_2 \otimes \sigma) \\ &= i(\lambda - \mu)(\sigma_1 \otimes \sigma - \sigma \otimes \sigma_1),\end{aligned} \tag{1.16}$$

where we have used the expression

$$P = \frac{1}{2}\left(I \otimes I + \sum_{a=1}^{3} \sigma_a \otimes \sigma_a\right) \tag{1.17}$$

and the multiplication formulae for the Pauli matrices. It follows that (1.9) will hold if $f(\lambda, \mu)$ is chosen to be

$$f(\lambda, \mu) = \frac{1}{\lambda - \mu}. \tag{1.18}$$

As a result, $L_n(\lambda)$ obeys the fundamental lattice Poisson brackets (1.9) with the r-matrix

$$r(\lambda, \mu) = r(\lambda - \mu) = \frac{P}{\lambda - \mu} \tag{1.19}$$

that already occurred for the NS model in Part I.

Introducing the monodromy matrix

$$T_N(\lambda)=\prod_{n=1}^{\overset{\curvearrowleft}{N}} L_n(\lambda), \tag{1.20}$$

we derive from (1.9) the corresponding Poisson brackets

$$\{T_N(\lambda) \underset{,}{\otimes} T_N(\mu)\}=[r(\lambda-\mu), T_N(\lambda)\otimes T_N(\mu)]. \tag{1.21}$$

As was already observed in § I.7, under the periodic boundary conditions the trace of the monodromy matrix is a generating function for integrals of the motion. In the quasi-periodic case

$$L_{N+1}(\lambda)=Q(c)L_1(\lambda)Q^{-1}(c), \tag{1.22}$$

where

$$Q(c)=\exp\frac{c\sigma_3}{2}, \tag{1.23}$$

a similar role is played by the function

$$F_N(\lambda)=\operatorname{tr} T_N(\lambda)Q^{-1}(c) \tag{1.24}$$

(cf. the NS model under the quasi-periodic boundary conditions in § I.2, Part I), which is a polynomial in λ of degree N,

$$F_N(\lambda)=e^{-\frac{c}{2}}\lambda^N+\sum_{n=1}^{N} I_n\lambda^{N-n}, \tag{1.25}$$

the coefficients I_n in turn being polynomials in p_j and $e^{\pm q_j}$. In particular, we have

$$I_1=e^{-\frac{c}{2}}\sum_{n=1}^{N} p_n, \tag{1.26}$$

$$\begin{aligned} I_2&=e^{-\frac{c}{2}}\left(\sum_{1\leqslant k<n\leqslant N} p_k p_n-\sum_{n=1}^{N-1} e^{q_{n+1}-q_n}-e^{q_1+c-q_N}\right)\\ &=e^{-\frac{c}{2}}\left(\sum_{1\leqslant k<n\leqslant N} p_k p_n-\sum_{n=1}^{N} e^{q_{n+1}-q_n}\right), \end{aligned} \tag{1.27}$$

so that

$$H = \frac{e^c}{2} I_1^2 - e^{\frac{c}{2}} I_2 . \tag{1.28}$$

Since $r(\lambda)$ commutes with $Q(c) \otimes Q(c)$, (1.21) yields

$$\{F_N(\lambda), F_N(\mu)\} = 0 , \tag{1.29}$$

hence $I_1, \ldots, I_N$ is an involutive family of integrals of the motion which contains the Hamiltonian of the model.

To conclude the proof of complete integrability of the Toda model it only remains to verify that the integrals I_n are functionally independent. Obviously,

$$I_n = e^{-\frac{c}{2}} S_n(p_1, \ldots, p_N) + I_n' , \tag{1.30}$$

where $S_n(p_1, \ldots, p_N)$ is the n-th elementary symmetric function and I_n' is a polynomial in $p_1, \ldots, p_N$ of degree not greater than $n-1$. Hence (1.6) holds for large p_n, and since everything is polynomial, it holds in the whole phase space $\mathscr{M}$ with the exception of an algebraic subvariety (relative to the coordinates p_n, e^{q_n}) of dimension less than N.

An explicit description of action-angle variables requires recourse to methods of algebraic geometry, which are not our concern in this book.

§ 2. The Auxiliary Linear Problem for the Toda Model in the Rapidly Decreasing Case

Here we shall introduce the principal characteristics of the auxiliary linear problem

$$F_{n+1} = L_n(\lambda) F_n \tag{2.1}$$

in the rapidly decreasing case

$$\lim_{n \to -\infty} q_n = 0 , \qquad \lim_{n \to +\infty} q_n = c , \qquad \lim_{|n| \to \infty} p_n = 0 . \tag{2.2}$$

We assume that the limiting values in (2.2) are attained sufficiently fast: the quantities q_n, $q_n - c$, p_n and their differencies of all orders decrease faster than any power of $|n|^{-1}$, as $|n| \to \infty$ (the lattice analogue of Schwartz's conditions).

1. The transition matrix and Jost solutions

The transition matrix $T(n, m, \lambda)$ is defined to be the solution of (2.1) with the initial condition

$$T(n, m, \lambda)|_{n=m} = I; \tag{2.3}$$

for $n>m$ it is given by

$$T(n, m, \lambda) = \overset{\curvearrowleft}{\prod_{k=m}^{n-1}} L_k(\lambda) \tag{2.4}$$

and for $n<m$

$$T(n, m, \lambda) = T^{-1}(m, n, \lambda) = \overset{\curvearrowright}{\prod_{k=n}^{m-1}} L_k^{-1}(\lambda). \tag{2.5}$$

The matrix $T(n, m, \lambda)$ is unimodular and is a polynomial in λ of degree $|n-m|$; it obeys the involution

$$\bar{T}(n, m, \lambda) = T(n, m, \bar{\lambda}). \tag{2.6}$$

As $n \to \pm\infty$, the auxiliary linear problem (2.1) simplifies and becomes

$$E_{n+1} = L_{\pm}(\lambda) E_n, \tag{2.7}$$

where

$$L_-(\lambda) = L(\lambda) = \begin{pmatrix} \lambda & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.8}$$

$$L_+(\lambda) = Q(c) L(\lambda) Q^{-1}(c). \tag{2.9}$$

For $\lambda \neq 2$, $L(\lambda)$ can be reduced to diagonal form

$$L(\lambda) = U(\lambda) \begin{pmatrix} \dfrac{1}{z(\lambda)} & 0 \\ 0 & z(\lambda) \end{pmatrix} U^{-1}(\lambda), \tag{2.10}$$

with

$$U(\lambda) = \begin{pmatrix} 1 & -z(\lambda) \\ -z(\lambda) & 1 \end{pmatrix}, \tag{2.11}$$

and $z(\lambda)$ is determined from

$$z+\frac{1}{z}=\lambda, \tag{2.12}$$

so that

$$z(\lambda)=\frac{\lambda+\sqrt{\lambda^2-4}}{2}. \tag{2.13}$$

The function $z(\lambda)$ is the analogue of $k(\lambda)$ for the NS model in the finite density case (see § I.8 of Part I) and is well defined on the Riemann surface of the function $\sqrt{\lambda^2-4}$. It is often advantageous to use z instead of the spectral parameter λ; in that case $F(z)$ will stand for $F(\lambda(z))$ for any function $F(\lambda)$.

The solution of (2.7) is given by

$$E_n^{(-)}(z)=E_n(z)=U(z)\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \tag{2.14}$$

and

$$E_n^{(+)}(z)=Q(c)\,E_n(z). \tag{2.15}$$

The matrix $E_n(z)$ obeys the involutions

$$\overline{E_n}(z)=E_n(\bar{z}), \tag{2.16}$$

$$E_n\left(\frac{1}{z}\right)=-\frac{1}{z}\,E_n(z)\sigma_1 \tag{2.17}$$

and the relation

$$\det E_n(z)=1-z^2. \tag{2.18}$$

On the circle $|z|=1$ the entries of $E_n(z)$ are bounded for all n, which corresponds to *the continuous spectrum* of the auxiliary linear problem (2.7). In terms of λ, the continuous spectrum fills in the interval $-2\leqslant\lambda\leqslant 2$. The matrix $E_n(z)$ degenerates at $z=\pm 1$ so that (2.7) has *virtual levels* on the boundary of the spectrum (cf. §§ I.8–I.9 of Part I). The interior and exterior of the unit circle relative to the variable z play a similar role to the upper and lower half-planes of the variable $k(\lambda)$ for the NS model in the finite density case. The analytic properties of $E_n(z)$ are similar to those of $E_\varrho(x,k)$ in § I.8, Part I.

The Jost solutions $T_\pm(n,z)$ for $|z|=1$ are defined to be the limits

$$T_{\pm}(n,z) = \lim_{m \to \pm\infty} T(n,m,z) E_m^{(\pm)}(z). \tag{2.19}$$

Alternatively, they can be identified as solutions of (2.1) with the asymptotic conditions

$$T_{\pm}(n,z) = E_n^{(\pm)}(z) + O(1) \tag{2.20}$$

as $n \to \pm\infty$.

The matrices $T_{\pm}(n,z)$ for $|z|=1$ obey the involutions

$$\overline{T_{\pm}}(n,z) = T_{\pm}(n,\bar{z}), \tag{2.21}$$

$$\overline{T_{\pm}}(n,z) = -\frac{1}{z} T_{\pm}(n,z)\sigma_1 \tag{2.22}$$

and the relation

$$\det T_{\pm}(n,z) = 1 - z^2. \tag{2.23}$$

Their analytic properties are as follows: the columns $T_-^{(1)}(n,z)$ and $T_+^{(2)}(n,z)$ can be analytically continued inside the unit circle, $|z| \leqslant 1$, whereas the columns $T_+^{(1)}(n,z)$ and $T_-^{(2)}(n,z)$ can be analytically continued outside it, $|z| \geqslant 1$, with the following asymptotic behaviour:

$$z^n T_-^{(1)}(n,z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(|z|), \tag{2.24}$$

$$|z| \leqslant 1$$

$$z^{-n} T_+^{(2)}(n,z) = \begin{pmatrix} 0 \\ e^{-\frac{c}{2}} \end{pmatrix} + O(|z|) \tag{2.25}$$

as $z \to 0$ and

$$z^n T_+^{(1)}(n,z) = -e^{-\frac{c}{2}} z \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(1), \tag{2.26}$$

$$|z| \geqslant 1$$

$$z^{-n} T_-^{(2)}(n,z) = -z \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(1) \tag{2.27}$$

as $|z| \to \infty$.

To prove the existence of the Jost solutions and to study their properties it is convenient to make a gauge transformation

$$F_n = \Omega_n \tilde{F}_n, \tag{2.28}$$

where

$$\Omega_n = \begin{pmatrix} e^{\frac{q_n}{2}} & 0 \\ 0 & -e^{-\frac{q_{n-1}}{2}} \end{pmatrix}, \tag{2.29}$$

which carries the inverse linear problem (2.1) into

$$\tilde{F}_{n+1} = \tilde{L}_n(\lambda)\tilde{F}_n, \tag{2.30}$$

with

$$\tilde{L}_n(\lambda) = \Omega_{n+1}^{-1} L_n(\lambda)\Omega_n = \begin{pmatrix} e^{\frac{q_n - q_{n+1}}{2}}(p_n + \lambda) & -e^{-\frac{q_{n+1} - 2q_n + q_{n-1}}{2}} \\ 1 & 0 \end{pmatrix}. \tag{2.31}$$

Setting

$$\tilde{F}_n = \begin{pmatrix} f_n \\ g_n \end{pmatrix} \tag{2.32}$$

we have $g_{n+1} = f_n$ and

$$c_{n+1} f_{n+1} - p_n f_n + c_n f_{n-1} = \lambda f_n, \tag{2.33}$$

with

$$c_n = e^{\frac{q_n - q_{n-1}}{2}}. \tag{2.34}$$

Thus the auxiliary linear problem (2.1) is equivalent to the eigenvalue problem (2.33) for an infinite Jacobi matrix $\mathscr{L}$,

$$\mathscr{L}_{nm} = c_n \delta_{n,m+1} - p_n \delta_{nm} + c_{n+1}\delta_{n+1,m}. \tag{2.35}$$

Let us show that this problem, for $|z| = 1$, has solutions $\psi_\pm(n, z)$ with the asymptotic behaviour

$$\psi_\pm(n, z) = z^n + o(1) \tag{2.36}$$

as $n \to \pm\infty$ $\left(\text{remind that } \lambda = z + \frac{1}{z}\right)$. We look for the solutions of the form

$$\psi_+(n, z) = z^n + \sum_{m=n}^{\infty} \Gamma(n, m) z^m \tag{2.37}$$

and

$$\psi_-(n,z)=z^n+\sum_{m=-\infty}^{n}\tilde{\Gamma}(n,m)z^m\,, \tag{2.38}$$

where

$$\lim_{n,m\to\infty}\Gamma(n,m)=\lim_{n,m\to-\infty}\tilde{\Gamma}(n,m)=0\,. \tag{2.39}$$

Consider, for definiteness, (2.37) and substitute it into (2.33). Collecting the coefficients of a given power of z we deduce

$$c_n(1+\Gamma(n-1,n-1))=1+\Gamma(n,n)\,, \tag{2.40}$$

$$c_n\Gamma(n-1,n)-p_n(1+\Gamma(n,n))=\Gamma(n,n+1) \tag{2.41}$$

and

$$\begin{aligned}\Gamma(n,m+1)+\Gamma(n,m-1)=c_{n+1}(\delta_{m-n,1}+\Gamma(n+1,m))\\ -p_n\Gamma(n,m)+c_n\Gamma(n-1,m)\end{aligned} \tag{2.42}$$

for $m>n$. In the class of kernels $\Gamma(n,m)$ satisfying (2.39), equations (2.40)–(2.42) are uniquely solvable. In fact, (2.40) allows to determine $\Gamma(n,n)$ whereas (2.41) gives $\Gamma(n,n+1)$ for all n, so that (2.42), a second order finite difference equation, has a unique solution in the region $m>n$. The limiting values in (2.39) are attained in the sense of Schwartz. This establishes the existence of the solution $\psi_+(n,z)$.

The existence of $\psi_-(n,z)$ is proved in a similar manner.

