6 Cross- and Autocorrelation in Multi-Period Credit Portfolio Models

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6.1 Introduction

For the risk assessment of credit portfolios single-period credit portfolio models are by now widely accepted and used in the practical analysis of loan respectively bonds books in the context of capital modeling. But already Finger (2000) pointed to the role of inter-period correlation in structural models and Thompson, McLeod, Teklos and Gupta (2005) strongly advocated that it is 'time for multi-period capital models'. With the emergence of structured credit products like CDOs the default-times/Gaussian-copula framework became standard for valuing and quoting liquid tranches at different maturities, Bluhm, Overbeck and Wagner (2002). Although it is known that the standard Gaussian-copula-default-times approach has questionable term structure properties the approach is quite often also used for the risk assessment by simply switching from a risk-neutral to a historical or subjective default measure.

From a pricing perspective Andersen (2006) investigates term structure effects and inter-temporal dependencies in credit portfolio loss models as these characteristics become increasingly important for new structures like forward-start CDOs. But the risk assessment is also affected by inter-temporal dependencies. For the risk analysis at different time horizons the standard framework is not really compatible with a single-period correlation structure; Morokoff (2003) highlighted the necessity for multi-period models in that case. Long-only investors in the bespoke tranche market with a risk-return and hold-to-maturity objective have built in the past CDO books with various vintage and maturity years, based only on a limited universe of underlying credits with significant overlap between the pools. A proper assessment of such a portfolio requires a consistent multi-period portfolio framework with reasonable inter-temporal dependence. Similarly, an investor with a large loan or bond book, enhanced with non-linear credit products, needs a reliable multi-period model with sensible inter-temporal properties as both bond or structured investments display different term structure characteristics.

In the following, we investigate several multi-period models, a CreditMetrics-type approach, i.e. a Markov chain Monte Carlo model with dependency introduced via a Gaussian copula, the well-known model for correlated default times, a continuous threshold model driven by time-changed correlated Wiener processes by Overbeck and Schmidt (2005), and a discrete barrier model (Finger (2000), Hull and White (2001)), also based on a driving Brownian motion. All models meet by construction the marginal default probability term structures. We then investigate the effect of a finer time discretization on the cumulative loss distribution at a given time horizon. The time-changed threshold model is invariant under this operation, whereas the credit migration approach converges to the limit of vanishing cross correlation, i.e. the correlation is 'discretized away'. Thompson et al. (2005) analyse the same problem for the discrete barrier model and observe decreasing loss volatility and tail risk. They conjecture that it converges to the limit of the 'true' portfolio loss distribution. We have similar findings but draw a different conclusion as we attribute the decreasing loss volatility to inherent features of the discrete model. Obviously, it is not congruent with a continuous-time default barrier model like the time-changed threshold model. Hence, the assumption that the time-changed model is the continuous limit of the discrete threshold model is wrong.

These findings imply that these types of credit portfolio models not only have to be calibrated to marginals, but also to a correlation structure for a given time horizon and time discretization in order to yield consistent valuation and risk assessment. We therefore turn to the problem of how to adjust the correlation structure, at least in the credit migration framework, while shortening the time steps such that the cumulative loss distribution is commensurate to a one-period setting at a given time horizon. This approach assumes that we are given a certain correlation structure for a fixed period, e.g. yearly correlations through time series estimation. We show that it suffices to compare the joint default probabilities and adapt the correlation parameter accordingly to obtain commensurate cumulative loss distribution at a given horizon.

Finally, we take a look at the autocorrelation of the different models and briefly highlight the different inter-temporal loss dependency of the models, as this plays an important part in risk-assessing books of CDOs.

