21

Stochastic Differential Equations

21.1 Existence of Strong Solutions to Stochastic Differential Equations

Stochastic differential equations arise when modeling prices of financial instruments, a variety of physical systems, and in many other branches of science. As we shall see in the next section, there is a deep relationship between stochastic differential equations and linear elliptic and parabolic partial differential equations.

As an example, let us try to model the motion of a small particle suspended in a liquid. Let us denote the position of the particle at time t by X_t. The liquid need not be at rest, and the velocity field will be denoted by $v(t, x)$, where t is time and x is a point in space. If we neglect the diffusion, the equation of motion is simply $dX_t/dt = v(t, X_t)$, or, formally, $dX_t = v(t, X_t)dt$.

On the other hand, if we assume that macroscopically the liquid is at rest, then the position of the particle can change only due to the molecules of liquid hitting the particle, and X_t would be modeled by the 3-dimensional Brownian motion, $X_t = W_t$, or, formally, $dX_t = dW_t$. More generally, we could write $dX_t = \sigma(t, X_t)dW_t$, where we allow σ to be non-constant, since the rate at which the molecules hit the particle may depend on the temperature and density of the liquid, which, in turn, are functions of space and time.

If both the effects of advection and diffusion are present, we can formally write the stochastic differential equation

$$dX_t = v(t, X_t)dt + \sigma(t, X_t)dW_t. \tag{21.1}$$

The vector v is called the drift vector, and σ, which may be a scalar or a matrix, is called the dispersion coefficient (matrix).

Now we shall specify the assumptions on v and σ, in greater generality than necessary for modeling the motion of a particle, and assign a strict meaning to the stochastic differential equation above. The main idea is to write the

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formal expression (21.1) in the integral form, in which case the right-hand side becomes a sum of an ordinary and a stochastic integral.

We assume that X_t takes values in the d-dimensional space, while W_t is an r-dimensional Brownian motion relative to a filtration $\mathcal{F}_t$. Let v be a Borel-measurable function from $\mathbb{R}^+ \times \mathbb{R}^d$ to $\mathbb{R}^d$, and σ a Borel-measurable function from $\mathbb{R}^+ \times \mathbb{R}^d$ to the space of $d \times r$ matrices. Thus, Eq. (21.1) can be re-written as

$$dX_t^i = v_i(t, X_t)dt + \sum_{j=1}^{r} \sigma_{ij}(t, X_t)dW_t^j, \quad 1 \le i \le d. \tag{21.2}$$

Let us further assume that the underlying filtration $\mathcal{F}_t$ satisfies the usual conditions and that we have a random d-dimensional vector ξ which is $\mathcal{F}_0$-measurable (and consequently independent of the Brownian motion W_t). This random vector is the initial condition for the stochastic differential equation (21.1).

Definition 21.1. *Suppose that the functions v and σ, the filtration, the Brownian motion, and the random variable ξ satisfy the assumptions above. A process X_t with continuous sample paths defined on the probability space $(\Omega, \mathcal{F}, \mathrm{P})$ is called a strong solution to the stochastic differential equation (21.1) with the initial condition ξ if:*

(1) X_t is adapted to the filtration $\mathcal{F}_t$.
(2) $X_0 = \xi$ almost surely.
(3) For every $0 \le t < \infty$, $1 \le i \le d$, and $1 \le j \le r$,

$$\int_0^t (|v_i(s, X_s)| + |\sigma_{ij}(s, X_s)|^2)ds < \infty \quad \textit{almost surely}$$

(which implies that $\sigma_{ij}(t, X_t) \in \mathcal{P}^*(W_t^j)$).
(4) For every $0 \le t < \infty$, the integral version of (21.2) holds almost surely:

$$X_t^i = X_0^i + \int_0^t v_i(s, X_s)ds + \sum_{j=1}^{r} \int_0^t \sigma_{ij}(s, X_s)dW_s^j, \quad 1 \le i \le d.$$

(Since the processes on both sides are continuous, they are indistinguishable.)

We shall refer to the solutions of stochastic differential equations as diffusion processes. Customarily the term "diffusion" refers to a strong Markov family of processes with continuous paths, with the generator being a second order partial differential operator (see Sect. 21.4). As will be discussed later in this chapter, under certain conditions on the coefficients, the solutions to stochastic differential equations form strong Markov families.

As with ordinary differential equations (ODE's), the first natural question which arises is that of the existence and uniqueness of strong solutions. As with ODE's, we shall require the Lipschitz continuity of the coefficients in the space variable, and certain growth estimates at infinity. The local Lipschitz continuity is sufficient to guarantee the local uniqueness of the solutions (as in the case of ODE's), while the uniform Lipschitz continuity and the growth conditions are needed for the global existence of solutions.

Theorem 21.2. *Suppose that the coefficients v and σ in Eq. (21.1) are Borel-measurable functions on $\mathbb{R}^+ \times \mathbb{R}^d$ and are uniformly Lipschitz continuous in the space variable. That is, for some constant c_1 and all $t \in \mathbb{R}^+$, $x, y \in \mathbb{R}^d$,*

$$|v_i(t,x) - v_i(t,y)| \le c_1 ||x - y||, \quad 1 \le i \le d, \tag{21.3}$$

$$|\sigma_{ij}(t,x) - \sigma_{ij}(t,y)| \le c_1 ||x - y||, \quad 1 \le i \le d, \ 1 \le j \le r. \tag{21.4}$$

Assume also that the coefficients do not grow faster than linearly, that is,

$$|v_i(t,x)| \le c_2(1 + ||x||), \quad |\sigma_{ij}(t,x)| \le c_2(1 + ||x||), \quad 1 \le i \le d, \ 1 \le j \le r, \tag{21.5}$$

for some constant c_2 and all $t \in \mathbb{R}^+$, $x \in \mathbb{R}^d$. Let W_t be a Brownian motion relative to a filtration $\mathcal{F}_t$, and ξ an $\mathcal{F}_0$-measurable $\mathbb{R}^d$-valued random vector that satisfies

$$\mathrm{E}||\xi||^2 < \infty.$$

Then there exists a strong solution to Eq. (21.1) with the initial condition ξ. The solution is unique in the sense that any two strong solutions are indistinguishable processes.

Remark 21.3. If we assume that (21.3) and (21.4) hold, then (21.5) is equivalent to the boundedness of

$$|v_i(t,0)|, \quad |\sigma_{ij}(t,0)|, \quad 1 \le i \le d, \ 1 \le j \le r,$$

as functions of t.