In terms of $\psi_\pm(n,z)$, the Jost solutions $T_\pm(n,z)$ are

$$T_\pm(n,z)=\Omega_n\begin{pmatrix}\psi_\pm\left(n,\frac{1}{z}\right) & -z\psi_\pm(n,z)\\ \psi_\pm\left(n-1,\frac{1}{z}\right) & -z\psi_\pm(n-1,z)\end{pmatrix} \tag{2.43}$$

and clearly satisfy the above requirements.

2. The reduced monodromy matrix and transition coefficients

The reduced monodromy matrix $T(z)$ is defined for $|z|=1$, $z\neq\pm1$, as a ratio of the Jost solutions,

$$T(z)=T_+^{-1}(n,z)\,T_-(n,z)\,, \tag{2.44}$$

and can be expressed as the limit

$$T(z)=\lim_{\substack{n\to\infty \\ m\to-\infty}} E_n^{-1}(z)Q^{-1}(c)\,T(n,m,z)\,E_m(z). \tag{2.45}$$

$T(z)$ is a unimodular matrix satisfying

$$\bar{T}(z)=\sigma_1 T(z)\sigma_1, \tag{2.46}$$

$$\bar{T}(z)=T(\bar{z}) \tag{2.47}$$

and can be written as

$$T(z)=\begin{pmatrix} a(z) & \bar{b}(z) \\ b(z) & \bar{a}(z) \end{pmatrix}, \tag{2.48}$$

where $a(z)$ and $b(z)$ are *the transition coefficients for the continuous spectrum.* These are defined for $|z|=1$, $z\neq\pm1$, satisfy the normalization condition

$$|a(z)|^2-|b(z)|^2=1 \tag{2.49}$$

and are symmetric,

$$\bar{a}(z)=a(\bar{z}), \qquad \bar{b}(z)=b(\bar{z}). \tag{2.50}$$

For the coefficient $a(z)$ we have

$$a(z)=\frac{1}{1-z^2}\det(T_-^{(1)}(n,z),\, T_+^{(2)}(n,z)), \tag{2.51}$$

which shows that it has an analytic continuation into the unit disk $|z|<1$ and

$$a(0)=e^{-\frac{c}{2}}. \tag{2.52}$$

A similar expression for $b(z)$,

$$b(z)=\frac{1}{1-z^2}\det(T_-^{(1)}(n,z),\, T_+^{(1)}(n,z)), \tag{2.53}$$

shows that in general it has no continuation off the circle $|z|=1$. Such a continuation is possible if there exists $N>0$ such that $c_n=1$, $p_n=0$ for $n>N$.

We shall now discuss the alternatives for the behaviour of $a(z)$ and $b(z)$ in the vicinity of $z=\pm 1$. If the columns $T_{-}^{(1)}(n,z)$ and $T_{+}^{(2)}(n,z)$ are linearly independent at $z=1$ or $z=-1$, then *$a(z)$ is singular and has the expansion*

$$a(z)=\frac{a_{\pm}}{z\mp 1}+O(1), \tag{2.54}$$

with $a_{\pm}$ nonzero and real (cf. the NS model in the case of finite density in § I.9, Part I). This is precisely what happens in a generic situation. In the special situation when $T_{-}^{(1)}(n,z)$ and $T_{+}^{(2)}(n,z)$ become linearly dependent at $z=1$ or $z=-1$, the coefficients a_{+} or a_{-} or both vanish, and $a(z)$ is nonsingular near the corresponding points. In that case $z=1$ or $z=-1$ or both values *are virtual levels*. They are located on the boundary, $\lambda=\pm 2$, of the continuous spectrum of the auxiliary linear problem.

The coefficient $b(z)$ is either singular or regular near $z=\pm 1$ simultaneously with $a(z)$. Indeed, we have

$$T_{+}^{(1)}(n,z)|_{z=\pm 1}=\mp T_{+}^{(2)}(n,z)|_{z=\pm 1}, \tag{2.55}$$

so that if a_{+} or a_{-} does not vanish, then

$$b(z)=\mp\frac{a_{\pm}}{z\mp 1}+O(1). \tag{2.56}$$

In particular, under this assumption we have

$$\lim_{z\to\pm 1}\frac{b(z)}{a(z)}=\mp 1. \tag{2.57}$$

(cf. the corresponding formulae in § I.9, Part I).

In view of the normalization condition, the zeros of $a(z)$ may only lie inside the circle $|z|=1$ and their number N is finite. If $a(z_j)=0$, then

$$T_{-}^{(1)}(n,z_j)=\gamma_j T_{+}^{(2)}(n,z_j), \qquad \gamma_j\neq 0 \tag{2.58}$$

and

$$\psi_{-}\left(n,\frac{1}{z_j}\right)=-z_j\gamma_j\psi_{+}(n,z_j), \quad j=1,\dots,N. \tag{2.59}$$

Thus $\lambda_j = z_j + \dfrac{1}{z_j}$ are *the discrete eigenvalues* of the self-adjoint operator $\mathscr{L}$, hence λ_j and consequently z_j are real, $-1 < z_j < 1$, $z_j \neq 0$, $j = 1, \ldots, N$. The associated *transition coefficients for the discrete spectrum* γ_j are also real.

Let us show that *the zeros* z_j *are simple*. From (2.43) we have

$$a(z) = -\frac{c_n z}{1-z^2}\left(\psi_+(n,z)\psi_-\left(n-1,\frac{1}{z}\right) - \psi_+(n-1,z)\psi_-\left(n,\frac{1}{z}\right)\right). \tag{2.60}$$

Differentiating this with respect to z and setting $z = z_j$ we find

$$\begin{aligned}\dot a(z_j) = \frac{c_n z_j}{1-z_j^2}\Bigg(&\dot\psi_+(n-1,z_j)\psi_-\left(n,\frac{1}{z_j}\right) - \dot\psi_+(n,z_j)\psi_-\left(n-1,\frac{1}{z_j}\right)\\ &-\frac{1}{z_j^2}\psi_+(n-1,z_j)\dot\psi_-\left(n,\frac{1}{z_j}\right) + \frac{1}{z_j^2}\psi_+(n,z_j)\dot\psi_-\left(n-1,\frac{1}{z_j}\right)\Bigg),\end{aligned} \tag{2.61}$$

where the dot indicates differentiation with respect to z. From (2.33) and

$$c_{n+1}\dot f_{n+1} - p_n \dot f_n + c_n \dot f_{n-1} = \left(z + \frac{1}{z}\right)\dot f_n + \left(1 - \frac{1}{z^2}\right) f_n \tag{2.62}$$

we deduce that the quantities

$$\phi_+(n,z) = c_n\left(\dot\psi_+(n,z)\psi_-\left(n-1,\frac{1}{z}\right) - \dot\psi_+(n-1,z)\psi_-\left(n,\frac{1}{z}\right)\right) \tag{2.63}$$

and

$$\phi_-(n,z) = \frac{c_n}{z^2}\left(\psi_+(n-1,z)\dot\psi_-\left(n,\frac{1}{z}\right) - \psi_+(n,z)\dot\psi_-\left(n-1,\frac{1}{z}\right)\right) \tag{2.64}$$

satisfy

$$\phi_\pm(n+1,z) = \phi_\pm(n,z) \pm \left(1 - \frac{1}{z^2}\right)\psi_+(n,z)\psi_-\left(n,\frac{1}{z}\right). \tag{2.65}$$

Setting $z = z_j$ and using (2.59) we obtain

$$\phi_+(n,z_j) = \frac{\gamma_j(z_j^2-1)}{z_j}\sum_{k=n}^{\infty}\psi_+^2(k,z_j) \tag{2.66}$$

and

$$\phi_-(n,z_j)=\frac{\gamma_j(z_j^2-1)}{z_j}\sum_{k=-\infty}^{n-1}\psi_+^2(k,z_j),\tag{2.67}$$

so that

$$\dot a(z_j)=\gamma_j\sum_{n=-\infty}^{\infty}\psi_+^2(n,z_j)\neq 0.\tag{2.68}$$

This equation also shows that

$$\operatorname{sign}\gamma_j=\operatorname{sign}\dot a(z_j),\qquad j=1,\dots,N\tag{2.69}$$

(cf. the corresponding arguments in § I.9, Part I).

The function $a(z)$ is uniquely determined by the coefficient $b(z)$ and the zeros $z_1,\dots,z_N$. To derive the corresponding dispersion relation consider Schwarz's formula

$$f(z)=\operatorname{Im}f(0)+\frac{1}{2\pi i}\int_{|\zeta|=1}\operatorname{Re}f(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta},\tag{2.70}$$

where $f(z)$ is analytic in the disk $|z|\leqslant 1$, and apply this to

$$f(z)=\log\prod_{j=1}^{N}\operatorname{sign}z_j\frac{zz_j-1}{z-z_j}a(z),\tag{2.71}$$

where the principal branch of the logarithm is taken. Using $a(0)>0$ and the normalization condition we find

$$a(z)=\prod_{j=1}^{N}\operatorname{sign}z_j\frac{z-z_j}{zz_j-1}\exp\left\{\frac{1}{4\pi i}\int_{|\zeta|=1}\log(1+|b(\zeta)|^2)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}\right\}.\tag{2.72}$$

Taking account of (2.50), we obtain the final expression for $a(z)$

$$a(z)=\prod_{j=1}^{N}\operatorname{sign}z_j\frac{z-z_j}{zz_j-1}\exp\left\{\frac{1}{2\pi i}\int_{C}\log(1+|b(\zeta)|^2)\frac{1-z^2}{(1-z\zeta)(\zeta-z)}d\zeta\right\},\tag{2.73}$$

where C is the semi-circle $|\zeta|=1$, $0\leqslant\arg\zeta\leqslant\pi$.

The data $b(z)$, z_j and c are not independent. Firstly, from (2.52) it follows that

$$e^{-\frac{c}{2}} = \prod_{j=1}^{N} |z_j| \exp\left\{\frac{1}{2\pi i} \int_C \log(1+|b(\zeta)|^2) \frac{d\zeta}{\zeta}\right\}. \tag{2.74}$$

This relation will be called the *condition* (*c*). *Secondly, in a generic situation when*

$$b(z) = \frac{b_\pm}{z \mp 1} + O(1) \tag{2.75}$$

near $z = \pm 1$, *we have*

$$\operatorname{sign} b_\pm = \prod_{j=1}^{N} (\mp \operatorname{sign} z_j) \tag{2.76}$$

(cf. the condition (θ) and the conditions for the determination of signs in § I.9, Part I).

To derive (2.76) we shall study the asymptotic behaviour of $a(z)$ as $z \to \pm 1$, $|z| < 1$ by using the dispersion relation (2.73). The dominant contribution into (2.73) comes from the singular term (2.75) and has the form

$$I_\pm = \frac{1}{2\pi i} \int_{C_\pm} \log \frac{|b_\pm|^2}{|\zeta \mp 1|^2} \frac{1-z^2}{(1-z\zeta)(\zeta - z)} \frac{d\zeta}{\zeta}, \tag{2.77}$$

where $C_\pm$ are small neighbourhoods of $\zeta = \pm 1$ on C. We have

$$\begin{aligned} I_+ &= \frac{1}{\pi i} \int_{C_+} \log \frac{|b_+| \cdot |\zeta + 1|}{2|\zeta - 1|} \cdot \frac{1-z^2}{(1-z\zeta)(\zeta - z)} \frac{d\zeta}{\zeta} + O(|z-1|) \\ &= \frac{1}{2\pi i} \int_{|\zeta|=1} \log \left| \frac{b_+}{2} \frac{\zeta+1}{\zeta-1} \right| \cdot \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} + O(|z-1|) \\ &= \log\left(-\frac{|b_+|}{2} \cdot \frac{z+1}{z-1}\right) + O(|z-1|), \end{aligned} \tag{2.78}$$

where the last equality made use of Schwarz's formula. This yields

$$a(z) = -\frac{|b_+| \prod_{j=1}^{N} (-\operatorname{sign} z_j)}{z-1} + O(1) \tag{2.79}$$

as $z \to 1$. Comparing this with (2.57) we arrive at (2.76) for the sign $+$.

The second formula in (2.76) is proved in a similar way.

Let us emphasize that, as for the NS model in the finite density case, complications in the analytic properties of the transition coefficients are caused by the fact that the continuous spectrum of the auxiliary linear problem has boundary points $\lambda = \pm 2$.

This completes our analysis of the mapping $\mathscr{F}: (p_n, q_n) \to (b(z), \bar{b}(z), z_j, \gamma_j, j=1, \ldots, N)$ from the initial data of the Toda model to the characteristics of the auxiliary linear problem (2.1).

3. Time evolution of the transition coefficients

We shall consider the evolution of the transition coefficients when $p_n(t)$ and $q_n(t)$ satisfy the Toda model equations of motion. Using the zero curvature representation (see § I.2) we obtain

$$\frac{dT}{dt}(n, m, z) = V_n(z)\,T(n, m, z) - T(n, m, z)\,V_m(z), \tag{2.80}$$

with

$$V_n(z) = \begin{pmatrix} 0 & -e^{q_n} \\ e^{q_{n-1}} & z + \dfrac{1}{z} \end{pmatrix}. \tag{2.81}$$

Letting in (2.80) $n \to \infty$, $m \to -\infty$ according to the definitions (2.19), (2.45) and using

$$\begin{aligned} &\lim_{n \to \pm\infty} (E_n^{(\pm)}(z))^{-1}\, V_n(z)\, E_n^{(\pm)}(z) \\ &= \lim_{n \to \pm\infty} (E_n^{(\pm)}(z))^{-1}\, L_{\pm}(z)\, E_n^{(\pm)}(z) = V(z), \end{aligned} \tag{2.82}$$

with

$$V(z) = \begin{pmatrix} z & 0 \\ 0 & \dfrac{1}{z} \end{pmatrix}, \tag{2.83}$$

we derive the evolution equations for the Jost solutions

$$\frac{dT_{\pm}(n, z)}{dt} = V_n(z)\,T_{\pm}(n, z) - T_{\pm}(n, z)\,V(z) \tag{2.84}$$

and for the reduced monodromy matrix,

$$\frac{dT}{dt}(z) = \frac{1}{2}\left(z - \frac{1}{z}\right)[\sigma_3, T(z)]. \tag{2.85}$$

These lead to the following *explicit time dependence of the transition coefficients*:

$$a(z,t) = a(z,0), \quad b(z,t) = e^{-\left(z-\frac{1}{z}\right)t} b(z,0), \tag{2.86}$$

$$z_j(t) = z_j(0), \quad \gamma_j(t) = e^{-\left(z_j-\frac{1}{z_j}\right)t} \gamma_j(0), \quad j = 1, \ldots, N. \tag{2.87}$$

As in the cases considered earlier, the coefficient $a(z)$ is a generating function for integrals of the motion. We close this section by describing a family of local integrals of the motion. The latter are understood to have the form

$$F = \sum_{n=-\infty}^{\infty} f_n, \tag{2.88}$$

where f_n is a polynomial in p_n, c_n and their higher differencies.

