Model of Multiple Lexicographical Programming Applied in Cervical Cancer Screening

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Abstract An important problem for the management of the screening program for cervical cancer is collecting the smears for women who live in remote areas. The issue is to plan the days in which the mobile unit will be used, and its route, such that the total cost and the testing time for all the eligible women are minimal. This paper presents a mathematical model for the Health-Economic problem and a Bellman type theorem for solving this model.

1 The Health-Economics Problem and a Mathematical Model

Screening, in Medicine, is a strategy used to identify diseases in an unsuspecting population. An important problem of the management of the screening program for cervical cancer is to take the smears for women from remote areas. For these women, a mobile unit equipped as a gynecological office is used. The unit goes in every village and the doctor takes the smears from eligible women who have been informed and invited. The unit also transports the smears to the cytological laboratory. The smears are processed and the laboratory provides the results within a maximum of given days (laboratory response time, usually equal to 21 days). From the Health Economics point of view, the problem is to plan the days in which the

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mobile unit will be used and its route such as the total cost and the testing time for all the eligible women to be, both of them, minimum.

Let m be the number of the villages. For every village $i \in \{1,...m\}$, we denote by n_i, the eligible number of women from the village i which will be tested. We know: the mean time interval for taking a smear, t_r; the maximum time that the mobile unit works every day, t_z; the total number of slides which can be read by laboratory in a day, z_l; the total number of resting slides that should be read by the laboratory besides the smears obtained by the mobile unit, z_l^0; the cost/day for the driver of the mobile unit, c_s; the cost/day for the medical doctor which is on the mobile unit, c_m; overhead/day for mobile unit, c_u; cost of fuel/ km, c_b; mean speed for the mobile unit, v; the laboratory response time, l_r; and the maximum number of the days when all the tests should be done, n_z.

The laboratory receives the slides in the evening of every day and, consequently has to give the answer in $l_r - 1$ days.

We assume that all routes to reach the villages and return to the base O are known. Let p be the number of these routes. For every route j, $j \in \{1,...,p\}$, the length, d_j, of the way and the villages for the mobile unit to pass through are known. In order to identify the affiliation of one village to one route, we introduce the following p vectors $\lambda^j = (\lambda_1^j,...,\lambda_m^j) \in \mathbf{R}^m$, $j \in \{1,...,p\}$, where $\lambda_k^j = 1$, if the route j passes through the village k, and $\lambda_k^j = 0$, if the route j does not pass through the village k.

The problem is to plan the days in which the mobile unit will be used and its route such as the total cost are minimal and, if we have several possibilities, to choose one for which the testing time for all the eligible women is also minimum.

We notice that in the literature, there is not such an approach for this problem. That's why we consider this problem as a dynamic system with finite horizon and vectorial total utility function. The mathematical model permits to obtain an algorithm which solves our problem. The number of steps of the dynamic system is chosen n_z (the maximum days when all the tests should be done). A step corresponds to a day. In each step $h \in \{1,...,n_z\}$, the stage of the system will be described by the vector of state variable $s^h \in \mathbf{N}^{m+1}$: the first component, s_1^h, gives the number of slides existing in the laboratory at the end of day h (this number is equal to the number of slides existing in the evening of the day $h-1$ minus the number of the slides which have been read in day h, plus the number of slides which have been taken in the day h); the following m components, s_i^h, $i \in \{2,...,m+1\}$, contain the number of untested women at the end of day h in the village i, respectively.

