14 Valuation of Multidimensional Bermudan Options

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14.1 Introduction

Multi-dimensional option pricing becomes an important topic in financial markets (Franke et al., 2008). Among which, the American-type derivative (e.g. the Bermudan option) pricing is a challenging problem. Unlike the European options which can only be exercised on the expiration date, the owner of a Bermudan option has the right to exercise early on a contractually specified finite set of dates. The dynamic programming approach is a practical and popular approach used to price the Bermudan option (Shreve, 2004, p.91). In that approach, the option value on each possible early exercise date is set to be the maximum of the payoff associated with immediate exercise, called the intrinsic value, and the discounted conditional expectation of the future option value, called the continuation value. The major problem of the approach lies in the computation of the continuation value.

In the literature, numerical methods, Barraquand and Martineau (1995) and Jeantheau (1998), and simulation based methods, Rust (1997), Tsitsiklis and Van Roy (1999), Longstaff and Schwartz (2001) and Broadie and Glasserman (2004), were proposed to solve this problem. Here we consider a dynamic semiparametric method to valuate multi-dimensional options. The proposed approach uses nonparametric step functions to approximate the option values on each possible early exercise date and evaluate the continuation values by parametric transition density. Unlike the simulation based method generating random sample paths, the proposed method selects the asset price points beforehand. And instead of numerically evaluating the multiple integral involved in computation of the continuation values, the proposed method provides closed form expressions for the integrals. Using this semiparametric technique, the proposed method provides a flexible and handy tool for multidimensional derivative pricing. Details of the dynamic semiparametric method are given in Section 14.3. The computational effort of the method is linear in the number of exercise opportunities and quadratic in the number of partition points. In addition, it is easily implemented when the multivariate joint distributions of the underlying assets are modeled by copula functions (Nelsen, 2006), which are to be introduced in the next section.

Section 14.2 defines the model assumptions. The proposed approach for valuing multidimensional Bermudan option is introduced in Section 14.3. One dimensional Bermudan option pricing of Black-Scholes model, multi-dimensional Bermudan option of multivariate geometric Brownian processes and a real example are demonstrated in Section 14.4. Section 14.5 concludes.

14.2 Model Assumptions

Consider a Bermudan option on d-dimensional underlying assets. Assume the price of each underlying asset $S_{\ell,t}$ follows a risk-neutral geometric Brownian process:

$$\frac{dS_{\ell,t}}{S_{\ell,t}} = (r - q_\ell)dt + \sigma_\ell dW_{\ell,t}, \ \ell = 1, \cdots, d, \tag{14.1}$$

where q_ℓ and σ_ℓ are the continuously compounded dividend yield and the instantaneous volatility of the ℓth asset, respectively, $W_{\ell,t}$'s are Wiener processes and the dependence among the $W_{\ell,t}$'s will be modeled by copula function introduced below. Let $\mathbf{X}_t = (X_{1,t}, \cdots, X_{d,t})^\top$ be the standardized log price per strike price, that is, $X_{\ell,t} = \log(S_{\ell,t}/K)$, $\ell = 1, \cdots, d$. Thus the (conditional) marginal distribution of $X_{\ell,t}$ is $\mathrm{N}(X_{\ell,0} + (r - q_\ell - \frac{1}{2}\sigma_\ell^2)t, \sigma_\ell^2 t)$. We will use copula functions to connect the asset marginals to their joint distribution. Since copula functions provide a flexible methodology for modeling of multivariate asset dependence, it has recently become a popular technique in financial markets, Sklar (1959), Cherubini et al. (2004), Nelsen (2006) and Giacomini et al. (2007). Let $F_\ell(x_\ell)$, $\ell = 1, \cdots, d$ denote the marginal distribution of X_ℓ, throughout we assume the joint distribution of $(X_1, \cdots, X_d)^\top$, $F(x_1, \cdots, x_d)$, is modeled by a copula function C, that is

$$F(x_1, \cdots, x_d) = C\{F_1(x_1), \cdots, F_d(x_d)\}. \tag{14.2}$$

For example, the Gaussian copula has the form $C(u_1, \cdots, u_d) = \Phi_R\{\Phi^{-1}(u_1), \cdots, \Phi^{-1}(u_d)\}$, where Φ is the distribution of $\mathrm{N}(0, 1)$ and Φ_R is the standardized multivariate normal distribution with correlation matrix R. When the univariate X_j's are normally distributed, the Gaussian copula is corresponding to the multivariate normal distribution. If the correlation matrix R is the identity matrix, then the Gaussian copula becomes the independence copula, implying that the random variables are independent. For example, in case $R = I_2$ the 2×2 identity matrix, by (14.2) and the definition of Gaussian copula, we have the joint distribution

$$\begin{aligned} F(x_1, x_2) &= \Phi_{I_2}\Big[\Phi^{-1}\{F_1(x_1)\}, \Phi^{-1}\{F_2(x_2)\}\Big] \\ &= \Phi\Big[\Phi^{-1}\{F_1(x_1)\}\Big]\Phi\Big[\Phi^{-1}\{F_2(x_2)\}\Big] \\ &= F_1(x_1)F_2(x_2). \end{aligned}$$

