Spanners of Additively Weighted Point Sets*

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Abstract. We study the problem of computing geometric spanners for (additively) weighted point sets. A weighted point set is a set of pairs (p, r) where p is a point in the plane and r is a real number. The distance between two points (p_i, r_i) and (p_j, r_j) is defined as $|p_i p_j| - r_i - r_j$. We show that in the case where all r_i are positive numbers and $|p_i p_j| \geq r_i + r_j$ for all i, j (in which case the points can be seen as non-intersecting disks in the plane), a variant of the Yao graph is a $(1 + \epsilon)$-spanner that has a linear number of edges. We also show that the Additively Weighted Delaunay graph (the face-dual of the Additively Weighted Voronoi diagram) has constant spanning ratio. The straight line embedding of the Additively Weighted Delaunay graph may not be a plane graph. Given the Additively Weighted Delaunay graph, we show how to compute a plane embedding with a constant spanning ratio in $O(n \log n)$ time.[1]

1 Introduction

Let G be a complete weighted graph where edges have positive weight. Given two vertices u, v of G, we denote by $\delta_G(u, v)$ the length of a shortest path in G between u and v. A spanning subgraph H of G is a *t-spanner* of G if $\delta_H(u, v) \leq t\delta_G(u, v)$ for all pair of vertices u and v. The smallest t having this property is called the *spanning ratio* of the graph H with respect to G. Thus, a graph with spanning ratio t approximates the $\binom{n}{2}$ distances between the vertices of G within a factor of t. Let P be a set of n points in the plane. A *geometric graph* with vertex set P is an undirected graph whose edges are line segments that are weighted by their length. The problem of constructing t-spanners of geometric graphs with $O(n)$ edges for any given point set has been studied extensively; see the book by Narasimhan and Smid [2] for an overview.

In this paper, we address the problem of computing geometric spanners with additive constraints on the points. More precisely, we define a weighted point set as a set of pairs (p, r) where p is a point in the plane and r is a real number. The distance between two points (p_i, r_i) and (p_j, r_j) is defined as $|p_i p_j| - r_i - r_j$. The problem we address is to compute a spanner of a complete graph on a weighted point set. To the best of our knowledge, the problem of constructing a geometric spanner in this context has not been previously addressed. We show how the Yao

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[1] Due to space constraints, some proofs have been omitted. All missing proofs can be found in the technical report version of this paper [1].

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graph can be adapted to compute a $(1+\epsilon)$-spanner in the case where all r_i are positive real numbers and $|p_i p_j| \geq r_i + r_j$ for all i, j (in which case the points can be seen as non-intersecting disks in the plane). In the same case, we also show how the Additively Weighted Delaunay graph (the face-dual of the Additively Weighted Voronoi diagram) provides a plane spanner that has the same spanning ratio as the Delaunay graph of a set of points. Since $|p_i p_j| < r_i + r_j$ implies that the distance is negative, we believe that the restriction $|p_i p_j| \geq r_i + r_j$ is reasonable because the t-spanner problem does not make sense when there are negative distances.

2 Related Work

Well known examples of geometric t-spanners include the Yao graph [3], θ-graphs [4], the Delaunay graph [5], and the Well-Separated Pair Decomposition [6]. Let $\theta < \pi/4$ be an angle such that $2\pi/\theta = k$, where k is an integer. The Yao graph with angle θ is defined as follows. For every point p, partition the plane into k cones $C_{p,1}, \ldots, C_{p,k}$ of angle θ and apex p. Then, there is an oriented edge from p to q if and only if q is the closest point to p in some cone $C_{p,i}$. For Yao graphs [3], the spanning ratio is at most $1/(\cos\theta - \sin\theta)$ provided that $\theta < \pi/4$. For θ-graphs, the spanning ratio is at most $1/(1 - 2\sin\frac{\theta}{2})$ provided that $\theta < \pi/3$ [4].

Given a set of points in the plane, there is an edge between p and q in the Delaunay graph if and only if there is an empty circle with p and q on its boundary [5]. The spanning ratio of the Delaunay triangulation is at most 2.42 [5]. The *Voronoi diagram* [7] of a finite set of points P is a partition of the plane into $|P|$ regions such that each region contains exactly those points having the same nearest neighbor in P. The points in P are also called *sites*. It is well known that the Voronoi diagram of a set of points is the face dual of the Delaunay graph of that set of points [7], i.e. two points have adjacent Voronoi regions if and only if they share an edge in the Delaunay graph.

