Matching for Graphs of Bounded Degree

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Abstract. We show that there exists a matching with $\frac{4m}{5k+3}$ edges in a graph of degree k and m edges.

Keywords: Matching, lower bound.

1 Introduction

Matching is an extensively studied topic. The question we want to address here is the size of a maximum matching in a graph. Earlier researchers studied the problem of the existence of a perfect matching (i.e. a matching of size $n/2$ with n vertices in a graph). Petersen [5] showed that a bridgeless cubic graph has a perfect matching. König [3] showed that there exists a perfect matching in any k-regular bipartite graph. Tutte [6] characterizes when a graph has a perfect matching. For graphs without a perfect matching the size of a maximum matching is studied. Nishizeki and Baybars [4] showed that any 3-connected planar graphs has a matching of size at least $(n+4)/3$ for $n \geq 22$. Biedl et al. [1] raised the question whether a bound better than $m/(2k-1)$ can be obtained for the size of a maximum matching in a graph of m edges and degree k. Recently Feng et al. [2] showed a lower bound of $2m/(3k-1)$ (for $k \geq 3$) for this problem.

In this paper we give a lower bound of a $4m/(5k+3)$-size matching for a graph of m edges and degree k. Here the degree of a graph is the maximum degree of any vertex of the graph.

2 The Bound

Assume that a maximum matching M is obtained for the input graph with m edges. Vertices incident to an edge in the matching are saturated vertices. Vertices not incident to any edge in the matching are unsaturated vertices. Without loss of generality we can assume that the input graph is connected. If the graph has no unsaturated vertex then for each edge e in the matching we can have at most $1+(2k-2)$ edges incident to e. Among which 1 is the edge in the matching and $2k-2$ are nonmatching edges. However, when we are counting this way the nonmatching edges are counted twice, once from each vertex they are incident to. Therefore we have that $|M|(2+(2k-2)) \geq 2m$, i.e. $|M| \geq m/k$. Therefore we assume that there is at least one unsaturated vertex.

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Since any alternating path starting from an unsaturated vertex a cannot end at another unsaturated vertex b (for otherwise an augmenting path exists), the only possibility for an alternating path to start from an unsaturated vertex and end at an unsaturated vertex is that the starting unsaturated vertex and the ending unsaturated vertex are the same vertex. If this happens and say such an alternating path is $a, a_1, a_2, ..., a_r, a$, then both a_1 and a_r have only one neighbor which is unsaturated (which is a in this case) for otherwise an augmenting path exists. In this case we remove edge (a_r, a) and therefore all alternating paths starting from an unsaturated vertex do not end at an unsaturated vertex. We have thus removed no more than $|M|$ nonmatching edges from the input graph. Let (a, b) be an edge in the matching, If a or b has a neighboring vertex which is unsaturated we say that (a, b) is an outer matching edge. Otherwise we say that (a, b) is an inner matching edge. Let M_1 be the set of outer matching edges and M_2 be the set of inner matching edges. For each matching edge in M_1 we will call the vertex of the edge which has an unsaturated neighbor the outer vertex and the vertex which has no unsaturated neighbors the inner vertex. There are no more than $|M_1|(k-1)$ nonmatching edges incident to outer vertices. Any nonmatching edge cannot be incident to two inner vertices for otherwise an augmenting path exists. We put nonmatching edges in three sets N_1, N_2, N_3. N_1 is the set of edges incident to an outer vertex. N_2 is the set of edges incident to an inner vertex and a vertex in M_2. N_3 is the set of edges incident to vertices in M_2 only. Then

$$|N_1| + |N_2| + |N_3| + |M| \geq m - |M|.$$

The number of nonmatching edges incident to vertices in M_2 is no more than $|M_2|(2k-2)$. Among which there are $2|N_3|$ edges incident to vertices in M_2 only and $|N_2|$ edges incident to vertices both in M_1 and M_2. Thus we have $|N_3| + |N_2|/2 \leq |M_2|(k-1)$. Also we have that $|N_1| \leq |M_1|(k-1)$ and $|M_1| + |M_2| = |M|$. Therefore we have that

$$|M_1|(k+1) + |M_2|(k+1) + |N_2|/2 \geq |N_1| + |N_2| + |N_3| + 2|M| \geq m.$$

We have that $|M_1|(k-1) \geq |N_2|$ (these edges cannot be incident to outer vertices). Now for each edge (a, b) in M_2 it cannot happen that one edge in N_2 is incident to a and an inner vertex c and another edge in N_2 is incident to b and another inner vertex d, for otherwise an augmenting path exists if $c \neq d$. If in this case $c = d$ then both a and b cannot be incident to other edges in N_2 besides (a, c) and (b, d). Therefore we partition M_2 into M_{21}, M_{22} and M_{23} with edges in M_{21} having only one vertex of the edge incident to edges in N_2 and edges in M_{22} having both vertices incident to edges in N_2 (i.e., an inner vertex is the only inner vertex neighbor of bother vertices of the edge in M_{22}) and edges in M_{23} has no vertex incident to edges in N_2 (i.e. they are incident to edges in N_3 only).

We have that $|M_{21}|(k-1) + 2|M_{22}| \geq |N_2|$. Therefore we have that $|M_2|(k-1) \geq |N_2|$ if $k \geq 3$. Now we have that $(1/2)|M_1|(k-1) + (1/2)|M_2|(k-1) \geq |N_2|$. Thus we have that

$$|M_1|(k+1)+|M_2|(k+1)+(1/4)|M_1|(k-1)+(1/4)|M_2|(k-1)=((5k+3)/4)|M|\geq m.$$

Therefore we obtain that $|M| \geq 4m/(5k+3)$ if $k \geq 3$.

For $k = 1, 2$, $|M| \geq 4m/(5k+3)$ obviously holds. Therefore we have

Theorem 1. For a graph of m edges and degree k there exists a matching of size $4m/(5k+3)$.

A reviewer of this paper posted the following question: Is there an upper bound of the form cm/k for some c? The answer to this question is negative. Consider the graph of vertex set $\{v_0, v_1, v_2, ..., v_k\}$ (assume that k is even) and edge set $\{(v_0, v_1), (v_0, v_2), ..., (v_0, v_k), (v_1, v_2), (v_3, v_4), ..., (v_{k-1}, v_k)\}$. This graph has $m = 3k/2$. The maximum matching has size $k/2 = m/3$. Thus cm/k cannot be used as an upper bound.

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