6.2 The Models

6.2.1 A Markov-Chain Credit Migration Model

The first model we investigate is essentially a CreditMetrics-type approach in a multi-period setting. A Markov state (rating) $Y \in \{1, \ldots, K\}$ is assigned to each single credit risky entity i, the absorbing state K is the default state. A default probability term-structure $F_i(t)$ exists for each initial credit state i together with a sequence of migration matrices M_{t_k} that is adapted to meet the term-structure. The migration matrix M_{t_n} defines a natural discretization of Y_t, but we can subdivide or refine the discretization arbitrarily through the introduction of a matrix square root $M_t^{1/2} = M_{t/2}$ or a generator matrix Q, $M_t = \exp(tQ)$, see Bluhm et al. (2002) for more details. The discrete Markov process Y_t with time-homogeneous migration matrix does not necessarily meet a given PD-term structure i.e., $\left(M_0^k\right)_{iK} \neq F_i(t_k), k = 1, 2, 3, \ldots,$ (with K as default state). This can easily be rectified by adapting the transition matrices recursively, i.e. the default column of the first matrix is set to the term structure and the remaining entries are renormalized.

With some linear algebra the next matrix can be adjusted accordingly, and so on. These transition matrices are chained together and create a discrete credit migration process for each credit entity, $Y_{t_k}^i$, on a time grid $0 = t_0 < t_1 < t_2 < t_3 < \ldots < t_n$. As migration matrix a Rating Agency's one-year transition matrix is typically used. In the multi-firm context we add a dependency structure between different credit entities, i.e. credit migrations are coupled through a Gaussian copula function with correlation matrix Σ in each step. There is no explicit interdependence between the steps apart from the autocorrelation generated by the migrations. From each migration matrix we can now calculate migration thresholds that separate the transition buckets. For some period t_j the thresholds c_{kl,t_j} are obtained from

$$
\begin{aligned}
c_{kl,t_j} &= \Phi^{-1}\left(\sum_{n=l}^{K} M_{kn,t_j}\right), \quad \text{for } k,l = 1,\ldots,K, \text{ with } \sum_{n=l}^{K} M_{kn,t_j} \neq 0,1 \\
c_{kl,t_j} &= -\infty, \qquad \text{for} \quad k,l = 1,\ldots,K, \quad \text{with} \quad \sum_{n=l}^{K} M_{kn,t_j} = 0 \\
c_{kl,t_j} &= +\infty, \qquad \text{for} \quad k,l = 1,\ldots,K, \quad \text{with} \quad \sum_{n=l}^{K} M_{kn,t_j} = 1.
\end{aligned}
\tag{6.1}
$$

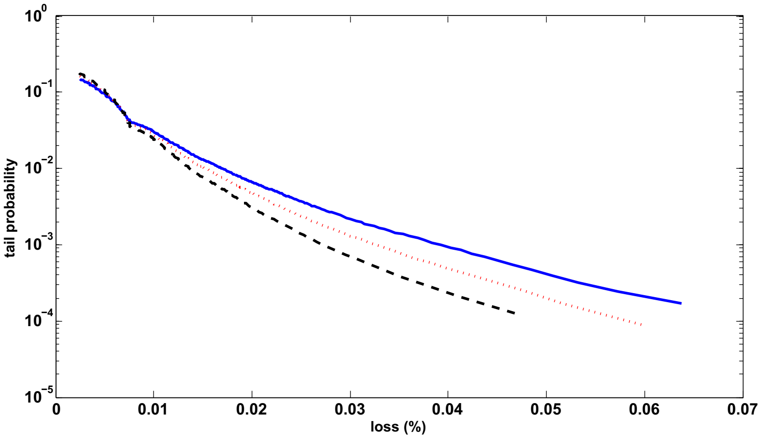

Figure 6.1. Refining Time-Discretization, migration model: annual(blue solid), semi-annual(red dotted), and quarterly(black dashed) discretization.

For each period $]t_{j-1}, t_j]$ correlated normal random variables are sampled, $\left(r_{i,t_j}\right)_{i=1,\dots,n} \sim \Phi(0,\Sigma)$, and credit i migrates from the initial state l to the final state k if

$$c_{lk-1,t_j} \leq r_{i,t_j} < c_{lk,t_j}.$$

As a remark, this type of correlated credit migration model is also the basis of the credit component in Moody's SIV Capital Model (Tabe and Rosa (2004)) and of Moody's KMV CDO Analyzer (Morokoff (2003)). MKMV's CDO Analyzer applies the migration technique to the MKMV Distance-to-Default-Indicator which is far more fine grained than usual rating classes.