4. Local integrals of the motion

We will show that *the expansion of* $\log a(z)$ *into a Taylor series at* $z = 0$,

$$\log a(z) = -\frac{c}{2} + \sum_{n=1}^{\infty} I_n z^n, \tag{2.89}$$

gives a sequence of local integrals of the motion for the Toda model including its Hamiltonian. In the previous examples of continuous models we were dealing with the asymptotic expansion of $\log a(\lambda)$ near the points $\lambda = \infty$ or $\lambda = 0$ where $b(\lambda)$ was rapidly decreasing. This enabled us to start with the asymptotic expansion of the transition matrix $T(x, y, \lambda)$ and then let $x \to +\infty$, $y \to -\infty$. For the Toda model, $b(z)$ in general is not defined near $z = 0$, so this method does not apply. We will outline another method for computing the coefficients I_n based on a direct analysis of the auxiliary linear problem (2.33) in the rapidly decreasing case.

Consider (2.60) for $|z| < 1$ and let $n \to +\infty$. Taking account of (2.36) we have

$$a(z)=\frac{z}{1-z^2}\lim_{n\to+\infty}(z^{n-1}\varphi(n,z)-z^n\varphi(n-1,z)), \tag{2.90}$$

where we have set $\varphi(n,z)=\psi_-\left(n,\frac{1}{z}\right)$. For z small, $\varphi(n,z)$ can be expressed as

$$\varphi(n,z)=z^{-n}\prod_{k=-\infty}^{n}\frac{\chi(k,z)}{c_k}. \tag{2.91}$$

Substituting this expression into (2.90) and using (2.2) and (2.34) yields

$$\begin{aligned} a(z) &= \frac{z}{1-z^2}\lim_{n\to+\infty}\left(\frac{1}{z}\prod_{k=-\infty}^{n}\frac{\chi(k,z)}{c_k}-z\prod_{k=-\infty}^{n-1}\frac{\chi(k,z)}{c_k}\right) \\ &= \prod_{n=-\infty}^{\infty}\frac{\chi(n,z)}{c_n}=e^{-\frac{c}{2}}\prod_{n=-\infty}^{\infty}\chi(n,z). \end{aligned} \tag{2.92}$$

We shall now present a procedure for computing $\chi(n,z)$. Substituting (2.91) into (2.33) gives the equation

$$\chi(n,z)(\chi(n+1,z)-1-zp_n-z^2)=-z^2c_n^2, \tag{2.93}$$

which has a solution of the form

$$\chi(n,z)=\sum_{m=0}^{\infty}\chi(n,m)z^m, \tag{2.94}$$

where

$$\chi(n,0)=1, \qquad \chi(n,1)=p_{n-1}, \tag{2.95}$$

$$\chi(n,2)=1-c_{n-1}^2 \tag{2.96}$$

and for $m>2$

$$\chi(n,m)=c_{n-1}^2\chi(n-1,m-2)-\sum_{k=3}^{m-1}\chi(n,k)\chi(n-1,m-k). \tag{2.97}$$

Formulae (2.92) and (2.94)-(2.97) allow us to express the I_m in terms of the p_n and c_n. In particular, we have

$$I_1=P=\sum_{n=-\infty}^{\infty}p_n \tag{2.98}$$

and

$$I_2 = -H = -\sum_{n=-\infty}^{\infty} \left(\frac{1}{2} p_n^2 + c_n^2 - 1\right). \tag{2.99}$$

By means of the dispersion relation (2.73), I_n can be expressed in terms of the transition coefficients and the discrete spectrum of the auxiliary linear problem. The corresponding *trace identities* are

$$I_n = \frac{1}{2\pi i} \int_C \log(1 + |b(\zeta)|^2)(\zeta^n + \zeta^{-n}) \frac{d\zeta}{\zeta} + \frac{1}{n} \sum_{j=1}^{N} (z_j^n - z_j^{-n}), \qquad n = 1, 2, \ldots \tag{2.100}$$

In § 4 we shall discuss whether the functionals I_n belong the algebra of observables on the phase space of our model.

This completes the analysis of the auxiliary linear problem and the mapping $\mathscr{F}$ for the Toda model.

§ 3. The Inverse Problem and Soliton Dynamics for the Toda Model in the Rapidly Decreasing Case

In this section we shall describe the mapping $\mathscr{F}^{-1}$, i.e. solve the inverse problem of reconstructing the p_n and q_n from the transition coefficients and the discrete spectrum. As before, we may take two routes, the matrix Riemann problem or the Gelfand-Levitan-Marchenko formulation. The presence of a boundary in the continuous spectrum of the auxiliary linear problem and the ensuing constraints on the transition coefficients and the discrete spectrum (the condition (*c*) etc.) lead to complications in the first approach (cf. the NS model in the case of finite density in § II.6, Part I). We shall therefore only deal with the Gelfand-Levitan-Marchenko formulation. At the end of this section it will be used to describe soliton dynamics for the Toda model.

1. The Gelfand-Levitan-Marchenko formulation

The formulation is based on the relationship between the Jost solutions for $|z| = 1$,

$$T_-(n, z) = T_+(n, z)\, T(z), \tag{3.1}$$

which in terms of $\psi_\pm(n, z)$ is written as

$$\frac{1}{a(z)}\psi_-\left(n,\frac{1}{z}\right)=\psi_+\left(n,\frac{1}{z}\right)+r(z)\psi_+(n,z) \tag{3.2}$$

and

$$\frac{1}{a(z)}\psi_+(n,z)=\psi_-(n,z)+\tilde{r}(z)\psi_-\left(n,\frac{1}{z}\right), \tag{3.3}$$

where

$$r(z)=-z\frac{b(z)}{a(z)}, \qquad \tilde{r}(z)=\frac{\bar{b}(z)}{za(z)}. \tag{3.4}$$

Let us consider, for definiteness, (3.2) and make the following operations. Insert (2.37)–(2.38) into (3.2), multiply by $\frac{1}{2\pi i}z^{m-1}$, $m\geqslant n$, and integrate over the circle $|z|=1$. Using (2.52), (2.59) and Cauchy's formula, we deduce

$$\delta_{n,m}+\Gamma(n,m)+K(n+m)+\sum_{l=n}^{\infty}\Gamma(n,l)K(l+m)=e^{\frac{c}{2}}\delta_{n,m}(1+\tilde{\Gamma}(n,n)), \tag{3.5}$$

where

$$K(n)=\frac{1}{2\pi i}\int_{|z|=1} r(z)z^n\frac{dz}{z}+\sum_{j=1}^{N}m_j z_j^n, \tag{3.6}$$

and

$$m_j=\frac{\gamma_j}{\dot{a}(z_j)}, \qquad j=1,\dots,N, \tag{3.7}$$

the dot indicating differentiation with respect to z.

In contrast to the previous examples of continuous models, (3.5) contains an additional term on the right hand side induced by the residue of $\frac{1}{a(z)}\psi_-(n,z)z^{m-1}$ at $z=0$. It can be expressed through $\Gamma(n,n)$ as follows. Consider (2.40); from (2.1), (2.34) and (2.39) we derive

$$1+\Gamma(n,n)=e^{\frac{q_n-c}{2}}. \tag{3.8}$$

An analogous equation for $\tilde{\Gamma}(n,n)$,

$$c_{n+1}(1+\tilde{\Gamma}(n+1,n+1))=1+\tilde{\Gamma}(n,n) \tag{3.9}$$

yields

$$1+\tilde{\Gamma}(n,n)=e^{-\frac{q_n}{2}}, \tag{3.10}$$

hence

$$e^{\frac{c}{2}}(1+\tilde{\Gamma}(n,n))=(1+\Gamma(n,n))^{-1}. \tag{3.11}$$

As a result, (3.5) becomes

$$\frac{\delta_{n,m}}{1+\Gamma(n,n)}=\delta_{n,m}+\Gamma(n,m)+K(n+m)+\sum_{l=n}^{\infty}\Gamma(n,l)K(l+m). \tag{3.12}$$

Unfortunately, this is a nonlinear equation for $\Gamma(n,m)$.

To reduce (3.12) to a linear equation, set

$$X(n,m)=\frac{\Gamma(n,m)}{1+\Gamma(n,n)}, \qquad m>n, \tag{3.13}$$

and multiply (3.12) by $(1+\Gamma(n,n))^{-1}$. For $m>n$ we obtain a linear equation,

$$X(n,m)+K(n+m)+\sum_{l=n+1}^{\infty}X(n,l)K(l+m)=0, \tag{3.14}$$

and for $m=n$

$$\frac{1}{(1+\Gamma(n,n))^2}=1+K(2n)+\sum_{l=n+1}^{\infty}X(n,l)K(l+n). \tag{3.15}$$

Equation (3.14) is precisely the Gelfand-Levitan-Marchenko equation from the right, and (3.15) allows to recover $\Gamma(n,n)$ from $X(n,m)$.

In a similar manner, (3.3) yields the Gelfand-Levitan-Marchenko equation from the left:

$$\tilde{X}(n,m)+\tilde{K}(n+m)+\sum_{l=-\infty}^{n-1}\tilde{X}(n,l)\tilde{K}(l+m)=0, \qquad n>m, \tag{3.16}$$

where

$$\tilde{K}(n)=\frac{1}{2\pi i}\int_{|z|=1}\tilde{r}(z)z^{-n}\frac{dz}{z}+\sum_{j=1}^{N}\tilde{m}_j z_j^{-n-2}, \tag{3.17}$$

$$\tilde{m}_j = \frac{1}{\gamma_j \dot{a}(z_j)}, \quad j=1,\dots,N, \tag{3.18}$$

and

$$\tilde{X}(n,m) = \frac{\tilde{\Gamma}(n,m)}{1+\tilde{\Gamma}(n,n)}, \tag{3.19}$$

with the relation

$$\frac{1}{(1+\tilde{\Gamma}(n,n))^2} = 1+\tilde{K}(2n)+\sum_{l=-\infty}^{n-1} \tilde{X}(n,l)\tilde{K}(l+n). \tag{3.20}$$

Let us now outline a procedure for solving the inverse problem.

The input data consist of functions $r(z), \tilde{r}(z)$ and of a set of real numbers $m_j, \tilde{m}_j, z_j, j=1,\dots,N$; c with the following properties.

I. *$r(z), \tilde{r}(z)$ are smooth functions on the circle $|z|=1$ and obey the involution*

$$\bar{r}(z)=r(\bar{z}), \quad \bar{\tilde{r}}(z)=\tilde{r}(\bar{z}) \tag{3.21}$$

and the relation

$$|r(z)|=|\tilde{r}(z)|\leqslant 1, \tag{3.22}$$

where equality in the estimate can only be attained at $z=\pm 1$, in which case

$$r(\pm 1)=-\tilde{r}(\pm 1)=1. \tag{3.23}$$

II. *The pairwise distinct numbers $z_j \neq 0$ lie in the interval $-1<z_j<1$, while m_j and $\tilde{m}_j$ are positive, $j=1,\dots,N$.*

III. *The condition (c) holds,*

$$e^{-\frac{c}{2}} = \prod_{j=1}^{N} |z_j| \exp\left\{\frac{i}{4\pi}\int_{|\zeta|=1} \log(1-|r(\zeta)|^2)\frac{d\zeta}{\zeta}\right\}. \tag{3.24}$$

IV. *The relations*

$$\frac{\tilde{r}(z)}{\bar{r}(z)} = -\frac{\bar{a}(z)}{a(z)} \tag{3.25}$$

hold and

$$m_j \tilde{m}_j = \frac{1}{\dot{a}^2(z_j)}, \quad j=1,\ldots,N, \tag{3.26}$$

where

$$a(z) = \prod_{j=1}^{N} \operatorname{sign} z_j \frac{z-z_j}{z z_j - 1} \times \exp\left\{\frac{1}{4\pi i} \int_{|\zeta|=1} \log(1-|r(\zeta)|^2) \frac{z+\zeta}{z-\zeta} \frac{d\zeta}{\zeta}\right\}. \tag{3.27}$$

Given these data, construct the kernels $K(n)$, $\tilde{K}(n)$ and consider (3.14), (3.16). We claim that

I′. *Equations* (3.14), (3.16) *are uniquely solvable in the spaces* $l_1(n+1, \infty)$ *and* $l_1(-\infty, n-1)$, *respectively. Their solutions* $X(n,m)$ *and* $\tilde{X}(n,m)$ *are rapidly decreasing as* $n, m \to +\infty$ *or* $n, m \to -\infty$, *respectively.*

II′. *The right hand sides of* (3.15) *and* (3.20) *are positive, and hence* $1+\Gamma(n,n)$ *and* $1+\tilde{\Gamma}(n,n)$ *can be taken positive.*

III′. *Let*

$$\Gamma(n,m) = (1+\Gamma(n,n)) X(n,m) \tag{3.28}$$

and

$$\tilde{\Gamma}(n,m) = (1+\tilde{\Gamma}(n,n)) \tilde{X}(n,m). \tag{3.29}$$

Then the functions $\psi_\pm(n,z)$ *defined by* (2.37)-(2.38) *satisfy*

$$c_{n+1}^{(\pm)} \psi_\pm(n+1,z) - p_n^{(\pm)} \psi_\pm(n,z) + c_n^{(\pm)} \psi_\pm(n-1,z) = \left(z+\frac{1}{z}\right) \psi_\pm(n,z), \tag{3.30}$$

with $c_n^{(\pm)}$ *positive,*

$$c_n^{(+)} = \frac{1+\Gamma(n,n)}{1+\Gamma(n-1,n-1)}, \quad c_n^{(-)} = \frac{1+\tilde{\Gamma}(n-1,n-1)}{1+\tilde{\Gamma}(n,n)} \tag{3.31}$$

and

$$p_n^{(+)} = \frac{c_n^{(+)} \Gamma(n-1,n) - \Gamma(n,n+1)}{1+\Gamma(n,n)}, \tag{3.32}$$

$$p_n^{(-)} = \frac{c_n^{(-)} \tilde{\Gamma}(n+1,n) - \tilde{\Gamma}(n,n-1)}{1+\tilde{\Gamma}(n,n)}. \tag{3.33}$$

IV′. *The relations*

$$\lim_{n\to\pm\infty} c_n^{(\pm)}=1, \qquad \lim_{n\to\pm\infty} p_n^{(\pm)}=0 \tag{3.34}$$

hold, where the limiting values are attained in the sense of Schwartz.