Copulae also provide a natural perspective to study the dependence in the tail of a multivariate distribution. For bivariate case, the lower tail dependence of X_1 and X_2 is defined as $\lambda_L = \lim_{v \to 0^+} \mathrm{P}\{F_2(X_2) \le v \mid F_1(X_1) \le v\} = \lim_{v \to 0^+} \frac{C(v,v)}{v}$. If $\lambda_L > 0$, then the two variables X_1 and X_2 are said to have lower tail dependence. Similarly, the upper tail dependence is defined as $\lambda_U = \lim_{v \to 1^-} \mathrm{P}\{F_2(X_2) > v \mid F_1(X_1) > v\} = \lim_{v \to 1^-} \frac{1-2v+C(v,v)}{1-v}$. If $\lambda_U > 0$, then there exists upper tail dependence. Archimedean copulae such as the Clayton and Gumbel copulae are two popular functions used to model the tail dependence of data. The Clayton copula has the form

$$C(u_1, \cdots, u_p) = \Big(\sum_{j=1}^{p} u_j^{-\theta} - p + 1\Big)^{\frac{-1}{\theta}}, \ \theta > 0,$$

and the Gumbel copula is

$$C(u_1, \cdots, u_p) = \exp\Big[-\Big\{\sum_{j=1}^{p}(-\log u_j)^{\theta}\Big\}^{\frac{1}{\theta}}\Big], \ \theta \ge 1.$$

The lower tail dependence of the bivariate Clayton copula is $\lambda_L = 2^{-1/\theta} > 0$, and the upper tail dependence of the Gumbel copula is $\lambda_U = 2 - 2^{1/\theta} > 0$, for $\theta > 1$. Thus the Clayton and Gumbel copulae are usable to model assets with lower and upper tail dependence, respectively. On the contrary, the Gaussian copula has neither upper nor lower dependence, unless the correlation coefficient $\rho = 1$. In Figure 21.1, we plot the random samples of four bivariate copulae, independent and correlated Gaussian, Clayton and Gumbel copulae with $\mathrm{N}(0, 1)$ marginals. Although the marginals are the same in the four cases, the plots display different tail dependence. The top-left is the independent Gaussian copula, denoted by Gaussian(0). The top-right is the Gaussian copula with correlation 0.5, which is the same as the bivariate normal distribution with zero mean, unit variance and correlation 0.5. The bottom-left and bottom-right are the Clayton and Gumbel copulae with parameter $\theta = 2$, which show a lower tail dependence and a upper tail dependence, respectively.

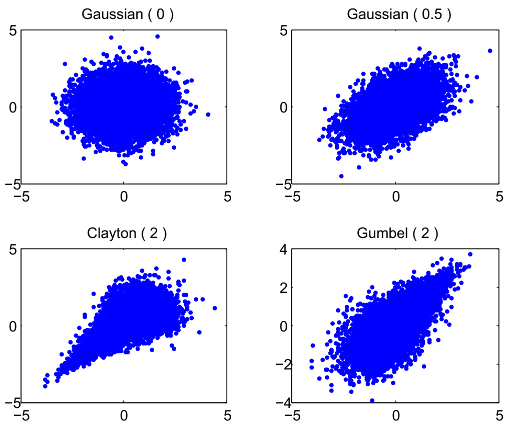

Figure 14.1. Bivariate copula plots with N(0, 1) marginals. Output of XFGbicopula .

14.3 Methodology

In this section, we introduce the proposed semiparametric method to valuate a d-dimensional Bermudan option with expiration date T. Assume the Bermudan option can only be exercised at time t_i, $i = 1, \cdots, n$, where $0 = t_0 < t_1 < \cdots < t_n = T$ and for simplicity we assume t_i's are equidistant with constant interval length $\Delta = t_i - t_{i-1}$. Let V_i denote the time t_i value of the Bermudan option, $\mathbf{S}_i = (S_{1,i}, \cdots, S_{d,i})^\top$ be the corresponding d underlying asset values, and $g(\mathbf{S}_i)$ be the option payoff function. Then the no arbitrage option values on possible early exercise dates are

$$\begin{cases} V_n(\mathbf{S}_n) = g(\mathbf{S}_n) \text{ and} \\ V_i(\mathbf{S}_i) = \max\{g(\mathbf{S}_i), e^{-r\Delta}\,\mathsf{E}(V_{i+1} \mid \mathbf{S}_i)\}, \text{ if } i < n \end{cases}, \tag{14.3}$$

where $r > 0$ is the riskless interest rate and $\mathsf{E}(\cdot|\mathbf{S}_i)$ is the conditional expectation under a risk-neutral probability measure given the information up to time t_i, Shreve (2004, p.91). In (14.3), the term $g(\mathbf{S}_i)$ is also called the *early exercise value* and $e^{-r\Delta}\,\mathsf{E}(V_{i+1}|\mathbf{S}_i)$ is the *continuation value* at time t_i. The Bermudan option will be exercised at time t_i if $g(\mathbf{S}_i) \geq e^{-r\Delta}\,\mathsf{E}(V_{i+1}|\mathbf{S}_i)$, and will be held continuously if $g(\mathbf{S}_i) < e^{-r\Delta}\,\mathsf{E}(V_{i+1}|\mathbf{S}_i)$.