But note one problem: The correlation structure of the model is not invariant under the refinement of the time discretization. Denote L the portfoli loss, then Figure 6.1 shows the tail probability $\mathrm{P}(L > x)$ for a sample portfolio with non-vanishing correlation at the one year horizon under annual, semi-annual, and quarterly discretization. For this, we have simply calculated appropriate square-roots of the migration matrices. The fatness in the tail of the loss distribution is significantly reduced for smaller migration intervals. As soon as we introduce correlation to the rating transitions a link between global correlation and discretization is generated. By this we mean that choosing the same local correlation parameter ρ for each time step, the joint arrival probability in the states m, n of two entities at time t, given they start at time 0 in states k, l

$$\mathrm{P}(Y_t^i = m, Y_t^j = n | Y_0^i = k, Y_0^j = l),$$

is a function of how fine we discretize the process, while keeping the local correlation constant. Smaller step-sizes de-correlate the processes Y_t^i and Y_t^j. This can easily be seen by the fact that for smaller step sizes the migration probabilities to the default state get smaller, but since the Gaussian copula has no tail dependence the correlation converges asymptotically to zero as we move the step size to zero. Obviously, this is an unpleasant feature when it comes to practical applications of the model, as e.g. the pricing or risk assessment of correlation sensitive product like a CDO depend then on the time discretization of the implementation.

In order to reconstitute the original correlation over a fixed time interval while halving the time step, one way is to adapt, i.e. increase, the local cross correlation. Suppose

$$\mathrm{P}(Y_1^i = K, Y_1^j = K | Y_0^i = k, Y_0^j = l)$$

is the joint default probability for one large step. Cutting the discretization in halves, the joint default probability is now

$$\begin{aligned}
&\mathrm{P}(Y_1^i = K, Y_1^j = K | Y_0^i = k, Y_0^j = l) = \\
&= \sum_{p,q} \mathrm{P}(Y_1^i = K, Y_1^j = K | Y_{1/2}^i = p, Y_{1/2}^j = q) \times \\
&\qquad \mathrm{P}(Y_{1/2}^i = p, Y_{1/2}^j = q | Y_0^i = k, Y_0^j = l).
\end{aligned} \tag{6.2}$$

Instead of trying to adjust the correlation for all pairs i, j we confine ourselves to a homogeneous state in the sense of a large pool approximation (Kalkbrener, Lotter and Overbeck (2004)). We obtain one adjustment factor and apply it to all names in the portfolio. For further discretization we simply nest the approach. Figure 6.2 shows the effect of the adjustment for an example. We use an inhomogeneous portfolio of 100 positions with exposures distributed uniformly in $[500, 1500]$, 1-year default probabilities in $[10bp, 100bp]$, and correlation between $[10\%, 30\%]$. As can be seen from the graph both loss distribution are now commensurable. From a risk perspective this degree of similarity seems sufficient, particularely if risk measures like expected shortfall are used. Further improvement can be achieved by computing adjustment factors for each rating state and for each matrix in the sequence of transition matrices (if they are different).

In case of the migration model we have so far chosen independent cross coupling mechanisms at each time step, so autocorrelation is solely introduced through the dispersion of the transition matrices. Defining a copula that couples transitions not only at one step but also between different steps