V′. *The relations*

$$p_n^{(+)}=p_n^{(-)}=p_n, \qquad c_n^{(+)}=c_n^{(-)}=e^{\frac{q_n-q_{n-1}}{2}} \tag{3.35}$$

hold, so that

$$\lim_{n\to-\infty} q_n=0, \qquad \lim_{n\to+\infty} q_n=c, \qquad \lim_{|n|\to\infty} p_n=0, \tag{3.36}$$

where the limiting values are attained in the sense of Schwartz.

VI′. *The functions* $a(z)$ *and* $b(z)=-\dfrac{a(z)r(z)}{z}$ *are the transition coefficients for the auxiliary linear problem*

$$F_{n+1}=L_n(\lambda)F_n, \tag{3.37}$$

where

$$L_n(\lambda)=\begin{pmatrix} p_n+\lambda & e^{q_n} \\ -e^{-q_n} & 0 \end{pmatrix}. \tag{3.38}$$

Its discrete spectrum consists of the eigenvalues $\lambda_j=z_j+\dfrac{1}{z_j}$ *with transition coefficients* $\gamma_j=m_j\dot{a}(z_j)$, $j=1,\dots,N$.

We will not give the proof of these assertions since it is a straightforward lattice transcription of the argument of § II.7, Part I. In conclusion, we only note that the Gelfand-Levitan-Marchenko formalism can be used to show that if the time dependence of the inverse problem data is given by (2.86)–(2.87), then the reconstructed $p_n(t)$ and $q_n(t)$ satisfy the Toda equations.

2. Soliton solutions

Soliton solutions of the Toda model correspond to

$$b(z)=0 \tag{3.39}$$

for all z on the circle $|z|=1$. In this case the requirements on the data $\{c, z_j, m_j, \tilde{m}_j, j=1,\dots,N\}$ simplify and amount to the following.

I. *The quantities* $z_j\neq 0$ *lie in the interval* $-1<z_j<1$ *and are pairwise distinct.*

II. *The condition* (c) *holds,*

$$e^{-c} = \prod_{j=1}^{N} z_j^2. \tag{3.40}$$

III. *The quantities* $m_j, \tilde{m}_j$ *are positive and related by*

$$m_j \tilde{m}_j = \frac{1}{\dot{a}^2(z_j)}, \quad j = 1, \ldots, N, \tag{3.41}$$

where

$$a(z) = \prod_{j=1}^{N} \operatorname{sign} z_j \frac{z - z_j}{z z_j - 1}. \tag{3.42}$$

For such data the Gelfand-Levitan-Marchenko equations (3.14)-(3.16) *reduce to linear algebraic equations and can be solved in closed form.*

Consider first the case $N = 1$. The kernel $K(n)$ of (3.14) has the form

$$K(n) = m_1 z_1^n \tag{3.43}$$

and is one-dimensional. Setting

$$X(n, m) = X(n) m_1 z_1^m \tag{3.44}$$

we find from (3.14)

$$X(n) + z_1^n + X(n) m_1 \sum_{l=n+1}^{\infty} z_1^{2l} = 0, \tag{3.45}$$

so that

$$X(n) = -\frac{z_1^n}{1 + |\gamma_1| z_1^{2n+2}}, \tag{3.46}$$

where we have used

$$m_1 = -\operatorname{sign} z_1 \gamma_1 (1 - z_1^2) = |\gamma_1| (1 - z_1^2). \tag{3.47}$$

Substituting (3.44) and (3.46) into (3.15) gives

$$\frac{1}{(1 + \Gamma(n, n))^2} = 1 + m_1 z_1^{2n} - \frac{m_1^2 z_1^{2n}}{1 + |\gamma_1| z_1^{2n+2}} \sum_{l=n+1}^{\infty} z_1^{2l} = \frac{1 + |\gamma_1| z_1^{2n}}{1 + |\gamma_1| z_1^{2n+2}}. \tag{3.48}$$

Now from (3.8) we find

$$e^{q_n} = e^c \frac{1+|\gamma_1| z_1^{2n+2}}{1+|\gamma_1| z_1^{2n}}. \tag{3.49}$$

(Remind that in this case $e^{-c} = z_1^2$.)

The time dependence is introduced by replacing γ_1 with $\gamma_1(t)$,

$$\gamma_1(t) = e^{-\left(z_1 - \frac{1}{z_1}\right)t} \gamma_1. \tag{3.50}$$

If we denote

$$z_1 = \varepsilon e^{-\alpha_1}, \qquad \alpha_1 > 0, \qquad \varepsilon = \pm 1, \tag{3.51}$$

the solutions $q_n(t)$ and $p_n(t)$ of the equations of motion for the Toda model are finally given by

$$q_n(t) = c + \log \frac{1+\exp\{-2\alpha_1(n+1-v_1 t + n_{01})\}}{1+\exp\{-2\alpha_1(n - v_1 t + n_{01})\}} \tag{3.52}$$

and

$$p_n(t) = \frac{dq_n}{dt}(t), \tag{3.53}$$

with

$$v_1 = \varepsilon_1 \frac{\operatorname{sh} \alpha_1}{\alpha_1}, \qquad n_{01} = -\frac{1}{2\alpha_1} \log|\gamma_1|. \tag{3.54}$$

The solution (3.52) represents a wave propagating along the lattice with velocity v_1, $|v_1| > 1$, whose center of inertia position at $t = 0$ is n_{01}. By the general definition of Part I, it should be called a *soliton for the Toda model.* The soliton is characterized by two real parameters, v_1 and n_{01}.

Let us now consider the general case of arbitrary N. As before, the kernel $K(n+m)$ is degenerate,

$$K(n+m) = \sum_{j=1}^{N} \sqrt{m_j}\, z_j^n \sqrt{m_j}\, z_j^m, \tag{3.55}$$

where $\sqrt{m_j} > 0$; we look for a solution of (3.14) of the form

$$X(n,m) = \sum_{j=1}^{N} X_j(n) \sqrt{m_j}\, z_j^m. \tag{3.56}$$

Substituting (3.56) into (3.14) yields a system of equations

$$M(n)X(n) = -Y(n), \tag{3.57}$$

where $X(n)$ is a column-vector with entries $X_j(n)$ and $Y(n)$ is one with entries $\sqrt{m_j}\, z_j^n$, $j=1, \ldots, N$, and $M(n)$ is an $N \times N$ matrix with entries given by

$$M(n)_{ij} = \delta_{ij} + \frac{\sqrt{m_i m_j}\,(z_i z_j)^{n+1}}{1 - z_i z_j}, \tag{3.58}$$

$i, j = 1, \ldots, N$.

From (3.56)-(3.57) we deduce

$$X(n, m) = -Y^\tau(n) M^{-1}(n) Y(m). \tag{3.59}$$

Substituting this into (3.15) gives

$$\begin{aligned} \frac{1}{(1+\Gamma(n,n))^2} &= 1 + Y^\tau(n)Y(n) + Y^\tau(n)(M(n) - I)X(n) \\ &= 1 - Y^\tau(n)X(n) = 1 + Y^\tau(n)M^{-1}(n)Y(n). \end{aligned} \tag{3.60}$$

The last formula can be simplified. Notice that (3.58) yields

$$M(n-1) - M(n) = Y(n)Y^\tau(n), \tag{3.61}$$

or

$$M(n-1)M^{-1}(n) = I + Y(n)Y^\tau(n)M^{-1}(n). \tag{3.62}$$

The matrix $B(n) = Y(n)Y^\tau(n)M^{-1}(n)$ is one-dimensional and

$$B^2(n) = \alpha(n)B(n), \qquad \alpha(n) = Y^\tau(n)M^{-1}(n)Y(n). \tag{3.63}$$

By comparing (3.60) and (3.62)-(3.63) it follows that

$$(1+\Gamma(n,n))^2 = \frac{\det M(n)}{\det M(n-1)}. \tag{3.64}$$

Introducing the time dependence by

$$\gamma_j(t) = e^{-\left(z_j - \frac{1}{z_j}\right)t}\gamma_j, \qquad j = 1, \ldots, N, \tag{3.65}$$

we derive from (3.64) *an expression for the N-soliton solution of the Toda model,*

$$q_n(t) = c + \log \frac{\det M(n,t)}{\det M(n-1,t)}. \tag{3.66}$$

The expression for $p_n(t)$ is given by (3.53) as usual.

As in the earlier examples, *the N-soliton solution describes a scattering process of N solitons. Specifically, for large $|t|$ the solution $q_n(t)$ can be expressed as the sum of one-soliton solutions,*

$$q_n(t) = \sum_{j=1}^{N} q_n^{(+j)}(t) + O(e^{-at}) \tag{3.67}$$

for $t \to +\infty$ and

$$q_n(t) = \sum_{j=1}^{N} q_n^{(-j)}(t) + O(e^{at}) \tag{3.68}$$

for $t \to -\infty$. Here $a = \min \alpha_j \min_{i \neq j} |v_i - v_j|$, and $q_n^{(\pm j)}(t)$ are solitons with parameters $c_j, v_j, n_{0j}^{(\pm)}$:

$$q_n^{(\pm j)}(t) = q_{c_j}(n - v_j t + n_{0j}^{(\pm)}), \tag{3.69}$$

where

$$c_j = -\log z_j^2, \qquad v_j = 2 \operatorname{sign} z_j \frac{\operatorname{sh} \frac{c_j}{2}}{c_j} \tag{3.70}$$

and

$$n_{0j}^{(+)} = n_{0j} + \frac{1}{c_j} \left(\sum_{v_k < v_j} \log \left| \frac{1 - z_j z_k}{z_j - z_k} \right| - \sum_{v_k > v_j} \log \left| \frac{1 - z_j z_k}{z_j - z_k} \right| \right), \tag{3.71}$$

$$n_{0j}^{(-)} = n_{0j} - \frac{1}{c_j} \left(\sum_{v_k < v_j} \log \left| \frac{1 - z_j z_k}{z_j - z_k} \right| - \sum_{v_k > v_j} \log \left| \frac{1 - z_j z_k}{z_j - z_k} \right| \right),$$

$$n_{0j} = \frac{1}{c_j} \log |\gamma_j|, \quad j = 1, \ldots, N. \tag{3.72}$$

The proof of these formulae is based on computations essentially analogous to those of § II.8, Part I.

As for the NS model in the finite density case, the N-soliton solution $q_n(t)$ with parameter c breaks up into solitons $q_n^{(\pm j)}(t)$ with distinct parameters c_j. Thus, only solitons with c_j distinct interact. The relation

$$c = \sum_{j=1}^{N} c_j \tag{3.73}$$

can be thought of as a conservation law. The interpretation of (3.67)-(3.72) in terms of scattering theory is similar to that for the previous examples.

This concludes our discussion of the inverse problem techniques and soliton dynamics for the Toda model.

§ 4. Complete Integrability of the Toda Model in the Rapidly Decreasing Case

In this section we shall consider the mapping $\mathscr{F}$ from the standpoint of canonical transformations in phase space. We shall see that, as in the finite density case of the NS model, the programme of constructing canonical action-angle variables for the Toda model reveals some interesting peculiarities connected with the presence of boundary points in the continuous spectrum of the auxiliary linear problem. We will demonstrate their effect on the Hamiltonian interpretation of soliton scattering theory.

1. The Poisson structure and the algebra of observables

The phase space $\mathscr{M}_c$ of the Toda model is parametrized by coordinates p_n, q_n subject to the rapidly decreasing boundary conditions

$$\lim_{n\to-\infty} q_n = 0, \qquad \lim_{n\to+\infty} q_n = c, \qquad \lim_{|n|\to\infty} p_n = 0. \tag{4.1}$$

The Poisson structure on $\mathscr{M}_c$ is given by the formal Poisson brackets

$$\{p_n, p_m\} = \{q_n, q_m\} = 0, \qquad \{p_n, q_m\} = \delta_{nm}. \tag{4.2}$$

The algebra of observables is composed of admissible functionals $F(p_n, q_n)$. A functional $F(p_n, q_n)$ is admissible if the induced Hamiltonian flow leaves $\mathscr{M}_c$ invariant. In particular, such a functional must satisfy

$$\lim_{|n|\to\infty} \frac{\partial F}{\partial p_n} = \lim_{|n|\to\infty} \frac{\partial F}{\partial q_n} = 0. \tag{4.3}$$

A simplest example of an inadmissible functional is

$$P = \sum_{n=-\infty}^{\infty} p_n, \tag{4.4}$$

which is the first coefficient in the expansion of $\log a(z)$ into a Taylor series at $z=0$ (see Subsection 4 of § 2). Its flow shifts all the q_n simultaneously and so violates the boundary conditions (4.1).

There is the following analogy with the NS model in the finite density case: the quantity c, as well as the phase θ, stands for the index of the phase space $\mathscr{M}_c$ and is related to the transition coefficients and the discrete spectrum by the condition (c). The functional P is analogous to N_ϱ in the finite density case of the NS model. This model has taught us that care is needed when studying the formal Poisson brackets of the transition coefficients on the boundary of the discrete spectrum. In what follows, we shall pay special attention to selecting admissible observables out of the family of local integrals of the motion I_n produced by the trace identities.