Considering that the numbers z_l, z_l^0, n_i, $i \in \{1,...,m\}$, n_z are known, the initial state of the system is described by the vector

$$s^0 = (z_l^0, n_1, ..., n_m). \tag{1}$$

Because s_1^h is the number of existing slides at the end of the day h, the maximum number of the slides which may be taken by laboratory at the end of the day $h+1$ is equal to $\max\{0, (l_r - 1) \cdot z_l - s_1^h\}$. Therefore, in the first day it may be taken only $\max\{0, (l_r - 1) \cdot z_l - z_l^0\}$ slides. In each step $h \in \{1,...,n_z\}$, the decision will be described by the decision vector $x^h \in \mathbf{N}^{m+1}$: the first component, x_1^h, indicates the number of the route done in step h (if this number is 0, in that step no movement exists); the following m components, x_i^h, $i \in \{2,...,m+1\}$, contain the number of women tested in the day h, in village i, respectively. d_j/v is the time necessary to go through the route j. Therefore the decisions set in the stage $h \in \{1,...,n_z\}$, if the system is in the state s^{h-1}, is the set $X_h(s^{h-1})$,

$$X_h(s^{h-1}) = \{0,1,...,p\} \times \tilde{X}_h, \tag{2}$$

where $\tilde{X}_h$ is the set of the solutions of the discrete system

$$\begin{cases} \sum\limits_{k=2}^{m} x_k^h \leq \max\{0, (l_r - 1)\cdot z_l - s_1^{h-1}\} \\ t_r \cdot \sum\limits_{k=2}^{m+1} \lambda_{k-1}^{x_1^h} \cdot x_k^h \leq t_z - \frac{d_{x_1^h}}{v} \\ x_k^h \leq \lambda_{k-1}^{x_1^h} \cdot s_k^{h-1} \cdot sgn(s_1^{h-1}), \ \forall k \in \{2,...,m+1\} \\ x_k^h \in \mathbf{N}, \forall k \in \{1,...,m+1\}. \end{cases} \tag{3}$$

The first inequality indicates that the number of slides taken in day h can not be greater than the number of slides which can be given to the laboratory in the evening. In the second inequality, the l.h.s. term gives the time necessary to take the slides and the r.h.s term gives the available time in a day minus the time spent on the route. In the third inequality, the l.h.s term gives the number of slides planed to be taken from village k in day h, which can not be greater then $\max\{0$, the number of slides remained to be taken in village $k\}$. The relation four indicates that the number of slides has to be a natural number.

The function $f_C : \{1,...,n_z\} \to \mathbf{R}$ describes the cost for each day. Thus the cost of day h is

$$f_C(x^h) = sgn x_1^h \cdot (c_u + c_s + c_m + c_b \cdot d_{x_1^h}), \tag{4}$$

where sgn denotes the function given by $sgn x = 0$, if $x = 0$, $sgn x = 1$, if $x > 0$ and $sgn x = -1$, if $x < 0$.
We remark that the cost is 0, if no movement is done; else it is equal with the sum of the costs.

The function $f_T : \{1,...,n_z\} \to \mathbf{R}$ indicates if in the day h, smears have been taken. Thus

$$f_T(h) = sgn x_1^h. \tag{5}$$

For all $h \in \{1,...,n_z\}$, the dynamic equations are

$$s_1^h = max\{0, s_1^{h-1} - z_l\} + \sum_{i=2}^{m+1} x_i^h, \quad s_i^h = s_i^{h-1} - x_i^h, \ \forall\, i \in \{2,...,m+1\}. \tag{6}$$

For all $h \in \{1,...,n_z\}$, the stationary equations are

$$s^h \in S_h = \{0, 1, ..., (l_r - 1) \cdot z_l\} \times \{0, 1, ..., n_1\} \times ... \times \{0, 1, ..., n_m\}, \tag{7}$$

and

$$x^h \in \{0, 1, ..., p\} \times \tilde{X}^h. \tag{8}$$

The total utility function is additive, having the value equal to the sum of the values of partial utility effect functions. By denoting this function with F, $F = (F_1, F_2) : \{0, 1, ..., n_z\} \to \mathbf{R}^2$, we have $F(0) = (0,0)$ and $F(h) = F(h-1) + (f_C(h), f_T(h))^T, \forall\, h \in \{1, 2, ..., n_z\}$.