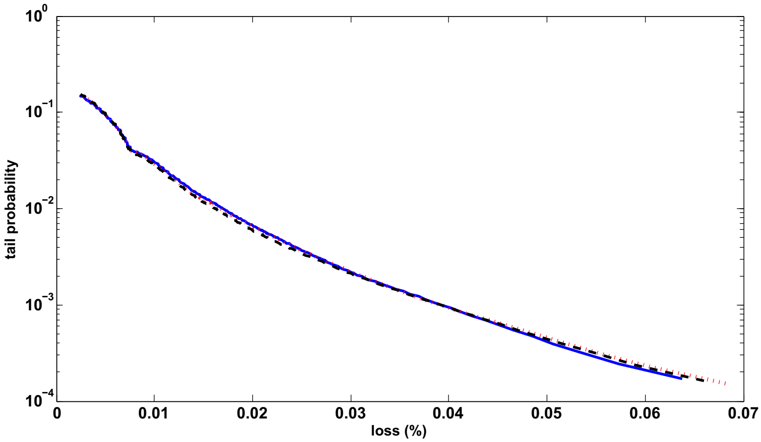

Figure 6.2. Refining Time-Discretization, migration model with adapted correlation: annual(blue solid), semi-annual(red dotted), and quarterly(black dashed) discretization.

would introduce explicit autocorrelation to the migration model (see Andersen (2006)), but we run into a heavy calibration problem, since (i) the calibration to the marginal default term structure becomes more involved, and (ii) the adjustment of the correlation structure while refining the discretization is much more difficult. If the cross dependency is formulated via a factor model we can also induce autocorrelation between the time steps by introducing an auto-regressive process for these factors, i.e. the factors Z_n that couple the transitions at each time step n are linked through $Z_{n+1} = \alpha Z_n + \sqrt{1-\alpha^2}\xi_n, \quad Z_1, \xi_n \sim \Phi(0,1)$, independent, and α is some coupling factor.

6.2.2 The Correlated-Default-Time Model

Another wide spread approach for a credit portfolio model is to generate *correlated default times* for the credit securities. This is done in analogy to a one-year-horizon asset value model by taking the credit curves of the securities as cumulative distributions of random default times and coupling these random variables by some copula function, usually the Normal copula, thus generating a multivariate dependency structure for the single default times. It is not by chance that this approach already has been used for the valuation of default baskets as the method focuses only on defaults and not on rating migrations.

From a simulation point of view, the default times approach involves much less random draws than a multi-step approach as we directly model the default times as continuous random variable. Time-consuming calculations in the default times approach could be expected in the part of the algorithm inverting the credit curves $F_i(t)$ in order to calculate default times according to the formula $\tau_i = F_i^{-1}\{\Phi(r_i)\}$. Fortunately, for practical applications the exact time when a default occurs is not relevant. Instead, the only relevant information is if an instrument defaults between two consecutive payment dates. Therefore, the copula function approach for default times can be easily discretized by calculating thresholds at each payment date $t_1 < t_2 < t_3 < \ldots < t_n$ according to

$$c_{i,t_k} \;=\; \Phi^{-1}\{F_i(t_k)\} \ ,$$

where F_i denotes the credit curve for some credit i. Clearly, one has

$$c_{i,t_1} \;<\; c_{i,t_2} \;<\; \ldots \;<\; c_{i,t_n} \ .$$

Setting $c_{i,t_0} = -\infty$, asset i defaults in period $]t_{k-1}, t_k]$ if and only if

$$c_{i,t_{k-1}} \;<\; r_i \;\leq c_{i,t_k} \ ,$$

where $(r_1, ..., r_m) \sim \Phi(0, \Sigma)$ denotes the random vector of standardized asset value log-returns with asset correlation matrix Σ. In a one-factor model setting r_i is typically represented as

$$r_i = \sqrt{\varrho}Y + \sqrt{1-\varrho}Z_i,$$

where Y, $Z_i \sim \Phi(0,1)$ are the systematic and specific risk components of the log-return r_i. Obviously, the discrete implementation of correlated default times is invariant to the refinement of the time discretization by construction.