2. The Poisson brackets of transition coefficients and discrete spectrum

Consider the Poisson brackets for the transition matrix $T(n, m, z)$ that follow from the fundamental Poisson brackets (1.9):

$$\{T(n,m,z) \underset{,}{\otimes} T(n,m,z')\} = [r(z,z'), T(n,m,z) \otimes T(n,m,z')], \qquad m<n, \tag{4.5}$$

where

$$r(z,z') = r(\lambda(z)-\lambda(z')), \tag{4.6}$$

and $r(\lambda)$ is given by (1.19), and let $n \to +\infty$, $m \to \pm\infty$ according to the definitions (2.19), (2.45). As a result, we obtain the following expressions for the Poisson brackets of the Jost solutions $T_\pm(n,z)$ and of the reduced monodromy matrix $T(z)$:

$$\begin{aligned}&\{T_\pm(n,z) \underset{,}{\otimes} T_\pm(n,z')\} \\ &= \mp r(z,z')\, T_\pm(n,z) \otimes T_\pm(n,z') \pm T_\pm(n,z) \otimes T_\pm(n,z')\, r_\pm(z,z'),\end{aligned} \tag{4.7}$$

$$\{T_+(n,z) \underset{,}{\otimes} T_-(n,z)\} = 0 \tag{4.8}$$

and

$$\{T(z) \underset{,}{\otimes} T(z')\} = r_+(z,z')\, T(z) \otimes T(z') - T(z) \otimes T(z')\, r_-(z,z'). \tag{4.9}$$

Here

$$r_{\pm}(z,z')=$$
$$\begin{pmatrix} \text{p.v.}\,\dfrac{zz'\alpha(z,z')}{(z-z')(zz'-1)} & 0 & 0 & 0 \\ 0 & \text{p.v.}\,\dfrac{zz'\beta(z,z')}{(z-z')(zz'-1)} & \mp\pi i\,\dfrac{\delta(zz'^{-1})z}{1-z^2} & 0 \\ 0 & \pm\pi i\,\dfrac{\delta(zz'^{-1})z}{1-z^2} & \text{p.v.}\,\dfrac{zz'\beta(z,z')}{(z-z')(zz'-1)} & 0 \\ 0 & 0 & 0 & \text{p.v.}\,\dfrac{zz'\alpha(z,z')}{(z-z')(zz'-1)} \end{pmatrix} \tag{4.10}$$

and

$$\alpha(z,z')=\frac{(zz'-1)^2}{(1-z^2)(1-z'^2)}, \quad \beta(z,z')=-\frac{(z-z')^2}{(1-z^2)(1-z'^2)}, \tag{4.11}$$

so that

$$\alpha(z,z')+\beta(z,z')=1, \tag{4.12}$$

and in view of the involutions (2.21) and (2.47), we assume that $|z|=|z'|=1$, $\operatorname{Im} z, \operatorname{Im} z'>0$ where $z, z'\neq \pm 1$; the delta function $\delta(zz'^{-1})$ is defined in a natural way,

$$\int_{|z'|=1} \delta(zz'^{-1})\, f(z')\, \frac{dz'}{z'} = f(z). \tag{4.13}$$

When deriving (4.10)–(4.11) we have also made use of

$$\lim_{n\to\pm\infty} \text{p.v.}\, \frac{(zz'^{-1})^n}{1-zz'^{-1}} = \mp\pi i\delta(zz'^{-1}), \tag{4.14}$$

where $|z|=|z'|=1$.

The Sochocki-Plemelj formula

$$\frac{1}{z-z'e^{-0}} = \lim_{\substack{\tilde z\to z' \\ |\tilde z|<1}} \frac{1}{z-\tilde z} = \text{p.v.}\,\frac{1}{z-z'} + \pi i\,\frac{\delta(zz'^{-1})}{z} \tag{4.15}$$

together with (4.7)–(4.11) leads to following Poisson brackets of the transition coefficients and the discrete spectrum:

$$\{a(z), a(z')\} = \{a(z), \bar a(z')\} = 0, \tag{4.16}$$

$$\{b(z), b(z')\} = 0, \tag{4.17}$$

$$\{b(z), \bar{b}(z')\} = 2\pi i \frac{z|a(z)|^2}{1-z^2} \delta(zz'^{-1}), \tag{4.18}$$

$$\{a(z), b(z')\} = \frac{zz'((1-zz')^2+(z-z')^2)a(z)b(z')}{(ze^{-0}-z')(1-zz')(1-z^2)(1-z'^2)}, \tag{4.19}$$

$$\{a(z), \bar{b}(z')\} = \frac{zz'((1-zz')^2+(z-z')^2)a(z)\bar{b}(z')}{(ze^{-0}-z')(1-zz')(1-z^2)(1-z'^2)} \tag{4.20}$$

and

$$\{a(z), \gamma_j\} = \frac{zz_j((1-zz_j)^2+(z-z_j)^2)a(z)\gamma_j}{(z-z_j)(1-zz_j)(1-z^2)(1-z_j^2)}, \tag{4.21}$$

$$\{b(z), z_j\} = \{b(z), \gamma_j\} = 0, \tag{4.22}$$

$$\{z_i, z_j\} = \{\gamma_i, \gamma_j\} = 0, \tag{4.23}$$

$$\{z_i, \gamma_j\} = -\frac{z_i^2}{1-z_i^2} \gamma_j \delta_{ij}, \qquad i,j=1,\ldots,N. \tag{4.24}$$

Due to the analyticity of $a(z)$, (4.19)-(4.21) remain valid for $|z|<1$ as well.

As in the case of the NS model, this gives a set of independent variables with simple Poisson brackets. Namely, consider (4.19) and (4.21) for $|z|<1$, let $|z|\to 1$ and split off the imaginary and real part, respectively. Then for $|z|=|z'|=1$, $\operatorname{Im} z \geqslant 0$, $\operatorname{Im} z' > 0$, we get

$$\{\log|a(z)|, \arg b(z')\} = -\frac{\pi\delta(zz'^{-1})z}{1-z^2} + \frac{\pi z'}{1-z'^2}(\delta(z)+\delta(-z)) \tag{4.25}$$

and

$$\{\log|a(z)|, \log|\gamma_j|\} = \frac{\pi i z_j}{1-z_j^2}(\delta(z)+\delta(-z)). \tag{4.26}$$

The terms containing $\delta(\pm z)$ result from the singular denominator $(1-z^2)^{-1}$ in (4.19) and (4.21); the delta function $\delta(\pm z)$ is defined by

$$\int_C \delta(\pm z) f(z) \frac{dz}{z} = \frac{1}{2} f(\pm 1), \tag{4.27}$$

where C is the semi-circle $|z|=1$, $0 \leqslant \arg z \leqslant \pi$. (Cf. the analogous formulae in § III.9, Part I.)

Let us introduce a set of variables

$$\varrho(\theta) = \frac{\sin\theta}{\pi}\log(1+|b(e^{i\theta})|^2), \qquad \varphi(\theta) = -\arg b(e^{i\theta}), \qquad 0 \leqslant \theta \leqslant \pi, \tag{4.28}$$

$$\tilde{p}_j = \lambda_j = z_j + \frac{1}{z_j}, \qquad \tilde{q}_j = \log|\gamma_j|, \qquad j = 1, \ldots, N, \tag{4.29}$$

with the following ranges

$$0 \leqslant \varrho(\theta) < \infty, \qquad 0 \leqslant \varphi(\theta) < 2\pi, \tag{4.30}$$

$$|\tilde{p}_j| > 2, \qquad -\infty < \tilde{q}_j < \infty. \tag{4.31}$$

Using (4.24)–(4.26) we see that the nonvanishing Poisson brackets of these variables are

$$\{\varrho(\theta), \varphi(\theta')\} = \delta(\theta-\theta') - \frac{\sin\theta}{\sin\theta'}(\delta(\theta)+\delta(\theta-\pi)), \tag{4.32}$$

$$\{\varrho(\theta), \tilde{q}_j\} = -\frac{2\sin\theta z_j}{z_j^2-1}(\delta(\theta)+\delta(\theta-\pi)), \tag{4.33}$$

$$\{\tilde{p}_i, \tilde{q}_j\} = \delta_{ij}, \qquad i,j = 1, \ldots, N. \tag{4.34}$$

These Poisson brackets would have canonical form if the right hand sides of (4.32)–(4.33) did not contain terms proportional to $\sin\theta(\delta(\theta)+\delta(\theta-\pi))$. *These additional terms should be interpreted in the same spirit as in § III.9, Part I. They must be taken into account every time we are dealing with functionals of the form*

$$F(\varrho) = \int_0^\pi \frac{\varrho(\theta)}{\sin\theta} f(\theta)\, d\theta, \tag{4.35}$$

where $f(\theta)$ *is smooth for* $0 \leqslant \theta \leqslant \pi$, $f(0) = f(\pi) \neq 0$. We shall encounter such functionals in the next subsection.

3. Hamiltonian dynamics and integrals of the motion in terms of the variables $\varrho(\theta)$, $\varphi(\theta)$, $\tilde{p}_j$, $\tilde{q}_j$

The variables introduced above may be regarded as coordinates on the phase space $\mathscr{M}_c$ in whose terms the Poisson structure (4.2) takes the form (4.32)–(4.34). *These variables, however, are not completely independent.* Specifically, we have the condition (*c*)

$$c = -\int_0^\pi \frac{\varrho(\theta)}{\sin\theta}\,d\theta - \sum_{j=1}^N \log z_j^2 . \tag{4.36}$$

Besides, in general position $\frac{\varrho(\theta)}{\sin\theta}$ has a singularity of the type $\log\frac{1}{|\sin\theta|}$ as $\theta \to 0, \pi$, whereas $\varphi(0)$ and $\varphi(\pi)$ are fixed and equal to 0 or π according to (2.76). If $\theta = 0$ or $\theta = \pi$ or both are virtual levels, $\frac{\varrho(\theta)}{\sin\theta}$ is finite at these points and $\varphi(\theta)$ takes the value 0 or π.

As an illustration *let us show that although at first sight the right hand side of* (4.36) *depends on the dynamical variables* $\varrho(\theta)$ *and* $\tilde{p}_j$, *it is actually in involution with* $\varphi(\theta)$ *and* $\tilde{q}_j$ (cf. § III.9, Part I). Indeed, from (4.32)-(4.34) we have

$$\{c, \varphi(\theta)\} = -\frac{1}{\sin\theta} + \frac{1}{\sin\theta}\int_0^\pi \frac{\sin\theta'}{\sin\theta'}(\delta(\theta') + \delta(\theta' - \pi))\,d\theta' = 0 \tag{4.37}$$

and

$$\{c, \tilde{q}_j\} = \frac{2z_j}{z_j^2 - 1} - \{\log z_j^2, \tilde{q}_j\} = 0 . \tag{4.38}$$

We will now show that *the introduction of the new variables trivializes the dynamics of the Toda model.* The Hamiltonian H and the equations of motion can be written as

$$H = -\int_0^\pi \frac{\cos 2\theta}{\sin\theta}\varrho(\theta)\,d\theta + \frac{1}{2}\sum_{j=1}^N \left(\frac{1}{z_j^2} - z_j^2\right), \tag{4.39}$$

$$\frac{\partial \varrho(\theta)}{\partial t} = \{H, \varrho(\theta)\} = 0, \quad \frac{d\tilde{p}_j}{dt} = \{H, \tilde{p}_j\} = 0, \tag{4.40}$$

$$\frac{\partial \varphi(\theta)}{\partial t} = \{H, \varphi(\theta)\} = -\frac{\cos 2\theta}{\sin\theta} + \frac{1}{\sin\theta} = 2\sin\theta, \tag{4.41}$$

$$\frac{d\tilde{q}_j}{dt} = \{H, \tilde{q}_j\} = \frac{z_j^4 + 1}{z_j(1 - z_j^2)} - \frac{2z_j}{1 - z_j^2} = -\left(z_j - \frac{1}{z_j}\right) \tag{4.42}$$

and their solution is a trivial matter. The result is equivalent to (2.86)-(2.87).

We emphasize that if the additional terms in the Poisson brackets (4.32)-(4.34) *were neglected, the time dependence of the transition coefficients would be incorrect.*

The trace identities (2.100) yield expressions for the local integrals I_n,

$$I_n = \int_0^{\pi} \frac{\cos n\theta}{\sin\theta} \varrho(\theta)\, d\theta + \frac{1}{n} \sum_{j=1}^{N} (z_j^n - z_j^{-n}), \tag{4.43}$$

so that these depend only on the variables $\varrho(\theta)$ and $\tilde{p}_j$. *These expressions show that the functionals I_{2n+1} are inadmissible.* In fact, the equations of motion

$$\frac{\partial\varphi(\theta)}{\partial t} = \{I_{2n+1}, \varphi(\theta)\} \tag{4.44}$$

have the form

$$\frac{\partial\varphi(\theta)}{\partial t} = \frac{\cos(2n+1)\theta}{\sin\theta} \tag{4.45}$$

(an additional term in the Poisson bracket (4.32) gives no contribution into (4.44)). The solution

$$\varphi(\theta, t) = \varphi(\theta, 0) + \frac{\cos(2n+1)\theta}{\sin\theta} t \tag{4.46}$$

for $t > 0$ is singular at $\theta = 0$ and $\theta = \pi$ and hence the dynamics induced by I_{2n+1} does not preserve the phase space $\mathscr{M}_c$. In particular, this shows once again that $P = -I_1$ is inadmissible.

The functionals I_{2n} are admissible and correspond to observables on the phase space $\mathscr{M}_c$. The induced equations of motion in the variables $\varrho(\theta), \varphi(\theta), \tilde{p}_j, \tilde{q}_j$ are

$$\frac{\partial\varphi(\theta)}{\partial t} = \{I_{2n}, \varphi(\theta)\} = \frac{\cos 2n\theta - 1}{\sin\theta} = -\frac{2\sin^2 n\theta}{\sin\theta}, \tag{4.47}$$

$$\frac{d\tilde{q}_j}{dt} = \{I_{2n}, \tilde{q}_j\} = \frac{(z_j^{2n} + z_j^{-2n})}{z_j - z_j^{-1}} - \frac{2}{z_j - z_j^{-1}} = \frac{(z_j^n - z_j^{-n})^2}{z_j - z_j^{-1}} \tag{4.48}$$

(now there is a contribution from the additional terms in (4.32)-(4.33)). The time evolution of the transition coefficients is given by

$$b(z,t)=\exp\left\{\frac{(z^n-z^{-n})^2}{z-z^{-1}}t\right\}b(z,0), \tag{4.49}$$

$$\gamma_j(t)=\exp\left\{\frac{(z_j^n-z_j^{-n})^2}{z_j-z_j^{-1}}t\right\}\gamma_j(0), \quad j=1,\dots,N. \tag{4.50}$$

For $n=1$ this gives (upon reversing the sign of time) the familiar expressions (2.86)-(2.87).

