7 Risk Measurement with Spectral Capital Allocation

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Spectral risk measures provide the framework to formulate the risk aversion of a firm specifically for each quantile of the loss distribution of a portfolio. More precisely the risk aversion is codified in a weight function, weighting each quantile. Since the basic coherent building blocks of spectral risk measures are expected shortfall measures, the most intuitive approach comes from combinations of those. For investment decisions the marginal risk or the capital allocation is the sensible approach. Since spectral risk measures are coherent there exists also a sensible capital allocation based on the notion of derivatives or more in the light of the coherency approach as an expectation under a generalized maximal scenario.

7.1 Introduction

Portfolio modeling has two main objectives: the quantification of portfolio risk, which is usually expressed as the economic capital of the portfolio, and its allocation to subportfolios and individual transactions. The standard approach in credit portfolio modeling is to define the economic capital in terms of a quantile of the portfolio loss distribution

$$q_\alpha(L) \;=\; F_L^{-1}(\alpha).$$

The capital charge of an individual transaction is traditionally based on a covariance technique and called volatility contribution. We refer to Bluhm et al. (2002) and Crouhy et al. (2000) for a survey on credit portfolio modeling and capital allocation.

Since the work by Artzner et al (1997) coherent risk measures are discussed intensively in finance and risk management. More recent is the question of a more coherent capital allocation. Especially the use of expected shortfall allocation as an allocation rule is recommend in Overbeck (2000),Denault

(2001), Bluhm et al. (2002), Kurth and Tasche (2003) and Kalkbrener et al. (2004).

Expected shortfall measures

$$\mathsf{E}\,S_\alpha(L) \;=\; \frac{1}{1-\alpha}\int_\alpha^1 q_u(L)du$$

are the building blocks of more general coherent risk measures, the spectral risk measure ρ. These are convex mixtures of expected shortfall measures. They can be represented by their spectral measure μ through

$$\rho = \rho_\mu = \int_0^t \mathsf{E}\,S_\alpha(1-\alpha)\mu(da) \tag{7.1}$$

or as a weighted sum of quantiles with $w(\alpha) = \mu([0,\alpha])$,

$$\rho = \rho_\mu = \rho_w = \int_0^1 q_\alpha(\cdot)w(\alpha)d\alpha. \tag{7.2}$$

In this paper we apply the allocation rules associated with a spectral risk measure to a credit portfolio and point out, which consequences to risk management the choice of the weight function w, the spectral measure μ or the measure

$$\widetilde{\mu} \stackrel{\text{def}}{=} (1-\alpha)\mu(d\alpha),$$

which we call mixing measure and thought to be the most easily one to calibrate and implement. The theoretical basis of the approach can be found in the basic papers Kalkbrener (2002), Kalkbrener et al (2004) and the explicit application to spectral capital allocation is provided by Overbeck (2005). We will first present the theoretical foundation of the proposed risk and allocation measures and then discuss general impact of the choice of the weight or mixing function and finally exhibits the differences on a concrete credit portfolio example.

7.2 Review of Coherent Risk Measures and Allocation

7.2.1 Coherent Risk Measures

It is well-known that the following four conditions define a coherent risk measure, Artzner et al (1997, 1999), Delbaen (2000).

Formally, a risk measure is nothing else as a positive real valued function r defined on the set of random variable (potential losses) V. The number $r(X)$ denotes the risk in portfolio X. r is called coherent if it obeys the following 4 rules.

- Subadditivity (Diversification)

$$r(X+Y) < r(X) + r(Y)$$

- Positive homogenous (Scaling)

$$r(aX) = ar(X), a > 0$$

- Monotone

$$r(X) < r(Y) \text{ if } X < Y \text{ (almost surely)}$$

- Translation property

$$r(X+a) = r(X) - a$$

Convex analysis gives already that a sub-additive positive homogenous function r can be point wise written as the maximal value of all linear functions which are below r (Delbaen (2000), Kalkbrener (2002), Kalkbrener et al (2004)). For risk measures this means that the first two axioms above lead to the following representation

$$r(X) \;=\; \max\{l(X) \mid l < r,\; l \;\text{ linear function }\} \tag{7.3}$$

The risk measure evaluate at a loss variable X takes the same value as the largest value of all linear function which lies below r on V evaluated on X.

Conceptually, this is similar to the gradient of the function r evaluated at the point X or as the best linear approximation of r which coincides with r at the point X. We will later see that this intuition gives rise to a sensible capital allocation.

A typical linear function for random variable is the expectation operator. Hence the basic result by Artzner et al (1997), Delbaen (2000)

$$r(X) \;=\; \sup\{E_Q[X] \mid Q \in \mathcal{Q}\} \tag{7.4}$$

$\mathcal{Q}, = \mathcal{Q}_r$, a suitable set of probability measures of absolutely continuous probability measures $Q << P$ with density dQ/dP , is similar to the representation (7.3).