Note further that the correlated-default-times approach with Gaussian-copula is in fact a static model. For this, we write the conditional joint default probability at different time horizons in a one-factor setting as

$$\begin{aligned} &\mathrm{P}[\tau_1 < s, \tau_2 < t | Y = y] = \\ &= \mathrm{P}[r_1 < \Phi^{-1}\{F_1(s)\}, r_2 < \Phi^{-1}\{F_2(t)\} | Y = y] \\ &= \mathrm{P}[Z_1 < \frac{\Phi^{-1}\{F_1(s)\} - \sqrt{\varrho}y}{\sqrt{1-\varrho}}, Z_2 < \frac{\Phi^{-1}\{F_2(t)\} - \sqrt{\varrho}y}{\sqrt{1-\varrho}}] \\ &= \Phi\left[\frac{\Phi^{-1}\{F_1(s)\} - \sqrt{\varrho}y}{\sqrt{1-\varrho}}\right] \Phi\left[\frac{\Phi^{-1}\{F_2(t)\} - \sqrt{\varrho}y}{\sqrt{1-\varrho}}\right]. \end{aligned} \tag{6.3}$$

The sample of the common factor Y is static for all time horizons, there is no dynamics through time.

6.2.3 A Discrete Barrier Model

Finger (2000) and Hull and White (2001) proposed a discrete multi-period barrier model on a time grid $t_0 < t_1 < \ldots < t_n$ based on correlated Brownian processes B_t^i where the default thresholds $c_i(t_k)$ are decreasing functions of time calibrated to satisfy the marginal term structure $F_i(t_k)$. Credit entity i defaults in period k if, for the first time, $B_{t_k}^i < c_i(t_k)$, i.e.

$$\tau_i = \min\left\{t_k \geq 0 : B_{t_k}^i < c_i(t_k), k = 0, \ldots, n\right\}.$$

The default barriers $c_i(t_k)$ are to be calibrated to match $F_i(t_k)$ such that

$$F_i(t_k) = \mathrm{P}(\tau_i < t_k).$$

Denote $\delta_k = t_k - t_{k-1}$, then from

$$\mathrm{P}\left\{B_{t_1}^i < c_i(t_1)\right\} = F_i(t_1)$$

follows that

$$c_i(t_1) = \sqrt{\delta_1}\Phi^{-1}\left\{F_i(t_1)\right\}.$$

The successive thresholds $c_i(t_k)$ are then found by solving

$$\begin{aligned} & F_i(t_k) - F_i(t_{k-1}) = \\ & = \mathrm{P}\left\{B_{t_1}^i > c_i(t_1) \cap \cdots \cap B_{t_{k-1}}^i > c_i(t_{k-1}) \cap B_{t_k}^i < c_i(t_k)\right\} \\ & = \int_{c_i(t_{k-1})}^{\infty} f_i(t_{k-1}, u)\Phi\left[\frac{c_i(t_k) - u}{\sqrt{\delta_k}}\right] du, \end{aligned}$$

where $f_i(t_k, x)$ is the density of $B_{t_k}^i$ given $B_{t_j}^i > c_i(t_j)$ for all $j < k$:

$$\begin{aligned} f_i(t_1, x) &= \frac{1}{\sqrt{2\pi\delta_1}} \exp\left(-\frac{x^2}{2\delta_1}\right) \\ f_i(t_k, x) &= \int_{c_i(t_{k-1})}^{\infty} f_i(t_{k-1}, u)\frac{1}{\sqrt{2\pi\delta_k}} \exp\left\{-\frac{(x-u)^2}{2\delta_k}\right\} du. \end{aligned}$$

Hence, the calibration of the default thresholds is an iterative process and requires the numerical evaluation of integrals with increasing dimension, which renders the model computationally very heavy. Another shortcoming of the model is that it is not invariant under the refinement of the time discretization, Thompson et al. (2005). Figure 6.3 shows the tail probability $\mathrm{P}(L > x)$ of a portfolio loss with different discretization (annual, semi-annual, quarterly) of the model. Obviously, the volatility and tail fatness of the loss distribution decreases with increasing refinement, and it is not clear where the limiting distribution is.

