22

Gibbs Random Fields

22.1 Definition of a Gibbs Random Field

The notion of Gibbs random fields was formalized by mathematicians relatively recently. Before that, these fields were known in physics, particularly in statistical physics and quantum field theory. Later, it was understood that Gibbs fields play an important role in many applications of probability theory. In this section we define the Gibbs fields and discuss some of their properties.

We shall deal with random fields with a finite state space X defined over $\mathbb{Z}^d$. The realizations of the field will be denoted by $\omega = (\omega_k, k \in \mathbb{Z}^d)$, where ω_k is the value of the field at the site k.

Let V and W be two finite subsets of $\mathbb{Z}^d$ such that $V \subset W$ and $\mathrm{dist}(V, \mathbb{Z}^d \setminus W) > R$ for a given positive constant R. We can consider the following conditional probabilities:

$$\mathrm{P}(\omega_k = i_k, k \in V | \omega_k = i_k, k \in W \setminus V), \quad \text{where } i_k \in X \quad \text{for } k \in W.$$

Definition 22.1. *A random field is called a Gibbs field with memory R if, for any finite sets V and W as above, these conditional probabilities (whenever they are defined) depend only on those of the values i_k for which* $\mathrm{dist}(k, V) \le R$.

Note that the Gibbs fields can be viewed as generalizations of Markov chains. Indeed, consider a Markov chain on a finite state space. The realizations of the Markov chain will be denoted by $\omega = (\omega_k, k \in \mathbb{Z})$. Let k_1, k_2, l_1 and l_2 be integers such that $k_1 < l_1 \le l_2 < k_2$. Consider the conditional probabilities

$$f(i_{k_1}, \ldots, i_{l_1 - 1}, i_{l_2+1}, \ldots, i_{k_2}) =$$

$$\mathrm{P}(\omega_{l_1} = i_{l_1}, \ldots, \omega_{l_2} = i_{l_2} | \omega_{k_1} = i_{k_1}, \ldots, \omega_{l_1-1} = i_{l_1-1}, \omega_{l_2+1} = i_{l_2+1}, \ldots, \omega_{k_2} = i_{k_2})$$

with $i_{l_1}, \ldots, i_{l_2}$ fixed. It is easy to check that whenever f is defined, it depends only on i_{l_1-1} and i_{l_2+1} (see Problem 12, Chap. 5). Thus, a Markov chain is a Gibbs field with $d = 1$ and $R = 1$.

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Let us introduce the notion of the interaction energy. Let $N_{d,R}$ be the number of points of $\mathbb{Z}^d$ that belong to the closed ball of radius R centered at the origin. Let U be a real-valued function defined on $X^{N_{d,R}}$. As arguments of U we shall always take the values of the field in a ball of radius R centered at one of the points of $\mathbb{Z}^d$. We shall use the notation $U(\omega_k; \omega_{k'}, 0 < |k'-k| \leq R)$ for the value of U on a realization ω in the ball centered at k and call U the interaction energy with radius R.

For a finite set $V \subset \mathbb{Z}^d$, its R-neighborhood will be denoted by V^R,

$$V^R = \{k : \mathrm{dist}(V, k) \leq R\}.$$

Definition 22.2. *A Gibbs field with memory* $2R$ *is said to correspond to the interaction energy* U *if*

$$\mathrm{P}(\omega_k = i_k, k \in V | \omega_k = i_k, k \in V^{2R} \backslash V)$$
$$= \frac{\exp(-\sum_{k \in V^R} U(i_k; i_{k'}, 0 < |k'-k| \leq R))}{Z(i_k, k \in V^{2R} \backslash V)}, \tag{22.1}$$

where $Z = Z(i_k, k \in V^{2R} \backslash V)$ *is the normalization constant, which is called the partition function,*

$$Z(i_k, k \in V^{2R} \backslash V) = \sum_{\{i_k, k \in V\}} \exp(-\sum_{k \in V^R} U(i_k; i_{k'}, 0 < |k'-k| \leq R)).$$

The equality (22.1) for the conditional probabilities is sometimes called the Dobrushin-Lanford-Ruelle or, simply, DLR equation after three mathematicians who introduced the general notion of a Gibbs random field. The minus sign is adopted from statistical physics.

Let $\Omega(V)$ be the set of configurations $(\omega_k, k \in V)$. The sum

$$\sum_{k \in V^R} U(\omega_k; \omega_{k'}, 0 < |k'-k| \leq R)$$

is called the energy of the configuration $\omega \in \Omega(V)$. It is defined as soon as we have the boundary conditions ω_k, $k \in V^{2R} \backslash V$.