The set $\mathcal{Q}$ is called the generalized scenarios associated with r. If the supremum is actually taken at some probability measure, this probability measure or its density with respect to P is called the generalized scenario associated with r. These approach also fits into the intuitive feature of risk measurement, namely scenario or stress analysis. For the interpretation in terms of scenarios the formulation with probability measure is more natural, but for the axiomatic approach to capital allocation the representation (7.3) is very useful.
The currently most prominent example of a coherent risk measure is Expected Shortfall (sometimes called Conditional VaR /tail conditional expectation). It is denoted by $\mathsf{E}\,S_\alpha$ and measures the average loss above the α-quantile of the loss distribution. The associated generalized scenarios can be explained as follows:
To each loss variable Y define the scenario as the "historical" calibrated objective scenario constraint on the condition that the loss variable exceeded its quantile. The expected shortfall coincides with the largest mean loss in these scenarios. Intuitively,

$$\mathsf{E}\{L|L > q_\alpha(L)\} = \max\{E\{L|Y > q_\alpha(Y)\}|\text{ all } Y \in L_\infty\}$$

Even if generalized scenarios are defined as a supremum, in the case of Expected Shortfall we can identify the density of the maximal "scenario". For this we need the formally correct definition of Expected Shortfall at level α. The problem with the intuitive definition above is the possible positive mass at the quantile itself. The exact definition of the Expected Shortfall at level α is therefore (Acerbi and Tasche (2002), Kalkbrener et al (2004):

DEFINITION 7.1

$$\mathsf{E}\,\mathrm{S}_\alpha(L) \stackrel{\text{def}}{=} (1-\alpha)^{-1}\big(\mathsf{E}[L\,\mathbf{1}\{L > \mathrm{q}_\alpha(L)\}] + \mathrm{q}_\alpha(L)\cdot[\mathrm{P}\{L \le \mathrm{q}_\alpha(L)\} - \alpha]\big).$$

Here we take the quantile defined by

$$q_u(L) = \inf\{x|P(L \le x) \ge u\}$$

the smallest u-quantile

Since $\mathsf{E}\,S_\alpha(L) = \mathsf{E}\{Lg_\alpha(L)\}$ with the function

$$g_\alpha(Y) \stackrel{\text{def}}{=} (1-\alpha)^{-1}[\mathbf{1}\{Y > \mathrm{q}_\alpha(Y)\} + \beta_Y\mathbf{1}\{Y = \mathrm{q}_\alpha(Y)\}], \tag{7.5}$$

where β_Y is a real number and

$$\beta_Y \stackrel{\text{def}}{=} \frac{\mathrm{P}\{Y \le \mathrm{q}_\alpha(Y)\} - \alpha}{\mathrm{P}\{Y = \mathrm{q}_\alpha(Y)\}} \quad \text{if} \quad \mathrm{P}\{Y = \mathrm{q}_\alpha(Y)\} > 0.$$

the density of the associated maximal scenario turns out to be the function g_α. Note that $\mathsf{E}\,S_\alpha(Y) = \mathsf{E}\{Y \cdot g(Y)\}$ and $\mathsf{E}\,S_\alpha(X) \geq \mathsf{E}\{X \cdot g(Y)\}$ for every $X, Y \in V$.

7.2.2 Spectral Risk Measures

For the interpretation of this density function (7.5) in terms of risk aversion as outlined in Acerbi (2002), let us reformulate the expected shortfall as an integral over the quantile function, the inverse of the distribution of L. It is well-known that

$$\mathsf{E}\,S_\alpha(L) = (1-\alpha)^{-1} \int_\alpha^1 q_u(L) du.$$

The implicit risk aversion with expected shortfall is, that all quantiles below α or all losses below the α quantile have no weights, i.e. there is no risk aversion and all losses above the α-quantile have the same risk aversion. Therefore the risk aversion weight function associated with $\mathsf{E}\,S_\alpha$ turns out to be

$$w_{\mathsf{E}\,S_\alpha}(u) \;=\; (1-\alpha)^{-1}\mathbf{1}(u > \alpha). \tag{7.6}$$

From a risk management point of view there might be many other weights given to some confidence levels u. If the weight function is increasing, which is reasonable since higher losses should have larger risk aversion weight, then we arrive at spectral risk measures.

DEFINITION 7.2 *Let w be an increasing function from $[0,1]$ such that $\int_0^1 w(u)du = 1$, then the map r_w defined by*

$$r_w(L) = \int_0^1 w(u) q_u(L) du$$

is called a spectral risk measure with weight function w.

The name spectral risk measure comes from the representation

$$r_w(X) = \int_0^1 \mathsf{E}\,S_\alpha(1-\alpha)\mu_u(da) \tag{7.7}$$

$$\text{with the spectral measure } \mu([0,b]) = w(b). \tag{7.8}$$

This representation is very useful when we want to find the scenario function representing a spectral risk measure r_w.

PROPOSITION 7.1 *The density of the scenario associated with the risk measure equals*

$$L_w \stackrel{\text{def}}{=} g_w(L) \stackrel{\text{def}}{=} \int_0^1 g_\alpha(L)(1-\alpha)\mu(d\alpha). \tag{7.9}$$

Here $g_\alpha(L)$ is defined in formula (7.5). In particular

$$r_w(L) = \mathsf{E}(LL_w) \tag{7.10}$$

Proof: We have

$$\begin{aligned} r_w(L) &= \int_0^1 \mathsf{E}\, S_\alpha(L)(1-\alpha)\mu(d\alpha) \\ &= \int_0^1 \mathsf{E}(LL_\alpha)(1-\alpha)\mu(d\alpha) \\ &= \int_0^1 \max[\mathsf{E}\{Lg_\alpha(Y)\}|Y \in L_\infty](1-\alpha)\mu(d\alpha) \\ &\geq \max[\int_0^1 \mathsf{E}\{L \int_0^1 g_\alpha(Y)(1-\alpha)\mu(d\alpha)\}|Y \in L_\infty] \\ &= \max[\mathsf{E}\{Lg_w(Y)\}|\forall Y \in L_\infty] \\ &\geq \mathsf{E}\{Lg_w(L)\} \end{aligned}$$

Hence

$$r_w(L) = \max[\mathsf{E}\{Lg_w(Y)\}|\forall Y \in L_\infty] = \mathsf{E}\{Lg_w(L)\}$$

$\diamond$.