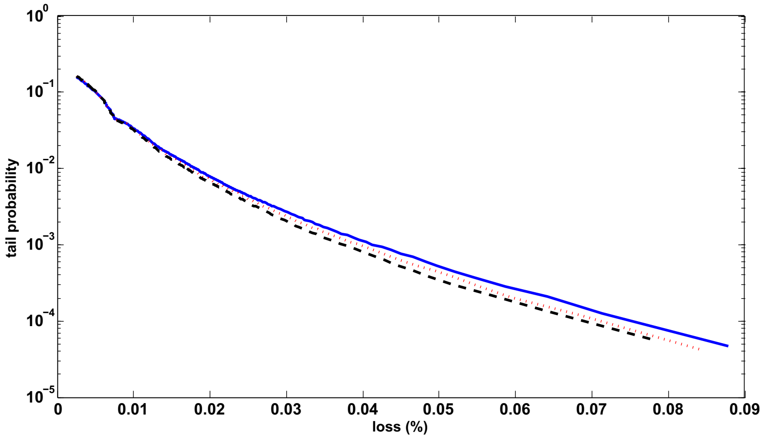

Figure 6.3. Refining Discretization, Hull-White Model: annual(blue solid), semi-annual(red dotted), and quarterly(black dashed) discretization.

6.2.4 The Time-Changed Barrier Model

The above mentioned discrete barrier model is drawn from a continuous version, i.e. correlated Brownian processes B_t^i with time-dependent barriers $c_i(t)$. The default time of credit i is then the first hitting time of the barrier $c_i(t)$ by the driving process B_t^i:

$$\tau_i = \inf\left\{t \geq 0 : B_t^i < c_i(t)\right\}.$$

If $c_i(t)$ is absolutely continuous, we can write

$$c_i(t) = c_i(0) + \int_0^t \mu_s^i ds,$$

and the default time τ_i is the first hitting time of the constant barrier $c_i(0)$ by a Wiener process with drift.

$$\begin{aligned} Y_t^i &= B_t^i - \int_0^t \mu_s^i ds \\ \tau_i &= \inf\left\{t \geq 0 : Y_t^i < c_i(0)\right\}. \end{aligned} \tag{6.4}$$

The problem is now to calibrate the model to the prescribed default term structure, $\mathrm{P}[\tau_i < t] = F_i(t)$. To this end, Overbeck and Schmidt (2005) put forward a barrier model based on Brownian processes B_t^i with suitably transformed time scales, (T_t^i), strictly increasing, $T_0^i = 0$. The first passage time to default τ_i of credit entity i is define through the process

$$Y_t^i = B^i_{T_t^i}$$

and

$$\tau_i = \inf\left\{s \geq 0 : Y_s^i < c_i\right\},$$

with a time independent barrier c_i. From the strong Markov property or the reflection principle of the Brownian motion follows that the first passage time of an untransformed Brownian motion with respect to a constant barrier c

$$\widetilde{\tau} = \inf\{t \geq 0 : B_t < c\}$$

is distributed as

$$\mathrm{P}(\widetilde{\tau} < t) = \mathrm{P}\left(\min_{0<s<t} B_s < c\right) = 2\Phi\left(\frac{c}{\sqrt{t}}\right). \qquad (6.5)$$

As T_t^i is strictly increasing we find that

$$\begin{aligned} \mathrm{P}(\tau_i < t) &= \mathrm{P}\left(\min_{0<s<t} B^i_{T_t^i} < c_i\right) = \mathrm{P}\left(\min_{0<s<T_t^i} B_t^i < c_i\right) \\ &= 2\Phi\left(\frac{c_i}{\sqrt{T_t^i}}\right) \qquad (6.6) \end{aligned}$$

Hence, given a default term structure $F_i(t)$ the model is calibrated to the marginals via the time transformation

$$T_t^i = \left[\frac{c_i}{\Phi^{-1}\left\{F_i(t)/2\right\}}\right]^2. \qquad (6.7)$$

Since $F(t)$ is strictly increasing this also follows for T_t. The constant default barrier c_i is then obtained by fixing a time t_0 with $T_{t_0}^i = t_0$ which implies

$$c_i = \Phi^{-1}\left\{F_i(t_0)/2\right\}\sqrt{t_0}. \qquad (6.8)$$

An obvious, but not necessarily the only sensible choice is to take t_0 as the final maturity. Dependency between credits is introduce here through the (local) instantaneous correlation matrix Σ of the Brownian processes B_t^i. The joint default probabilities $\mathrm{P}[\tau_i < t, \tau_j < t]$ can be written in analytical, but rather technical form, which allows the calibration of the model to prescribed joint default probabilities.

