18 Simulation Based Option Pricing

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Here we develop an approach for efficient pricing discrete-time American and Bermudan options which employs the fact that such options are equivalent to the European ones with a consumption, combined with analysis of the market model over a small number of steps ahead. This approach allows constructing both upper and lower bounds for the true price by Monte Carlo simulations. An adaptive choice of local lower bounds and use of the kernel interpolation technique enhance efficiency of the whole procedure, which is supported by numerical experiments.

18.1 Introduction

The valuation of high-dimensional American and Bermudan options is one of the most difficult numerical problems in financial engineering. Several approaches have recently been proposed for pricing such options using Monte Carlo simulation technique (see, e.g. Andersen and Broadie (2004), Bally, Pagès, and Printems (2005), Belomestny and Milstein (2004), Boyle, Broadie, and Glasserman (1997), Broadie and Glasserman (1997), Clément, Lamberton and Protter (2002), Glasserman (2004), Haugh and Kogan (2004), Jamshidian (2003), Kolodko and J. Schoenmakers (2004), Longstaff and Schwartz (2001), Rogers (2001) and references therein). In some papers, procedures are proposed that are able to produce upper and lower bounds for the true price and hence allow for evaluating the accuracy of price estimates.

In Belomestny and Milstein (2004) we develop the approach for pricing American options both for discrete-time and continuous-time models. The approach is based on the fact that any American option is equivalent to the European one with a consumption process involved. This approach allows us, in principle, to construct iteratively a sequence $v^1,\ V^1,\ v^2,\ V^2,\ v^3, ...,$ where $v^1,\ v^2,\ v^3,$ ... is an increasing (at any point) sequence of lower bounds and $V^1,\ V^2, ...,$ is a decreasing sequence of upper bounds. Unfortunately, the complexity of the procedure increases dramatically with any new iteration step. Even V^2 is too expensive for the real construction.

Let us consider a discrete-time financial model and let

$$(B_{t_i}, X_{t_i}) = (B_{t_i}, X^1_{t_i}, ..., X^d_{t_i}), \ i = 0, 1, ..., L,$$

be the vector of prices at time t_i, where B_{t_i} is the price of a scalar riskless asset (we assume that B_{t_i} is deterministic and $B_{t_0} = 1$) and $X_{t_i} = (X^1_{t_i}, ..., X^d_{t_i})^\top$ is the price vector process of risky assets (along with index t_i we shall use below the index i and instead of (t_i, X_{t_i}) we will write (t_i, X_i)). Let $f_i(x)$ be the profit made by exercising an American option at time t_i if $X_{t_i} = X_i = x$.

Here we propose to use an increasing sequence of lower bounds for constructing an upper bound and lower bound for the initial position (t_0, X_0). It is supposed that the above sequence is not too expensive from the computational point of view. This is achieved by using local lower bounds which take into account a small number of exercise dates ahead.

Let $(t_i, X_{i,m}), \ i = 0, 1, ..., L; \ m = 1, ..., M$, be M independent trajectories all starting from the point (t_0, X_0) and let $v^1 \leq v^2 \leq ... \leq v^l$ be a finite sequence of lower bounds which can be calculated at any position (t_i, x). Clearly, these lower bounds are also ordered according to their numerical complexities and a natural number l indicates the maximal such complexity as well as the quality of the lower bound v^l. Any lower bound gives a lower bound for the corresponding continuation value (lower continuation value) and an upper bound for the consumption process (upper consumption process). If the payoff at $(t_i, X_{i,m})$ is less or equal to the lower continuation value, then the position $(t_i, X_{i,m})$ belongs to the continuation region and the consumption at $(t_i, X_{i,m})$ is equal to zero. Otherwise the position $(t_i, X_{i,m})$ can belong either to the exercise region or to the continuation region. In the latter cases we compute the upper consumption at $(t_i, X_{i,m})$ as a difference between the payoff and the lower continuation value.

It is important to emphasize that the lower bounds are applied adaptively. It means that if, for instance, using the lower bound v^1 (which is the cheapest one among $v^1, v^2, \ ..., \ v^l$) at the position $(t_i, X_{i,m})$, we have found that this position belongs to the continuation region (i.e., the corresponding upper consumption process is equal to zero), we do not calculate any further bounds. Similarly, if the upper consumption process is positive but comparatively small, we can stop applying further bounds at $(t_i, X_{i,m})$ because a possible error will not be large. Finally, if the upper consumption process is not small enough after applying lower bounds $v^1, ..., \ v^j$ but changes not significantly after applying v^{j+1}, we can stop applying further bounds as well. The lower bounds are prescribed to every position $(t_i, X_{i,m})$ and are, as a rule, local. Applying them means, in some sense, a local analysis of the considered financial market at any position. Such a local analysis for all positions $(t_i, X_{i,m})$, $i = 0, 1, ..., L$; $m = 1, ..., M$, yields some global lower bound and upper bound at the original position (t_0, X_0). If we detect that the difference between the global upper and lower bounds is large, we can return to the deeper local analysis. It is clear that, in principle, this analysis can give exhaustive results in a finite number of steps (it suffices to take the following sequence of American options at $(t_i, X_{i,m})$: v^1 is the price of the American option on the time interval $[t_i, t_{i+1}]$, v^2 is the price on $[t_i, t_{i+2}]$ and so on, in a way that v^{L-i} is the price on $[t_i, t_L]$). Thus, we have no problems with convergence of the algorithms based on the approach considered.