We shall prove the uniqueness part of Theorem 21.2 and indicate the main idea for the proof of the existence part. To prove uniqueness we need the Gronwall Inequality, which we formulate as a separate lemma (see Problem 1).

Lemma 21.4. *If a function $f(t)$ is continuous and non-negative on $[0, t_0]$, and*

$$f(t) \le K + L \int_0^t f(s) ds$$

holds for $0 \le t \le t_0$, with K and L positive constants, then

$$f(t) \le K e^{Lt}$$

for $0 \le t \le t_0$.

Proof of Theorem 21.2 (uniqueness part). Assume that both X_t and Y_t are strong solutions relative to the same Brownian motion, and with the same initial condition. We define the sequence of stopping times as follows:

$$\tau_n = \inf\{t \geq 0 : \max(||X_t||, ||Y_t||) \geq n\}.$$

For any t and t_0 such that $0 \leq t \leq t_0$,

$$\begin{aligned}
&\mathrm{E}||X_{t\wedge\tau_n} - Y_{t\wedge\tau_n}||^2 \\
&= \mathrm{E}|| \int_0^{t\wedge\tau_n} (v(s, X_s) - v(s, Y_s))ds \\
&+ \int_0^{t\wedge\tau_n} (\sigma(s, X_s) - \sigma(s, Y_s))dW_s||^2 \leq 2\mathrm{E}[\int_0^{t\wedge\tau_n} ||v(s, X_s) - v(s, Y_s)||ds]^2 \\
&+2\mathrm{E}\sum_{i=1}^{d}[\sum_{j=1}^{r} \int_0^{t\wedge\tau_n} (\sigma_{ij}(s, X_s) - \sigma_{ij}(s, Y_s))dW_s^j]^2 \\
&\quad \leq 2t\mathrm{E} \int_0^{t\wedge\tau_n} ||v(s, X_s) - v(s, Y_s)||^2 ds \\
&\quad +2\mathrm{E}\sum_{i=1}^{d}\sum_{j=1}^{r} \int_0^{t\wedge\tau_n} |\sigma_{ij}(s, X_s) - \sigma_{ij}(s, Y_s)|^2 ds \\
&\quad \leq (2dt + 2rd)c_1^2 \int_0^{t\wedge\tau_n} \mathrm{E}||X_{s\wedge\tau_n} - Y_{s\wedge\tau_n}||^2 ds \\
&\quad \leq (2dt_0 + 2rd)c_1^2 \int_0^{t} \mathrm{E}||X_{s\wedge\tau_n} - Y_{s\wedge\tau_n}||^2 ds.
\end{aligned}$$

By Lemma 21.4 with $K = 0$ and $L = (2dt_0 + 2rd)c_1^2$,

$$\mathrm{E}||X_{t\wedge\tau_n} - Y_{t\wedge\tau_n}||^2 = 0 \quad \text{for } 0 \leq t \leq t_0,$$

and, since t_0 can be taken to be arbitrarily large, this equality holds for all $t \geq 0$. Thus, the processes $X_{t\wedge\tau_n}$ and $Y_{t\wedge\tau_n}$ are modifications of one another, and, since they are continuous almost surely, they are indistinguishable. Now let $n \to \infty$, and notice that $\lim_{n\to\infty} \tau_n = \infty$ almost surely. Therefore, X_t and Y_t are indistinguishable. □

The existence of strong solutions can be proved using the Method of Picard Iterations. Namely, we define a sequence of processes $X_t^{(n)}$ by setting $X_t^{(0)} \equiv \xi$ and

$$X_t^{(n+1)} = \xi + \int_0^t v(s, X_s^{(n)})ds + \int_0^t \sigma(s, X_s^{(n)})dW_s, \quad t \geq 0$$

for $n \geq 0$. It is then possible to show that the integrals on the right-hand side are correctly defined for all n, and that the sequence of processes $X_t^{(n)}$

converges to a process X_t for almost all ω uniformly on any interval $[0, t_0]$. The process X_t is then shown to be the strong solution of Eq. (21.1) with the initial condition ξ.

Example (Black and Scholes formula). In this example we consider a model for the behavior of the price of a financial instrument (a share of stock, for example) and derive a formula for the price of an option. Let X_t be the price of a stock at time t. We assume that the current price (at time $t = 0$) is equal to P. One can distinguish two phenomena responsible for the change of the price over time. One is that the stock prices grow on average at a certain rate r, which, if we were to assume that r was constant, would lead to the equation $dX_t = rX_t dt$, since the rate of change is proportional to the price of the stock.

Let us, for a moment, assume that $r = 0$, and focus on the other phenomenon affecting the price change. One can argue that the randomness in X_t is due to the fact that every time someone buys the stock, the price increases by a small amount, and every time someone sells the stock, the price decreases by a small amount. The intervals of time between one buyer or seller and the next are also small, and whether the next person will be a buyer or a seller is a random event. It is also reasonable to assume that the typical size of a price move is proportional to the current price of the stock. We described intuitively the model for the evolution of the price X_t as a random walk, which will tend to the process defined by the equation $dX_t = \sigma X_t dW_t$ if we make the time step tend to zero. (This is a result similar to the Donsker Theorem, which states that the measure induced by a properly scaled simple symmetric random walk tends to the Wiener measure.) Here, σ is the volatility which we assumed to be a constant.

When we superpose the above two effects, we obtain the equation

$$dX_t = rX_t dt + \sigma X_t dW_t \tag{21.6}$$

with the initial condition $X_0 = P$. Let us emphasize that this is just a particular model for the stock price behavior, which may or may not be reasonable, depending on the situation. For example, when we modeled X_t as a random walk, we did not take into account that the presence of informed investors may cause it to be non-symmetric, or that the transition from the random walk to the diffusion process may be not justified if, with small probability, there are exceptionally large price moves.

Using the Ito formula (Theorem 20.27), with the martingale W_t and the function $f(t, x) = P \exp(\sigma x + (r - \frac{1}{2}\sigma^2)t)$, we obtain

$$f(t, W_t) = P \exp(\sigma W_t + (r - \frac{1}{2}\sigma^2)t) =$$

$$P + \int_0^t rP \exp(\sigma W_s + (r - \frac{1}{2}\sigma^2)s)ds + \int_0^t \sigma P \exp(\sigma W_s + (r - \frac{1}{2}\sigma^2)s)dW_s.$$

This means that

$$X_t = P\exp(\sigma W_t + (r - \frac{1}{2}\sigma^2)t)$$

is the solution of (21.6).