The discretization of the time-changed model for practical applications is straight forward and simply obtained by discretizing the SDE of the correlated Brownian motion while taking into account that the different dimensions evolve at different time scales. Figure 6.4 shows the behavior of the time-changed model under a refinement of the time discretization. Obviously, the model is within sampling errors invariant under this operation.

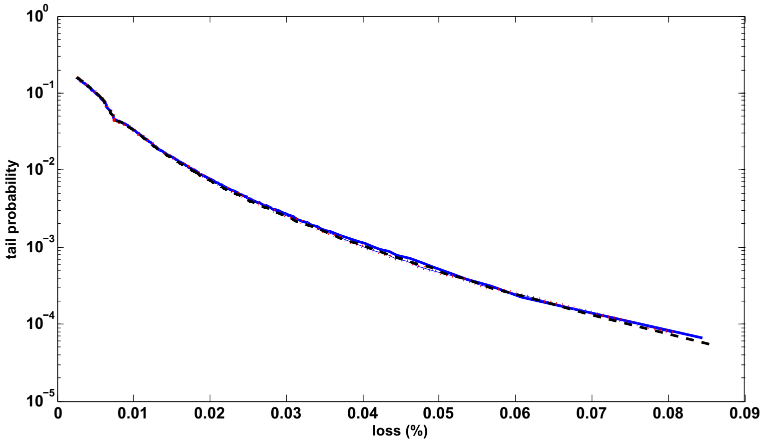

Figure 6.4. Refining Discretization, time-change Model: annual(blue solid), semi-annual(red dotted), and quarterly(black dashed) discretization.

6.3 Inter-Temporal Dependency and Autocorrelation

Finally, let us take a look at the inter-temporal dependency of the various models. All models are set up to meet by construction the default-term structure, hence produce the same first order loss moments through time, and they are calibrated to the same loss volatility at maturity (4 years). Figures 6.5-6.7 now serve to demonstrate the different inter-temporal characteristics of the four models. The graphs show the different joint loss distributions at the 2- and 4-year horizon, depicted as heat map. The upper triangle is empty as $L(4 \text{ years}) \geq L(2 \text{ years})$. Clearly, the migration model has the least autocorrelation as joint losses accumulate at the edges of the triangle. In contrast, the correlated-default-times model shows the highest inter-temporal dependency between losses, as joint losses accumulate in the middle of the triangle. This comes not as a surprise and reflects the fact that the model is essentially a static one where static factors drive the dependency through the whole time. Due to the driving Brownian motion it is also obvious that the two barrier models show similar inter-temporal dependency that lie somewhere between the first two extreme cases.

The control of inter-temporal dependency is not so much a problem if we only model a single plain vanilla CDO, but as soon as we have a structure with significant default-timing feature or if we want to assess the risk of a portfolio of non-linear credit products the inter-temporal dependency plays indeed an important role. For risk assessment the dependency through time should also be estimated from credit data, but these estimates seem not to support the high degree of inter-temporal dependency as generated by the barrier models.