7.2.3 Coherent Allocation Measures

Starting with the representation (7.3) one can now find for each Y a linear function $h_Y = h_Y^r$ which satisfies

$$r(Y) = h_Y(Y) \text{ and } h_Y(X) \leq r(X), \forall X. \tag{7.11}$$

A "diversifying" capital allocation associated with r is given by

$$\Lambda_r(X, Y) = h_Y(X). \tag{7.12}$$

The function Λ_r is then *linear* in the first variable and *diversifying* in the sense that the capital allocated to a portfolio X is always bounded by the capital of X viewed as its own subportfolio

$$\Lambda(X, Y) \leq \Lambda(X, X). \tag{7.13}$$

$\Lambda(X, X)$ can be called the standalone capital or risk measure of X. In general we have the following two theorems: A linear and diversifying capital allocation Λ, which is continuous, i.e. $\lim_{\epsilon \to 0} \Lambda(X, Y + \epsilon X) = \Lambda(X, Y)\ \forall X$, at a portfolio Y, is uniquely determined by its associated risk measure, i.e. the diagonal values of Λ. More specifically, given the portfolio Y then the capital allocated to a subportfolio X of Y is the derivative of the associated risk measure ρ at Y in the direction of X.

PROPOSITION 7.2 *Let Λ be a linear, diversifying capital allocation. If Λ is continuous at $Y \in V$ then for all $X \in V$*

$$\Lambda(X, Y) = \lim_{\epsilon \to 0} \frac{r(Y + \epsilon X) - \rho(Y)}{\epsilon}.$$

The following theorem states the equivalence between positively homogeneous, sub-additive risk measures and linear, diversifying capital allocations.

PROPOSITION 7.3 **(a)** *If there exists a linear, diversifying capital allocation Λ with associated risk measure r, i.e. $r(X) = \Lambda(X, X)$, then r is positively homogeneous and sub-additive.*
(b) *If r is positively homogeneous and sub-additive then Λ_r as defined in (7.12) is a linear, diversifying capital allocation with associated risk measure r.*

7.2.4 Spectral Allocation Measures

Since in the case of spectral risk measures r_w the maximal linear functional in (7.11) can be identified as an integration with respect to the probability measure with density (7.9) from Theorem 1, we obtain $h_Y(X) = \mathsf{E}\{X g_w(Y)\}$ and therefore the following capital allocation

$$\Lambda_w(X, Y) = \mathsf{E}\{X g_w(Y)\} = \int_0^1 \mathsf{E}\, SC_\alpha(X, Y)(1 - \alpha)\mu(d\alpha) \quad (7.14)$$

$$= \int_0^1 \mathsf{E}\, SC_\alpha(X, Y)\widetilde{\mu}(d\alpha) \quad (7.15)$$

$$\text{where} \quad \mathsf{E}\, SC_\alpha(X, Y) = \mathsf{E}\{X g_\alpha(Y)\} \quad (7.16)$$

is the Expected Shortfall Contribution and $\widetilde{\mu}$ is defined in (7.17). Intuitively, the capital allocated to transaction or subportfolio X in a portfolio Y equals its expectation under the generalized maximal scenario associated with w.

7.3 Weight Function and Mixing Measure

One might try to base the calibration or determination of the spectral risk measure based on the spectral measure μ or the weight function w. Since the weight function w is nothing else as the distribution function of μ, there is also a 1-1 correspondence to the more intuitive mixing measure

$$\widetilde{\mu}(d\alpha) = (1-\alpha)\mu(d\alpha). \tag{7.17}$$

If we define more generally for an arbitrary measure $\widetilde{\mu}$ the functional

$$\widetilde{\rho} = \int_0^1 \mathsf{E}\, S_\alpha \widetilde{\mu}(da) \tag{7.18}$$

then $\widetilde{\rho}$ is coherent iff $\widetilde{\mu}$ is a probability measure. Since

$$\begin{aligned} 1 &= \widetilde{\mu}([0,1]) = \int_0^1 (1-u)\mu(du) \\ &= \int_0^1 \int_0^1 \mathbf{1}[u,1](v)dv\mu(du) = \int_0^1 \int_0^1 \mathbf{1}[0,v](u)\mu(du)dv \\ &= \int_0^1 w(v)dv. \end{aligned}$$

If we have now a probability measure $\widetilde{\mu}$ on $[0,1]$ the representing μ and w in (7.1,7.2) can be obtained by

$$\frac{d\mu}{d\widetilde{\mu}} = \frac{1}{1-\alpha} \tag{7.19}$$

$$w(b) = \mu([0,b]) = \int_0^b \frac{1}{1-\alpha}\widetilde{\mu}(d\alpha). \tag{7.20}$$

7.4 Risk Aversion

If we assume a discrete measure

$$\widetilde{\mu} = \sum_{i=1}^n p_i \delta_{\alpha_i} \tag{7.21}$$

then the risk aversion function w is an increasing step function with step size of $p_i/(1-\alpha_i)$ at the points α_i

$$w(b) \;=\; \sum_{\alpha_i \leq b} \frac{p_i}{1-\alpha_i}. \tag{7.22}$$

This has to be kept in mind. If we assume equal weights for the two expected shortfall at 99% and 90% then the increase in risk aversion at the first quantile 90% is $0.5/0.1 = 5$ and $0.5/0.01 = 50$. The risk aversion against losses above the 99% is therefore 11 times higher than against those between the 90% and 99% quantile. It is therefore sensible to assume quite small weights on $\mathsf{E}\,S_\alpha$ with large αs.