From practical point of view, our purpose is to obtain a plan of taking the smears such that the function F_1 to be minimum and, if we have possibilities to choose which one assures the minimum for F_2, too. Therefore we obtain the following type of dynamic programming problem:

$$\begin{cases} \left(\sum_{h=1}^{n_z} f_T(x^h),\, \sum_{h=1}^{n_z} f_C(x^h)\right) \to lex - min \\ s_1^h = max\{0, s_1^{h-1} - z_l\} + \sum\limits_{i=2}^{m+1} x_i^h, \forall h \in \{1, ..., n_z\}, \\ s_i^h = s_i^{h-1} - x_i^h, \forall\, i \in \{2, ..., m+1\}, \forall h \in \{1, ..., n_z\}, \\ s^0 \text{ given}, \quad s^h \in S_h, \quad \text{and} \quad x^h \in X_h(s^{h-1}), \forall h \in \{1, ..., n_z\}, \end{cases} \tag{9}$$

where S_h is given by by (7) and $X_h(s^{h-1})$ by (2) and (3).

By analogy with the definition of lexicographic optimality used in the general context of vectorial programming problem (see [2]) we call this type of problem as *lexicographic dynamic programming problem.*

Remark 1. The subject of dynamic programming problem, when the total utility function is a vectorial function, is discussed in [4]. In [5] fundamental dynamic programming recursive equations are extended to the multi-criteria framework. In that paper, a more detailed procedure for a general recursive solution scheme for the multi-criteria discrete mathematical programming problem is developed. A short note about multi-criteria dynamic programming problem is given in [6]. Recently, multi-criteria dynamic programming is extended for solving variously practical problem. This implies some sort of generalization of Belman's theorem . In [3], an application in Pharmacoeconomics is given. In our paper, we show how the problem (9) can be solved using dynamic programming. But firstly we have to give a generalization of Belman's theorem.

2 Belman's Theorem for Lexicographical Dynamic Programming

Let be a discrete finite stages decision problem, with n stages, with the static equations

$$\begin{cases} s^0 \text{ given}, \\ s^h \in S_h,\ h \in \{1,...,n\}, \\ x^h \in X_h(s^{h-1}),\ h \in H = \{1,...,n\}, \end{cases} \tag{10}$$

and the dynamic equations

$$s^h = g_h(s^{h-1}, x^h),\ h \in H, \tag{11}$$

with s^0 the initial state of the system. S_h denotes the set of the states of system in the stage h and $X_h(s^{h-1})$ denotes the set of the decisions which may be taken in the stage h, if the system is in the state s^{h-1}.

A sequence $(x^1,...,x^n)$, where $x^h \in X_j(s^{h-1})$, for every $h \in \{1,...,n\}$, is called a policy of the system. The set of all the policies of the system will be denoted by Pol. In each stage $h \in H$, if we take the decision $x^h \in X_h(s^{h-1})$, the obtained utility is denoted by $f_h(s^{h-1}, x^h)$. It is a vector in $\mathbf{R}^p$, where $p \in \mathbf{N}$, $p \geq 1$. The total utility is given by the function $F = (F_1,...,F_p) : Pol \to \mathbf{R}^p$.

Analogously to the classical dynamic programming, for the discrete finite dynamic system with n stages, having the static equation (10) and dynamic equation (11), we build the sets

$$\hat{S}_n := S_n, \quad \hat{S}_{h-1} = \{s \in S_{h-1} \,|\, \exists x \in X_h(s) \text{ such that } g_h(s,x) \in \hat{S}_h\}, \tag{12}$$

for $h = n$, $h = n-1,...,h = 1$. Again, for $h = n$, $h = n-1,...,h = 1$ and for each $s \in \hat{S}_{h-1}$ we build the set

$$\hat{X}_h(s) = \{x \in X_h(s) \,|\, g_h(s,x) \in \hat{S}_h\}. \tag{13}$$

Using the new notations, the lex-min dynamic problem can be rewritten as:

$$(\text{DLP}) \quad \begin{cases} F(x^1,...,x^n) \to lex-\min \\ s^k = g_k(s^{k-1}, x^k),\ k \in \{1,...,n\}, \\ s^0 \text{ given}, \\ s^k \in \hat{S}_k,\ k \in \{1,...,n\}, \\ x^k \in \hat{X}_k(s^{k-1}),\ k \in \{1,...,n\}. \end{cases} \tag{14}$$