A *European call option* is the right to buy a share of the stock at an agreed price S (strike price) at an agreed time $t > 0$ (expiration time). The value of the option at time t is therefore equal to $(X_t - S)^+ = (X_t - S)\chi_{\{X_t \geq S\}}$ (if $X_t \leq S$, then the option becomes worthless). Assume that the behavior of the stock price is governed by (21.6), where r and σ were determined empirically based on previous observations. Then the expected value of the option at time t will be

$$V_t = \mathrm{E}(P\exp(\sigma W_t + (r - \frac{1}{2}\sigma^2)t) - S)^+$$

$$= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2t}}(Pe^{\sigma x + (r - \frac{1}{2}\sigma^2)t} - S)^+ dx.$$

The integral on the right-hand side of this formula can be simplified somewhat, but we leave this as an exercise for the reader.

Finally, the current value of the option may be less than the expected value at time t. This is due to the fact that the money spent on the option at the present time could instead be invested in a no-risk security with an interest rate γ, resulting in a larger buying power at time t. Therefore the expected value V_t should be discounted by the factor $e^{-\gamma t}$ to obtain the current value of the option. We obtain the Black and Scholes formula for the value of the option

$$V_0 = \frac{e^{-\gamma t}}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2t}}(Pe^{\sigma x + (r - \frac{1}{2}\sigma^2)t} - S)^+ dx.$$

Example (A Linear Equation). Let W_t be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ relative to a filtration $\mathcal{F}_t$. Let ξ be a square-integrable random variable measurable with respect to $\mathcal{F}_0$.

Consider the following one-dimensional stochastic differential equation with time-dependent coefficients

$$dX_t = (a(t)X_t + b(t))dt + \sigma(t)dW_t. \tag{21.7}$$

The initial data is $X_0 = \xi$. If $a(t), b(t)$, and $\sigma(t)$ are bounded measurable functions, by Theorem 21.2 this equation has a unique strong solution. In order to find an explicit formula for the solution, let us first solve the homogeneous ordinary differential equation

$$y'(t) = a(t)y(t)$$

with the initial data $y(0) = 1$. The solution to this equation is $y(t) = \exp(\int_0^t a(s)ds)$, as can be verified by substitution. We claim that the solution of (21.7) is

$$X_t = y(t)(\xi + \int_0^t \frac{b(s)}{y(s)} ds + \int_0^t \frac{\sigma(s)}{y(s)} dW_s). \tag{21.8}$$

Note that if $\sigma \equiv 0$, we recover the formula for the solution of a linear ODE, which can be obtained by the method of variation of constants. If we formally differentiate the right-hand side of (21.8), we obtain the expression on the right-hand side of (21.7). In order to justify this formal differentiation, let us apply Corollary 20.28 to the pair of semimartingales

$$X_t^1 = y(t) \text{ and } X_t^2 = \xi + \int_0^t \frac{b(s)}{y(s)} ds + \int_0^t \frac{\sigma(s)}{y(s)} dW_s.$$

Thus,

$$\begin{aligned} X_t &= y(t)(\xi + \int_0^t \frac{b(s)}{y(s)} ds + \int_0^t \frac{\sigma(s)}{y(s)} dW_s) \\ &= \xi + \int_0^t y(s) d(\int_0^s \frac{b(u)}{y(u)} du) + \int_0^t y(s) d(\int_0^s \frac{\sigma(u)}{y(u)} dW_u) + \int_0^t X_s^2 dy(s) \\ &= \xi + \int_0^t b(s) ds + \int_0^t \sigma(s) dW_s + \int_0^t a(s) X_s ds, \end{aligned}$$

where we used (20.11) to justify the last equality. We have thus demonstrated that X_t is the solution to (21.7) with initial data $X_0 = \xi$.

Example (the Ornstein-Uhlenbeck Process). Consider the stochastic differential equation

$$dX_t = -aX_t dt + \sigma dW_t, \quad X_0 = \xi. \tag{21.9}$$

This is a particular case of (21.7) with $a(t) \equiv -a$, $b(t) \equiv 0$, and $\sigma(t) \equiv \sigma$. By (21.8), the solution is

$$X_t = e^{-at}\xi + \sigma \int_0^t e^{-a(t-s)} dW_s.$$

This process is called the Ornstein-Uhlenbeck Process with parameters (a, σ) and initial condition ξ. Since the integrand $e^{-a(t-s)}$ is a deterministic function, the integral is a Gaussian random process independent of ξ (see Problem 1, Chap. 20). If ξ is Gaussian, then X_t is a Gaussian process. Its expectation and covariance can be easily calculated:

$$m(t) = \mathrm{E}X_t = e^{-at}\mathrm{E}\xi,$$

$$\begin{aligned} b(s,t) = \mathrm{E}(X_s X_t) &= e^{-as} e^{-at} \mathrm{E}\xi^2 + \sigma^2 \int_0^{s \wedge t} e^{-a(s-u) - a(t-u)} du \\ &= e^{-a(s+t)}(\mathrm{E}\xi^2 + \sigma^2 \frac{e^{2as\wedge t} - 1}{2a}). \end{aligned}$$

In particular, if ξ is Gaussian with $\mathrm{E}\xi = 0$ and $\mathrm{E}\xi^2 = \frac{\sigma^2}{2a}$, then

$$b(s,t) = \frac{\sigma^2 e^{-a|s-t|}}{2a}.$$

Since the covariance function of the process depends on the difference of the arguments, the process is wide-sense stationary, and since it is Gaussian, it is also strictly stationary.

21.2 Dirichlet Problem for the Laplace Equation

In this section we show that solutions to the Dirichlet problem for the Laplace equation can be expressed as functionals of the Wiener process.

Let D be an open bounded domain in $\mathbb{R}^d$, and let $f : \partial D \to \mathbb{R}$ be a continuous function defined on the boundary. We shall consider the following partial differential equation

$$\Delta u(x) = 0 \quad \text{for } x \in D \tag{21.10}$$

with the boundary condition

$$u(x) = f(x) \quad \text{for } x \in \partial D. \tag{21.11}$$

This pair, Eq. (21.10) and the boundary condition (21.11), is referred to as the Dirichlet problem for the Laplace equation with the boundary condition $f(x)$.

By a solution of the Dirichlet problem we mean a function u which satisfies (21.10), (21.11) and belongs to $C^2(D) \cap C(\overline{D})$.