3 Definitions and Notation

Definition 1. *A set $P = \{(p_1, r_1), \ldots, (p_n, r_n)\}$ of ordered pairs, where each p_i is a point in the plane and each r_i is a real number, is called a* weighted point set*. The notation $p_i \in P$ means that there exists an ordered pair (p_i, r_i) such that $(p_i, r_i) \in P$. The* additive distance *from a point $p \notin P$ in the plane to a point $p_i \in P$, noted $d(p, p_i)$, is defined as $|pp_i| - r_i$, where $|pp_i|$ is the Euclidean distance from p to p_i. The* additive distance *between two points $p_i, p_j \in P$, noted $d(p_i, p_j)$, is defined as $|p_i p_j| - r_i - r_j$, where $|p_i p_j|$ is the Euclidean distance from p_i to p_j.*

The problem we address in this paper is the following:

Problem 1. Let P be a weighted point set and let $K(P)$ be the complete weighted graph with vertex set P and edges weighted by the additive distance between

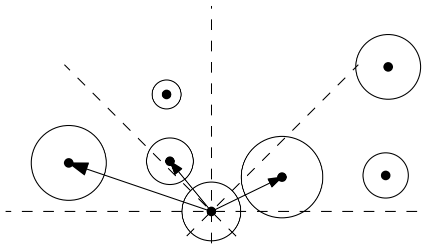

Fig. 1. A straightforward generalization of the Yao graph

their endpoints. Compute a t-spanner with $O(n)$ edges of $K(P)$ for a fixed constant $t > 1$.

Notice that in the case where all r_i are positive numbers, the pairs (p_i, r_i) can be viewed as disks D_i in the plane. If, for all i, j we also have $d(p_i, p_j) \geq 0$, then the disks are disjoint. In that case, the distance $d(D_i, D_j) = d(p_i, p_j) = |p_i p_j| - r_i - r_j$ is also equal to $\min\{|q_i q_j| : q_i \in D_i \text{ and } q_j \in D_j\}$, where the notation $q_i \in D_i$ means $|p_i q_i| \leq r_i$. To compute a spanner of an additively weighted point set is then equivalent to computing a spanner of a set of disks in the plane. **From now to the end of this paper, it is assumed that all r_i are positive numbers and $d(p_i, p_j) \geq 0$ for all i, j.** If $\mathcal{D}$ is a set of disks in the plane, then a *spanner* of $\mathcal{D}$ is a spanner of the complete graph whose vertex set is $\mathcal{D}$ and whose edges (D_i, D_j) are given weights equal to $d(D_i, D_j)$.

Notice also that the additive distance may not be a metric since the triangle inequality does not necessarily hold. Although this may seem counter-intuitive, this makes sense in some networks, since a direct communication is not always easier than routing through a common neighbor. For example, in wireless networks, the amount of energy that is needed to transmit a message is a power of the Euclidean distance between the sender and the receiver. Therefore, using several small hops can be more energy efficient than a direct communication over one long-distance link.

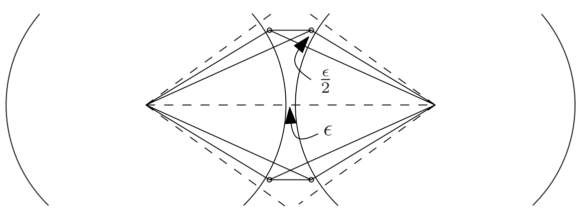

Fig. 2. The straightforward generalization of the Yao graph does not have constant spanning ratio

Figure 1 shows how the Yao graph can be generalized using the additive distance: every node keeps an outgoing edge with the closest disk that intersects each cone. However, this graph is not a spanner. Figure 2 shows how to construct an example with four disks that has an arbitrarily large spanning ratio. Nonetheless, in Section 4, we see that a minor adjustment to the Yao graph can be made in order to compute a $(1+\epsilon)$-spanner of a set of disjoint disks that has $O(n)$ edges.

The Delaunay graph in the additively weighted setting is computable in time $O(n \log n)$ [8]. To the best of our knowledge, its spanning properties have not been previously studied. In Section 6, we show that it is a spanner and that its spanning ratio is the same as that of the standard Delaunay graph.

4 The Additively Weighted Yao Graph

As we saw in the previous section, a straightforward generalization of the Yao graph fails to provide a graph with bounded spanning ratio. In this section, we show how a few subtle modifications to the construction, provide an approach to build a $(1+\epsilon)$-spanner. We define the modified Yao construction below.