If $p = 1$, a policy $x \in Pol$ is called optimal, if there is no other policy $y \in Pol$ such that $F(y) < F(x)$. For $p = 1$, an optimal policy can be find using classical Bellman's theorem. For every $h \in \{1,...,n\}$, let's consider the problem

$$(\text{DLPM}_h) \quad \begin{cases} F_h(s^{h-1}, x^h, x^{h+1},...,x^n) \to \min \\ s^k = g_k(s^{k-1}, x^k),\ k \in \{h,...,n\}, \\ s^{h-1} \text{ given}, \\ s^k \in \hat{S}_k,\ k \in \{h,...,n\}, \\ x^k \in \hat{X}_k(s^{k-1}),\ k \in \{h,...,n\}, \end{cases} \tag{15}$$

where F_h denotes the total utility function if the process begins only at the stage h, the system being in the state s^{h-1}. For all $h \in \{1,...,n-1\}$, let us denote by $Pol_h(s^{h-1})$ the set of the policies of the above problems.

Theorem 1. *(Bellman's theorem [1]). A policy $x = (x^{h-1}, x^h, ..., x^n) \in Pol_{h-1}$ is an optimal policy of the problem $(DLPM_{h-1})$ only if $(x^h, ..., x^n)$ is an optimal policy of the problem $(DLPM_h)$.*

For our problem the classical Bellman's theorem does not work because our function is a vectorial one and not a scalar function. Therefore we have to give a generalization of it.

We say that a policy $x \in Pol$ is *lexicographically minimal* if there is no $y \in Pol$ such that $F(y) <_{lex} F(x)$, where $<_{lex}$ denotes the lexicographical ordering.

We remember that if $u = (u_1, ..., u_p)$ and $v = (v_1, ..., v_p)$ are two points in $\mathbf{R}^p$, then we set:

$$u <_{lex} v, \quad \text{if} \quad \begin{array}{l} \text{there is } i \in \{1,...,p\} \text{ such that } u_i < v_i \text{ and,} \\ \text{if } i > 1, \text{ then } u_j = v_j,\ \forall\, j \in \{1,...,i-1\}. \end{array} \tag{16}$$

We call *lex-min dynamic problem*, the problem of determining a lexicographically minimal policy.

Definition 1. The total utility function is said to be lexicographic prospective increasing separable if there are $n-1$ vectorial functions $\alpha_i : \mathbf{R}^p \times \mathbf{R}^p \to \mathbf{R}^p$, $i \in \{1,...,n-1\}$, such that

$$\begin{aligned} &F(x^1, ..., x^n) \\ &\quad = \alpha_1(f_1(s^0, x^1), \alpha_2(f_2(s^1, x^2), \alpha_3(...\alpha_{n-2}(f_{n-2}(\\ &\qquad s^{n-1}, x^{n-2}), \alpha_{n-1}(f_{n-1}(s^{n-2}, x^{n-1}), f_n(s^{n-1}, x^n)))...))), \end{aligned} \tag{17}$$

for all $(x^1, ..., x^n) \in Pol$, and if for all $i \in \{1,...,n-1\}$, the function α_i is lexicographic increasing in the second argument:

$$\alpha_i(u,v) <_{lex} \alpha_i(u,v'), \text{ for all } (u,v),\ (u,v') \in \mathbf{R}^p \times \mathbf{R}^p \text{ with } v \le v'. \tag{18}$$

It is easy to see that if

$$F(x) = \sum_{j=1}^{n} f_h(s^{h-1}, x^h), \text{ for all } x \in Pol, \tag{19}$$

then F is lexicographic prospective increasing separable.