Let W_t be a d-dimensional Brownian motion relative to a filtration $\mathcal{F}_t$. Without loss of generality we may assume that $\mathcal{F}_t$ is the augmented filtration constructed in Sect. 19.4. Let $W_t^x = x + W_t$. For a point $x \in \overline{D}$, let τ^x be the first time the process W_t^x reaches the boundary of D, that is

$$\tau^x(\omega) = \inf\{t \geq 0 : W_t^x(\omega) \notin D\}.$$

In Sect. 19.6 we showed that the function

$$u(x) = \mathrm{E}f(W_{\tau^x}^x) \tag{21.12}$$

defined in $\overline{D}$ is harmonic inside D, that is, it belongs to $C^2(D)$ and satisfies (21.10). From the definition of $u(x)$ it is clear that it satisfies (21.11). It remains to study the question of continuity of $u(x)$ at the points of the boundary of D.

Let

$$\sigma^x(\omega) = \inf\{t > 0 : W_t^x(\omega) \notin D\}.$$

Note that here $t > 0$ on the right-hand side, in contrast to the definition of τ^x. Let us verify that σ^x is a stopping time. Define an auxiliary family of stopping times

$$\tau^{x,s}(\omega) = \inf\{t \geq s : W_t^x(\omega) \notin D\}$$

(see Lemma 13.15). Then, for $t > 0$,

$$\{\sigma^x \leq t\} = \bigcup_{n=1}^{\infty} \{\tau^{x,\frac{1}{n}} \leq t\} \in \mathcal{F}_t.$$

In addition,

$$\{\sigma^x = 0\} = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \{\tau^{0,\frac{1}{n}} \leq 1/m\} \in \bigcap_{m=1}^{\infty} \mathcal{F}_{1/m} = \mathcal{F}_{0+} = \mathcal{F}_0,$$

where we have used the right-continuity of the augmented filtration. This demonstrates that σ^x is a stopping time. Also note that since $\{\sigma^x = 0\} \in \mathcal{F}_0$, the Blumenthal Zero-One Law implies that $\mathrm{P}(\sigma^x = 0)$ is either equal to one or to zero.

Definition 21.5. *A point $x \in \partial D$ is called regular if $\mathrm{P}(\sigma^x = 0) = 1$, and irregular if $\mathrm{P}(\sigma^x = 0) = 0$.*

Regularity means that a typical Brownian path which starts at $x \in \partial D$ does not immediately enter D and stay there for an interval of time.

Example. Let $D = \{x \in \mathbb{R}^d, 0 < ||x|| < 1\}$, where $d \geq 2$, that is, D is a punctured unit ball. The boundary of D consists of the unit sphere and the origin. Since Brownian motion does not return to zero for $d \geq 2$, the origin is an irregular point for Brownian motion in D.

Similarly, let $D = B^d \setminus \{x \in \mathbb{R}^d : x_2 = \ldots = x_d = 0\}$. ($D$ is the set of points in the unit ball that do not belong to the x_1-axis.) The boundary of D consists of the unit sphere and the segment $\{x \in \mathbb{R}^d : -1 < x_1 < 1, x_2 = \ldots = x_d = 0\}$. If $d \geq 3$, the segment consists of irregular points.

Example. Let $x \in \partial D$, $y \in \mathbb{R}^d$, $||y|| = 1$, $0 < \theta \leq \pi$, and $r > 0$. The cone with vertex at x, direction y, opening θ, and radius r is the set

$$C_x(y, \theta, r) = \{z \in \mathbb{R}^d : ||z - x|| \leq r, (z - x, y) \geq ||z - x|| \cos\theta\}.$$

We shall say that a point $x \in \partial D$ satisfies the exterior cone condition if there is a cone $C_x(y, \theta, r)$ with y, θ, and r as above such that $C_x(y, \theta, r) \subseteq \mathbb{R}^d \setminus D$. It is not difficult to show (see Problem 8) that if x satisfies the exterior cone condition, then it is regular. In particular, if D is a domain with a smooth boundary, then all the points of ∂D are regular.

The question of regularity of a point $x \in \partial D$ is closely related to the continuity of the function u given by (21.12) at x.

Theorem 21.6. *Let D be a bounded open domain in $\mathbb{R}^d$, $d \geq 2$, and $x \in \partial D$. Then x is regular if and only if for any continuous function $f : \partial D \to \mathbb{R}$, the function u defined by (21.12) is continuous at x, that is*

$$\lim_{y \to x, y \in \overline{D}} \mathrm{E} f(W^y_{\tau^y}) = f(x). \tag{21.13}$$

Proof. Assume that x is regular. First, let us show that, with high probability, a Brownian trajectory which starts near x exits D fast. Take ε and δ such that $0 < \varepsilon < \delta$, and define an auxiliary function

$$g^\delta_\varepsilon(y) = \mathrm{P}(W^y_t \in D \ \text{ for } \ \varepsilon \leq t \leq \delta).$$

This is a continuous function of $y \in \overline{D}$, since the indicator function of the set $\{\omega : W^y_t(\omega) \in D \ \text{ for } \ \varepsilon \leq t \leq \delta\}$ tends to the indicator function of the set $\{\omega : W^{y_0}_t(\omega) \in D \ \text{ for } \ \varepsilon \leq t \leq \delta\}$ almost surely as $y \to y_0$. Note that

$$\lim_{\varepsilon \downarrow 0} g^\delta_\varepsilon(y) = \mathrm{P}(W^y_t \in D \ \text{ for } \ 0 < t \leq \delta) = \mathrm{P}(\sigma^y > \delta),$$

which implies that the right-hand side is an upper semicontinuous function of y, since it is a limit of a decreasing sequence of continuous functions. Therefore,

$$\limsup_{y \to x, y \in \overline{D}} \mathrm{P}(\tau^y > \delta) \leq \limsup_{y \to x, y \in \overline{D}} \mathrm{P}(\sigma^y > \delta) \leq \mathrm{P}(\sigma^x > \delta) = 0,$$

since x is a regular point. We have thus demonstrated that

$$\lim_{y \to x, y \in \overline{D}} \mathrm{P}(\tau^y > \delta) = 0$$

for any $\delta > 0$.

Next we show that, with high probability, a Brownian trajectory which starts near x exits D through a point on the boundary which is also near x. Namely, we wish to show that for $r > 0$,

$$\lim_{y \to x, y \in \overline{D}} \mathrm{P}(||x - W^y_{\tau^y}|| > r) = 0. \tag{21.14}$$

Take an arbitrary $\varepsilon > 0$. We can then find $\delta > 0$ such that

$$\mathrm{P}(\max_{0 \leq t \leq \delta} ||W_t|| > r/2) < \varepsilon/2.$$

We can also find a neighborhood U of x such that $||y - x|| < r/2$ for $y \in U$, and

$$\sup_{y \in \overline{D} \cap U} \mathrm{P}(\tau^y > \delta) < \varepsilon/2.$$

Combining the last two estimates, we obtain

$$\sup_{y \in \overline{D} \cap U} \mathrm{P}(||x - W^y_{\tau^y}|| > r) < \varepsilon,$$

which justifies (21.14).