In Subsection 18.2 we recall the basic notions related to the pricing of American and Bermudan options and sketch the approach developed in Belomestny and Milstein (2004). The developed method is presented in Subsection 18.3 . Two numerical examples are given in Subsection 18.4 .

18.2 The Consumption Based Processes

To be self-contained, let us briefly recall the approach to pricing American options that has been developed in Belomestny and Milstein (2004).

18.2.1 The Snell Envelope

We assume that the modelling is based on the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_i)_{0\leq i\leq L}, Q)$, where the probability measure Q is the risk-neutral pricing measure for the problem under consideration, and X_i is a Markov chain with respect to the filtration $(\mathcal{F}_i)_{0\leq i\leq L}$.

The discounted process $\widetilde{X}_i \stackrel{\text{def}}{=} X_i/B_i$ is a martingale with respect to the Q and the price of the corresponding discrete American option at (t_i, X_i) is given by

$$u_i(X_i) = \sup_{\tau\in\mathcal{T}_{i,L}} B_i \,\mathsf{E}\left\{\frac{f_\tau(X_\tau)}{B_\tau}|\mathcal{F}_i\right\}, \tag{18.1}$$

where $\mathcal{T}_{i,L}$ is the set of stopping times τ taking values in $\{i, i+1, ..., L\}$. The value process u_i (Snell envelope) can be determined by the dynamic programming principle:

$$\begin{aligned} u_N(x) &= f_N(x), \qquad (18.2)\\ u_i(x) &= \max\left\{f_i(x), B_i\,\mathsf{E}\left\{\frac{u_{i+1}(X_{i+1})}{B_{i+1}}|X_i = x\right\}\right\},\ i = L-1, ..., 0. \end{aligned}$$

We see that theoretically the problem of evaluating $u_0(x)$, the price of the discrete-time American option, is easily solved using iteration procedure (18.2). However, if X is high dimensional and/or L is large, the above iteration procedure is not practical.

18.2.2 The Continuation Value, the Continuation and Exercise Regions

For the considered American option, let us introduce the continuation value

$$C_i(x) = B_i \mathsf{E}\left\{ \frac{u_{i+1}(X_{i+1})}{B_{i+1}} | X_i = x \right\}, \tag{18.3}$$

the continuation region $\mathcal{C}$ and the exercise (stopping) region $\mathcal{E}$:

$$\begin{aligned} \mathcal{C} &= \{(t_i, x) : f_i(x) < C_i(x)\}, \\ \mathcal{E} &= \{(t_i, x) : f_i(x) \geq C_i(x)\}. \end{aligned} \tag{18.4}$$

Let $X_j^{i,x}$, $j = i, i+1, ..., L$, be the Markov chain starting at time t_i from the point x : $X_i^{i,x} = x$, and $X_{j,m}^{i,x}$, $m = 1, ..., M$, be independent trajectories of the Markov chain. The Monte Carlo estimator $\widehat{u}_i(x)$ of $u_i(x)$ (in the case when $\mathcal{E}$ is known) has the form

$$\widehat{u}_i(x) = \frac{1}{M} \sum_{m=1}^{M} \frac{B_i}{B_\tau} f(X_{\tau,m}^{i,x}), \tag{18.5}$$

where τ is the first time at which $X_j^{i,x}$ gets into $\mathcal{E}$ (of course, τ in (18.5) depends on i, x, and m : $\tau = \tau_m^{i,x}$). Thus, for estimating $u_i(x)$, it is sufficient to examine sequentially the position $(t_j, X_{j,m}^{i,x})$ for $j = i, i+1, ..., L$, whether it belongs to $\mathcal{E}$ or not. If $(t_j, X_{j,m}^{i,x}) \in \mathcal{E}$, then we stop at the instant $\tau = t_j$ on the trajectory considered. If $(t_j, X_{j,m}^{i,x}) \in \mathcal{C}$, we move one step more along the trajectory.

Let v be any lower bound, i.e. $u_i(x) \geq v_i(x)$, $i = 0, 1, ..., L$. Clearly, $f_i(x)$ is a lower bound. If $v_i^1, ..., v_i^l$ are some lower bounds then the function $v_i(x) = \max_{1 \leq k \leq l} v_i^k(x)$ is also a lower bound. Henceforth we consider lower bounds satisfying the inequality $v_i(x) \geq f_i(x)$. Introduce the set

$$\mathcal{C}_v = \left\{ (t_i, x) : f_i(x) \leq B_i \mathsf{E}\left\{ \frac{v_{i+1}(X_{i+1})}{B_{i+1}} | X_i = x \right\} \right\}.$$

Since $\mathcal{C}_v \subset \mathcal{C}$, any lower bound provides us with a sufficient condition for moving along the trajectory: if $(t_j, X_{j,m}^{i,x}) \in \mathcal{C}_v$, we do one step ahead.