Definition 2. *Let $\mathcal{D}$ be a finite set of disjoint disks and $\theta \leq 0.228$ be an angle such that $2\pi/\theta = k$, where k is an integer. The* Yao$(\theta, \mathcal{D})$ *graph is defined as follows. For every disk $D = (p, r)$, partition the plane into k cones $C_{p,1}, \ldots, C_{p,k}$ of angle θ and apex p. A disk* blocks *a cone $C_{p,i}$ provided that the disk intersects both rays of $C_{p,i}$. Let $F \in \mathcal{D}$ be a disk different from D with center in $C_{p,j}$. Add an edge from D to F in* Yao$(\theta, \mathcal{D})$ *if and only if one of the two following conditions is met:*

1. *among all blocking disks that have their center in $C_{p,j}$, F is the one that is the closest to D;*
2. *among all disks that have their center in $C_{p,j}$ and are at a distance of at least r from D, F is the one that is the closest to D.*

Notice that there are two main changes. Within each cone, we now add potentially two edges as opposed to only one edge in the case of unweighted points. Next, in the second condition to add an edge, we do not add an edge to the closest disk within a cone but to the closest disk whose distance is at least r from the disk centered at the apex with radius r. We now prove that these two modifications imply that the resulting graph is a $(1+\epsilon)$-spanner.

Lemma 1. *Let p_1, p_2, p_3 such that the angle $\angle p_3p_1p_2 = \alpha \leq \theta < \pi/4$ and $|p_1p_3| \leq |p_1p_2|$. Then $|p_2p_3| \leq |p_1p_2| - (\cos\theta - \sin\theta)|p_1p_3|$.*

Theorem 1. *Let $\mathcal{D}$ be a finite set of disjoint disks and $\theta \leq 0.228$. Then $Y(\theta, \mathcal{D})$ is a t-spanner of $\mathcal{D}$, where $t = 1/(\cos 2\theta - \sin 2\theta - 2\sin(\theta/2))$.*

Proof: We proceed by induction on the rank of the weighted distances between the pairs of disks D_1 and D_2.

Base case: The disks D_1 and D_2 form a closest pair. In that case, the edge (D_1, D_2) is in Yao$(\theta, \mathcal{D})$. To see this, let $r_1 \leq r_2$. If D_2 is blocking the cone centered at p_1 that contains it, then it is in Yao$(\theta, \mathcal{D})$ by Case 1 of Definition 2. Otherwise, then it is at distance at least r_1 from D_1 and therefore it is in Yao$(\theta, \mathcal{D})$ by Case 2 of Definition 2.

Induction case: Let $D_1 = (p_1, r_1)$ and $D_2 = (p_2, r_2)$. Without loss of generality, $r_1 \leq r_2$. If the edge (D_1, D_2) is in Yao$(\theta, \mathcal{D})$, then there is nothing to prove. Otherwise, there are two cases to consider depending on whether or not the shortest path from D_1 to D_2 in the complete graph on $\mathcal{D}$ is the edge (D_1, D_2). If the shortest path is not the edge (D_1, D_2), then all edges on the shortest path must have length less than $d(D_1, D_2)$. By applying the induction hypothesis on each of those edges, we conclude that the distance from D_1 to D_2 in Yao$(\theta, \mathcal{D})$ is at most t times the length of the shortest path D_1 to D_2 in the complete graph on $\mathcal{D}$, as required.

We now consider the case when the edge (D_1, D_2) 1) is not in Yao$(\theta, \mathcal{D})$ and 2) is the shortest path from D_1 to D_2 in the complete graph. Observe that the conjunction of those two facts imply that the disk D_2 does not block the cone whose apex is p_1 and contains p_2: If D_2 was blocking the cone, then since (D_1, D_2) is not an edge in Yao$(\theta, \mathcal{D})$, there must be a disk D_3 that is also blocking the cone and is closer to D_1 than D_2. However, this implies that the shortest path from D_1 to D_2 in the complete graph is not the edge (D_1, D_2) (see Figure 3).