Now let f be a continuous function defined on the boundary, and let $\varepsilon > 0$. Take $r > 0$ such that $\sup_{z \in \partial D, ||z-x|| \le r} |f(x) - f(z)| < \varepsilon$. Then

$$|\mathrm{E} f(W^y_{\tau^y}) - f(x)| \le$$

$$\sup_{z \in \partial D, ||z-x|| \le r} |f(x) - f(z)| + 2\mathrm{P}(||x - W^y_{\tau^y}|| > r) \sup_{z \in \partial D} |f(y)|.$$

The first term on the right-hand side here is less than ε, while the second one tends to zero as $y \to x$ by (21.14). We have thus demonstrated that (21.13) holds.

Now let us prove that x is regular if (21.13) holds for every continuous f. Suppose that x is not regular. Since $\sigma^x > 0$ almost surely, and a Brownian trajectory does not return to the origin almost surely for $d \ge 2$, we conclude that $||W^x_{\sigma^x} - x|| > 0$ almost surely. We can then find $r > 0$ such that

$$\mathrm{P}(||W^x_{\sigma^x} - x|| \ge r) > 1/2.$$

Let S_n be the sphere centered at x with radius $r_n = 1/n$. We claim that if $r_n < r$, there is a point $y_n \in S_n \cap D$ such that

$$\mathrm{P}(||W^{y_n}_{\tau^{y_n}} - x|| \ge r) > 1/2. \tag{21.15}$$

Indeed, let τ^x_n be the first time the process W^x_t reaches S_n. Let μ_n be the measure on $S_n \cap D$ defined by $\mu_n(A) = \mathrm{P}(\tau^x_n < \sigma^x; W^x_{\tau^x_n} \in A)$, where A is a Borel subset of $S_n \cap D$. Then, due to the Strong Markov Property of Brownian motion,

$$\begin{aligned} 1/2 < \mathrm{P}(||W^x_{\sigma^x} - x|| \ge r) &= \int_{S_n \cap D} \mathrm{P}(||W^y_{\tau^y} - x|| \ge r) d\mu_n(y) \\ &\le \sup_{y \in S_n \cap D} \mathrm{P}(||W^y_{\tau^y} - x|| \ge r), \end{aligned}$$

which justifies (21.15). Now we can take a continuous function f such that $0 \le f(y) \le 1$ for $y \in \partial D$, $f(x) = 1$, and $f(y) = 0$ if $||y - x|| \ge r$. By (21.15),

$$\limsup_{n \to \infty} \mathrm{E} f(W^{y_n}_{\tau^{y_n}}) \le 1/2 < f(x),$$

which contradicts (21.13). □

Now we can state the existence and uniqueness result.

Theorem 21.7. *Let D be a bounded open domain in $\mathbb{R}^d$, $d \ge 2$, and f a continuous function on ∂D. Assume that all the points of ∂D are regular. Then the Dirichlet problem for the Laplace equation (21.10)–(21.11) has a unique solution. The solution is given by (21.12).*

Proof. The existence follows from Theorem 21.6. If u_1 and u_2 are two solutions, then $u = u_1 - u_2$ is a solution to the Dirichlet problem with zero boundary condition. A harmonic function which belongs to $C^2(D) \cap C(\overline{D})$ takes the maximal and the minimal values on the boundary of the domain. This implies that u is identically zero, that is $u_1 = u_2$. □

Probabilistic techniques can also be used to justify the existence and uniqueness of solutions to more general elliptic and parabolic partial differential equations. However, we shall now assume that the boundary of the domain is smooth, and thus we can bypass the existence and uniqueness questions, instead referring to the general theory of PDE's. In the next section we shall demonstrate that the solutions to PDE's can be expressed as functionals of the corresponding diffusion processes.

21.3 Stochastic Differential Equations and PDE's

First we consider the case in which the drift and the dispersion matrix do not depend on time. Let X_t be the strong solution of the stochastic differential equation

$$dX_t^i = v_i(X_t)dt + \sum_{j=1}^{r} \sigma_{ij}(X_t)dW_t^j, \quad 1 \leq i \leq d, \tag{21.16}$$

with the initial condition $X_0 = x \in \mathbb{R}^d$, where the coefficients v and σ satisfy the assumptions of Theorem 21.2. In fact, Eq. (21.16) defines a family of processes X_t which depend on the initial point x and are defined on a common probability space. When the dependence of the process on the initial point needs to be emphasized, we shall denote the process by X_t^x. (The superscript x is not to be confused with the superscript i used to denote the i-th component of the process.)

Let $a_{ij}(x) = \sum_{k=1}^{r} \sigma_{ik}(x)\sigma_{jk}(x) = (\sigma\sigma^*)_{ij}(x)$. This is a square non-negative definite symmetric matrix which will be called the diffusion matrix corresponding to the family of processes X_t^x. Let us consider the differential operator L which acts on functions $f \in C^2(\mathbb{R}^d)$ according to the formula

$$Lf(x) = \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}(x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} v_i(x)\frac{\partial f(x)}{\partial x_i}. \tag{21.17}$$

This operator is called the infinitesimal generator of the family of diffusion processes X_t^x. Let us show that for $f \in C^2(\mathbb{R}^d)$ which is bounded together with its first and second partial derivatives,

$$Lf(x) = \lim_{t \downarrow 0} \frac{\mathrm{E}[f(X_t^x) - f(x)]}{t}. \tag{21.18}$$

In fact, the term "infinitesimal generator" of a Markov family of processes X_t^x usually refers to the right-hand side of this formula. (The Markov property of the solutions to SDE's will be discussed below.) By the Ito Formula, the expectation on the right-hand side of (21.18) is equal to

$$\mathrm{E}[\int_0^t Lf(X_s^x)ds + \int_0^t \sum_{i=1}^d \sum_{j=1}^r \frac{\partial f(X_s^x)}{\partial x_i}\sigma_{ij}(X_s^x)dW_s^j] = \mathrm{E}[\int_0^t Lf(X_s^x)ds],$$

since the expectation of the stochastic integral is equal to zero. Since Lf is bounded, the Dominated Convergence Theorem implies that

$$\lim_{t\downarrow 0} \frac{\mathrm{E}[\int_0^t Lf(X_s^x)ds]}{t} = Lf(x).$$

The coefficients of the operator L can be obtained directly from the law of the process X_t instead of the representation of the process as a solution of the stochastic differential equation. Namely,

$$v_i(x) = \lim_{t\downarrow 0} \frac{\mathrm{E}[(X_t^x)^i - x_i]}{t}, \quad a_{ij}(x) = \lim_{t\downarrow 0} \frac{\mathrm{E}[((X_t^x)^i - x_i)((X_t^x)^j - x_j)]}{t}.$$

We leave the proof of this statement to the reader.