The objective is to derive the initial option value $V_0(S_0)$, the main difficulty arises from evaluation of the continuation value. For instance, consider a univariate Bermudan put option on an underlying asset without paying dividend, $q = 0$, with payoff function $(K - S_n)^+$. Under geometric Brownian motion model assumption, the continuation value at time t_{n-1} can be obtained by the following Black and Scholes (1973) formula,

$$e^{-r\Delta}\,\mathsf{E}[(K-S_n)^+ \mid S_{n-1}] = Ke^{-r\Delta}\Phi(-d_2) - S_{n-1}\Phi(-d_1),$$

where $d_1 = \frac{\log(S_{n-1}/K)+(r+\frac{\sigma^2}{2})\Delta}{\sigma\sqrt{\Delta}}$ and $d_2 = d_1 - \sigma\sqrt{\Delta}$. Thus the continuation value at time t_{n-2} is

$$\begin{aligned} & e^{-r\Delta}\,\mathsf{E}(V_{n-1} \mid S_{n-2}) \\ = \;& e^{-r\Delta}\,\mathsf{E}\Big(\max\{(K-S_{n-1})^+, Ke^{-r\Delta}\Phi(-d_2) - S_{n-1}\Phi(-d_1)\} \mid S_{n-2}\Big), \end{aligned}$$

which is difficult to evaluate and has no closed-form solution. As the time move backwards to time t_0, the problem becomes more knotty. To handle the problem, we use step functions to approximate the option value at time t_n, $V_n(\mathbf{S}_n)$ defined in (14.3). Since the conditional joint distribution of $\mathbf{X}_t$ given $\mathbf{X}_{t-1}$ is modeled by the copula function $C\{F_1(X_{1,t}|X_{1,t-1}), \cdots, F_d(X_{d,t}|X_{d,t-1})\}$, it is relatively easy to evaluate the continuation value at time t_{n-1}. Accordingly, we define the approximate option value at time t_{n-1} to be the maximum of the intrinsic value and this continuation value. Continue the procedure backwards to t_0, we can obtain the initial option value. The proposed procedure uses a dynamic semiparametric approach, which incorporates nonparametric step function approximation and parametric model assumption, to tackle the difficult multiple integral computation involved in the high-dimensional derivative pricing problem. The details of the procedure is given below.

First, we confine the space of $\mathbf{X}_T$ to a proper finite region, say ± 5 standard deviation region of a given initial value $\mathbf{X}_0$, and then partition the region with equidistant grid points, denoted by $\mathbf{x}^{(j)} = (x_1^{(j)}, \cdots, x_d^{(j)})^\top$, $j = 1, \cdots, N$. The distance between two adjacent points in each dimension is denoted by Δ_x (see Figure 21.2 for the two-dimensional case). We keep the partition length Δ_x constant throughout the time. Start from the time point $i = n$, we use $\widetilde{V}_i(\cdot)$ to denote the approximate option function at time t_i, and set $\widetilde{V}_n(\mathbf{x}^{(j)}) = g(\mathbf{x}^{(j)})$ on the expiration date. The proposed steps to compute the d-dimensional Bermudan option are:

(1) Set the grid $\mathbf{A}^{(j)} = \prod_{\ell=1}^{d}[x_\ell^{(j)} - (1-c)\Delta_x, x_\ell^{(j)} + c\Delta_x], j = 1, \cdots, N$ (see Figure 21.2 for the two-dimensional case) and c is a pre-chosen constant. Based on the grids $\{\mathbf{A}^{(j)}\}_{j=1}^{N}$, define the step function

$$\widehat{V}_i = \sum_{j=1}^{N} \widetilde{V}_i(\mathbf{x}^{(j)})\mathbf{1}\{\mathbf{X}_i \in \mathbf{A}^{(j)}\}.$$

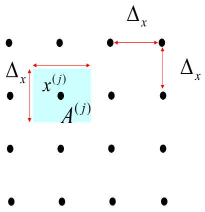

Figure 14.2. Two-dimensional grid points.

The constant c is chosen to meet the criterion that the European option values derived from this scheme are close to the benchmarks. In which the European benchmark option values can either be obtained analytically or by Monte Carlo simulation. For instance, the option on a geometric average for multivariate normal distributed underlying assets, the benchmark can be obtained by Black-Scholes formula since it can be reduced to a one-dimensional problem (for details see example 14.2).