The conjunction of the following three facts:

1. $r_1 \leq r_2$;
2. $\theta \leq 0.228 < \sin^{-1}(1/3)$ and
3. D_2 does not block the cone,

imply that $d(D_1, D_2) > r_1$. Since (D_1, D_2) is not an edge, there is another disk whose distance is at least r that is closer to D_1. Let $D_3 = (p_3, r_3)$ be the closest disk to D_1 such that p_3 is in the same θ-cone with apex at p_1 as p_2 and $d(D_1, D_3) \geq r_1$. By definition, the edge (D_1, D_3) is in Yao$(\theta, \mathcal{D})$. Observe that $d(D_2, D_3) < d(D_1, D_2)$. To see this, let $a := d(D_1, D_2) - r_1$. We have that

$$d(D_2, D_3) \leq a + 4r_1 \sin(\theta/2) \leq a + 4r_1 \sin(0.114) < a + r_1 = d(D_1, D_2).$$

Let p_1' be the point of D_1 that is the closest to D_3, p_1'' be the point of D_1 that is

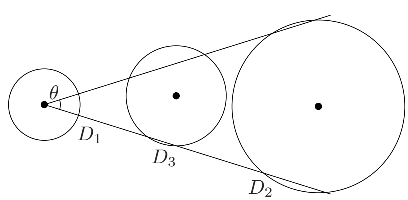

Fig. 3. If D_2 blocks the cone but the edge (D_1, D_2) is not in Yao$(\theta, \mathcal{D})$, then there exists D_3 such that $d(D_1, D_3) + d(D_3, D_2) < d(D_1, D_2)$

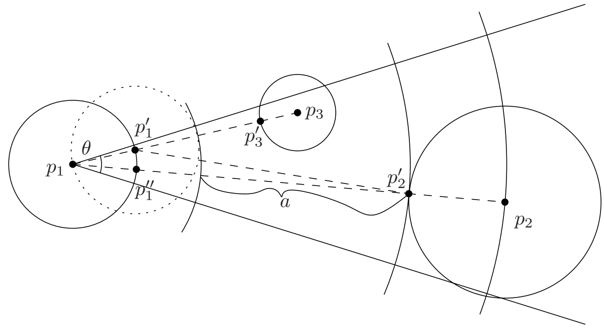

Fig. 4. Illustration of the proof of Theorem 1

the closest to D_2, p_2' be the point of D_2 that is the closest to D_1, and p_3' be the point of D_3 that is the closest to D_1 (see Figure 4). Notice that $|p_1'p_3'| \leq |p_1'p_2'|$ and that since $d(D_1, D_2) \geq d(D_1, D_3) \geq r_1$, then the angle $\angle p_2'p_1'p_3'$ is at most $2\theta < \pi/4$. Therefore, we can apply Lemma 1 to conclude that

$$|p_2'p_3'| \leq |p_1'p_2'| - (\cos 2\theta - \sin 2\theta)|p_1'p_3'|,$$

which implies that

$$d(D_2, D_3) \leq d(D_1, D_2) + |p_1'p_1''| - (\cos 2\theta - \sin 2\theta)d(D_1, D_3).$$

Also, since $|p_1'p_1''| \leq 2\sin(\theta/2)r_1 \leq 2\sin(\theta/2)d(D_1, D_3)$, we have

$$d(D_2, D_3) \leq d(D_1, D_2) - (\cos 2\theta - \sin 2\theta - 2\sin(\theta/2))d(D_1, D_3).$$

Finally, since $d(D_2, D_3) < d(D_1, D_2)$, the induction hypothesis tells us that Yao$(\theta, \mathcal{D})$ contains a path from D_2 to D_3 whose length is at most $td(D_2, D_3)$. This means that the distance from D_1 to D_2 in Yao$(\theta, \mathcal{D})$ is at most

$$d(D_1, D_3) + td(D_2, D_3) \leq d(D_1, D_3) + t(d(D_1, D_2) - \frac{1}{t}d(D_1, D_3)) = td(D_1, D_2).$$

Using Maple, we verified that the value 0.228 is an upper bound on the values of θ such that $t > 0$. □

Corollary 1. *For any $\epsilon > 0$ and any set $\mathcal{D}$ of n disjoint disks, it is possible to compute a $(1 + \epsilon)$-spanner of $\mathcal{D}$ that has $O(n)$ edges.*

*P*roof: The bound on the number of edges comes from the fact that each cone contains at most two edges, and the stretch factor of $1 + \epsilon$ comes from the fact that $\lim_{\theta \to 0} 1/(\cos 2\theta - \sin 2\theta - 2\sin(\theta/2)) = 1$. □