18.2.3 Equivalence of American Options to European Ones with Consumption Processes

For $0 \le i \le L-1$ the equation (18.2) can be rewritten in the form

$$u_i(x) = B_i E\left\{\frac{u_{i+1}(X_{i+1})}{B_{i+1}}|X_i = x\right\} + \left[f_i(x) - B_i E\left\{\frac{u_{i+1}(X_{i+1})}{B_{i+1}}|X_i = x\right\}\right]^+. \tag{18.6}$$

Introduce the functions

$$\gamma_i(x) = \left[f_i(x) - B_i \, \mathsf{E}\left\{\frac{u_{i+1}(X_{i+1})}{B_{i+1}}|X_i = x\right\}\right]^+, \ i = L-1, ..., 0. \tag{18.7}$$

Due to (18.6), we have

$$u_{L-1}(X_{L-1}) = B_{L-1}\, \mathsf{E}\left\{\frac{f_L(X_L)}{B_L}|\mathcal{F}_{L-1}\right\} + \gamma_{L-1}(X_{L-1}),$$

$$u_{L-2}(X_{L-2}) = B_{L-2}\, \mathsf{E}\left\{\frac{u_{L-1}(X_{L-1})}{B_{L-1}}|\mathcal{F}_{L-2}\right\} + \gamma_{L-2}(X_{L-2})$$

$$= B_{L-2}\, \mathsf{E}\left\{\frac{f_L(X_L)}{B_L}|\mathcal{F}_{L-2}\right\} + B_{L-2}\, \mathsf{E}\left\{\frac{\gamma_{L-1}(X_{L-1})}{B_{L-1}}|\mathcal{F}_{L-2}\right\} + \gamma_{L-2}(X_{L-2}).$$

Analogously, one gets

$$u_i(X_i) = B_i\, \mathsf{E}\left\{\frac{f_L(X_L)}{B_L}|\mathcal{F}_i\right\} + B_i \sum_{k=1}^{L-(i+1)} \mathsf{E}\left\{\frac{\gamma_{L-k}(X_{L-k})}{B_{L-k}}|\mathcal{F}_i\right\} \tag{18.8}$$
$$+\gamma_i(X_i), \ i = 0, ..., L-1.$$

Putting $X_0 = x$ and recalling that $B_0 = 1$, we obtain

$$u_0(x) = \mathsf{E}\left\{\frac{f_L(X_L)}{B_L}\right\} + \gamma_0(x) + \sum_{i=1}^{L-1} \mathsf{E}\left\{\frac{\gamma_i(X_i)}{B_i}\right\}. \tag{18.9}$$

Formula (18.9) gives us the price of the European option with the payoff function $f_i(x)$ in the case when the underlying price process is equipped with the consumption γ_i defined in (18.7).

18.2.4 Upper and Lower Bounds Using Consumption Processes

The results about the equivalence of the discrete-time American option to the European one with the consumption process cannot be used directly because

$u_i(x)$ and consequently $\gamma_i(x)$ are unknown. We take the advantage of this connection in the following way (see Belomestny and Milstein (2004)).

Let $v_i(x)$ be a lower bound on the true option price $u_i(x)$. Introduce the function (upper consumption process)

$$\gamma_{i,v}(x) = \left[f_i(x) - B_i \, \mathsf{E} \left\{ \frac{v_{i+1}(X_{i+1})}{B_{i+1}} | X_i = x \right\} \right]^+, \ i = 0, ..., L-1. \quad (18.10)$$

Clearly,

$$\gamma_{i,v}(x) \geq \gamma_i(x).$$

Hence the price $V_i(x)$ of the European option with payoff function $f_i(x)$ and upper consumption process $\gamma_{i,v}(x)$ is an upper bound: $V_i(x) \geq u_i(x)$.

Conversely, if $V_i(x)$ is an upper bound on the true option price $u_i(x)$ and

$$\gamma_{i,V}(x) = \left[f_i(x) - B_i \, \mathsf{E} \left\{ \frac{V_{i+1}(X_{i+1})}{B_{i+1}} | X_i = x \right\} \right]^+, \ i = 0, ..., L-1, \quad (18.11)$$

then the price $v_i(x)$ of the European option with lower consumption process $\gamma_{i,V}(x)$ is a lower bound.

Thus, starting from a lower bound $v_i^1(x)$, one can construct the sequence of lower bounds $v_i^1(x) \leq v_i^2(x) \leq v_i^3(x) \leq ... \leq u_i(x)$, and the sequence of upper bounds $V_i^1(x) \geq V_i^2(x) \geq ... \geq u_i(x)$. All these bounds can be, in principle, evaluated by the Monte Carlo simulations. However, each further step of the procedure requires labor-consuming calculations and in practice it is possible to realize only a few steps of this procedure. In this connection, much attention in Belomestny and Milstein (2004) is given to variance reduction technique and some constructive methods for reducing statistical errors are proposed there.