7.5 Implementation

There are several ways to implement a spectral contribution in a portfolio model. According to Acerbi(2002) a Monte-Carlo-based implementation of the spectral risk measure would work as follows:

Let L^n be the n-th realization of the portfolio loss. If we have generated N loss distribution scenario, let us denote by $n:N$ index of the n-th largest loss which itself is then denote by $L^{n:N}$, i.e. the indices $1:N, 2:N, .., N:N \in \mathrm{N}$ are defined by the property that

$$L^{1:N} < L^{2:N} < ... < L^{N:N}$$

The approximative spectral risk measure is then defined by

$$\sum_{n=1}^{N} L^{n:N} w(n/N) / \sum_{k=1}^{N} w(k/N)$$

Therefore a natural way to approximate the spectral contribution of another random variable L_i, which specifically might be a transaction in the portfolio represented by L or a subportfolio of L, is

$$\sum_{n=1}^{N} L_i^{n:N} \frac{w(n/N)}{\sum_{k=1}^{N} w(k/N)}, \tag{7.23}$$

where $L_i^{n,N}$ denotes the loss in transaction i in the scenario $n:N$, i.e. in the scenario where the portfolio loss was the n-th largest. It is then expected that

$$\mathsf{E}(L_i L_w) = \lim_{N\to\infty} \sum_{n=1}^{N} L_i^{n:N} \frac{w(n/N)}{\sum_{k=1}^{N} w(k/N)}.$$

As in most applications we assume that

$$L = \sum_i L_i$$

with the transaction loss variable L_i and in the example later we will actually calculate within a multi-factor Merton-type credit portfolio model.

7.5.1 Mixing Representation

Let us review the standard implementation of the expected shortfall contribution. In the setting of the previous setting we can see that for $w(u) = \frac{1}{1-\alpha}\mathbf{1}[\alpha,1](u)$ the weights for all scenarios with $\frac{n}{N} < \alpha$ is 0 and for all others it is

$$\begin{aligned} & \frac{\frac{1}{1-\alpha}}{\sum_{k=\{(\alpha)N\}}^{N} \frac{1}{1-\alpha}} \\ \cong \; & \frac{1}{(1-\alpha)N} \end{aligned}$$

(Here $[\cdot]$ denote the Gauss brackets.) Therefore the expected shortfall contribution equals

$$\frac{1}{\{(1-\alpha)N\}} \sum_{n=(\alpha N)}^{N} L_i^{n:N} \tag{7.24}$$

or more intuitively the average of the counterparty i losses in all scenarios where the portfolio losses was higher or equal than the $[\alpha N]$ largest portfolio loss.

Due to the fact that we have chosen a finite convex combination of Expected Shortfall , i.e. the mixing measure

$$\widetilde{\mu}(du) \;=\; \sum_{k=1}^{K} p_i \delta_{\alpha_i}$$

and formulae (7.24) and (7.18) we will take for a transaction L_i the approximation

$$\mathrm{SCA}(L_i, L)_{\mathrm{vec}p,\mathrm{vec}\alpha,N} \;=\; \sum_{k=1}^{K} p_i \Big[\frac{1}{\{(1-\alpha_i)N\}} \sum_{n=[\alpha_i N]}^{N} L_i^{n:N} \Big] \tag{7.25}$$

as the Spectral Capital Allocation with discrete mmixing measure μ represented by the vectors vec$p = (p_1, .., p_K)$, vec$\alpha = (\alpha_1, .., \alpha_K)$ for a Monte-Carlo-Sample of length N.

7.5.2 Density Representation

Another possibility is to rely on the approximation of the Expected Shortfall Contribution as in Kalkbrener et al (2004) and to integrate over the spectral measure μ:

$$\mathsf{E}(L_i L_w) = \lim_{N\to\infty} \int_0^1 \Big\{ \sum_{n=1}^N L_i^{n:N} \frac{w_\alpha(i/N)}{\sum_{k=1}^N w_\alpha(k/N)} (1-\alpha) \Big\} \mu(da) \quad (7.26)$$

If L has a continuous distribution than we have that

$$\begin{aligned} \mathsf{E}(L_i L_w) &= \mathsf{E}\{L_i \int_0^1 L_\alpha \mu(d\alpha)\} \\ &= \int_0^1 \mathsf{E}[L_i \mathbf{1}\{L > q_\alpha(L)\}](1-\alpha)^{-1}\mu(d\alpha) \\ &= \lim_{N\to\infty} N^{-1} \sum_{n=1}^N L_i^n \int_0^1 \mathbf{1}\{L^n > q_\alpha(L)\}(1-\alpha)^{-1}\mu(d\alpha) \quad (7.27) \end{aligned}$$

If L has not a continuous distribution we have to use the density function (7.9) and might approximate the spectral contribution by

$$\mathsf{E}(L_i L_w) \sim N^{-1} \sum_{n=1}^N L_i^n g_w(L^n). \quad (7.28)$$

The actual calculation of the density g_w in (7.28) might be quite involved. On the other hand the integration with respect to μ in (7.26) and (7.27) is also not easy. If w is a step function as in the example 1 above, then μ is a sum of weighted Dirac-measure and the implementation of spectral risk measure as in (7.23) is straightforward.