(2) Compute the continuation value at time t_{i-1} given $\mathbf{X}_{i-1} = \mathbf{x}^{(h)}$ by

$$\mathsf{E}(\widehat{V}_i|\mathbf{X}_{i-1}) = \sum_{j=1}^{N} \widetilde{V}_i(\mathbf{x}^{(j)})\mathrm{P}(\mathbf{X}_i \in \mathbf{A}^{(j)} \mid \mathbf{X}_{i-1} = \mathbf{x}^{(h)}) = \mathbf{P}_h\widetilde{\mathbf{V}}_i,$$

where $\mathbf{P}_h$ is the hth row of the transition matrix $\mathbf{P} = (p_{hj})_{N\times N}$ with

$$\begin{aligned} p_{hj} &= \mathrm{P}(\mathbf{X}_i \in \mathbf{A}^{(j)} \mid \mathbf{X}_{i-1} = \mathbf{x}^{(h)}) \\ &= \sum_{i_1=1}^{2}\cdots\sum_{i_d=1}^{2}(-1)^{i_1+\cdots+i_d}C(u_{1i_1},\cdots,u_{di_d}), \end{aligned}$$

where C is the copula, $u_{\ell 1} = F_\ell(x_\ell^{(j)} - (1-c)\Delta_x \mid x_\ell^{(h)})$ and $u_{\ell 2} = F_\ell(x_\ell^{(j)} + c\Delta_x \mid x_\ell^{(h)})$ for all $\ell = 1,\cdots,d$, and $\widetilde{\mathbf{V}}_i = \left(\widetilde{V}_i(\mathbf{x}^{(1)}),\cdots,\widetilde{V}_i(\mathbf{x}^{(N)})\right)^\top$ is the approximate option value at time t_i, McNeil et al. (2005). Note that the transition matrix $\mathbf{P}$ is the same for $i = 1,\cdots,n$.

(3) The approximate option value at time t_{i-1} given $\mathbf{X}_{i-1} = \mathbf{x}^{(h)}$ is obtained by $\widetilde{V}_{i-1}(\mathbf{x}^{(h)}) = \max\{g(\mathbf{x}^{(h)}), e^{-r\Delta}\mathbf{P}_h\widetilde{\mathbf{V}}_i\}$. Note that if the interest is to valuate a European option, then just set $\widetilde{V}_{i-1}(\mathbf{x}^{(h)}) = e^{-r\Delta}\mathbf{P}_h\widetilde{\mathbf{V}}_i$.

(4) If $i-1 = 0$, then stop; otherwise set $i = i-1$ and return to (1).

Since the proposed method performs iterative matrix vector multiplication at each time t_i, its computational effort is linear in the number of exercise

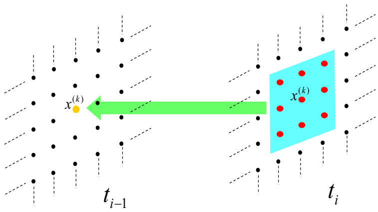

Figure 14.3. Non-zero elements in the two-dimensional case.

opportunities n. At each time t_i, on account of the matrix size $(N \times N)$, the computational work of matrix multiplication is quadratic in the total number of grid points N. Although the size $(N \times N)$ of the transition matrix $\mathbf{P}$ gets large as either the maturity time T or the dimension d of the underlying assets increases, lots (most) of it elements are zeros. This is due to the reason that the transition probabilities are negligible for far apart grid points, say more greater than five standard deviations (see Figure 21.3 for the two-dimensional example). Specifically, the row length (N) of the transition matrix $\mathbf{P}$ is of order $\mathcal{O}(T^{d/2})$ and the number of nonzero entries of each row is of order $\mathcal{O}(\Delta^{d/2}) = \mathcal{O}((\frac{T}{n})^{d/2})$, as a result the ratio of non-zero elements of $\mathbf{P}$ is of order $\mathcal{O}(n^{-d/2})$. In another word, the transition matrix $\mathbf{P}$ is a sparse matrix populated primarily with zeros.

When storing and manipulating sparse matrices on a computer, we can utilize specialized algorithms and data structures, eg. the SPARSE routine of MATLAB, to save the computation time and to consume less memory. Furthermore, since the partition grid points of the d-dimensional asset price space are determined in advance and kept fixed, the transition matrix remains unchanged throughout the time, which contrasts sharply with the time varying transition matrix used in simulation based approach. In the simulation based method, e.g. see Rust (1997) and Broadie and Glasserman (2004), random samples are generated by Monte Carlo method at each time t_i, and the continuation value at time t_{i-1} given the kth random sample, $\mathbf{S}_{i-1}^{(k)}$, is approximated by $\sum_{j=1}^{N} w_i^{(k,j)} V_i(\mathbf{S}_i^{(j)})$, where $w_i^{(k,j)}$ determines the stochastic weights of the sample at time t_i. In Figure 21.4 and 21.5, we illustrate the design grid points of the proposed scheme and the random samples of the simulation based method, respectively. Let $\mathbf{P}^{(i)} = (w_i^{(k,j)})_{k,j}$, the matrix of stochastic weights, then the continuation value at time t_{i-1} can be viewed as a matrix multiplication of the option value at time t_i, and the matrix $\mathbf{P}^{(i)}$ varies as time changes.

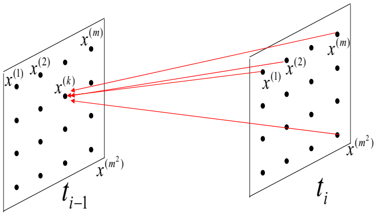

Figure 14.4. The designed points of the proposed scheme.