Theorem 22.3. *For any interaction energy* U *with finite radius, there exists at least one Gibbs field corresponding to* U*.*

Proof. Take a sequence of cubes V_i centered at the origin with sides of length $2i$. Fix arbitrary boundary conditions $\{\omega_k, k \in V_i^{2R} \backslash V_i\}$, for example $\omega_k = x$ for all $k \in V_i^{2R} \backslash V_i$, where x is a fixed element of X, and consider the probability distribution $\mathrm{P}_{V_i}(\cdot | \omega_k, k \in V_i^{2R} \backslash V_i)$ on the finite set $\Omega(V_i)$ given by (22.1) (with V_i instead of V).

Fix V_j. For $i > j$, the probability distribution $\mathrm{P}_{V_i}(\cdot | \omega_k, k \in V_i^{2R} \backslash V_i)$ induces a probability distribution on the set $\Omega(V_j)$. The space of such probability distributions is tight. (The set $\Omega(V_j)$ is finite, and we can consider an arbitrary metric on it. The property of tightness does not depend on the particular metric.)

Take a subsequence $\{j_s^{(1)}\}$ such that the induced probability distributions on $\Omega(V_1)$ converge to a limit $\mathrm{Q}^{(1)}$. Then find a subsequence $\{j_s^{(2)}\} \subseteq \{j_s^{(1)}\}$ such that the induced probability distributions on the space $\Omega(V_2)$ converge to a limit $\mathrm{Q}^{(2)}$. Since $\{j_s^{(2)}\} \subset \{j_s^{(1)}\}$, the probability distribution induced by $\mathrm{Q}^{(2)}$ on the space $\Omega(V_1)$ coincides with $\mathrm{Q}^{(1)}$. Arguing in the same way, we can find a subsequence $\{j_s^{(m)}\} \subseteq \{j_s^{(m-1)}\}$, for any $m \geq 1$, such that the probability distributions induced by $\mathrm{P}_{V_{j(m)}}(\cdot|\omega_k, k \notin V_{j(m)})$ on $\Omega(V_m)$ converge to a limit, which we denote by $\mathrm{Q}^{(m)}$. Since $\{j_s^{(m)}\} \subseteq \{j_s^{(m-1)}\}$, the probability distribution on $\Omega(V_{m-1})$ induced by $\mathrm{Q}^{(m)}$ coincides with $\mathrm{Q}^{(m-1)}$.

Then, for the sequence of probability distributions

$$\mathrm{P}_{V_{j_m^{(m)}}}(\cdot|\omega_k, k \in V^{2R}_{j_m^{(m)}} \backslash V_{j_m^{(m)}})$$

corresponding to the diagonal subsequence $\{j_m^{(m)}\}$, we have the following property:

For each m, the restrictions of the probability distributions to the set $\Omega(V_m)$ converge to a limit $\mathrm{Q}^{(m)}$, and the probability distribution induced by $\mathrm{Q}^{(m)}$ on $\Omega(V_{m-1})$ coincides with $\mathrm{Q}^{(m-1)}$. The last property is a version of the Consistency Conditions, and by the Kolmogorov Consistency Theorem, there exists a probability distribution Q defined on the natural σ-algebra of subsets of the space Ω of all possible configurations $\{\omega_k, k \in \mathbb{Z}^d\}$ whose restriction to each $\Omega(V_m)$ coincides with $\mathrm{Q}^{(m)}$ for any $m \geq 1$.

It remains to prove that Q is generated by a Gibbs random field corresponding to U. Let V be a finite subset of $\mathbb{Z}^d$, W a finite subset of $\mathbb{Z}^d$ such that $V^{2R} \subseteq W$, and let the values $\omega_k = i_k$ be fixed for $k \in W \backslash V$. We need to consider the conditional probabilities

$$q = \mathrm{Q}\{\omega_k = i_k, k \in V | \omega_k = i_k, k \in W \backslash V\}.$$

In fact, it is more convenient to deal with the ratio of the conditional probabilities corresponding to two different configurations, $\omega_k = \bar{i}_k$ and $\omega_k = \bar{\bar{i}}_k$, $k \in V$, which is equal to

$$\begin{aligned} q_1 &= \frac{\mathrm{Q}\{\omega_k = \bar{i}_k, k \in V | \omega_k = i_k, k \in W \backslash V\}}{\mathrm{Q}\{\omega_k = \bar{\bar{i}}_k, k \in V | \omega_k = i_k, k \in W \backslash V\}} \\ &= \frac{\mathrm{Q}\{\omega_k = \bar{i}_k, k \in V, \omega_k = i_k, k \in W \backslash V\}}{\mathrm{Q}\{\omega_k = \bar{\bar{i}}_k, k \in V, \omega_k = i_k, k \in W \backslash V\}}. \end{aligned} \tag{22.2}$$