One should not be misled into thinking that the other "half" of the local integrals are inadmissible. In fact, the quantities

$$\tilde{I}_n=I_n-I_{n-2}, \quad n>1, \tag{4.51}$$

with $I_0=-c$ are already admissible. They may be expressed as

$$\tilde{I}_n=\int_0^\pi \frac{\cos n\theta-\cos(n-2)\theta}{\sin\theta}\varrho(\theta)\,d\theta+\sum_{j=1}^N\left(\frac{z_j^n-z_j^{-n}}{n}-\frac{z_j^{n-2}-z_j^{2-n}}{n-2}\right), \tag{4.52}$$

and the integrand $\dfrac{\cos n\theta-\cos(n-2)\theta}{\sin\theta}$ is nonsingular at $\theta=0$ and $\theta=\pi$. *Hence the functionals $\tilde{I}_n$ correspond to observables on the phase space $\mathcal{M}_c$, and when writing down the induced equations of motion one may neglect the additional terms in* (4.32)-(4.33). A similar regularization was made for the NS model in the finite density case. The only quantity that does not admit this kind of regularization is P (cf. § III.9, Part I).

Hamilton's equations of motion

$$\frac{dp_n}{dt}=\{\tilde{I}_l,p_n\}, \quad \frac{dq_n}{dt}=\{\tilde{I}_l,q_n\}, \tag{4.53}$$

$n=-\infty,\dots,\infty$, are naturally called *the higher Toda equations. All of them are exactly solvable.*

The above results imply that the Toda model and all its higher analogues are completely integrable Hamiltonian systems. The variables $\varrho(\theta)$, $\varphi(\theta)$, $\tilde{p}_j$, and $\tilde{q}_j$ are effectively their action-angle variables.

The regularized integrals of the motion $\tilde{I}_l$ display separation of modes in a natural manner. Thus for $\tilde{H}=-\tilde{I}_2=H-c$ we have

$$\tilde{H}=2\int_0^\infty \sin\theta\,\varrho(\theta)\,d\theta+\frac{1}{2}\sum_{j=1}^N(z_j^{-2}-z_j^2+2\log z_j^2), \tag{4.54}$$

which can be interpreted as a sum over independent modes. The continuous spectrum mode with index θ has positive energy given by

$$h(\theta)=2\sin\theta, \qquad 0\leqslant\theta\leqslant\pi, \tag{4.55}$$

and the discrete spectrum mode (soliton) also has positive energy

$$h(z)=\tfrac{1}{2}z^{-2}-\tfrac{1}{2}z^{2}+\log z^{2}, \qquad -1<z<1. \tag{4.56}$$

4. Soliton dynamics

The Poisson brackets (4.32)-(4.33) show that in general *soliton dynamics cannot be decoupled from the continuous spectrum modes dynamics in a Hamiltonian manner.* In other words, the constraint $\varrho(\theta)=0$ is inconsistent with these Poisson brackets. Nevertheless (cf. the NS model in the finite density case in § III.9, Part I), the equations of motion generated by the regularized functionals $\tilde{I}_l$ have an independent Hamiltonian formulation in the N-soliton submanifold of the phase space. Namely, on the phase space with coordinates $\tilde{p}_j$, $\tilde{q}_j$, $j=1,\dots,N$, subject to $|\tilde{p}_j|>2$ endowed with the Poisson structure

$$\{\tilde{p}_i,\tilde{q}_j\}=\delta_{ij}, \qquad i,j=1,\dots,N, \tag{4.57}$$

the Hamiltonians

$$\tilde{I}_l^{(\mathrm{sol})}=\tilde{I}_l\big|_{\varrho(\theta)=0} \tag{4.58}$$

induce an evolution that coincides with soliton dynamics governed by the higher Toda equations.

Just as in the finite density case of the NS model, soliton scattering given by (3.67)-(3.72) *is not described by a canonical transformation* if the asymptotic variables $\tilde{p}_j$, $\tilde{q}_j^{(\pm)}=\tilde{q}_j\pm\Delta\tilde{q}_j$, where

$$\Delta\tilde{q}_j=\sum_{v_k<v_j}\log\left|\frac{1-z_jz_k}{z_j-z_k}\right|-\sum_{v_k>v_j}\log\left|\frac{1-z_jz_k}{z_j-z_k}\right|, \tag{4.59}$$

are supposed to have the same Poisson brackets as $\tilde{p}_j$, $\tilde{q}_j$.

In fact, for the two-soliton scattering we have

$$\Delta\tilde{q}_1=-\Delta\tilde{q}_2=\log\frac{|\tilde{p}_1\tilde{p}_2+\sqrt{\tilde{p}_1^2-4}\sqrt{\tilde{p}_2^2-4}-4|}{2(\tilde{p}_1-\tilde{p}_2)} \tag{4.60}$$

for $\tilde{p}_1>\tilde{p}_2$, and this is clearly not a function of the difference $\tilde{p}_1-\tilde{p}_2$ only.

This means, of course, that the a priori hypothesis that the $\tilde{p}_j$, $\tilde{q}_j^{(\pm)}$ form a canonical set is false. The correct choice of canonical asymptotic variables for soliton dynamics (and also for continuous spectrum modes) requires a separate analysis and is not our concern here.

The example of the Toda model shows that the inverse scattering method for lattice models is no less efficient than for continuous ones. The fundamental Poisson brackets (1.9) for $L_n(\lambda)$ are of crucial importance for the Hamiltonian interpretation of the method. This ends our description of the Toda model.

§ 5. The Lattice LL Model as a Universal Integrable System with Two-Dimensional Auxiliary Space

In § II.8 we have seen that the LL model is in some sense universal among integrable models with two-dimensional phase space for fixed x, which admit a zero curvature representation with two-dimensional auxiliary space. In particular, the SG, NS and HM models were interpreted as its limiting cases. Here we shall introduce a lattice analogue of the LL model, the LLL model, and consider the corresponding limiting cases. In this way, besides the LHM and LNS_1 models described in § I.2 we shall obtain a natural lattice analogue of the SG model - the LSG model.

As was observed in § I.2, the easiest thing to define when passing from continuous to discrete models is the matrix $L_n(\lambda)$ of the zero curvature representation. It is a more direct descendant of its continuous counterpart, the matrix $U(x,\lambda)$ of the auxiliary linear problem, than other entities such as $V_n(\lambda)$ and the associated equations of motion, or the Poisson structure and the Hamiltonian. We shall therefore proceed as follows: first, guided by natural requirements we shall define $L_n(\lambda)$ and then describe the LLL model itself.

The principal condition is that $L_n(\lambda)$ should satisfy the fundamental lattice Poisson brackets

$$\{L_n(\lambda) \underset{,}{\otimes} L_m(\mu)\} = [r(\lambda-\mu), L_n(\lambda) \otimes L_n(\mu)]\,\delta_{nm}. \tag{5.1}$$

The significance of these relations was illustrated above by the Toda model. We shall take $r(\lambda)$ to be the r-matrix of the LL model

$$r(\lambda) = -\frac{1}{2}\sum_{a=1}^{3} u_a(\lambda)\sigma_a \otimes \sigma_a, \tag{5.2}$$

where

$$u_1(\lambda)=\varrho\,\frac{1}{\operatorname{sn}(\lambda,k)}\,,\qquad u_2(\lambda)=\varrho\,\frac{\operatorname{dn}(\lambda,k)}{\operatorname{sn}(\lambda,k)}\,,\qquad u_3(\lambda)=\varrho\,\frac{\operatorname{cn}(\lambda,k)}{\operatorname{sn}(\lambda,k)}\,, \tag{5.3}$$

and

$$\varrho=\frac{1}{2}\sqrt{J_3-J_1}\,,\qquad 0<k=\sqrt{\frac{J_2-J_1}{J_3-J_1}}<1 \tag{5.4}$$

with $J_1<J_2<J_3$ (see § II.8). This is quite natural since (5.1) may be interpreted as the Poisson brackets of the one step transition matrix to the next lattice site, i.e. for a small interval Δ in the corresponding continuous model (see § III.1 of Part I and § 1).

Using this analogy, we can approximately write $L_n(\lambda)$ as

$$\begin{aligned} L_n(\lambda)&=I+\int_{\Delta_n} U(x,\lambda)\,dx+O(\Delta^2)\\ &=I+\frac{1}{i}\sum_{a=1}^{3} u_a(\lambda)\int_{\Delta_n} S_a(x)\,dx+O(\Delta^2) \end{aligned} \tag{5.5}$$

(see the expression (II.8.2) for $U(x,\lambda)$). Terms of order $O(\Delta^2)$ in (5.5) are not specified by the initial continuous model. The discussion of the LHM and LNS_1 models in § I.2 shows that these terms are determined from the zero curvature representation. We will presently see that they are also uniquely determined by the fundamental Poisson brackets (5.1). As suggested by (5.5), it is natural to look for $L_n(\lambda)$ in the form

$$L_n(\lambda)=\mathscr{S}_0^{(n)}I+\frac{1}{i}\sum_{a=1}^{3} u_a(\lambda)\,\mathscr{S}_a^{(n)}\sigma_a\,, \tag{5.6}$$

where $\mathscr{S}_\alpha^{(n)}$, $\alpha=0,1,2,3$, are some new dynamical variables. To recover the LL model in the continuum limit, these must have the asymptotic behaviour

$$\mathscr{S}_0^{(n)}=1+O(\Delta^2)\,,\qquad \mathscr{S}_a^{(n)}=\Delta\,S_a(x)+O(\Delta^3)\,, \tag{5.7}$$

with $\Delta n=x$, $\Delta\to 0$, $S_1^2(x)+S_2^2(x)+S_3^2(x)=1$.

Remarkably, the fundamental Poisson brackets (5.1) *with the r-matrix* (5.2)-(5.3) *are satisfied for* $L_n(\lambda)$ *of the form* (5.6) *if* $\mathscr{S}_0^{(n)}$, $\mathscr{S}_a^{(n)}$ *have the following Poisson brackets*

$$\{\mathscr{S}_a^{(n)},\mathscr{S}_0^{(m)}\}=J_{bc}\,\mathscr{S}_b^{(n)}\mathscr{S}_c^{(n)}\delta_{nm} \tag{5.8}$$

and

$$\{\mathscr{S}_a^{(n)}, \mathscr{S}_b^{(m)}\} = -\mathscr{S}_0^{(n)}\mathscr{S}_c^{(n)}\delta_{nm}. \tag{5.9}$$

Here and below (a, b, c) is a cyclic permutation of the indices 1, 2, 3, and we have set

$$J_{bc} = \tfrac{1}{4}(J_c - J_b). \tag{5.10}$$

To derive (5.8)–(5.9) one should use (II.8.8) and the identities

$$u_a(\lambda-\mu)u_b(\lambda)u_a(\mu) - u_b(\lambda-\mu)u_a(\lambda)u_b(\mu) = J_{ab}u_c(\lambda), \tag{5.11}$$

which are a consequence of addition theorems for the Jacobi elliptic functions. They can also be verified directly by comparing the poles in λ on the right and on the left of (5.11) and using the Liouville theorem.

Let us discuss the Poisson brackets (5.8)–(5.9).

1. These Poisson brackets are ultralocal: the variables $\mathscr{S}_\alpha^{(n)}$ that belong to different sites are in involution. Hence we can first consider (5.8)–(5.9) in one site (suppressing the dependence on n) as the Poisson brackets on $\mathbb{R}^4$

$$\{\mathscr{S}_a, \mathscr{S}_0\} = J_{bc}\mathscr{S}_b\mathscr{S}_c, \tag{5.12}$$

$$\{\mathscr{S}_a, \mathscr{S}_b\} = -\mathscr{S}_0\mathscr{S}_c. \tag{5.13}$$

2. The Jacobi identity for the Poisson brackets (5.8)–(5.9) and (5.12)–(5.13) is ensured by the equation (II.8.12) for $r(\lambda)$. However, it can easily be verified directly by making use of the obvious relation

$$J_{12} + J_{23} + J_{31} = 0. \tag{5.14}$$

3. Unlike the Lie-Poisson brackets that occur for the HM and LHM models (see §§ I.1–I.2), *the Poisson brackets* (5.12)–(5.13) *are quadratic in the generators* $\mathscr{S}_0, \mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3$. In a natural sense, they are a deformation of the Poisson brackets for the LHM model. In particular, in the continuum limit (5.7) they go over into the Lie-Poisson brackets for the HM model.

4. The Poisson structure (5.12)–(5.13) is degenerate. *Its annihilator is generated by two polynomials*,

$$\mathscr{C}_0 = \sum_{a=1}^{3} \mathscr{S}_a^2 \tag{5.15}$$

and

$$\mathscr{C}_1 = \mathscr{S}_0^2 - \frac{1}{4}\sum_{a=1}^{3} J_a\mathscr{S}_a^2. \tag{5.16}$$

The equations

$$\mathscr{C}_0 = c_0, \quad \mathscr{C}_1 = c_1, \tag{5.17}$$

where c_0 and c_1 are real, define a symplectic submanifold $\Gamma = \Gamma(J_a, c_0, c_1)$ in $\mathbb{R}^4$.

5. The manifold Γ is in general disconnected. Under the condition

$$c_1 > -\frac{J_1}{4} c_0 \tag{5.18}$$

Γ is homeomorphic to a disjoint union of two spheres $\mathbf{S}^2$. The additional requirement $\mathscr{S}_0 > 0$ selects one of them; the corresponding phase space will be denoted by Γ_0. If

$$-\frac{J_3}{4} c_0 < c_1 < -\frac{J_2}{4} c_0, \tag{5.19}$$

Γ is homeomorphic to the union of two spheres as before. However, if $-\frac{J_2}{4} c_0 < c_1 < -\frac{J_1}{4} c_0$, Γ is connected and homeomorphic to the torus $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$; this type of phase space will occur later on when describing the LSG model. If $c_1 < -\frac{J_3}{4} c_0$, (5.17) has no solution in $\mathbb{R}^4$.