18.2.5 Bermudan Options

As before, let us consider the discrete-time model

$$(B_i, X_i) = (B_i, X_i^1, ..., X_i^d), \quad i = 0, 1, ..., L.$$

Suppose that an investor can exercise only at an instant from the set of stopping times $S = \{s_1, ..., s_l\}$ within $\{0, 1, ..., L\}$, where $s_l = L$. The price $u_i(X_i)$ of the so called Bermudan option is given by

$$u_i(X_i) = \sup_{\tau \in \mathcal{T}_{S \cap [i,L]}} B_i \, \mathsf{E} \left\{ \frac{f_\tau(X_\tau)}{B_\tau} | \mathcal{F}_i \right\},$$

where $\mathcal{T}_{S\cap[i,L]}$ is the set of stopping times τ taking values in $\{s_1, ..., s_l\} \cap \{i, i+1, ..., L\}$ with $s_l = L$.

The value process u_i is determined as follows:

$$u_L(x) = f_L(x),$$
$$u_i(x) = \begin{cases} \max\left\{f_i(x),\ B_i \mathsf{E}\left\{\dfrac{u_{i+1}(X_{i+1})}{B_{i+1}}|X_i = x\right\}\right\},\ i \in S, \\ B_i \left\{\dfrac{u_{i+1}(X_{i+1})}{B_{i+1}}|X_i = x\right\},\ i \notin S. \end{cases}$$

Similarly to American options, any Bermudan option is equivalent to the European one with the payoff function $f_i(x)$ and the consumption process γ_i defined as

$$\gamma_i(x) = \begin{cases} \left[f_i(x) - B_i \mathsf{E}\left\{\dfrac{u_{i+1}(X_{i+1})}{B_{i+1}}|X_i = x\right\}\right]^+,\ i \in S, \\ 0,\ i \notin S. \end{cases}$$

Thus, all the results obtained in this section for discrete-time American options can be carried over to Bermudan options. For example, if $v_i(x)$ is a lower bound on the true option price $u_i(x)$, the price $V_i(x)$ of the European option with the payoff function $f_i(x)$ and with the consumption process

$$\gamma_{i,v}(x) = \begin{cases} \left[f_i(x) - B_i \mathsf{E}\left\{\dfrac{v_{i+1}(X_{i+1})}{B_{i+1}}|X_i = x\right\}\right]^+,\ i \in S, \\ 0,\ i \notin S. \end{cases}$$

is an upper bound: $V_i(x) \geq u_i(x)$.

18.3 The Main Procedure

The difficulties mentioned in Subsection 2.4 can be avoided by using an increasing sequence of simple lower bounds.

18.3.1 Local Lower Bounds

The trivial lower bound is $f_i(x)$ and the simplest nontrivial one is given by

$$v_i^{i+1}(x) = \max\left\{f_i(x), B_i \mathsf{E}\left\{\frac{f_{i+1}(X_{i+1})}{B_{i+1}}|X_i = x\right\}\right\}.$$

The function $v_i^{i+1}(x)$ is the price of the American option at the position (t_i, x) on the time interval $[t_i, t_{i+1}]$. It takes into account the behavior of assets at one step ahead. Let $v_i^{i+k}(x)$ be the price of the American option at the position (t_i, x) on the time interval $[t_i, t_{i+k}]$. The function $v_i^{i+k}(x)$ corresponds to an analysis of the market over k steps ahead. The calculation of $v_i^{i+k}(x)$ can be done iteratively. Indeed, the price of the American option on the interval $[t_i, t_{i+k+1}]$ with $k+1$ exercise periods can be calculated using the American options on the interval $[t_{i+1}, t_{i+k+1}]$ with k exercise periods

$$v_i^{i+k+1}(x) = \max\left\{ f_i(x), B_i \,\mathsf{E}\left\{ \frac{v_{i+1}^{i+k+1}(X_{i+1})}{B_{i+1}} | X_i = x \right\}\right\}. \qquad (18.12)$$

We see that $v_i^{i+k+1}(x)$ is, as a rule, much more expensive than $v_i^{i+k}(x)$. The direct formula (18.12) can be too laborious even for $k \geq 3$. As an example of a simpler lower bound, let us consider the maximum of the American option on the interval $[t_i, t_{i+k}]$ and the European option on the interval $[t_i, t_{i+k+1}]$:

$$\bar{v}_i^{i+k}(x) = \max\left\{ v_i^{i+k}(x), B_i \,\mathsf{E}\left\{ \frac{f_{i+k+1}(X_{i+k+1})}{B_{i+k+1}} | X_i = x \right\}\right\}.$$

This lower bound is not so expensive as $v_i^{i+k+1}(x)$. Clearly

$$v_i^{i+k}(x) \leq \bar{v}_i^{i+k}(x) \leq v_i^{i+k+1}(x).$$

Different combinations consisting of European, American, and Bermudan options can give other simple lower bounds.

The success of the main procedures (see below) exceedingly depends on a choice of lower bounds. Therefore their efficient construction is of great importance. To this aim one can use the known methods and among them the method from Belomestny and Milstein (2004).

We emphasize again (see Introduction) that if after using some lower bound it is established that the position belongs to $\mathcal{C}$, then this position does not need any further analysis. Therefore, at the beginning the simplest nontrivial lower bound $v_i^{i+1}(x)$ should be applied and then other lower bounds should be used adaptively in the order of increasing complexity.