5 Quotient Graphs and Quotient Spanners

The main idea in the remainder of this paper is the following: we show how to compute a set of points from each D_i such that the (standard) Delaunay graph of those points is *equivalent* to the Additively Weighted Delaunay graph. By choosing the appropriate equivalence relation as well as the appropriate point set, we can then show that the spanning ratio of the Additively Weighted Delaunay graph is bounded by the spanning ratio of the standard Delaunay graph. The reduction of one graph to another is done by means of a quotient:

Definition 3. *Let P_1 and P_2 be non-empty sets of points in the plane. The* distance *between P_1 and P_2, denoted by $|P_1P_2|$, is defined as the minimum $|p_1p_2|$ over all pairs of points such that $p_1 \in P_1$ and $p_2 \in P_2$.*

Definition 4. *Let $G = (V, E)$ be a geometric graph and $\mathcal{V}$ be a partition of V. The* quotient graph *of G by $\mathcal{V}$, denoted $G/\mathcal{V}$, is the graph having $\mathcal{V}$ as vertices and there is an edge (U, W) (where U and W are in $\mathcal{V}$) if and only if there exists an edge $(u, w) \in E$ with $u \in U$ and $w \in W$. The weight of the edge (U, W) is equal to $|UW|$.*

If P is a (non-weighted) point set and $\mathcal{P}$ is a partition of P, then the notation $P/\mathcal{P}$ designates the quotient of the complete Euclidean graph on P by $\mathcal{P}$. If $\mathcal{S}$ is a set of pairwise disjoint sets of points in the plane such that $P \subseteq \bigcup \mathcal{S}$, then the notation $P/\mathcal{S}$ designates the quotient of the complete Euclidean graph on P by the partition of P induced by $\mathcal{S}$.

Lemma 2. *Let $G = (V, E)$ be a complete geometric graph, $\mathcal{V}$ be a partition of V and S be a t-spanner of G. Then $S/\mathcal{V}$ is a t-spanner of $G/\mathcal{V}$.*

6 The Additively Weighted Delaunay Graph

Lee and Drysdale [9] studied a variant of the Voronoi diagram called the Additively Weighted Voronoi diagram, which is defined as follows: Let P be a weighted point set. The *Additively Weighted Voronoi diagram* of P is a partition of the plane into $|P|$ regions such that each region contains exactly the points in the plane having the same closest neighbor in P according to the additive distance. In other words, the Voronoi cell of a pair (p_i, r_i) contains the points p such that $d(p, p_i)$ is minimum over all other pairs in P. The *Additively Weighted Delaunay graph* (AW-Delaunay graph) is defined as the face-dual of the Additively Weighted Voronoi diagram.

Alternatively, if all r_i are positive and for all i, j, we have $|p_ip_j| \geq r_i + r_j$, then the pairs (p_i, r_i) can be seen as disks D_i of radius r_i centered at p_i and $d(p, D_i)$ is the minimum $|pq|$ over all $q \in D_i$. For a set $\mathcal{D}$ of disks in the plane, we denote the AW-Delaunay graph computed from $\mathcal{D}$ as $\mathrm{Del}(\mathcal{D})$. When no two disks intersect, the AW-Delaunay graph is a natural generalization of the Delaunay graph of a set of points. We say that two intersecting disks A and B *properly* intersect if $|A \cap B| > 1$ (i.e. they are not tangent).

Proposition 1. *Let $\mathcal{D}$ be a set of disjoint disks in the plane, and $A, B \in \mathcal{D}$. The edge (A, B) is in* Del($\mathcal{D}$) *if and only if there is a disk C that is tangent to both A and B and does not properly intersect any other disk in $\mathcal{D}$.*

*P*roof: Suppose (A, B) is in Del($\mathcal{D}$), and let c be a point on the boundary of the Voronoi cells of A and B and r be the distance from c to A. Since c is equidistant from A and B, it is also at distance r from B. This means that the disk C centered at c is tangent to both A and B. This disk cannot properly intersect any other disk of $\mathcal{D}$, since this would contradict the fact that c is in the Voronoi cells of A and B. Similarly, if there is a disk that is tangent to both A and B but does not properly intersect any other disk of $\mathcal{D}$, then A and B are Voronoi neighbors. □