6. Let us return to the lattice Poisson brackets (5.8)–(5.9). Their natural domain is the product of N copies of $\mathbb{R}^4$, N being the number of lattice sites. The phase space $\mathscr{M}$ of the LLL model will be the product of the Γ_0's where c_0 and c_1 do not depend on the index n, so that the model is homogeneous in space. In the continuum limit, with

$$c_0 = \Delta^2, \quad c_1 = 1, \tag{5.20}$$

$\mathscr{M}$ goes into the phase space of the LL model.

We thus have defined the phase space $\mathscr{M}$ of the LLL model and the matrix $L_n(\lambda)$ of the corresponding auxiliary linear problem

$$F_{n+1} = L_n(\lambda) F_n. \tag{5.21}$$

The latter leads to the monodromy matrix

$$T_N(\lambda) = \prod_{n=1}^{\overset{\curvearrowleft}{N}} L_n(\lambda), \tag{5.22}$$

whose Poisson brackets have the same form as for $L_n(\lambda)$,

$$\{T_N(\lambda) \underset{,}{\otimes} T_N(\mu)\} = [r(\lambda-\mu), T_N(\lambda) \otimes T_N(\mu)]. \tag{5.23}$$

It follows that the functions

$$F_N(\lambda) = \operatorname{tr} T_N(\lambda) \tag{5.24}$$

form an involutive family of observables on $\mathscr{M}$,

$$\{F_N(\lambda), F_N(\mu)\} = 0. \tag{5.25}$$

The choice of the family (5.24) corresponds to the periodic boundary conditions

$$\mathscr{S}_\alpha^{(n+N)} = \mathscr{S}_\alpha^{(n)}, \qquad \alpha = 0, 1, 2, 3. \tag{5.26}$$

Let us show that this family contains local observables representable as the sum over lattice sites

$$G_k = \sum_{n=1}^{N} g(\mathscr{S}_\alpha^{(n)}, \dots, \mathscr{S}_\alpha^{(n+k)}), \tag{5.27}$$

with $k<N$. We shall say that G_k describes the interaction of $k+1$ nearest neighbours on the lattice. In particular, H will describe the interaction of two nearest neighbours.

To define it we proceed as follows. Observe that the expression (5.24) for $F_N(\lambda)$ simplifies if $\lambda=\lambda_0$, where λ_0 is a point at which $L_n(\lambda)$ degenerates. In fact, we have

$$L_n(\lambda_0) = \alpha_n \beta_n^\tau \tag{5.28}$$

where α_n and β_n are column-vectors and τ indicates transposition. It follows that

$$F_N(\lambda_0) = \prod_{n=1}^{N} \beta_{n+1}^\tau \alpha_n, \qquad \beta_{N+1} = \beta_1, \tag{5.29}$$

so that $\log F_N(\lambda_0)$ is a local observable describing a nearest neighbour interaction. Unfortunately, this quantity is complex in general. To define a real-valued observable one should use two involutions satisfied by $L_n(\lambda)$,

$$\overline{L_n}(\lambda) = \sigma_2 L_n(\bar{\lambda}) \sigma_2 \tag{5.30}$$

and

$$L_n(-\lambda) = \sigma_2 L_n^\tau(\lambda) \sigma_2 \tag{5.31}$$

(these are immediate consequences of (5.3) and (5.6)). The first one implies $\overline{F_N(\lambda)} = F_N(\bar{\lambda})$, so that

$$H = \log \frac{|F_N(\lambda_0)|^2}{2} \tag{5.32}$$

also belongs to the involutive family generated by $F_N(\lambda)$. The second involution allows us to calculate H explicitly.

Indeed, from

$$\det L_n(\lambda_0) = c_1 + c_0\left(u_1^2(\lambda_0) + \frac{J_1}{4}\right) = 0 \tag{5.33}$$

and (5.18) it follows that λ_0 can be taken pure imaginary, which yields

$$L_n(\bar{\lambda}_0) = L_n(-\lambda_0) = \sigma_2 \beta_n \alpha_n^\tau \sigma_2 , \tag{5.34}$$

hence

$$F_N(\bar{\lambda}_0) = F_N(-\lambda_0) = \prod_{n=1}^{N} \alpha_{n+1}^\tau \beta_n , \qquad \alpha_{N+1} = \alpha_1 . \tag{5.35}$$

This implies

$$H = \sum_{n=1}^{N} \log \frac{\beta_{n+1}^\tau \alpha_n \alpha_{n+1}^\tau \beta_n}{2} = \sum_{n=1}^{N} \log h(\mathcal{S}_\alpha^{(n)}, \mathcal{S}_\alpha^{(n+1)}), \tag{5.36}$$

with

$$\begin{aligned} h(\mathcal{S}_\alpha^{(n)}, \mathcal{S}_\alpha^{(n+1)}) &= \frac{1}{2} \operatorname{tr} L_{n+1}(\lambda_0) L_n(\lambda_0) \\ &= \mathcal{S}_0^{(n)} \mathcal{S}_0^{(n+1)} + \sum_{a=1}^{3} \left(\frac{c_1}{c_0} + \frac{J_a}{4}\right) \mathcal{S}_a^{(n)} \mathcal{S}_a^{(n+1)} . \end{aligned} \tag{5.37}$$

To derive the last equation we have made use of (5.6), (5.15)-(5.17) and (5.33).

The quantity H is what we shall take to be the Hamiltonian of the LLL model. The corresponding equations of motion

$$\frac{d\mathcal{S}_\alpha^{(n)}}{dt} = \{H, \mathcal{S}_\alpha^{(n)}\}, \qquad \alpha = 0, 1, 2, 3 , \tag{5.38}$$

are not so very instructive and will not be stated here explicitly. We will rather discuss their general properties.

1) *In the continuum limit defined by (5.7) and (5.20), H goes into the Hamiltonian of the LL model (see § I.1)*

$$-2H+2N\log 2=\frac{\Delta}{2}\int\left(\left(\frac{d\vec{S}}{dx}\right)^2-J(\vec{S})\right)dx+O(\Delta^2) \tag{5.39}$$

and (5.38) gives the LL equation.

2) *The LLL model is a completely integrable Hamiltonian system.* Indeed, a family of $N-1$ independent integrals of the motion in involution comprising H may be produced as follows:

$$I_k=\frac{d^k}{d\lambda^k}\log|F_N(\lambda)|^2\Big|_{\lambda=\lambda_0},\qquad k=0,\ldots,N-2. \tag{5.40}$$

The quantities I_k are local and describe the interaction of $k+2$ nearest neighbours. The missing integral can be taken in the form $\arg F_N(\lambda_0)$.

3) *The equations of motion (5.38) are representable as a zero curvature condition,*

$$\frac{dL_n}{dt}(\lambda)=V_{n+1}(\lambda)L_n(\lambda)-L_n(\lambda)V_n(\lambda). \tag{5.41}$$

In fact, arguing in complete analogy to § III.3, we find

$$\{\log F_N(\mu), L_n(\lambda)\}=V_{n+1}(\lambda,\mu)L_n(\lambda)-L_n(\lambda)V_n(\lambda,\mu), \tag{5.42}$$

where

$$V_n(\lambda,\mu)=\frac{1}{F_N(\mu)}\operatorname{tr}_1\left(\left(\prod_{k=n}^{\overset{\curvearrowleft}{N}}L_k(\mu)\otimes I\right)r(\mu-\lambda)\left(\overset{\curvearrowleft}{\prod_{k=1}^{n-1}}L_k(\mu)\otimes I\right)\right) \tag{5.43}$$

and we have used the notation tr_1 explained there. It follows from (5.32) that $\{H, L_n(\lambda)\}$ coincides with the right hand side of (5.41) where

$$V_n(\lambda)=\tfrac{1}{2}(V_n(\lambda,\lambda_0)+V_n(\lambda,\bar{\lambda}_0)). \tag{5.44}$$

Thus the equations of motion

$$\frac{dL_n}{dt}(\lambda)=\{H, L_n(\lambda)\} \tag{5.45}$$

are representable in the form (5.41).

The expression for $V_n(\lambda)$ can be simplified by using (5.29), (5.35) and (5.37). We have

$$V_n(\lambda)=\frac{\operatorname{tr}_1(\alpha_{n-1}\beta_n^\tau\otimes I)r(\lambda_0-\lambda)}{2\beta_n^\tau\alpha_{n-1}}-\frac{\operatorname{tr}_1(\sigma_2\beta_{n-1}\alpha_n^\tau\sigma_2\otimes I)r(\lambda_0+\lambda)}{2\alpha_n^\tau\beta_{n-1}}$$

$$=-\frac{1}{h(S_\alpha^{(n-1)},S_\alpha^{(n)})}\operatorname{tr}_1((L_{n-1}(\lambda_0)L_n(\lambda_0)\otimes I)r(\lambda-\lambda_0)$$

$$+(L_{n-1}(-\lambda_0)L_n(-\lambda_0)\otimes I)r(\lambda+\lambda_0)). \tag{5.46}$$

The last formula shows that $V_n(\lambda)$ depends only on the two nearest neighbours.

We emphasize that, as in the continuum case, *the fundamental Poisson brackets can replace the zero curvature representation*. This is yet another demonstration of the utility and universality of the notion of r-matrix.

This concludes our description of the LLL model. We shall now consider its limiting cases obtained by degenerating the elliptic curve E (see § II.8).

The simplest limit corresponds to $k\to 0$ so that $J_1=J_2<J_3$. The corresponding $L_n(\lambda)$ becomes

$$L_n(\lambda)=\mathscr{S}_0^{(n)}I+\frac{\varrho}{i\sin\lambda}(\mathscr{S}_1^{(n)}\sigma_1+\mathscr{S}_2^{(n)}\sigma_2+\cos\lambda\,\mathscr{S}_3^{(n)}\sigma_3), \tag{5.47}$$

where the variables $\mathscr{S}_\alpha^{(n)}$ satisfy the Poisson brackets (5.8)-(5.9) with $J_{12}=0$, $J_{13}=J_{23}=\varrho^2$. In this case there is an explicit expression for the $\mathscr{S}_\alpha$ (in each site) in terms of the usual variables S_1, S_2, S_3 on a sphere of radius R in $\mathbb{R}^3$

$$S_1^2+S_2^2+S_3^2=R^2 \tag{5.48}$$

with the Lie-Poisson brackets

$$\{S_a,S_b\}=-S_c. \tag{5.49}$$

Namely, we set

$$\mathscr{S}_0=\operatorname{ch}(\varrho S_3),\qquad \mathscr{S}_3=\frac{1}{\varrho}\operatorname{sh}(\varrho S_3),$$
$$\mathscr{S}_1=\frac{1}{\varrho}F(S_3)S_1,\qquad \mathscr{S}_2=\frac{1}{\varrho}F(S_3)S_2, \tag{5.50}$$

where

$$F(x)=\sqrt{\frac{\operatorname{sh}^2\varrho R-\operatorname{sh}^2\varrho x}{R^2-x^2}}. \tag{5.51}$$

The variables $\mathscr{S}_\alpha$ then have the Poisson brackets (5.12)-(5.13); the invariants $\mathscr{C}_0$ and $\mathscr{C}_1$ take the values

$$c_0 = \frac{\operatorname{sh}^2 \varrho R}{\varrho^2}, \qquad c_1 = 1 - \frac{J_1}{4} c_0 . \tag{5.52}$$

Substituting (5.50)-(5.51) into (5.47) gives the matrix $L_n(\lambda)$ for a model which for obvious reasons will be called *the partially anisotropic LHM model*; its r-matrix results from (5.2)-(5.3) in the limit as $k \to 0$ and is given by

$$r(\lambda) = -\frac{\varrho}{2\sin\lambda}(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \cos\lambda\, \sigma_3 \otimes \sigma_3). \tag{5.53}$$

The same r-matrix serves for the partially anisotropic HM model (see § I.8), which is a continuum limit of our model upon the naive replacement

$$S_a^{(n)} = \Delta S_a(x), \qquad R = \Delta . \tag{5.54}$$

The partially anisotropic LHM model admits further degeneration. Specifically, in the limit as $\varrho \to 0$ $\left(\text{and replacing } \lambda \text{ by } \frac{2\varrho}{\lambda}\right)$ we come down to the isotropic case $J_1 = J_2 = J_3$ that corresponds to the LHM model of § I.2. The associated r-matrix results from (5.53) in this limit and coincides with the r-matrix for the HM model in § II.3. As was explained in § I.2, this also gives the LNS_1 model.

Let us now describe a lattice analogue of the SG model, the LSG model. It is in essence just another real form of the partially anisotropic LHM model considered above. More precisely, we interchange the roles of J_1, J_2, J_3 and assume that $J_1 = J_2 > J_3$, whereas (5.3)-(5.4) (with $k = 0$) and the form of $L_n(\lambda)$ (5.47) are left intact. The constraints (5.19) become

$$-\frac{J_1}{4} c_0 < c_1 < -\frac{J_3}{4} c_0 \tag{5.55}$$

and the phase space of the model in one lattice site is homeomorphic to the torus $\mathbf{T}^2$. The variables $\mathscr{S}_\alpha$ are expressed as functions of the canonical variables π and φ on the torus,

$$\{\pi, \varphi\} = 1 . \tag{5.56}$$

Namely, let

$$\mathscr{S}_0 = s\cos\frac{\beta\varphi}{2}, \qquad \mathscr{S}_3 = \frac{s}{\gamma}\sin\frac{\beta\varphi}{2},$$
$$\mathscr{S}_1 = -\frac{f(\varphi)}{\gamma}\sin\frac{\beta\pi}{4}, \quad \mathscr{S}_2 = -\frac{f(\varphi)}{\gamma}\cos\frac{\beta\pi}{4}, \tag{5.57}$$

where

$$f(x) = \sqrt{1 + \frac{s^2\cos\beta x}{2}}, \tag{5.58}$$

with $\gamma = \frac{\beta^2}{8} > 0$ and $s > 0$ arbitrary constants. Then the $\mathscr{S}_\alpha$ have the Poisson brackets (5.12)-(5.13) with parameters

$$J_{12} = 0, \qquad J_{13} = J_{23} = -\gamma^2 \tag{5.59}$$

where $J_1 = J_2 = 4\gamma^2$, $J_3 = 0$, and the invariants $\mathscr{C}_0$ and $\mathscr{C}_1$ take the values

$$c_0 = \frac{s^2+2}{2\gamma^2}, \qquad c_1 = \frac{s^2-2}{2}. \tag{5.60}$$

Consider now the matrix $L_n^{\mathrm{SG}}(\alpha)$ of the form

$$L_n^{\mathrm{SG}}(\alpha) = \frac{1}{i}\operatorname{sh}\alpha\,\sigma_2 L_n(i\alpha), \tag{5.61}$$

where $L_n(\lambda)$ is given by (5.47) with $\varrho = i\gamma$. Expressing $\mathscr{S}_\alpha^{(n)}$ through π_n and φ_n according to (5.57)-(5.58) we find

$$L_n^{\mathrm{SG}}(\alpha) = f(\varphi_n)\cos\frac{\beta\pi_n}{4}I + \frac{1}{i}f(\varphi_n)\sin\frac{\beta\pi_n}{4}\sigma_3$$
$$+ \frac{s}{i}\left(\operatorname{ch}\alpha\sin\frac{\beta\varphi_n}{2}\sigma_1 + \operatorname{sh}\alpha\cos\frac{\beta\varphi_n}{2}\sigma_2\right). \tag{5.62}$$

The matrix $L_n^{\mathrm{SG}}(\alpha)$ satisfies the fundamental Poisson brackets (5.1) with the r-matrix given by (5.53) for $\lambda = i\alpha$, $\varrho = i\gamma$. The matrix $r(\alpha)$ coincides (up to an irrelevant summand proportional to $I \otimes I$) with the r-matrix for the SG model of § II.6.