18.3.2 The Main Procedure for Constructing Upper Bounds for the Initial Position (Global Upper Bounds)

Aiming to estimate the price of the American option at a fixed position (t_0, x_0), we simulate the independent trajectories $X_{i,m},\ i = 1, ..., L,\ m =$

$1, ..., M$, of the process X_i, starting at the instant $t = t_0$ from $x_0 : \ X_0 = x_0$. Let $v_i(x)$ be a lower bound and $(t_i, X_{i,m})$ be the position on the m-th trajectory at the time instant t_i. We calculate the lower continuation value

$$c_{i,v}(X_{i,m}) = B_i \, \mathsf{E} \left\{ \frac{v_{i+1}(X_{i+1,m})}{B_{i+1}} | \mathcal{F}_i \right\} \tag{18.13}$$

at the position $(t_i, X_{i,m})$. If

$$f_i(X_{i,m}) < c_{i,v}(X_{i,m}), \tag{18.14}$$

then $(t_i, X_{i,m}) \in \mathcal{C}$ (see (18.4)) and we move one step ahead along the trajectory to the next position $(t_{i+1}, \ _m X_{i+1})$. Otherwise if

$$f_i(X_{i,m}) \geq c_{i,v}(X_{i,m}), \tag{18.15}$$

then we cannot say definitely whether the position $(t_i, X_{i,m})$ belongs to $\mathcal{C}$ or to E. In spite of this fact we do one step ahead in this case as well. Let us recall that the true consumption at (t_i, x) is equal to

$$\gamma_i(x) = [f_i(x) - C_i(x)]^+ \tag{18.16}$$

(see (18.7) and (18.3)). Thus, it is natural to define the upper consumption $\gamma_{i,v}$ at any position $(t_i, X_{i,m})$ by the formula

$$\gamma_{i,v}(X_{i,m}) = [f_i(X_{i,m}) - c_{i,v}(X_{i,m})]^+. \tag{18.17}$$

Obviously, $c_{i,v} \leq C_i$ and hence $\gamma_{i,v} \geq \gamma_i$. Therefore, the price $V_i(x)$ of the European option with payoff function $f_i(x)$ and upper consumption process $\gamma_{i,v}$ is an upper bound on the price $u_i(x)$ of the original American option. In the case (18.14) $\gamma_{i,v}(X_{i,m}) = \gamma_i(X_{i,m}) = 0$ and we do not get any error. If (18.15) holds and besides $c_{i,v}(X_{i,m}) < C_i(X_{i,m})$, we get an error. If $\gamma_{i,v}(X_{i,m})$ is large, then it is in general impossible to estimate this error, but if $\gamma_{i,v}(X_{i,m})$ is small, the error is small as well.

Having found $\gamma_{i,v}$, we can construct an estimate $\widehat{V}_0(x_0)$ of the upper bound $V_0(x_0)$ for $u_0(x_0)$ by the formula

$$\widehat{V}_0(x_0) = \frac{1}{M} \sum_{m=1}^{M} \frac{f_L(X_{L,m})}{B_L} + \frac{1}{M} \sum_{i=0}^{L-1} \sum_{m=1}^{M} \frac{\gamma_{i,v}(X_{i,m})}{B_i}. \tag{18.18}$$

Note that for the construction of an upper bound V_0 one can use different local lower bounds depending on a position. This opens various opportunities for adaptive procedures. For instance, if $\gamma_{i,v}(X_{i,m})$ is large, then it is reasonable to use a more powerful local instrument at the position $(t_i, X_{i,m})$.

18.3.3 The Main Procedure for Constructing Lower Bounds for the Initial Position (Global Lower Bounds)

Let us proceed to the estimation of a lower bound $v_0(x_0)$. We stress that both $V_0(x_0)$ and $v_0(x_0)$ are estimated for the initial position $\{t_0, x_0\}$ only. Since we are interested in obtaining as large as possible lower bound, it is reasonable to calculate different not too expensive lower bounds at the position $\{t_0, x_0\}$ and to take the largest one. Let us fix a local lower bound v. We denote by $t_0 \leq \tau_1^{(m)} \leq L$ the first time when either (18.15) is fulfilled or $\tau_1^{(m)} = L$. The second time $\tau_2^{(m)}$ is defined in the following way. If $\tau_1^{(m)} < L$, then $\tau_2^{(m)}$ is either the first time after $\tau_1^{(m)}$ for which (18.15) is fulfilled or $\tau_2^{(m)} = L$. So, $t_0 \leq \tau_1^{(m)} < \tau_2^{(m)} \leq L$. In the same way we can define θ times

$$0 \leq \tau_1^{(m)} < \tau_2^{(m)} < ... < \tau_\theta^{(m)} = L. \tag{18.19}$$

The number θ depends on the m-th trajectory: $\theta = \theta^{(m)}$ and can vary between 1 and $L+1 : 1 \leq \theta \leq L+1$. We put by definition $\tau_{\theta+1}^{(m)} = \tau_\theta^{(m)} = L$, $\tau_{\theta+2}^{(m)} = ... = \tau_{L+1}^{(m)} = L$. Thus, we get times $\tau_1, ..., \tau_{L+1}$ which are connected with the considered process X_i. For any $1 \leq k \leq L+1$ the time τ_k does not anticipate the future because at each point X_i at time t_i the knowledge of $X_j,\ j = 0, 1, ..., i$, is sufficient to define it uniquely. So, the times $\tau_1, ..., \tau_{L+1}$ are stopping rules and the following lower bound can be proposed

$$v_0(x_0) = \max_{1 \leq k \leq L+1} \mathsf{E} \frac{f_{\tau_k}(X_{\tau_k})}{B_{\tau_k}}$$

which can be in turn estimated as

$$\widehat{v}_0(x_0) = \max_{1 \leq k \leq L+1} \frac{1}{M} \sum_{m=1}^{M} \frac{f_{\tau_k^{(m)}}(X_{\tau_k^{(m)},m})}{B_{\tau_k^{(m)}}}.$$

Of course, $v_0(x_0)$ depends on the choice of the local lower bound v. Clearly, increasing the local lower bound implies increasing the global lower bound $v_0(x_0)$.