7.6 Credit Portfolio Model

In the examples below we apply the presented concepts to a standard default only type model with a normal copula based on an industry and region factor model, with 27 factors mainly based on MSCI equity indices. We assume fixed recovery and exposure-at-default. For a specification of such a model, we could refer to Bluhm et al. (2002) or other text books on credit risk modeling.

7.7 Examples

7.7.1 Weighting Scheme

Lets take 5 quantile 50%, 90%, 95%, 99%, 99.9% and the 99.98% quantile. We like now to find weighting scheme for Expected Shortfall, which still gives a nice risk aversion function. Or inversely we start with a sensible risk aversion as in (7.29) and then solve for the suitable convex combination of expected shortfall measures.

As a first step in the application of spectral risk measures one might think to give to different loss probability levels different weight. This is a straightforward extension of expected shortfall. One might view Expected Shortfall at the 99%-level view as a risk aversion which ignores losses below the 99%-quantile and all losses above the 99%-quantile have the same influence. From an investors point of view this means that only senior debts are cushioned by risk capital. One might on the other hand also be aware of losses which occur more frequently, but of course with a lower aversion than those appearing rarely.

As a concrete example one might set that losses up to the 50% confidence level should have zero weights, losses between 50% and 99% should have a weight w_0 and losses above the 99%-quantile should have a weight of $k_1 w_0$ and above the 99.9% quantile it should have a weight of $k_2 w_0$. The first tranch from 50% to 99% correspond to an investor in junior debt, and the tranch from 99% to 99.9% to a senior investor and above the 99.9% a super senior investor or the regulators are concerned. This gives a step function for w:

$$\begin{aligned} w(u) &= w_0 \mathbf{1}(0.99 > u > 0.5) + k_1 w_0 \mathbf{1}(0.999 > u > 0.99) \\ &\quad + k_2 w_0 \mathbf{1}(1 > u > 0.999) \end{aligned} \tag{7.29}$$

The parameter w_0 should be chosen such that the integral over w is still 1.

7.7.2 Concrete Example

The portfolio consists of 279 assets with total notional EUR 13.7bn and the following industry and regions breakdown:

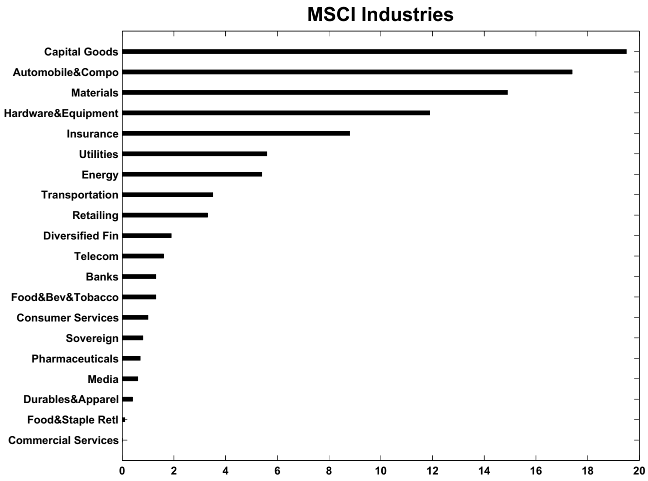

Figure 7.1. MSCI industry breakdown XFGIndustryBreakdown

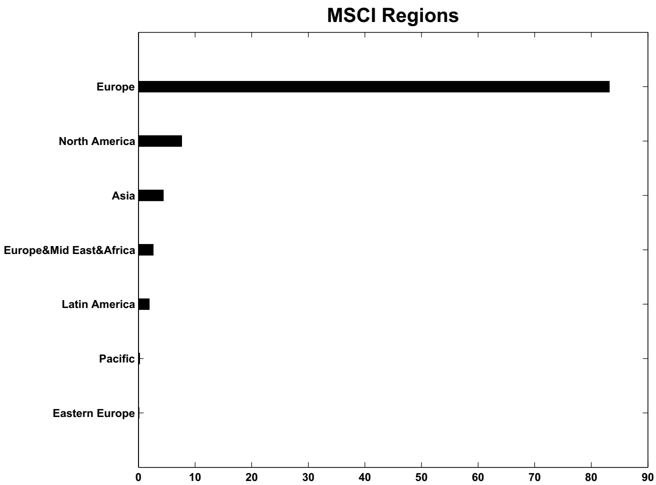

Figure 7.2. MSCI region breakdown XFGRegionsBreakdown

The portfolio correlation structure is obtained from the R^2 and the correlation structure of the industry and regional factors. The R^2 is the R^2 of the one-dimensional regression of the asset returns with respect to its composite factor, modeled as the sum of industry and country factor. The underlying factor model is based on 24 MSCI Industries and 7 MSCI Regions. The weighted average R^2 is 0.5327.