Now let L be any differential operator given by (21.17). Let D be a bounded open domain in $\mathbb{R}^d$ with a smooth boundary ∂D. We shall consider the following partial differential equation

$$Lu(x) + q(x)u(x) = g(x) \quad \text{for } x \in D, \tag{21.19}$$

with the boundary condition

$$u(x) = f(x) \quad \text{for } x \in \partial D. \tag{21.20}$$

This pair, Eq. (21.19) and the boundary condition (21.20), is referred to as the Dirichlet problem for the operator L with the potential $q(x)$, the right-hand side $g(x)$, and the boundary condition $f(x)$. We assume that the coefficients $a_{ij}(x)$, $v_i(x)$ of the operator L, and the functions $q(x)$ and $g(x)$ are continuous on the closure of D (denoted by $\overline{D}$), while $f(x)$ is assumed to be continuous on ∂D.

Definition 21.8. *An operator L of the form (21.17) is called uniformly elliptic on D if there is a positive constant k such that*

$$\sum_{i=1}^d \sum_{j=1}^d a_{ij}(x) y_i y_j \geq k||y||^2 \tag{21.21}$$

for all $x \in D$ and all vectors $y \in \mathbb{R}^d$.

We shall use the following fact from the theory of partial differential equations (see "Partial Differential Equations" by A. Friedman, for example).

Theorem 21.9. *If a_{ij}, v_i, q, and g are Lipschitz continuous on $\overline{D}$, f is continuous on ∂D, the operator L is uniformly elliptic on D, and $q(x) \leq 0$ for $x \in \overline{D}$, then there is a unique solution $u(x)$ to (21.19)–(21.20) in the class of functions which belong to $C^2(D) \cap C(\overline{D})$.*

Let $\sigma_{ij}(x)$ and $v_i(x)$, $1 \leq i \leq d$, $1 \leq j \leq r$, be Lipschitz continuous on $\overline{D}$. It is not difficult to see that we can then extend them to bounded Lipschitz continuous functions on the entire space $\mathbb{R}^d$ and define the family of processes X_t^x according to (21.16). Let τ_D^x be the stopping time equal to the time of the first exit of the process X_t^x from the domain D, that is

$$\tau_D^x = \inf\{t \geq 0 : X_t^x \notin D\}.$$

By using Lemma 20.18, we can see that the stopped process $X_{t \wedge \tau_D^x}^x$ and the stopping time τ_D^x do not depend on the values of $\sigma_{ij}(x)$ and $v_i(x)$ outside of $\overline{D}$.

When L is the generator of the family of diffusion processes, we shall express the solution $u(x)$ to (21.19)–(21.20) as a functional of the process X_t^x. First, we need a technical lemma.

Lemma 21.10. *Suppose that $\sigma_{ij}(x)$ and $v_i(x)$, $1 \leq i \leq d$, $1 \leq j \leq r$, are Lipschitz continuous on $\overline{D}$, and the generator of the family of processes X_t^x is uniformly elliptic in D. Then*

$$\sup_{x \in \overline{D}} \mathrm{E}\tau_D^x < \infty.$$

Proof. Let B be an open ball so large that $\overline{D} \subset B$. Since the boundary of D is smooth and the coefficients $\sigma_{ij}(x)$ and $v_i(x)$ are Lipschitz continuous in $\overline{D}$, we can extend them to Lipschitz continuous functions on $\overline{B}$ in such a way that L becomes uniformly elliptic on B. Let $\varphi \in C^2(B) \cap C(\overline{B})$ be the solution of the equation $L\varphi(x) = 1$ for $x \in B$ with the boundary condition $\varphi(x) = 0$ for $x \in \partial B$. The existence of the solution is guaranteed by Theorem 21.9. By the Ito Formula,

$$\mathrm{E}\varphi(X_{t \wedge \tau_D^x}^x) - \varphi(x) = \mathrm{E}\int_0^{t \wedge \tau_D^x} L\varphi(X_s^x)ds = \mathrm{E}t \wedge \tau_D^x.$$

(The use of the Ito Formula is justified by the fact that φ is twice continuously differentiable in a neighborhood of $\overline{D}$, and thus there is a function $\psi \in C_0^2(\mathbb{R}^2)$, which coincides with φ in a neighborhood of $\overline{D}$. Theorem 20.27 can now be applied to the function ψ.)

Thus,

$$\sup_{x \in \overline{D}} \mathrm{E}\,(t \wedge \tau_D^x) \leq 2 \sup_{x \in \overline{D}} |\varphi(x)|,$$

which implies the lemma if we let $t \to \infty$. □

Theorem 21.11. *Suppose that $\sigma_{ij}(x)$ and $v_i(x)$, $1 \le i \le d$, $1 \le j \le r$, are Lipschitz continuous on $\overline{D}$, and the generator L of the family of processes X_t^x is uniformly elliptic on D. Assume that the potential $q(x)$, the right-hand side $g(x)$, and the boundary condition $f(x)$ of the Dirichlet problem (21.19)–(21.20) satisfy the assumptions of Theorem 21.9. Then the solution to the Dirichlet problem can be written as follows:*

$$u(x) = \mathrm{E}[f(X^x_{\tau^x_D})\exp(\int_0^{\tau^x_D} q(X^x_s)ds) - \int_0^{\tau^x_D} g(X^x_s)\exp(\int_0^s q(X^x_u)du)ds].$$