REMARK 18.1 It is reasonable instead of the stopping criterion (18.15) to use the following criterion

$$\gamma_{i,v}(X_{i,m}) \geq \varepsilon \tag{18.20}$$

for some $\varepsilon > 0$. On the one hand, $\gamma_{i,v} \geq \gamma_i$ and hence the stopping criterion with $\varepsilon = 0$ can lead to earlier stopping and possibly to a large error when

$\gamma_{i,v} > 0$ but $\gamma_i = 0$. On the other hand, if $0 < \gamma_{i,v}(X_{i,m}) < \varepsilon$ we can make an error using criterion (18.20). Indeed, in this case we continue and if $\gamma_i > 0$ then $(t_i, X_{i,m}) \in \mathcal{E}$ and the true decision is to stop. Since the price of the option at $(t_i, X_{i,m})$ upon the continuation is $C_i(X_{i,m})$ and

$$f_i(X_{i,m}) - C_i(X_{i,m}) = \gamma_i \leq \gamma_{i,v} < \varepsilon,$$

the error due to the wrong decision at $(t_i, X_{i,m})$ is small as long as ε is small. It is generally difficult to estimate the influence of many such wrong decisions on the global lower bound. Fortunately, any $\varepsilon > 0$ leads to a sequence of stopping times (18.19) and, consequently, to a global lower bound $v_0(x_0)$. What the global upper bound is concerned, we have $0 \leq \gamma_{i,v} - \gamma_i < \varepsilon$ when $\gamma_{i,v} < \varepsilon$ and hence the error in estimating V_0 is small due to (18.18). The choice of ε can be based on some heuristics and the empirical analysis of overall errors in estimating true γ_i's.

18.3.4 Kernel Interpolation

The computational complexity of the whole procedure can be substantially reduced by using methods from the interpolation theory. As discussed in the previous sections, the set of independent paths

$$\mathcal{P}_M \stackrel{\text{def}}{=} \{X_{i,m},\, i = 1, ..., L,\, m = 1, ..., M\}$$

and the sequence of local lower bounds $\{v_i^1, ..., v_i^l\}$ deliver the set of the upper consumption values $\{\gamma_{i,v}({}_mX_i),\ i = 0, ..., L,\ m = 1, ..., M\}$, where $v_i \stackrel{\text{def}}{=} \max\{v_i^1, ..., v_i^l\}$. If M is large one may take a subset $\mathcal{P}_{\widetilde{M}}$ of $\mathcal{P}_M$ containing first $\widetilde{M} \ll M$ trajectories

$$\mathcal{P}_{\widetilde{M}} \stackrel{\text{def}}{=} \{X_{i,m},\, i = 1, ..., L,\, m = 1, ..., \widetilde{M}\} \tag{18.21}$$

and compute $\{\gamma_{i,v}(X_{i,m}),\ i = 0, ..., L,\ m = 1, ..., \widetilde{M}\}$. The remaining consumption values $\gamma_{i,v}({}_nX_i)$ for $n = \widetilde{M}+1, ..., M$ can be approximated by

$$\widehat{\gamma}_{i,v}(X_{i,n}) \stackrel{\text{def}}{=} \sum_{\left\{m: X_{i,m} \in \mathcal{B}^k_{\mathcal{P}_{\widetilde{M}}}({}_nX_i)\right\}} w_{n,m} \gamma_{i,v}({}_mX_i),$$

where $\mathcal{B}^k_{\mathcal{P}_{\widetilde{M}}}({}_nX_i)$ is the set of k nearest neighbors of ${}_nX_i$ lying in the $\mathcal{P}_{\widetilde{M}}$ for fixed exercise date t_i and

$$w_{n,m} \stackrel{\text{def}}{=} \frac{K(\|{}_nX_i - X_{i,m}\|/h)}{\sum_{\left\{m: X_{i,m} \in \mathcal{B}^k_{\mathcal{P}_{\widetilde{M}}}({}_nX_i)\right\}} K(\|{}_nX_i - X_{i,m}\|/h)}$$

with $K(\cdot)$ being a positive kernel. A bandwidth h and the number of nearest neighbors k are chosen experimentally. Having found $\widehat{\gamma}_{i,v}({}_nX_i)$, we get the global upper bound at (t_0, x_0) according to (18.18) by plugging estimated values $\widehat{\gamma}_{i,v}({}_mX_i)$ with $m = \widetilde{M}+1, ..., M$ in place of the corresponding $\gamma_{i,v}({}_mX_i)$.