For every $h \in \{1,...,n\}$, by $\varphi_h : (\mathbf{R}^m \times \mathbf{R}^q)^{n+1-h} \to \mathbf{R}^p$ we denote a continuously function which satisfied the condition:

i) if $h \in \{1,...,n-1\}$, then

$$\begin{aligned} &\varphi_h(s^{h-1}, x^h, \ldots, s^{n-1}, x^n) \\ &= \alpha_h(f_h(s^{h-1}, x^h),\ \alpha_{h+1}(f_{h+1}, (...(\alpha_{n-1}(f_{n-1}(s^{n-2}, x^{n-1}),\ f_n(s^{n-1}, x^n)))...))), \end{aligned} \tag{20}$$

for all $(s^{h-1},x^h,...,s^{n-1},x^n) \in \hat{S}_{h-1} \times \hat{X}_h(s^{h-1}) \times ... \times \hat{S}_{n-1} \times \hat{X}_n(s^{n-1})$;
ii) if $h = n$, then

$$\varphi_n(s^{n-1},x^n) = f_n(s^{n-1},x^n), \text{ for all } (s^{n-1},x^n) \in \hat{S}_{n-1} \times \hat{X}_n(s^{n-1}). \tag{21}$$

We remark that

$$\varphi_h(s^{h-1},x^h,...,s^{n-1},x^n) = \alpha_h(f_h(s^{h-1},x^h), \varphi_{h+1}(s^h,x^{h+1},...,s^{n-1},x^n)), \tag{22}$$

for all $(s^{h-1},x^h,...,s^{n-1},x^n) \in \hat{S}_{h-1} \times \hat{X}_h(s^{h-1}) \times ... \times \hat{S}_{n-1} \times \hat{X}_n(s^{n-1})$. Also, for every $h \in \{1,...,n\}$, we consider the problem

$$\begin{cases} \varphi_h(s^{h-1},x^h,s^h,x^{h+1},...,s^{n-1},x^n) \rightarrow lex-\min \\ s^k = g_k(s^{k-1},x^k),\ k \in \{h,...,n\}, \\ s^{h-1} \text{ given}, \\ s^k \in \hat{S}_k,\ k \in \{h,...,n\}, \\ x^k \in \hat{X}_h(s^{k-1}),\ k \in \{h,...,n\}. \end{cases} \tag{23}$$

This problem could be rewritten as

$$(\mathrm{DLPM}_h) \quad \begin{cases} \alpha_h(f_h(s^{h-1},x^h), \varphi_{h+1}(s^h,x^{h+1},...,s^{n-1},x^n)) \rightarrow lex-\min \\ s^k = g_k(s^{k-1},x^k),\ k \in \{h,...,n\}, \\ s^{h-1} \text{ given}, \\ s^k \in \hat{S}_k,\ k \in \{h,...,n\}, \\ x^k \in \hat{X}_h(s^{k-1}),\ k \in \{h,...,n\}. \end{cases} \tag{24}$$

For all $h \in \{1,...,n-1\}$, let us denote by $Pol_h(s^{h-1})$ the set of the policies of (24).

Theorem 2. *If the total utility function F is lexicographic prospective increasing separable, then the policy $(x^{h-1},x^h,...,x^n) \in Pol_{h-1}$ is a lexicographically minimal policy of the problem $(DLPM_{h-1})$ only if $(x^h,...,x^n)$ is a lexicographically minimal policy of the problem $(DLPM_h)$.*

Proof. Let $(x^{h-1},x^h,...,x^n) \in Pol_{h-1}$ be a lexicographically minimal policy of the problem (DLPM$_{h-1}$). If we suppose that $(x^h,...,x^n)$ is not a lexicographically minimal policy of the problem (DLPM$_h$), then there is $(y^h,...,y^n) \in Pol_h$ such that

$$\varphi_h(s^{h-1},y^h,...,s^{n-1}y^n) <_{lex} \varphi_h(s^{h-1},x^h,...,s^{n-1}x^n). \tag{25}$$

As $(y^h,...,y^n) \in Pol_h$ and $(x^{h-1},x^h,...,x^n) \in Pol_{h-1}$, obviously we have

$$(x^{h-1},y^h,...,y^n) \in Pol_{h-1}.$$

The monotony of the function α_{h-1} implies

$$\begin{aligned} &\alpha_{h-1}(f_{h-1}(s^{h-2},x^{h-1}), \varphi_h(s_{h-1},y^h,...,s_{n-1},y^n) <_{lex} \\ &\alpha_{h-1}(f_{h-1}(s^{h-2},x^{h-1}, \varphi_h(s_{h-1},x^h,...,s_{n-1},x^n). \end{aligned} \tag{26}$$