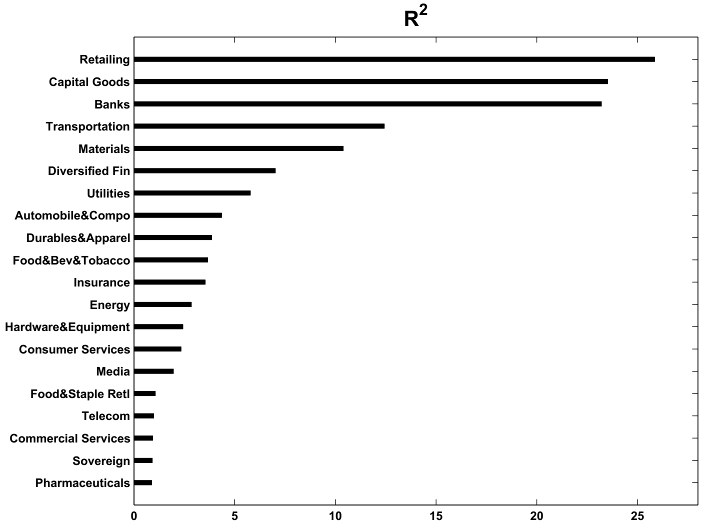

Figure 7.3. R^2 values of different MSCI industries. XFGRsquared

The risk contributions are calculated at quantiles $50\%, 90\%, 95\%, 99\%, 99.9\%$ and 99.98%.

Figure 7.4 shows the total Expected Shortfall Contributions allocated to the industries normalized with respect to automobile industry risk contributions and ordered by $\mathrm{ESC}_{99\%}$.

In order to capture all risks of the portfolio a risk measure, which combines few quantile levels, is needed. As one can see, Hardware and Materials have mainly tail exposure (largest consumption of ESC at the 99.98%-quantile), where Transportation, Diversified Finance and Sovereign have the second to fourth largest consumption of ESC at the 50%-quantile, i.e. are considerable more exposed to events happening roughly every second year as Hardware and Materials.

The spectral risk measure as a convex combination of Expected Shortfall risk measures at the following quantiles $50\%, 90\%, 95\%, 99\%, 99.9\%$ and 99.98% can capture both effects, at the tail and at the median of the loss distribution.

Four spectral risk measures are calculated. The first three are calibrated in

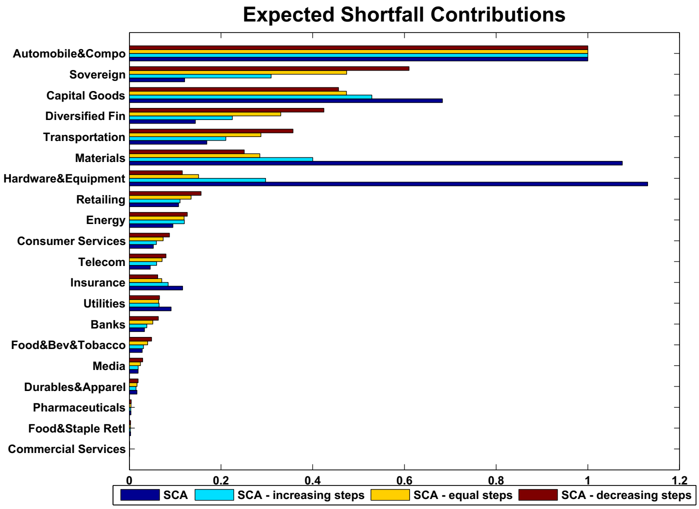

Figure 7.4. Expected shortfall contributions for different industries at different quantiles. XFGESC

terms of increase of the risk aversion function at each considered quantile as in Figure 7.5. The least conservative one is "SCA - decreasing steps" in which the risk aversion increases at each quantile by half the size it has increased at the quantile before. "SCA -equal steps" increases in risk aversion by the same amount at each quantile, "SCA -increasing steps" increases in risk aversion at each quantile by doubling the increase at each quantile. The last most conservative one is SCA - 0.1/0.1/0.1/0.15/0.15/0.4, in which the weights of $\widetilde{\mu}$ are directly set to 0.1 at the 50%, 90%, 95%- quantiles, 0.15 at the 99% and 99.9%- quantiles and 0.4 at the 99.98%-quantile as in Figure 7.6. The last one has a very steep increase in the risk aversion at the extreme quantiles.

As a comparison to the expected shortfall, the chart below shows the Spectral risk allocation allocated to industries ordered by SCA - equal steps and normalized with respect to automobile industry SCA as in Figure 7.7.

All tables so far were based on the risk allocated to the industries. Much of the displayed effects are just driven by exposure, i.e. "Automotive" is by far the largest exposure in that portfolio and all sensible risk measure should mirror this concentration. Interestingly enough the most tail emphasizing measures are the exceptions. There the largest contributors Hardware and Materials have actually less than 10% of the entire exposure.

Usually one uses as well percentage figures and risk return figures for portfolio management. On the chart "RC/TRC" the percentage of total risk (TRC)

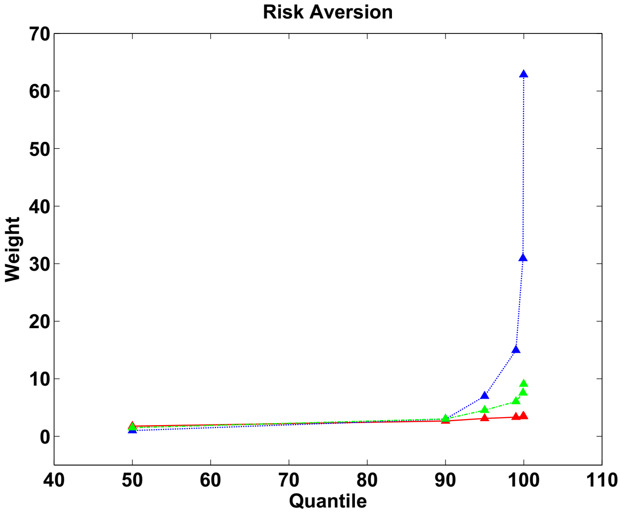

Figure 7.5. Risk aversion calculated with respect to different methods. The dotted blue, dashed-dotted and solid lines represent “SCA - decreasing steps”, “SCA - equal steps” and “SCA - incresing steps” correspondingly. XFGriskaversion