It follows from our construction that the probabilities Q in this ratio are the limits found with the help of probability distributions $\mathrm{P}_{V_{j_m^{(m)}}}(\cdot|\omega_k, k \in V^{2R}_{j_m^{(m)}} \backslash V_{j_m^{(m)}})$. We can express the numerator in (22.2) as follows:

$$\begin{aligned}
&\mathrm{Q}\{\omega_k = \bar{i}_k, k \in V, \omega_k = i_k, k \in W \backslash V\} \\
&= \lim_{m\to\infty} \mathrm{P}_{V_{j_m^{(m)}}}(\omega_k = \bar{i}_k, k \in V, \omega_k = i_k, k \in W\backslash V | \omega_k = i_k, k \in V^{2R}_{j_m^{(m)}} \backslash V_{j_m^{(m)}}) \\
&= \lim_{m\to\infty} \frac{\sum_{\{i_k, k \in V_{j_m^{(m)}} \backslash W\}} \exp(-\sum_{k\in V^R_{j_m^{(m)}}} U(i_k; i_{k'}, 0 < |k'-k| \le R))}{Z(i_k, k \in V^{2R}_{j_m^{(m)}} \backslash V_{j_m^{(m)}})}.
\end{aligned}$$

A similar expression for the denominator in (22.2) is also valid. The difference between the expressions for the numerator and the denominator is that, in the first case, $i_k = \bar{i}_k$ for $k \in V$, while in the second, $i_k = \bar{\bar{i}}_k$ for $k \in V$.

Therefore, the corresponding expressions $U(i_k; i_{k'}, |k'-k| \le R)$ for k such that dist $(k, V) > R$ coincide in both cases, and

$$q_1 = \frac{r_1}{r_2},$$

where

$$r_1 = \exp(-\sum_{k\in V^R} U(i_k; i_{k'}, 0 < |k'-k| \le R)), \quad i_k = \bar{i}_k \;\text{ for } k \in V,$$

while

$$r_2 = \exp(-\sum_{k\in V^R} U(i_k; i_{k'}, 0 < |k'-k| \le R)), \quad i_k = \bar{\bar{i}}_k \;\text{ for } k \in V.$$

This is the required expression for q_1, which implies that Q is a Gibbs field. □

22.2 An Example of a Phase Transition

Theorem 22.3 is an analogue of the theorem on the existence of stationary distributions for finite Markov chains. In the ergodic case, this distribution is unique. In the case of multi-dimensional time, however, under very general conditions there can be different random fields corresponding to the same function U. The related theory is connected to the theory of phase transitions in statistical physics.

If $X = \{-1, 1\}$, $R = 1$ and $U(i_0; i_k, |k| = 1) = \pm\beta\sum_{|k|=1}(i_0 - i_k)^2$, the corresponding Gibbs field is called the Ising model with inverse temperature β (and zero magnetic field). The plus sign corresponds to the so-called ferromagnetic model; the minus sign corresponds to the so-called anti-ferromagnetic model. Again, the terminology comes from statistical mechanics. We shall consider here only the case of the ferromagnetic Ising model and prove the following theorem.

Theorem 22.4. *Consider the following interaction energy over $\mathbb{Z}^2$:*

$$U(\omega_0; \omega_k, |k| = 1) = \beta \sum_{|k|=1} (\omega_0 - \omega_k)^2.$$

If β is sufficiently large, there exist at least two different Gibbs fields corresponding to U.

Proof. As before, we consider the increasing sequence of squares V_i and plus-minus boundary conditions, i.e., either $\omega_k \equiv +1$, $k \notin V_i$, or $\omega_k \equiv -1$, $k \notin V_i$. The corresponding probability distributions on $\Omega(V_i)$ will be denoted by $\mathrm{P}^+_{V_i}$ and $\mathrm{P}^-_{V_i}$ respectively. We shall show that $\mathrm{P}^+_{V_i}(\omega_0 = +1) \geq 1 - \varepsilon(\beta)$, $\mathrm{P}^-_{V_i}(\omega_0 = -1) \geq 1 - \varepsilon(\beta)$, $\varepsilon(\beta) \to 0$ as $\beta \to \infty$. In other words, the Ising model displays strong memory of the boundary conditions for arbitrarily large i. Sometimes this kind of memory is called the long-range order. It is clear that the limiting distributions constructed with the help of the sequences $\mathrm{P}^+_{V_i}$ and $\mathrm{P}^-_{V_i}$ are different, which constitutes the statement of the theorem.

We shall consider only $\mathrm{P}^+_{V_i}$, since the case of $\mathrm{P}^-_{V_i}$ is similar. We shall show that a typical configuration with respect to $\mathrm{P}^+_{V_i}$ looks like a "sea" of $+1$'s surrounding small "islands" of -1's, and the probability that the origin belongs to this "sea" tends to 1 as $\beta \to \infty$ uniformly in i.