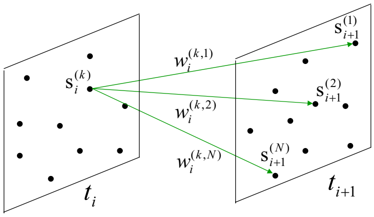

Figure 14.5. The random samples of the simulation based method.

14.4 Examples

In this section, we demonstrate three simulated examples (example 14.1-14.3) and one real application (example 14.4) to valuate Bermudan options by the proposed scheme.

EXAMPLE 14.1 *Suppose the underlying asset satisfies the following risk-neutral geometric Brownian motion*

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t, \tag{14.4}$$

where $r = 0.08$, $\sigma = 0.2$ *and* $q = 0$, 0.04, 0.08 *or* 0.12. *Consider a one dimensional Bermudan put option with strike price* $K = 100$, *time to maturity* $T = 3$, *length of time interval* $\Delta = \frac{1}{52}$ *(i.e.* $n = 156$*) and payoff function* $g(S_t) = (K - S_t)^+$. *In Figure 21.6, we plot a simulated path of* $\{S_t\}$ *satisfying (14.4) with* $r = 0.08$, $\sigma = 0.2$, $q = 0$, $T = 3$, $\Delta = \frac{1}{52}$ *and the initial stock price* $S_0 = K = 100$. *The stock price at the maturity date is* 117. *Thus the payoff is* 17 *at time* T *in this realization. At each time* $t < T$, *the owner of this option would exercise early only when the payoff is positive, i.e.* $S_t > K$, *and would hold the option continuously when* $S_t < K$. *If* $S_t > K$, *then she needs to compute the continuation value of her option in order to decide whether exercising immediately or not.*

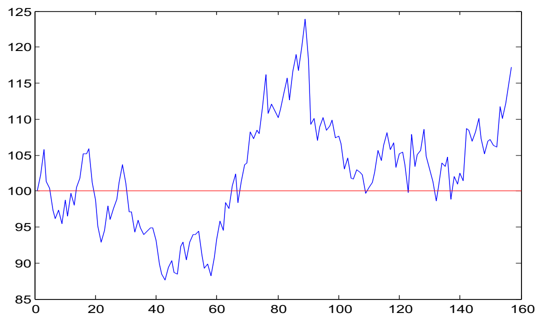

Figure 14.6. A simulated path of the stock price process satisfying (14.4) with $r = 0.08$, $\sigma = 0.2$, $q = 0$, $T = 3$, $\Delta = \frac{1}{52}$ and the initial stock price $S_0 = K = 100$. Output of XFGstock .

Let $X_t = \log(S_t/K)$ denote the standardized log price per strike price and let $\{x^{(j)}\}_{j=1}^{401}$ denote the 401 pre-chosen equidistant grid points of X_t, where $x^{(1)} = X_0 - 5\sigma\sqrt{T}$ and $x^{(401)} = X_0 + 5\sigma\sqrt{T}$, that is the distance between two adjacent points is $\Delta_x = 0.0087$ and $X_0 = x^{(201)}$. In the following, we illustrate the procedure to compute the approximate option values backwards from time t_{155} to t_{154}. First the continuation values of $x^{(j)}$ at time t_{155}, $e^{-0.0015}\,\mathsf{E}(V_{155} \mid x^{(j)})$, are derived by the Black-Scholes formula, and the option values of $x^{(j)}$'s are obtained by

$$\widetilde{V}_{155}(x^{(j)}) = \max\{100 - 100\exp(x^{(j)}), e^{-0.0015}\,\mathsf{E}(V_{156} \mid x^{(j)})\},$$

$j = 1, \cdots, 401$. In Figure 21.7, we show the evolution of the intrinsic and continuation values at t_i in (a) to the approximate option value at t_{i-1} in (b). In Figure 21.7 (a), the green line is the intrinsic value, the red dash curve is the continuation value and the intersection of the green line and the red curve represents the early exercised boundary of the Bermuda option. In Figure 21.7 (b), the blue curve is the approximate option value, $\widetilde{V}_{i-1}$. Define the following step function

$$\widehat{V}_{155} = \sum_{j=1}^{401} \widetilde{V}_{155}(x^{(j)})\mathbf{1}_{\{X_{155}\in A^{(j)}\}},$$

where $A^{(j)} = [x^{(j)} - (1-c)\Delta_x, x^{(j)} + c\Delta_x]$ and c is chosen to let the European option price of X_0 computed by proposed method meets that of the Black-

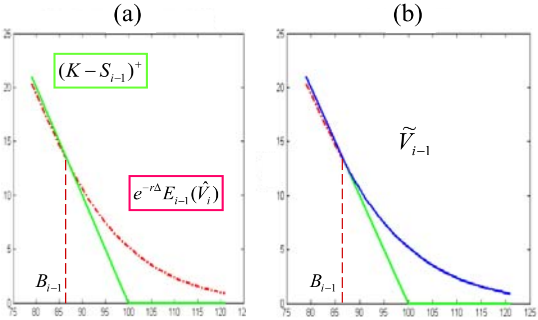

Figure 14.7. (a) The intrinsic and continuation values at t_i (b) The approximate option value, $\widetilde{V}_{i-1}$, at t_{i-1}.