This contradicts the hypotheses that $(x^{h-1}, x^h, \ldots, x^n) \in Pol_{h-1}$ is a lexicographically minimal policy of the problem (DLPM$_{h-1}$). $\square$

3 Practical Approach and Conclusions

Let s_i be the number of the routes which connect the base O with a village $i \in \{1,\ldots,m\}$, and d^i_j, $j \in \{1,\ldots,m\}$, their lengths. If $\min\{d^i_j/v \,|\, j \in \{1,\ldots,s_i\}\} + t_r \leq t_z$, then the medical problem has no solution because the time t_z is not enough for the mobile unit to go to village i, to take at least one smears and to come back. In the following we consider that $\min\{d^i_j/v \,|\, j \in \{1,\ldots,s_i\}\} + t_r > t_z$, is true for all $i \in \{1,\ldots,m\}$.

In the same way that a classical dynamic programming problem can be solved using Bellman's theorem, it is possible to solve the problem (9) using Theorem 2. First we take G_{n_z+1} equal to the null function and $\hat{S}_n = \{(s_1^{n_z}, 0, \ldots, 0) \,|\, s_1^{n_z} \in \{0, 1, \ldots, (l_r - 1)\cdot z_l - z_l^0\}\}$. Then, setting $k = n_z$, $k = n_z - 1$,...,$k = 1$, we solve, for each $s^{k-1} \in \hat{S}_{k-1}$, the problem

$$(P_k)\ \{F_k(s^{k-1}, x^k) + G_{k+1}(g_k(s^{k-1}, x^k)) | x^k \in \hat{X}_k(s^{k-1})\} \rightarrow \text{lex-min}, \tag{27}$$

where $F_k(s^{k-1}, x^k) = (\sum_{h=k}^{n_z} f_T(x^h),\ \sum_{h=k}^{n_z} f_C(x^h))$,
$g_k(s^{k-1}, x^k) = (\max\{0, s_1^{h-1} - z_l\} + \sum\limits_{i=2}^{m+1} x_i^h,\ s_2^{h-1} - x_2^h, \ldots, s_{m+1}^{h-1} - x_{m+1}^h)$, and
$G_k(s^{k-1}) = \text{lex-min}\{F_k(s^{k-1}, x^k) + G_{k+1}(g_k(s^{k-1}, x^k)) | x^k \in \hat{X}_k(s^{k-1})\}$. An optimal policy of the problem (9) is $(\hat{u}^1(x^0), \ldots, \hat{u}^{n_z}(x^{n_z-1}))$, where $\hat{u}^k(x^{k-1})$ denotes a lex-min solution of (P_k), $k \in \{1, \ldots, k_{n_z}\}$.

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References

1. Bellman, R.: Dynamic Programming. Princeton University Press, Prinseton, New Jersey (1957)
2. Ehrgott, M.: Multicriteria Optimization. Springer, Berlin-Heidelberg (2005)
3. Lupşa, L.: Use of dynamic programming for the supply of a pharmacy. In: E. Popoviciu (ed.) Proc. of the Tiberiu Popoviciu Intinerant Seminar of Functional Equations, Approximation and Convexity, pp. 237–239. SRIMA Publishing House, Cluj-Napoca (2001)
4. Mitten, L.G.: Synthesis of optimal multistage processes. J. Optim. Theory Appl. **12**, 610–619 (1964)
5. Villarreal, B., Karwan, M.H.: Quasimonotonicity, regularity and duality for nonlinear systems of partial differential equations. J. Optim. Theory Appl. **38**(1), 43–69 (1982)
6. Yu, P.L.: Multiple-Criteria Decision Making. Concepts, Techniques, and Extensions. Plenum Press, New York and London (1985)