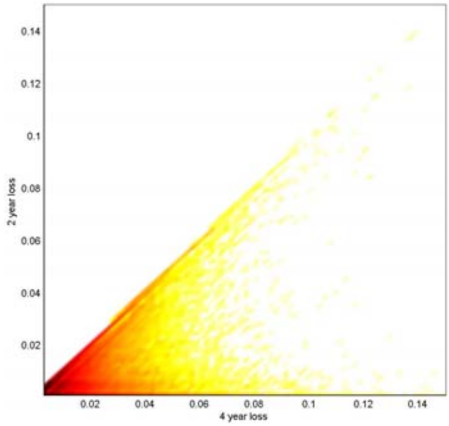

Figure 6.5. Joint Loss Distribution (2-4 years), Credit migration model

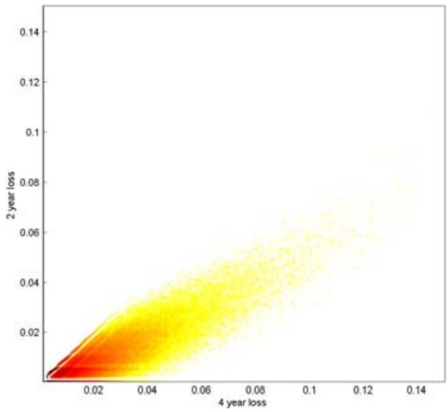

Figure 6.6. Joint Loss Distribution (2-4 years), Correlated-default-times model

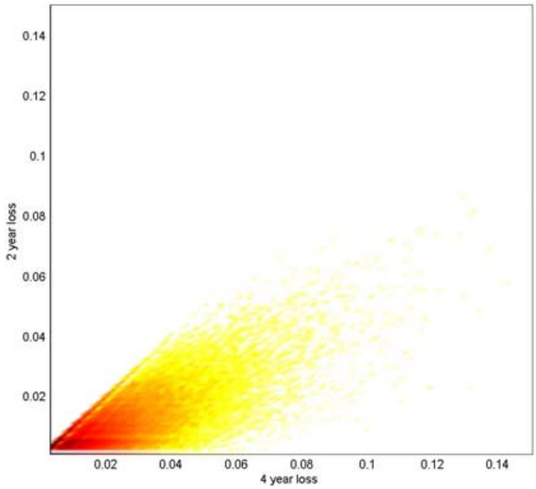

Figure 6.7. Joint Loss Distribution (2-4 years), Time-change barrier model

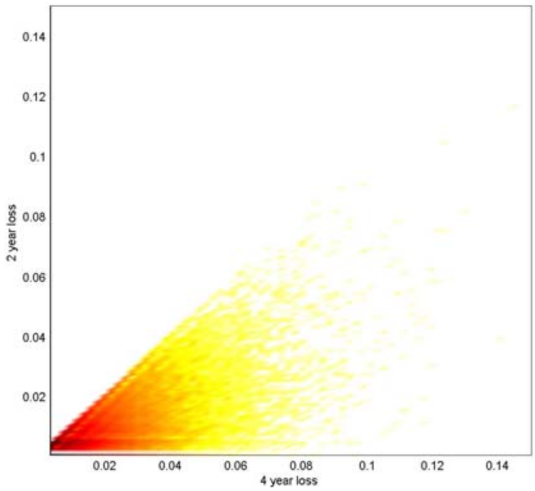

Figure 6.8. Joint Loss Distribution (2-4 years), Discrete barrier model

6.4 Conclusion

For an assessment of a portfolio of structured credit products a multi-period model with known cross- and autocorrelation is necessary. We investigate implementations of four different multi-period credit portfolio model and show that not for all of the models the correlation structure is invariant under the operation of a refined time discretization. Hence one should not blindly use these tpye of models at different periods and discretizations. In case of the discrete barrier model the continuous limit is unclear, but it is definitely not congruent to the time-changed barrier model. In case of a Markov Chain migration framework we argue that the cumulative loss distribution converges in the limit to a loss distribution with zero correlation as the time discretization is refined towards zero.

We then show how to correct the correlation structure while refining the discretization to obtain a congruent loss distribution at a given horizon. Finally, we analyse the inter temporal dependency of the different models and find that the correlated-default times model has the highest degree of inter-temporal dependency, the migration model relatively little and that the models driven by a Brownian motion are in between these two cases. We therefore conclude that before applying a multi-period model for risk assessment to a structured credit book the properties of the model in terms of inter-temporal and cross correlations should be fully understood, as different models have obviously different properties and will lead to differing results.

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