Scholes formula. The continuation value of $x^{(h)}$ at time t_{154} is given by

$$\begin{aligned} e^{-0.0015}\,\mathsf{E}(\widehat{V}_{155} \mid x^{(h)}) &= e^{-0.0015}\sum_{j=1}^{401}\widetilde{V}_{155}(x^{(j)})\mathrm{P}(X_{155} \in A^{(j)} \mid x^{(h)}) \\ &= e^{-0.0015}\mathbf{P}_h\widetilde{\mathbf{V}}_{155}, \end{aligned}$$

where $\mathbf{P}_h$ is the hth row of the transition matrix $\mathbf{P} = (p_{hj})_{401\times 401}$ with

$$\begin{aligned} p_{hj} &= \mathrm{P}(X_{155} \in A^{(j)} \mid x^{(h)}) \\ &= \Phi(\tfrac{x^{(j)}+c\Delta_x - x^{(h)} - (r-q-0.5\sigma^2)\Delta}{\sigma\sqrt{\Delta}}) - \Phi(\tfrac{x^{(j)}-(1-c)\Delta_x - x^{(h)} - (r-q-0.5\sigma^2)\Delta}{\sigma\sqrt{\Delta}}), \end{aligned}$$

$\Phi(\cdot)$ is the standard normal cumulative distribution function and $\widetilde{\mathbf{V}}_{155} = \Big(\widetilde{V}_{155}(x^{(1)}), \cdots, \widetilde{V}_{155}(x^{(401)})\Big)^\top$ are the approximate option values at time t_{155}. Therefore, the approximate option values of $x^{(j)}$'s at time t_{154} are $\widetilde{V}_{154}(x^{(j)}) = \max\{100 - 100\exp(x^{(j)}), e^{-0.0015}\,\mathsf{E}(\widehat{V}_{155} \mid x^{(j)})\}$, $j = 1, \cdots, 401$. Note that the transition matrix $\mathbf{P}$ remains unchanged throughout the time. Proceeding the above procedure backwards to time zero, one obtains the desired option value.

Table 14.1 presents the simulation results for different initial stock prices, $S_0 = 90, 100, 110$. In the table, we give the option prices obtained by the proposed method and the methods by Ju (1998), denoted as EXP3, by and Lai and AitSahalia (2001), denoted as LSP4. In approximating the early exercise boundary of the Bermuda option, Ju (1998) adopts multipiece exponential function and Lai and AitSahalia (2001) adopt a linear spline method. The

S_0		Bin.	LSP4	EXP3	Alg.
90	(1)	20.08	20.08	20.08	20.09
100	$q = 0.12$	15.50	15.51	15.50	15.50
110		11.80	11.81	11.80	11.81
90	(2)	16.21	16.20	16.20	16.21
100	$q = 0.08$	11.70	11.70	11.70	11.71
110		8.37	8.37	8.36	8.37
90	(3)	13.50	13.49	13.49	13.50
100	$q = 0.04$	8.94	8.94	8.93	8.95
110		5.91	5.91	5.90	5.92
90	(4)	11.70	11.70	11.69	11.69
100	$q = 0.00$	6.93	6.93	6.92	6.93
110		4.16	4.15	4.15	4.16

Table 14.1. Bermudan put values of example 14.1 with parameters $r = 0.08$, $\sigma = 0.20$, $K = 100$, $T = 3$ and $\Delta = 1/52$. Output of XFGBP1 .

values based on 10,000 steps of the binomial method are taken as the benchmark option prices. The results show that our approach is competitive and comparable with the LSP4 and EXP3 methods.

EXAMPLE 14.2 *Assume now two underlying assets satisfying (14.1), i.e.* $d = 2$*, with parameters* $r = 0.05$*,* $q_1 = q_2 = 0$*,* $\sigma_1 = \sigma_2 = 0.2$ *and the joint distribution of the two log stock price processes is bivariate normal with correlation coefficient* $\rho = 0.3$*. Consider a Bermudan put option on a geometric average with* $K = 100$*,* $T = 1$*,* $\mathbf{S}_0 = 100$*,* $\Delta = 1/12$ *(i.e.* $n = 12$*) and payoff function* $g(\mathbf{S}_t) = (K - \sqrt{S_{1,t}S_{2,t}})^+$*. First, we confine the space of* $\mathbf{X}_{12}$ *to* $[-0.57, 0.63]^2$*, and partition the region with 25 equidistance partition points in each dimension, that is we have 625 two-dimensional grid points, denoted by* $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(625)}$*. The transition matrix* $\mathbf{P} = (p_{hj})_{625\times 625}$ *has entries*

$$p_{hj} = \mathrm{P}(\mathbf{X}_i \in \mathbf{A}^{(j)} \mid \mathbf{x}^{(h)}) = \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}(-1)^{i_1+i_2} C_{0.3}^{Ga}(u_{1i_1}, u_{2i_2}),$$