The simulations show that an essential reduction of computational time can be sometimes achieved at small loss of precision. The reason for the success of kernel methods is that the closeness of the points in the state space implies the closeness of the corresponding consumption values.

18.4 Simulations

18.4.1 Bermudan Max Calls on d Assets

This is a benchmark example studied in Broadie and Glasserman (1997), Haugh and Kogan (2004) and Rogers (2001) among others. Specifically, the model with d identical assets is considered where each underlying has dividend yield δ. The risk-neutral dynamic of assets is given by

$$\frac{dX_t^k}{X_t^k} = (r-\delta)dt + \sigma dW_t^k, \quad k = 1, ..., d, \tag{18.22}$$

where W_t^k, $k = 1, ..., d$, are independent one dimensional Brownian motions and r, δ, σ are constants. At any time $t \in \{t_0, ..., t_L\}$ the holder of the option may exercise it and receive the payoff

$$f(X_t) = (\max(X_t^1, ..., X_t^d) - K)^+.$$

In applying the method developed in this paper we take $t_i = iT/L$, $i = 0, ..., L$, with $T = 3$, $L = 9$ and simulate $M = 50000$ trajectories

$$\mathcal{P}_M = \{X_{i,m},\, i = 0, ..., L\}_{m=1}^M$$

using Euler scheme with a time step $h = 0.1$. Setting $\widetilde{M} = 500$, we define the set $\mathcal{P}_{\widetilde{M}}$ as in (18.21) and compute adaptively the lower continuation values for every point in $\mathcal{P}_{\widetilde{M}}$. To this end we simulate $N = 100$ points

$${}_nX_{i+1}^{(t_i,\, X_{i,m})}, \quad 1 \le n \le N,$$

from each point $(t_i,\, X_{i,m})$ with $i < L$ and $m \le \widetilde{M}$. For any natural l such that $0 \le l \le L - i - 1$, values

$$v_{i+1}^{(j)}\big({}_nX_{i+1}^{(t_i,\, X_{i,m})}\big), \quad 0 \le j \le l,$$

based on local lower bounds of increasing complexity, can be constructed as follows. First, $v_{i+1}^{(0)}\left({}_n X_{i+1}^{(t_i, X_{i,m})}\right) = f\left({}_n X_{i+1}^{(t_i, X_{i,m})}\right)$ and $v_{i+1}^{(j)}$ for $j = 1, 2$ are values of the American option on the intervals $[t_{i+1}, t_{i+1+j}]$. If $j > 2$ then $v_{i+1}^{(j)}$ is defined as value of the Bermudan option with three exercise instances at time points $\{t_{i+1}, t_{i+j}, t_{i+j+1}\}$. Now, we estimate the corresponding lower continuation value by

$$\widehat{c}_{i,l}(X_{i,m}) = \frac{e^{-r(t_{i+1}-t_i)}}{N} \sum_{n=1}^{N} \max_{0 \le j \le l} \left\{ v_{i+1}^{(j)}\left({}_n X_{i+1}^{(t_i, X_{i,m})}\right) \right\}.$$

Clearly, $\widehat{c}_{i,l}$ is the Monte-Carlo estimate of $c_{i,v}$, where $v = \max_{0 \le j \le l} v_{i+1}^{(j)}$. Let us fix a maximal complexity l^*. Sequentially increasing l from 0 to $l_i^* = \min\{l^*, L - i - 1\}$, we compute $\widehat{c}_{i,l}$ until $l \le l_*$, where

$$l_* \stackrel{\text{def}}{=} \min\{l : f_i(X_{i,m}) < \widehat{c}_{i,l}(X_{i,m})\}$$

or $l_* \stackrel{\text{def}}{=} l_i^*$ if

$$f_i(X_{i,m}) \ge \widehat{c}_{i,l}(X_{i,m}), \quad l = 1, \dots, l_i^*.$$

Note, that in the case $l_* < l_i^*$ the numerical costs are reduced as compared to the non-adaptive procedure while the quality of the estimate $\widehat{c}_{i,v_*}$, where $v_* = \max_{0 \le j \le l_*} v_{i+1}^{(j)}$ is preserved. The estimated values $\widehat{c}_{i,v_*}(X_{i,m})$ allow us, in turn, to compute the estimates for the corresponding upper consumptions $\gamma_{i,v_*}(X_{i,m})$ with $m = 1, \dots, \widetilde{M}$. The upper consumptions values for $m = \widetilde{M} + 1, \dots M$ are estimated using kernel interpolation with an exponential kernel (see Subsection 3.4). In Table 18.1 the corresponding results are presented in dependence on l^* and x_0 with $X_0 = (X_0^1, \dots, X_0^d)^T$, $X_0^1 = ... = X_0^d = x_0$. The true values are quoted from Glasserman (2004). We see that while the quality of bounds increases significantly from $l^* = 1$ to $l^* = 3$, the crossover to $l^* = 6$ has a little impact on it. It means that either the true value is achieved (as for $x_0 = 90$) or deeper analysis is needed (as for $x_0 = 100$).