Take an arbitrary configuration $\omega \in \Omega(V_i)$. For each $k \in V_i$ such that $\omega_k = -1$ we construct a unit square centered at k with sides parallel to the coordinate axes, and then we slightly round off the corners of the square.

The union of these squares with rounded corners is denoted by $I(\omega)$. The boundary of $I(\omega)$ is denoted by $B(\omega)$. It consists of those edges of the squares where ω takes different values on different sides of the edge. A connected component of $B(\omega)$ is called a contour.

It is clear that each contour is a closed non-self-intersecting curve. If $B(\omega)$ does not have a contour containing the origin inside the domain it bounds, then $\omega_0 = +1$.

Given a contour Γ, we shall denote the domain it bounds by $\mathrm{int}(\Gamma)$. Let a contour Γ be such that the origin is contained inside $\mathrm{int}(\Gamma)$. The number of such contours of length n does not exceed $n3^{n-1}$.

Indeed, since the origin is inside $\mathrm{int}(\Gamma)$, the contour Γ intersects the semi-axis $\{z_1 = 0\} \cap \{z_2 < 0\}$. Of all the points belonging to $\Gamma \cap \{z_1 = 0\}$, let us select that with the smallest z_2 coordinate and call it the starting point of the contour. Since the contour has length n, there are no more than n choices for its starting point. Once the starting point of the contour is fixed, the edge of Γ containing it is also fixed. It is the horizontal segment centered at the starting point of the contour. Counting from the segment connected to the right end-point of this edge, there are no more than three choices for each next edge, since the contour is not self-intersecting. Therefore, there are no more than $n3^{n-1}$ contours in total.

Lemma 22.5 (Peierls Inequality). *Let Γ be a closed curve of length n. Then,*

$$\mathrm{P}^+_{V_i}(\{\omega \in \Omega(V_i) : \Gamma \subseteq B(\omega)\}) \le e^{-8\beta n}.$$

We shall prove the Peierls Inequality after completing the proof of Theorem 22.4.

Due to the Peierls Inequality, the probability $\mathrm{P}^+_{V_i}$ that there is at least one contour Γ with the origin inside $\mathrm{int}(\Gamma)$, is estimated from above by

$$\sum_{n=4}^{\infty} n 3^{n-1} e^{-8\beta n},$$

which tends to zero as $\beta \to \infty$. Therefore, the probability of $\omega_0 = -1$ tends to zero as $\beta \to \infty$. Note that this convergence is uniform in i. □

Proof of the Peierls Inequality. For each configuration $\omega \in \Omega(V_i)$, we can construct a new configuration $\omega' \in \Omega(V_i)$, where

$$\omega'_k = -\omega_k \quad \text{if} \quad k \in \mathrm{int}(\Gamma),$$

$$\omega'_k = \omega_k \quad \text{if} \quad k \notin \mathrm{int}(\Gamma).$$

For a given Γ, the correspondence $\omega \leftrightarrow \omega'$ is one-to-one.

Let $\omega \in \Omega(V_i)$ be such that $\Gamma \subseteq \mathcal{B}(\omega)$. Consider the ratio

$$\frac{\mathrm{P}^+_{V_i}(\omega)}{\mathrm{P}^+_{V_i}(\omega')} = \frac{\exp(-\beta \sum_{k:\mathrm{dist}(k,V_i)\le 1} \sum_{k':|k'-k|=1} (\omega_k - \omega_{k'})^2)}{\exp(-\beta \sum_{k:\mathrm{dist}(k,V_i)\le 1} \sum_{k':|k'-k|=1} (\omega'_k - \omega'_{k'})^2)}.$$

Note that all the terms in the above ratio cancel out, except those in which k and k' are adjacent and lie on the opposite sides of the contour Γ. For those terms, $|\omega_k - \omega_{k'}| = 2$, while $|\omega'_k - \omega'_{k'}| = 0$. The number of such terms is equal to $2n$ (one term for each side of each of the edges of Γ). Therefore,

$$\mathrm{P}^+_{V_i}(\omega) = e^{-8\beta n}\mathrm{P}^+_{V_i}(\omega').$$

By taking the sum over all $\omega \in \Omega(V_i)$ such that $\Gamma \subseteq \mathcal{B}(\omega)$, we obtain the statement of the lemma. □

One can show that the Gibbs field is unique if β is sufficiently small. The proof of this statement will not be discussed here.

The most difficult problem is to analyze Gibbs fields in neighborhoods of those values β_{cr} where the number of Gibbs fields changes. This problem is related to the so-called critical percolation problem and to conformal quantum field theory.