Proof. As before, we can extend $\sigma_{ij}(x)$ and $v_i(x)$ to Lipschitz continuous bounded functions on $\mathbb{R}^d$, and the potential $q(x)$ to a continuous function on $\mathbb{R}^d$, satisfying $q(x) \le 0$ for all x. Assume at first that $u(x)$ can be extended as a C^2 function to a neighborhood of $\overline{D}$. Then it can be extended as a C^2 function with compact support to the entire space $\mathbb{R}^d$. We can apply the integration by parts (20.23) to the pair of semimartingales $u(X_t^x)$ and $\exp(\int_0^t q(X_s^x)ds)$. In conjunction with (20.11) and the Ito formula,

$$\begin{aligned} u(X^x_t)\exp(\int_0^t q(X^x_s)ds) = u(x) &+ \int_0^t u(X^x_s)\exp(\int_0^s q(X^x_u)du)q(X^x_s)ds \\ &+ \int_0^t \exp(\int_0^s q(X^x_u)du)Lu(X^x_s)ds \\ &+ \sum_{i=1}^d\sum_{j=1}^r \int_0^t \exp(\int_0^s q(X^x_u)du)\frac{\partial u}{\partial x_i}(X^x_s)\sigma_{ij}(X^x_s)dW^j_s. \end{aligned}$$

Notice that, by (21.19), $Lu(X_s^x) = g(X_s^x) - q(X_s^x)u(X_s^x)$ for $s \le \tau_D^x$. Therefore, after replacing t by $t \wedge \tau_D^x$ and taking the expectation on both sides, we obtain

$$\mathrm{E}(u(X^x_{t\wedge\tau^x_D})\exp(\int_0^{t\wedge\tau^x_D} q(X^x_s)ds)) = u(x)$$

$$+\mathrm{E}\int_0^{t\wedge\tau^x_D} g(X^x_s)\exp(\int_0^s q(X^x_u)du)ds.$$

By letting $t \to \infty$, which is justified by the Dominated Convergence Theorem, since $\mathrm{E}\tau_D^x$ is finite, we obtain

$$u(x) = \mathrm{E}[u(X^x_{\tau^x_D})\exp(\int_0^{\tau^x_D} q(X^x_s)ds) - \mathrm{E}\int_0^{\tau^x_D} g(X^x_s)\exp(\int_0^s q(X^x_u)du)ds]. \tag{21.22}$$

Since $X^x_{\tau^x_D} \in \partial D$ and $u(x) = f(x)$ for $x \in \partial D$, this is exactly the desired expression for $u(x)$.

At the beginning of the proof, we assumed that $u(x)$ can be extended as a C^2 function to a neighborhood of $\overline{D}$. In order to remove this assumption, we consider a sequence of domains $D_1 \subseteq D_2 \subseteq \ldots$ with smooth boundaries, such that $\overline{D}_n \subset D$ and $\bigcup_{n=1}^\infty D_n = D$. Let $\tau^x_{D_n}$ be the stopping times corresponding to the domains D_n. Then $\lim_{n\to\infty} \tau^x_{D_n} = \tau^x_D$ almost surely for all $x \in D$. Since u is twice differentiable in D, which is an open neighborhood of $\overline{D}_n$, we have

$$u(x) = \mathrm{E}[u(X^x_{\tau^x_{D_n}}) \exp(\int_0^{\tau^x_{D_n}} q(X^x_s)ds) - \mathrm{E}\int_0^{\tau^x_{D_n}} g(X^x_s) \exp(\int_0^s q(X^x_u)du)ds]$$

for $x \in D_n$. By taking the limit as $n \to \infty$ and using the dominated convergence theorem, we obtain (21.22). □

Example. Let us consider the partial differential equation

$$Lu(x) = -1 \quad \text{for } x \in D$$

with the boundary condition

$$u(x) = 0 \quad \text{for } x \in \partial D.$$

By Theorem 21.11, the solution to this equation is simply the expectation of the time it takes for the process to exit the domain, that is $u(x) = \mathrm{E}\tau^x_D$.

Example. Let us consider the partial differential equation

$$Lu(x) = 0 \quad \text{for } x \in D$$

with the boundary condition

$$u(x) = f \quad \text{for } x \in \partial D.$$

By Theorem 21.11, the solution of this equation is

$$u(x) = \mathrm{E}f(X^x_{\tau^x_D}) = \int_{\partial D} f(y)d\mu_x(y),$$

where $\mu_x(A) = \mathrm{P}(X^x_{\tau^x_D} \in A)$, $A \in \mathcal{B}(\partial D)$, is the measure on ∂D induced by the random variable $X^x_{\tau^x_D}$.

Now let us explore the relationship between diffusion processes and parabolic partial differential equations. Let L be a differential operator with time-dependent coefficients, which acts on functions $f \in C^2(\mathbb{R}^d)$ according to the formula

$$Lf(x) = \frac{1}{2}\sum_{i=1}^d \sum_{j=1}^d a_{ij}(t,x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^d v_i(t,x)\frac{\partial f(x)}{\partial x_i}.$$

We shall say that L is uniformly elliptic on $D \subseteq \mathbb{R}^{1+d}$ (with t considered as a parameter) if

$$\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}(t,x)y_i y_j \geq k||y||^2$$

for some positive constant k, all $(t,x) \in D$, and all vectors $y \in \mathbb{R}^d$. Without loss of generality, we may assume that a_{ij} form a symmetric matrix, in which case $a_{ij}(t,x) = (\sigma\sigma^*)_{ij}(t,x)$ for some matrix $\sigma(t,x)$.

Let $T_1 < T_2$ be two moments of time. We shall be interested in the solutions to the backward parabolic equation

$$\frac{\partial u(t,x)}{\partial t} + Lu(t,x) + q(t,x)u(t,x) = g(t,x) \quad \text{for } (t,x) \in (T_1,T_2) \times \mathbb{R}^d \quad (21.23)$$

with the terminal condition

$$u(T_2,x) = f(x) \quad \text{for } x \in \mathbb{R}^d. \quad (21.24)$$

The function $u(t,x)$ is called the solution to the Cauchy problem (21.23)–(21.24). Let us formulate an existence and uniqueness theorem for the solutions to the Cauchy problem (see "Partial Differential Equations" by A. Friedman, for example).

Theorem 21.12. *Assume that $q(t,x)$ and $g(t,x)$ are bounded, continuous, and uniformly Lipschitz continuous in the space variables on $(T_1,T_2] \times \mathbb{R}^d$, and that $\sigma_{ij}(t,x)$ and $v_i(t,x)$ are continuous and uniformly Lipschitz continuous in the space variables on $(T_1,T_2] \times \mathbb{R}^d$. Assume that they do not grow faster than linearly, that is (21.5) holds, and that $f(x)$ is bounded and continuous on $\mathbb{R}^d$. Also assume that the operator L is uniformly elliptic on $(T_1,T_2] \times \mathbb{R}^d$.*

Then there is a unique solution $u(t,x)$ to the problem (21.23)–(21.24) in the class of functions which belong to $C^{1,2}((T_1,T_2) \times \mathbb{R}^d) \cap C_b((T_1,T_2] \times \mathbb{R}^d)$. (These are the functions which are bounded and continuous in $(T_1,T_2] \times \mathbb{R}^d$, and whose partial derivative in t and all second order partial derivatives in x are continuous in $(T_1,T_2) \times \mathbb{R}^d$.)