where $u_{\ell 1} = \Phi(\frac{x_\ell^{(j)}+c\Delta_x - x_\ell^{(h)} - 0.03\Delta}{0.2\sqrt{\Delta}})$*,* $u_{\ell 2} = \Phi(\frac{x_\ell^{(j)}+c\Delta_x - x_\ell^{(h)} - 0.03\Delta}{0.2\sqrt{\Delta}})$*, for* $\ell = 1, 2$*, and* $C_{0.3}^{Ga}$ *is the Gaussian copula with correlation coefficient 0.3. Obviously, the the entries of* p_{hj} *of the transition matrix are independent of the time index. To decide the adjusting coefficient* c *of the grids, we demonstrate the*

	Copula	T (year)	European Ben. (std.)	European Alg.	Bermudan Ben.	Bermudan Alg.
2-dim.	Gaussian(0)	0.25	2.33	2.34	2.38	2.39
		0.5	3.02	3.02	3.18	3.18
		1	3.75	3.75	4.13	4.13
	Gaussian(0.3)	0.25	2.69	2.69	2.74	2.75
		0.5	3.50	3.50	3.67	3.67
		1	4.38	4.37	4.79	4.79
	Clayton(5)	0.25	3.30 (0.004)	3.30		3.37
		0.5	4.33 (0.006)	4.33		4.52
		1	5.46 (0.006)	5.48		5.94
	Gumbel(5)	0.25	3.33 (0.004)	3.33		3.39
		0.5	4.36 (0.005)	4.36		4.55
		1	5.50 (0.008)	5.49		5.96
3-dim.	Gaussian(0)	0.25	1.86	1.86	1.91	1.92
		0.5	2.38	2.37	2.53	2.53
	Gaussian(0.3)	0.25	2.41	2.41	2.47	2.47
		0.5	3.13	3.11	3.29	3.28
	Clayton(5)	0.25	3.27 (0.002)	3.27		3.35
		0.5	4.29 (0.004)	4.29		4.49

Table 14.2. Multi-dimensional put option prices on a geometric average with parameters $r = 0.05$, $\sigma = 0.2$, $\mathbf{S_0}$=K=100 and $\Delta = 1/12$ (year). Gaussian(ρ): ρ denotes the equi-correlation among securities. Clayton(α) and Gumbel(α): α is the parameter of Clayton and Gumbel copulae. The Ben. values of the Gaussian cases are computed by `XFGBPgmeanR1` , while the Ben. values of the Clayton and Gumbel cases are from `XFGEPmean2MC` (2 dimensional case) and `XFGEPmean3MC` (3 dimensional case). The 2 and 3 dimensional Alg. values are obtained by `XFGBPgmean2` and `XFGBPgmean3` , respectively.

Gaussian copula case. In the case of Gaussian copula, this problem can also be considered as a one-dimensional option pricing problem. Let $\bar{S}_t$ denote the geometric mean of $S_{1,t}$ and $S_{2,t}$, that is $\bar{S}_t = \sqrt{S_{1,t}S_{2,t}}$. Since $S_{1,t}$ and $S_{2,t}$ both are geometric Brownian motions, thus by Ito's lemma we have

$$d \log \bar{S}_t = (\widetilde{r} - \frac{1}{2}\widetilde{\sigma}^2)dt + \widetilde{\sigma} dW_t,$$

where W_t is a Wiener process, $\widetilde{\sigma}^2 = \frac{1}{4}(\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)$, which is due to the bivariate normal distributed assumption, and $\widetilde{r} = r + \frac{1}{2}\widetilde{\sigma}^2 - \frac{1}{4}(\sigma_1^2 + \sigma_2^2)$. Consequently, the European put option values can be obtained by the following

		European		Bermudan
Copula	T (year)	Ben. (std.)	Alg.	Alg.
Gaussian(0)	2/3	8.46 (0.006)	8.45	8.65
	1	9.55 (0.009)	9.55	10.04
Gaussian(0.3)	2/3	7.90 (0.011)	7.89	8.07
	1	8.93 (0.013)	8.93	9.36
Clayton(5)	2/3	6.73 (0.014)	6.74	6.92
	1	7.66 (0.013)	7.61	8.02

Table 14.3. Multi-dimensional max call option prices with parameters $r = 0.05$, $q = 0.1$, $\sigma = 0.2$, $\mathbf{S_0}$=K=100 and $\Delta =$ 1/3 (year). The Ben. values are computed by `XFGECmax2MC` and the Alg. values are from `XFGBCmax2` .

formula

$$V_0(\mathbf{S}_0) = e^{-rT}\,\mathsf{E}[(K - \bar{S}_T)^+ \mid \mathbf{S}_0] = e^{-(r-\widetilde{r})T}\{Ke^{-\widetilde{r}T}\Phi(-d_2) - \bar{S}_0\Phi(-d_1)\}, \tag{14.5}$$

where $d_1 = \frac{\log(\bar{S}_0/K)+(\widetilde{r}+0.5\widetilde{\sigma}^2)T}{\widetilde{\sigma}\sqrt{T}}$, $d_2 = d_1 - \widetilde{\sigma}\sqrt{T}$ *and the second equality is due to the Black-Scholes formula. The above result can also be extended to d-dimensional European option on geometric average.*