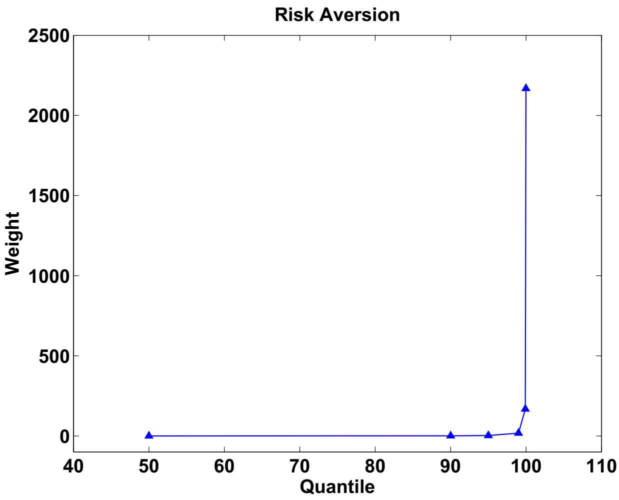

Figure 7.6. Risk aversion when the weights are directly set to 0.1 at the 50%, 90%, 95%- quantiles, 0.15 at the 99% and 99.9%- quantiles and 0.4 at the 99.98%-quantile. XFGriskaversion2

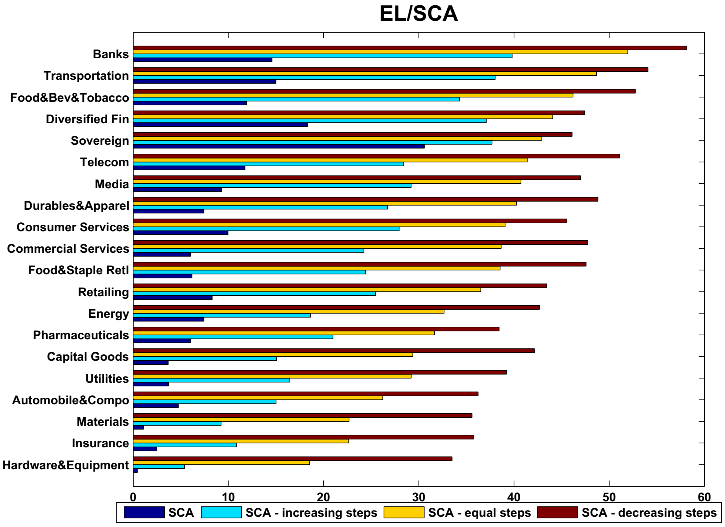

Figure 7.7. Different risk contributions with respect to different SCA methods. XFGSCA

allocated to the specific industries is displayed in Figure 7.8.

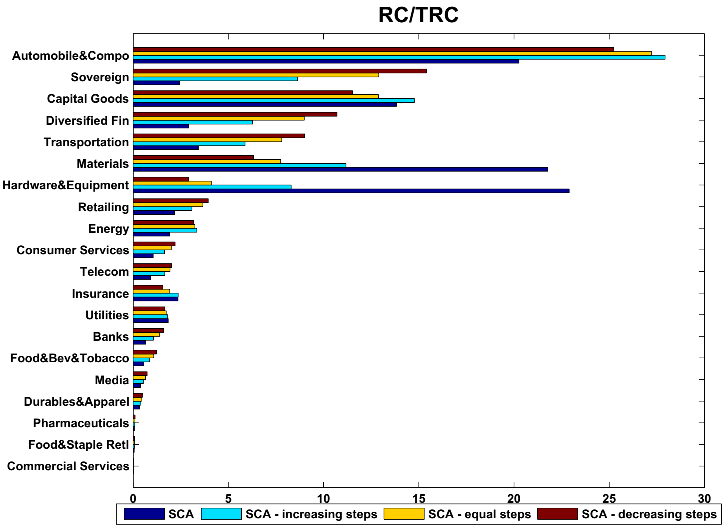

Figure 7.8. Total risk contributions with respect to different SCA methods. XFGRCTRC

For the risk management the next table showing allocated risk capital per exposure is very useful. It compares the riskiness of the industry normalized by their exposure. Intuitively it means that if you increase the exposure in "transportation" by a small amount like 100.000 Euro than the additionally capital measured by SCA-increasing steps will increase by 2.5%, i.e. by 2.5000 Euro. In that sense it gives the marginal capital rate in each industry class. Here the sovereign class is the most risky one. In that portfolio the sovereign exposure was a single transaction with a low rated country and it is therefore no surprise that "sovereign" performance worst in all risk measures.