The Hamiltonian of the LSG model is

$$H^{\mathrm{LSG}} = \sum_{n=1}^{N} \log\left(\frac{2s^2\gamma^2}{s^2+2}(-\mathscr{S}_1^{(n)}\mathscr{S}_1^{(n+1)}+\mathscr{S}_2^{(n)}\mathscr{S}_2^{(n+1)}) + \frac{2-s^2}{2+s^2}\gamma^2\mathscr{S}_3^{(n)}\mathscr{S}_3^{(n+1)}+\mathscr{S}_0^{(n)}\mathscr{S}_0^{(n+1)}\right), \tag{5.63}$$

where the $\mathscr{S}_\alpha^{(n)}$ should be replaced by their expressions (5.57)–(5.58) in terms of the π_n, φ_n; H results from (5.36)–(5.37) by taking account of (5.60)–(5.61). Notice that the opposite sign of $\mathscr{S}_1^{(n)}\mathscr{S}_1^{(n+1)}$ and $\mathscr{S}_3^{(n)}\mathscr{S}_3^{(n+1)}$ as compared to (5.37) agrees with (5.61) which may be interpreted as the operation of alternating the sign,

$$\mathscr{S}_1^{(n)} \to (-1)^n\mathscr{S}_1^{(n)}, \qquad \mathscr{S}_3^{(n)} \to (-1)^n\mathscr{S}_3^{(n)} \tag{5.64}$$

(cf. the argument of § I.2).

The auxiliary linear problem (upon the replacement $\lambda = e^\alpha$), the Hamiltonian H, and other characteristics of the LSG model in the continuum limit

$$\pi_n = \Delta\pi(x), \qquad \varphi_n = \varphi(x), \qquad s = \frac{m\Delta}{2} \tag{5.65}$$

turn into the corresponding entities for the SG model, see § I.1. This is the reason for referring to the above completely integrable lattice model as the lattice SG model.

We point out that although the LSG and the partially anisotropic LHM models are essentially equivalent on the lattice, their continuous counterparts lie far apart, since they result from different continuum limits.

The list of models generated by the LLL model is in no way exhausted by the above examples. We may consider the higher analogues of the LLL model with the Hamiltonians I_k, their contractions, and also other values of the parameters J_a and of the invariants $\mathscr{C}_0$, $\mathscr{C}_1$. We may, moreover, vary the structure of $L_n(\lambda)$ by replacing

$$L_n(\lambda) \to A\,L_n(\lambda), \tag{5.66}$$

where $A \otimes A$ commutes with the r-matrix. In this way one can derive the Toda model as well.

Still, we have chosen the Toda model to be our basic example of a lattice model because its investigation is technically simpler. At the same time, it gives a fairly satisfactory illustration of the main features of the inverse scattering formalism for lattice models.

§ 6. Notes and References

1. The complete integrability of the Toda model in the periodic case was established by S. V. Manakov [M 1974a] and H. Flaschka [F 1974a, b], who made use of the Lax representation with the matrix $\mathscr{L}$ given by (2.35),

$$\mathscr{L}_{nm} = c_n \delta_{n,m+1} - p_n \delta_{nm} + c_m \delta_{n,m-1}, \tag{6.1}$$

with $\delta_{n+N,m} = \delta_{n,N+m} = \delta_{nm}$. The functions $\operatorname{tr}\mathscr{L}^k$, $k=1, \ldots, N$, form an involutive family on the phase space of the model, and $H = \frac{1}{2}\operatorname{tr}\mathscr{L}^2$. The matrix $L_n(\lambda)$ of the form (1.8) was introduced in [TF 1979].

2. The general solution of the periodic Toda model in terms of theta functions was derived in [K 1978]. The corresponding canonical action-angle variables were introduced in [FM 1976].

3. The auxiliary linear problem (2.33) (for $N=\infty$),

$$\mathscr{L} f = \lambda f, \tag{6.2}$$

and the related inverse problem were investigated in [M 1974a], [F 1974a, b] (without discussing the peculiarities connected with the boundary of the continuous spectrum and the condition (*c*)); see also the books [ZMNP 1980] and [T 1981]. The latter contains various physical applications of the Toda model.

4. For $p_n = 0$, (6.2) turns into the auxiliary linear problem for the Volterra model introduced in § I.2,

$$c_{n+1} f_{n+1} + c_n f_{n-1} = \lambda f_n \tag{6.3}$$

where $c_n = \sqrt{u_n}$ (see [M 1974a]). In that case the transition coefficients $a(z)$ and $b(z)$ obey an additional involution

$$a(z) = a(-z), \quad b(z) = -b(-z). \tag{6.4}$$

Equation (6.3) is a lattice analogue of the one-dimensional Schrödinger equation and was studied for that reason in [CK 1973].

5. In the continuum limit, the Toda equations of motion go over into the equation of a nonlinear string,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^4 u}{\partial x^4}, \tag{6.5}$$

which can be solved by the inverse scattering method as well (see [Z 1973]), while the Volterra equations of motion go over into the KdV equation (see

[M 1974a], [ZMNP 1980]). Also, the zero curvature representations for lattice models go over into their continuous counterparts.

6. The action-angle variables of § 4 (but ignoring the additional terms in the Poisson brackets (4.32)-(4.33)) were introduced in [M 1974b] (see also [E 1981]) which from the very beginning used the Hamiltonian $\tilde{H}$.

7. It would be interesting to describe the topology of the phase space $\mathcal{M}_c$ in terms of the variables $\varrho(\theta)$, $\varphi(\theta)$, $\tilde{p}_j$, $\tilde{q}_j$ and the associated algebra of observables.

8. For models whose continuous spectrum has a boundary, the correct choice of canonical asymptotic variables for soliton dynamics (and of continuous spectrum modes) presents a nontrivial problem (cf. the NS model in the case of finite density). For the KdV equation, the problem was solved in [BFT 1986]. The method of this paper can in principle be applied to the Toda model and the NS model in the case of finite density.

9. As in the case of the NS and KdV models, a hierarchy of Poisson structures can be defined for the Toda model, starting with the Poisson brackets (1.3). The second Poisson structure for the Toda model was defined in [A 1979]; in terms of the variables p_n, $u_n = e^{q_n - q_{n-1}}$ it is given by

$$\begin{aligned} \{p_n, u_m\} &= -p_n u_m (\delta_{nm} - \delta_{n-1,m}), \\ \{p_n, p_m\} &= u_m^2 \delta_{n,m+1} - u_n^2 \delta_{n,m-1}, \\ \{u_n, u_m\} &= -u_n u_m (\delta_{n+1,m} - \delta_{n-1,m}). \end{aligned} \tag{6.6}$$

Unlike the Poisson structure (1.3), the Poisson brackets (6.6) have a nontrivial restriction to the submanifold $p_n = 0$. The resulting Poisson structure

$$\{u_n, u_m\} = -u_n u_m (\delta_{n+1,m} - \delta_{n-1,m}) \tag{6.7}$$

gives rise to the equations of motion for the Volterra model

$$\frac{du_n}{dt} = u_n (u_{n+1} - u_{n-1}), \tag{6.8}$$

if the Hamiltonian is set to be

$$H = \sum_n u_n . \tag{6.9}$$

In the continuum limit $u_n \to 1 - \Delta u(x)$, the Poisson brackets (6.7) go over into the Poisson brackets (I.3.15) for the KdV model.

The Poisson brackets (I.2.18) for the Volterra model result from the third Poisson structure for the Toda model by restricting to the submanifold

$p_n=0$. In the continuum limit, as $u_n \to 1-\Delta^2 u(x)$, this gives the second Poisson structure for the KdV model (see § III.10 of Part I).

10. The LLL model and the quadratic algebra of Poisson brackets were introduced by E. K. Sklyanin [S 1982]. The rapidly decreasing case of the model was investigated in [V 1985] where action-angle variables were also described.

11. The LSG model was stated in [IK 1981 a, b]. In [T 1982] the rapidly decreasing case of the model was shown to be completely integrable and action-angle variables were reported.

12. In [T 1982] and [V 1985] it was shown that the action-angle variables for the LLL and LSG models coincide with their analogues for the LL and SG models, and it was pointed out that this fact is due the coincidence of the associated r-matrices.

13. The scheme for constructing the local Hamiltonians for lattice models outlined above was developed in [IK 1982 a, b]. We note that the Hamiltonian defined in [IK 1982 a, b] and considered in [T 1982] differs from the one discussed in § 5.

14. Our derivation of the lattice analogue of the LL model was based on the construction of the matrix $L_n(\lambda)$ of the auxiliary linear problem which satisfies the fundamental Poisson brackets. We believe this to be the most fruitful principle for deriving integrable lattice analogues of continuous models.

References

[A 1979] Adler, M.: On a trace functional for formal pseudodifferential operators and symplectic structure of the Korteweg-de Vries type equations. Invent. Math. *50*, 219–248 (1979)

[BFT 1986] Buslaev, V. S., Faddeev, L. D., Takhtajan, L. A.: Scattering theory for the Korteweg-de Vries equation and its Hamiltonian interpretation. Physica *D 18*, 255–266 (1986)

[CK 1973] Case, K. M., Kac, M.: A discrete version of the inverse scattering problem. J. Math. Phys. *14*, 594–603 (1973)

[E 1981] Eilenberger, G.: Solitons. Mathematical method for Physicists. Berlin, Springer 1981

[F 1974 a] Flaschka, H.: The Toda lattice. I. Existence of integrals. Phys. Rev. *B 9*, 1924–1925 (1974)

[F 1974 b] Flaschka, H.: On the Toda lattice. II. Inverse scattering solution. Prog. Theor. Phys. *51*, 703–716 (1974)

[FM 1976] Flaschka, H., McLaughlin, D.: Canonically conjugate variables for the Korteweg-de Vries equation and the Toda lattice with periodic boundary conditions. Prog. Theor. Phys. *55*, 438–456 (1976)

[IK 1981 a] Izergin, A. G., Korepin, V. E.: The lattice sine-Gordon model. Vestn. Leningr. Univ. Ser. Fiz. Khim. *22*, 84–87 (1981) [Russian]

[IK 1981 b] Izergin, A. G., Korepin, V. E.: The lattice quantum Sine-Gordon model. Lett. Math. Phys. *5*, 199–205 (1981)

[IK 1982a] Isergin, A. G., Korepin, V. E.: Lattice regularizations of two-dimensional quantum field models. In: Problems in quantum field theory and statistical physics. 3. Zap. Nauchn. Semin. LOMI *120*, 75-91 (1982) [Russian]

[IK 1982b] Isergin, A. G., Korepin, V. E.: Lattice versions of quantum field theory models in two dimensions. Nucl. Phys. *B205*, 401-413 (1982)

[K 1978] Krichever, I. M.: Algebraic curves and nonlinear difference equations. Usp. Mat. Nauk *33* (4), 215-216 (1978) [Russian]

[M 1974a] Manakov, S. V.: Complete integrability and stochastization of discrete dynamical systems. Zh. Exp. Teor. Fiz. *67*, 543-555 (1974) [Russian]; English transl. in Sov. Phys. JETP *40*, 269-274 (1975)

[M 1974b] Manakov, S. V.: Inverse scattering method as applied to problems of the physics of waves in nonlinear media. Candidate thesis, Chernogolovka 1974 [Russian]

[S 1982] Sklyanin, E. K.: Algebraic structures connected with the Yang-Baxter equation. Funk. Anal. Prilož. *16* (4), 27-34 (1982) [Russian]; English transl. in Funct. Anal. Appl. *16*, 263-270 (1982)

[T 1981] Toda, M.: Theory of non-linear lattices. Springer series in solid state physics 20. Berlin-Heidelberg-New York, Springer 1981

[T 1982] Tarasov, V. O.: Classical version of the lattice sine-Gordon model. In: Problems in quantum field theory and statistical physics. 3. Zap. Nauchn. Semin. LOMI *120*, 173-187 (1982) [Russian]

[TF 1979] Takhtajan, L. A., Faddeev, L. D.: The quantum inverse problem method and the XYZ Heisenberg model. Usp. Mat. Nauk *34* (5), 13-63 (1979) [Russian]; English transl. in Russian Math. Surveys *34* (5), 11-68 (1979)

[V 1985] Volkov, A. Yu.: A discrete version of the Landau-Lifshitz equation. In: Problems in quantum field theory and statistical physics. 5. Zap. Nauchn. Semin. LOMI *145*, 62-71 (1985) [Russian]

[Z 1973] Zakharov, V. E.: On stochastization of one-dimensional chains of nonlinear oscillators. Zh. Exp. Teor. Fiz. *65*, 219-225 (1973) [Russian]; English transl. in Sov. Phys. JETP *38*, 108-110 (1974)

[ZMNP 1980] Zakharov, V. E., Manakov, S. V., Novikov, S. P., Pitaievski, L. P.: Theory of Solitons. The Inverse Problem Method. Moscow, Nauka 1980 [Russian]; English transl.: New York, Plenum 1984