Note that the Additively Weighted Delaunay graph is not necessarily isomorphic to the Delaunay graph of the centers of the disks. When all radii are equal, however, the two graphs coincide. We now show that if $\mathcal{D}$ is a set of disks in the plane, then Del($\mathcal{D}$) is a spanner of $\mathcal{D}$. The intuition behind the proof is the following: we show the existence of a finite set of points P such that $K(P)/\mathcal{D}$ (where $K(P)$ is the complete graph with vertex set P) is isomorphic (i.e. there is a one-to-one relation between the nodes that preserves the lengths of the edges) to the complete graph on $\mathcal{D}$ and Del(P)/$\mathcal{D}$ is a subgraph of Del($\mathcal{D}$). Then, we use Lemma 2 to prove that Del(P)/$\mathcal{D}$ is a spanner of $\mathcal{D}$, which implies that Del($\mathcal{D}$) is a spanner of $\mathcal{D}$.

Definition 5. *Let A, B be disjoint disks and S a set of points such that $A \cap S = \emptyset$ and $B \cap S = \emptyset$. A set of points R* represents *S with respect to A and B if for every disk F that is tangent to both A and B, we have $F \cap S \neq \emptyset \Rightarrow F \cap R \neq \emptyset$. If $\mathcal{D}$ is a set of disjoint disks, then a set of points $\mathcal{R}$* represents *$\mathcal{D}$ if for all $A, B, C \in \mathcal{D}$, there is a subset of $\mathcal{R}$ that represents C with respect to A and B.*

Lemma 3. *Let $\mathcal{D}$ be a set of n disjoint disks. There exists a set of at most $2\binom{n}{3}$ points that represents $\mathcal{D}$.*

Lemma 4. *Let A and B be two disjoint disks and C be a disk intersecting both of them. Then there exists a disk G inside C that is tangent to both A and B.*

*P*roof: We show how to construct G. Let a, b, c and r_A, r_B, r_C respectively be the centers and radii of A, B and C. Without loss of generality, assume $|ac| - r_C \leq |bc| - r_B$. Let F be the disk centered at c and having radius $r_F = |bc| - r_B$. The disk F is tangent to B. If F is also tangent to A, then let $G = F$ and we are done. Otherwise, F is properly intersecting A. In that case, let p be the tangency point of F and B, l be the line through b and c, and G be the disk through p having its center on l and tangent to A. The result follows from the fact that G is tangent to B and inside C. □

Definition 6. *Let A and B be two disks in the plane. The* distance points *of A and B are the two ends of the shortest line segment between A and B.If $\mathcal{D}$ is a set of disjoint disks, then the set of* distance points *of $\mathcal{D}$ is the set containing the distance points of every pair of disks in $\mathcal{D}$.*

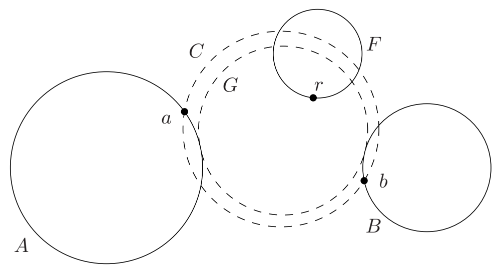

Fig. 5. Illustration of the proof of Theorem 2

Theorem 2. *Let $\mathcal{D}$ be a set of n disjoint disks. Then* $\text{Del}(\mathcal{D})$ *is a t-spanner of $\mathcal{D}$, where t is the spanning ratio of the Delaunay triangulation of a set of points.*

*P*roof: By Lemma 3, let R be a set of size at most $2\binom{n}{3}$ that represents $\mathcal{D}$, let S be the set of distance points of $\mathcal{D}$, and let $P = R \cup S$. Since $\text{Del}(P)$ is a t-spanner of P, by Lemma 2, we have $\text{Del}(P)/\mathcal{D}$ is a t-spanner of $K(P)/\mathcal{D}$, where $K(P)$ is the complete graph with vertex set P. Since P contains the distance points of $\mathcal{D}$, $K(P)/\mathcal{D}$ is isomorphic to the complete graph defined on $\mathcal{D}$. We show that each edge (A, B) of $\text{Del}(P)/\mathcal{D}$ is in $\text{Del}(\mathcal{D})$. Let (A, B) be an edge of $\text{Del}(P)/\mathcal{D}$. This means that in P, there are two points a and b with $a \in A, b \in B$ such that there is an empty circle C through a and b. By Lemma 4, C contains a disk G that is tangent to both A and B. The disk G is a witness of the presence of the edge (A, B) in $\text{Del}(\mathcal{D})$. If that was not the case, this would mean that there exists a disk $F \in \mathcal{D}$ such that $G \cap F \neq \emptyset$. By definition of R, this implies that $G \cap R \neq \emptyset$ and thus $C \cap P \neq \emptyset$, which contradicts the fact that C is an empty circle. Therefore, the edge (A, B) is in $\text{Del}(\mathcal{D})$. Since $\text{Del}(P)/\mathcal{D}$ is a t-spanner of $\mathcal{D}$ and a subgraph of $\text{Del}(\mathcal{D})$, $\text{Del}(\mathcal{D})$ is a t-spanner of $\mathcal{D}$. □