Assume that the random vector $(\log S_{1,t}, \cdots, \log S_{d,t})^\top$ *has a multivariate normal distribution with covariance matrix* $t{\cdot}\Sigma = t{\cdot}(\sigma_{jk})$. *Let* $\bar{S}_t = (S_{1,t}\cdots S_{d,t})^{1/d}$, *thus* $\log \bar{S}_t$ *given* $\mathbf{S}_0$ *is normally distributed with mean* $\log \bar{S}_0 + (\widetilde{r} - \frac{1}{2}\widetilde{\sigma}^2)t$ *and variance* $\widetilde{\sigma}^2 t$, *where* $\widetilde{\sigma} = \frac{1}{d}\sqrt{\sum_{j,k}\sigma_{jk}}$ *and* $\widetilde{r} = r + \frac{1}{2}\widetilde{\sigma}^2 - \frac{1}{2d}\sum_j \sigma_{jj}$. *Thus the European put option values on a d-dimensional geometric average can be obtained by (14.5) analogously. And the corresponding Bermudan option can also be valuated using this reduced one-dimensional version. Thus for Gaussian copula, we can use (14.5) to obtain the benchmarks of the European and Bermudan geometric option prices and the adjusting coefficient c can then be determined.*

Table 14.2 presents the results of several expiration dates T for Gaussian, Clayton and Gumbel copulae. For Clayton and Gumbel copulae, since no closed-form solutions exit, thus the benchmarks of European options are obtained by Monte Carlo simulation. For the Gaussian cases, the estimated option values are all close to the benchmarks, which shows the proposed scheme provides a promising approach for multi-dimensional options on a geometric average.

EXAMPLE 14.3 *Suppose two underlying assets satisfying (14.1) with* $r = 0.05$, $q_1 = q_2 = 0.1$ *and* $\sigma_1 = \sigma_2 = 0.2$. *Consider a Bermudan max-call*

	$T = 1/4$		$T = 1/2$		$T = 1$	
S_0/K	Euro.	Berm.	Euro.	Berm.	Euro.	Berm.
0.9	7.86	20.49	4.01	20.49	1.18	20.49
1.0	2.41	3.90	1.45	4.27	0.48	4.42
1.1	0.62	0.84	0.49	1.11	0.19	1.24

Table 14.4. European and Bermudan put values of example 14.4 with parameters $r = 0.5736$, $\sigma = 0.304$, $\Delta = 1/12$ and $S_0 = 184.375$. Output of XFGBP1 .

option with $K = 100$*,* $T = 2/3, 1$*,* $\mathbf{S}_0 = 100$*,* $\Delta = 1/3$ *(i.e.* $n = 3$*) and payoff function* $g(\mathbf{S}_t) = \{\max(S_{1,t}, S_{2,t}) - K\}^+$*. Table 14.3 gives the results of Bermudan max call option prices for Gaussian copula with* $\rho = 0, 0.3$*, and Clayton copula with* $\alpha = 5$*. Since no closed-form solutions of European max-call option exit for Gaussian and Clayton copulae, the European benchmarks are obtained by Monte Carlo simulation. The simulation results show that all the Bermudan options are more valuable than their European counterparts.*

EXAMPLE 14.4 *Consider a standard Bermuda put option on the IBM shares. In (Tsay, 2005, p.*$259 - 260$*) a geometric Brownian motion process (14.4) is fitted to the 252 daily IBM stock prices of* 1998*. The parameters' estimated values are* $r = 0.5732$*,* $q = 0$ *and* $\sigma = 0.304$*. The stock price of IBM on Dec.* 31*,* 1998 *is* $S_0 = 184.375$*. Assume the possible early exercise dates are at the end of each month, that is the length of the time interval is* $\Delta = 1/12$*. Table 14.4 presents the European and Bermudan put option values on Dec. 31, 1998, for different* $S_0/K = 0.9, 1, 1.1$*, where* K *is the strike price, and maturity time* $T = 1/4, 1/2, 1$*. The European put values are computed by the Black-Scholes formula and the Bermudan put values are computed by the proposed method with 401 pre-chosen equidistant grid points as in example 14.1. The results show the Bermudan options are all more valuable than their European counterparts.*

14.5 Conclusion

The proposed method gives an innovative semiparametric approach to multidimensional Bermudan option pricing. The method is applicable to use copula functions modeling multivariate asset dependence. The simulation results show that the proposed approach is very tractable for numerical implementation and provides an accurate method for pricing Bermudan options. Although the transition matrix of the proposed method is a sparse matrix containing lots of zeros, the geometrically increasing rate (in time) of the matrix size still impedes its application. To tackle this problem, Huang and Guo (2007) apply important sampling idea to re-weight the grid probabilities and keep the matrix size constant throughout the time.

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