18.4.2 Bermudan Basket-Put

In this example we consider again the model with d identical assets driven by independent identical geometrical Brownian motions (see (18.22)) with $\delta = 0$. Defining the basket at any time t as $\bar{X}_t = (X_t^1 + ... + X_t^d)/d$, let us consider the Bermudan basket put option granting the holder the right to sell this basket for a fixed price K at time $t \in \{t_0, ..., t_L\}$ getting the profit given by $f(\bar{X}_t) = (K - \bar{X}_t)^+$. We apply our method for constructing lower and upper bounds on the true value of this option at the initial point (t_0, X_0). In

l^*	x_0	Lower Bound $v_0(X_0)$	Upper Bound $V_0(X_0)$	True Value
	90	7.892±0.1082	8.694±0.0023	8.08
1	100	12.872±0.1459	15.2568±0.0042	13.90
	110	19.275±0.1703	23.8148±0.0062	21.34
	90	8.070±0.1034	7.900±0.0018	8.08
3	100	13.281±0.1434	14.241±0.0038	13.90
	110	19.526±0.1852	21.807±0.0058	21.34
	90	8.099±0.1057	7.914±0.0018	8.08
6	100	13.196±0.1498	13.844±0.0038	13.90
	110	19.639±0.1729	21.411±0.0056	21.34

Table 18.1. Bounds (with 95% confidence intervals) for the 2-dimensional Bermudan max call with parameters $K = 100$, $r = 0.05$, $\sigma = 0.2$, $L = 9$ and l^* varying as shown in the table.

order to construct local lower bounds we need to compute the prices of the corresponding European style options $v_t^{t+\theta}(x) = e^{-r\theta}\,\mathsf{E}(f(\bar{X}_{t+\theta})|X_t = x)$ for different θ and t. It can be done in principle by Monte-Carlo method since the closed form expression for $v_t^{t+\theta}(x)$ is not known. However, in this case it is more rational to use the so-called moment-matching procedure from Brigo, Mercurio, Rapisarda and Scotti (2002) and to approximate the distribution of the basket $\bar{X}_{t+\theta}$ by a log-normal one with parameters $\widetilde{r} - \widetilde{\sigma}^2/2$ and $\widetilde{\sigma}\theta^{1/2}$, where $\widetilde{r}$ and $\widetilde{\sigma}$ are chosen in a such way that the first two moments of the above log-normal distribution coincide with the true ones. In our particular example $\widetilde{r} = r$ and

$$\widetilde{\sigma}^2 = \frac{1}{\theta}\log\left\{\frac{\sum_{i,j=1}^d X_t^i X_t^j \exp(\mathbf{1}_{\{i=j\}}\sigma^2\theta)}{\left[\sum_{i=1}^d X_t^i\right]^2}\right\}. \tag{18.23}$$

In Table 18.2 the results of simulations for different maximal complexity l^* and initial values $x_0 = X_0^1 = ... = X_0^d$ are presented. Here, overall $M = 50000$ paths are simulated and on the subset of $\widetilde{M} = 500$ trajectories the local analysis is conducted. Other trajectories are handled with the kernel interpolation method as described in Subsection 3.4. Similar to the previous example, significant improvements are observed for $l^* = 2$ and $l^* = 3$. The difference between the upper bound and lower bound for $l^* > 3$ is less than 5%.

l	x_0	Lower Bound $v_0(X_0)$	Upper Bound $V_0(X_0)$	True Value
	100	2.391±0.0268	2.985±0.0255	2.480
1	105	1.196±0.0210	1.470±0.0169	1.250
	110	0.594±0.0155	0.700±0.0105	0.595
	100	2.455±0.0286	2.767±0.0238	2.480
2	105	1.210±0.0220	1.337±0.0149	1.250
	110	0.608±0.0163	0.653±0.0094	0.595
	100	2.462±0.0293	2.665±0.0228	2.480
3	105	1.208±0.0224	1.295±0.0144	1.250
	110	0.604±0.0166	0.635±0.0090	0.595
	100	2.473±0.0200	2.639±0.0228	2.480
6	105	1.237±0.0231	1.288±0.0142	1.250
	110	0.611±0.0169	0.632±0.0089	0.595
	100	2.479±0.0300	2.627±0.0226	2.480
9	105	1.236±0.0232	1.293±0.0144	1.250
	110	0.598±0.0167	0.627±0.0087	0.595

Table 18.2. Bounds (with 95% confidence intervals) for the 5-dimensional Bermudan basket put with parameters $K = 100$, $r = 0.05$, $\sigma = 0.2$, $L = 9$ and different l^*.

18.5 Conclusions

In this paper a new Monte-Carlo approach towards pricing discrete American and Bermudan options is presented. This approach relies essentially on the representation of an American option as the European one with the consumption process involved. The combination of the above representation with the analysis of the market over a small number of time steps ahead provides us with a lower as well an upper bound on the true price at a given point. Additional ideas concerning adaptive computation of the continuation values and the use of interpolation techniques help reducing the computational complexity of the procedure. In summary, the approach proposed has following features:

- It is Monte-Carlo based and is applicable to the problems of medium dimensionality.
- The propagation of errors is transparent and the quality of final bounds can be easily assessed.
- It is adaptive that is its numerical complexity can be tuned to the accuracy needed.

⊡ Different type of sensitivities can be efficiently calculated by combining the current approach with the method developed in Milstein and Tretyakov (2005).

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