7 Computing a Plane Embedding

Note that the embedding of the AW-Delaunay graph that consists of straight line segments between the centers of the disks is not necessarily a plane graph. However, the Voronoi diagram of a set of disks $\mathcal{D}$, denoted $\text{Vor}(\mathcal{D})$, is planar [10]. Since $\text{Del}(\mathcal{D})$ is the face-dual of $\text{Vor}(\mathcal{D})$, it is also planar. An important characteristic of the Delaunay graph of a set of points regarded as a spanner is that it is a plane graph. Therefore, a natural question is whether $\text{Del}(\mathcal{D})$ has a plane embedding that is also a spanner.

The proof of Theorem 2 suggests the existence of an algorithm allowing to compute such an embedding: compute the Delaunay triangulation of the set P that contains the distance points and the representative of $\mathcal{D}$. The graph $\text{Del}(P)$ can be regarded as a multigraph whose vertex set is $\mathcal{D}$. Then, for each pair of disks that share one or more edges, just keep the shortest of those edges. This simple algorithm allows us to compute a plane embedding of $\text{Del}(\mathcal{D})$ that is also a spanner of $\mathcal{D}$. However, its running time is $O(n^3 \log n)$.

On the other hand, it is also possible to compute in time $O(n \log n)$ a plane spanner of $\mathcal{D}$ whose spanning ratio is t^2, where t is the spanning ratio of the Delaunay graph of a set of points. Here is how to do this: First, compute Del($\mathcal{D}$). Then, let P be the set of distance points of all pairs of disks that share an edge in Del($\mathcal{D}$). Compute Del(P). Since P has size $O(n)$, this can be done in time $O(n \log n)$. Also, Del(P) is a plane graph. As in the above paragraph, the graph Del(P) can be regarded as a multigraph whose vertex set is $\mathcal{D}$. Again, for each pair of disks that share one or more edges, just keep the shortest of those edges. All that remains to explain is why the resulting graph is a t^2-spanner of $\mathcal{D}$. Let $D_1, D_2 \in \mathcal{D}$. The straight line embedding of Del($\mathcal{D}$) contains a t-spanning path between D_1 and D_2. The endpoints of the edges of that path are the distance points between the disks. In Del(P), each of those edges is approximated within a factor of t, leading to a spanning ratio of t^2. Therefore, we showed the following:

Theorem 3. *Let $\mathcal{D}$ be a set of n disjoint disks and t be the spanning ratio of the Delaunay triangulation of a set of points. In time $O(n^3 \log n)$, it is possible to compute a plane t-spanner of $\mathcal{D}$, and in time $O(n \log n)$, it is possible to compute a plane t^2-spanner of $\mathcal{D}$.*

8 Conclusion

In this paper, we showed how, given a weighted point set where weights are positive and $|p_i p_j| \geq r_i + r_j$ for all $i \neq j$, it is possible to compute a $(1+\epsilon)$-spanner of that point set that has a linear number of edges. We also showed that the Additively Weighted Delaunay graph is a t-spanner of an additively weighted point set in the same case. The constant t is the same as for the Delaunay triangulation of a point set (the best current value is 2.42 [5]). We could not see how the Well-Separated Pair Decomposition (WSPD) can be adapted to solve that problem. The first difficulty resides in the fact that it is not even clear that, given a weighted point set, a WSPD of that point set always exists. Other obvious open questions are whether our results still hold when some weights are negative or $|p_i p_j| < r_i + r_j$ for some $i \neq j$. Also, we did not verify whether our variant of the Yao graph can be computed in time $O(n \log n)$. Finally, whether or not it is possible to compute a plane embedding of Del($\mathcal{D}$) that has the same spanning ratio than the Delaunay graph of a set of points in time $O(n \log n)$ remains a open question.

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