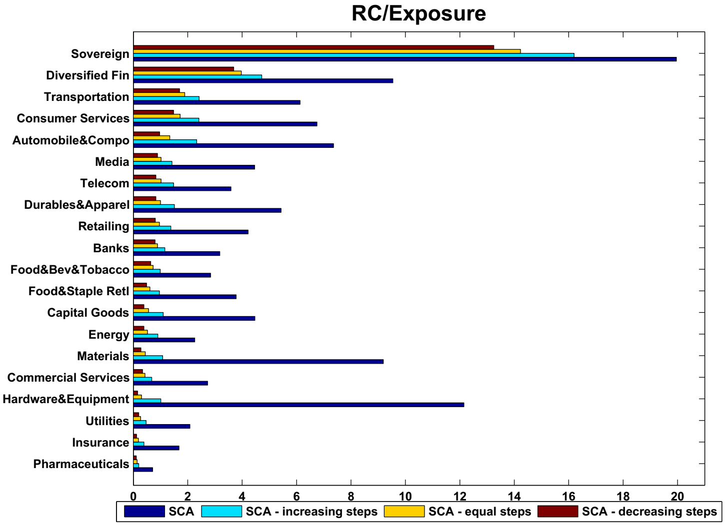

Figure 7.9. Allocated risk capital per exposure with respect to different SCA methods. XFGRCExposure

With that information one should now be in the position to judge about the possible choice of the most sensible spectral risk measure among the four presented. The measure denoted by SCA based on the weights 0.1,0.1,0.1, 0.15,0.15, 0.4, overemphasis tail risk and ignores volatility risk like the 50%-quantile. From the other three spectral risk measures, also the risk aversion function of the one with increasing steps, does emphasis too much the higher quantiles. SCA decreasing steps seems to punished counterparties with a low rating very much, it seems to a large extend expected loss driven, which can be also seen in the following table on the RAROC-type Figures 7.10. On that table "decreasing steps" does not show much dispersion. One could in summary therefore recommend SCA-equal steps.

For information purpose we have also displayed the Expected Loss/Risk Ratio for the Expected Shortfall Contribution in Figure 7.11. Here the dispersion for the ESC at the 50% quantile is even lower as for the SCA-decreasing steps.

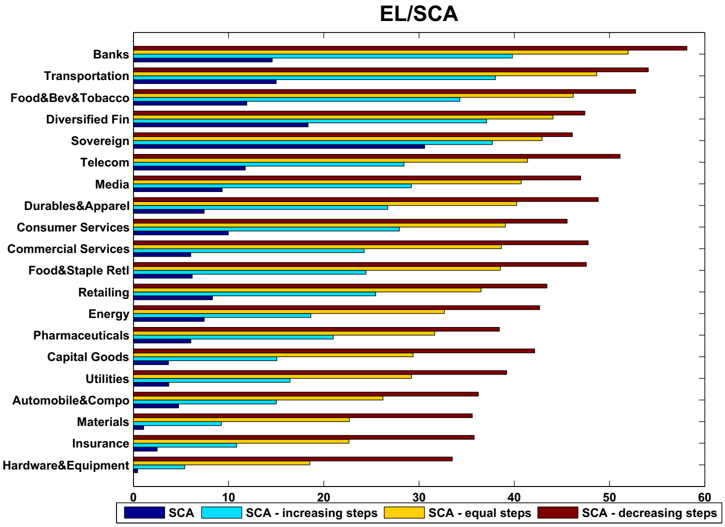

Figure 7.10. EL/SCA with respect to different SCA methods. XFGELSCA

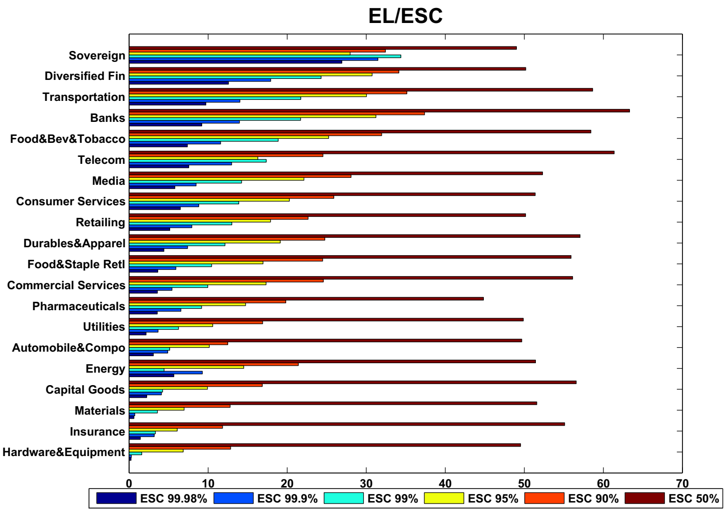

Figure 7.11. Expected Loss/Risk Ratio for the Expected Shortfall Contribution at different quantiles. XFGELESC

7.8 Summary

In order to combine different loss levels in one risk measure spectral risk measures provide a sensible tool. Weighting of the quantiles is usually be done by the risk aversion function. Starting from an implementation point of view it looks more convenient to write a spectral risk measure as a convex combination of expected shortfall measures. However one has to be careful in the effects on the risk aversion function. All this holds true and become even more important if capital allocation is considered, which finally serves as a decision tool to differentiate sub-portfolios with respect to their riskiness. We analyze an example portfolio with respect to the risk impact of the industries invested in. Our main focus are the different specification of the spectral risk measure and we argue in favour for the spectral risk measure based on a risk aversion which has the same magnitude of increase at each considered quantile, namely the 50%, 90%, 95%, 99%, 99.9%, and 99.98% quantile. This risk measure exhibits a proper balance between tail risk and more volatile risk.

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