Remark 21.13. In textbooks on PDE's, this theorem is usually stated under the assumption that σ_{ij} and v_i are bounded. As will be explained below, by using the relationship between PDE's and diffusion processes, it is sufficient to assume that σ_{ij} and v_i do not grow faster than linearly.

Let us now express the solution to the Cauchy problem as a functional of the corresponding diffusion process. For $t \in (T_1,T_2]$, define $X_s^{t,x}$ to be the solution to the stochastic differential equation

$$dX_s^i = v_i(t+s,X_s)ds + \sum_{j=1}^{r} \sigma_{ij}(t+s,X_s)dW_s^j, \quad 1 \leq i \leq d, \quad s \leq T_2 - t, \quad (21.25)$$

with the initial condition $X_0^{t,x} = x$. Let

$$a_{ij}(t,x) = \sum_{k=1}^{r} \sigma_{ik}(t,x)\sigma_{jk}(t,x) = (\sigma\sigma^*)_{ij}(t,x).$$

Theorem 21.14. *Suppose that the assumptions regarding the operator L and the functions $q(t,x)$, $g(t,x)$, and $f(x)$, formulated in Theorem 21.12, are satisfied. Then the solution to the Cauchy problem can be written as follows:*

$$u(t,x) = \mathrm{E}[f(X_{T_2-t}^{t,x}) \exp(\int_0^{T_2-t} q(t+s, X_s^{t,x})ds)$$
$$- \int_0^{T_2-t} g(t+s, X_s^{t,x}) \exp(\int_0^s q(t+u, X_u^{t,x})du)ds].$$

This expression for $u(t,x)$ is called the Feynman-Kac formula.

The proof of Theorem 21.14 is the same as that of Theorem 21.11, and therefore is left to the reader.

Remark 21.15. Let us assume that we have Theorems 21.12 and 21.14 only for the case in which the coefficients are bounded.

Given $\sigma_{ij}(t,x)$ and $v_i(t,x)$, which are continuous, uniformly Lipschitz continuous in the space variables, and do not grow faster than linearly, we can find continuous functions $\sigma_{ij}^n(t,x)$ and $v_i^n(t,x)$ which are uniformly Lipschitz continuous in the space variables and bounded on $(T_1, T_2] \times \mathbb{R}^d$, and which coincide with $\sigma_{ij}(t,x)$ and $v_i(t,x)$, respectively, for $||x|| \leq n$.

Let $u^n(t,x)$ be the solution to the corresponding Cauchy problem. By Theorem 21.14 for the case of bounded coefficients, it is possible to show that u^n converge point-wise to some function u, which is a solution to the Cauchy equation with the coefficients which do not grow faster than linearly, and that this solution is unique. The details of this argument are left to the reader.

In order to emphasize the similarity between the elliptic and the parabolic problems, consider the processes $Y_t^{x,t_0} = (t+t_0, X_t^x)$ with values in $\mathbb{R}^{1+d}$ and initial conditions (t_0, x). Then the operator $\partial/\partial t + L$, which acts on functions defined on $\mathbb{R}^{1+d}$, is the infinitesimal generator for this family of processes.

Let us now discuss fundamental solutions to parabolic PDE'a and their relation to the transition probability densities of the corresponding diffusion processes.

Definition 21.16. *A non-negative function $G(t,r,x,y)$ defined for $t < r$ and $x, y \in \mathbb{R}^d$ is called a fundamental solution to the backward parabolic equation*

$$\frac{\partial u(t,x)}{\partial t} + Lu(t,x) = 0, \tag{21.26}$$

if for fixed t, r, and x, the function $G(t,r,x,y)$ belongs to $L^1(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda)$, where λ is the Lebesgue measure, and for any $f \in C_b(\mathbb{R}^d)$, the function

$$u(t,x) = \int_{\mathbb{R}^d} G(t,r,x,y)f(y)dy$$

belongs to $C^{1,2}((-\infty,r)\times\mathbb{R}^d)\cap C_b((-\infty,r]\times\mathbb{R}^d)$ *and is a solution to (21.26) with the terminal condition* $u(r,x)=f(x)$.

Suppose that $\sigma_{ij}(t,x)$ and $v_i(t,x)$, $1\le i\le d$, $1\le j\le r$, $(t,x)\in\mathbb{R}^{1+d}$, are continuous, uniformly Lipschitz continuous in the space variables, and do not grow faster than linearly. It is well-known that in this case the fundamental solution to (21.26) exists and is unique (see "Partial Differential Equations of Parabolic Type" by A. Friedman). Moreover, for fixed r and y, the function $G(t,r,x,y)$ belongs to $C^{1,2}((-\infty,r)\times\mathbb{R}^d)$ and satisfies (21.26). Let us also consider the following equation, which is formally adjoint to (21.26)

$$-\frac{\partial\widetilde{u}(r,y)}{\partial r}+\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{\partial^2}{\partial y_i\partial y_j}[a_{ij}(r,y)\widetilde{u}(r,y)]-\sum_{i=1}^{d}\frac{\partial}{\partial y_i}[v_i(r,y)\widetilde{u}(r,y)]=0, \tag{21.27}$$

where $\widetilde{u}(r,y)$ is the unknown function. If the partial derivatives

$$\frac{\partial a_{ij}(r,y)}{\partial y_i},\ \frac{\partial^2 a_{ij}(r,y)}{\partial y_i\partial y_j},\ \frac{\partial v_i(r,y)}{\partial y_i},\quad 1\le i,j\le d, \tag{21.28}$$

are continuous, uniformly Lipschitz continuous in the space variables, and do not grow faster than linearly, then for fixed t and x the function $G(t,r,x,y)$ belongs to $C^{1,2}((t,\infty)\times\mathbb{R}^d)$ and satisfies (21.27).

Let $X_s^{t,x}$ be the solution to Eq. (21.25), and let $\mu(t,r,x,dy)$ be the distribution of the process at time $r>t$. Let us show that under the above conditions on σ and v, the measure $\mu(t,r,x,dy)$ has a density, that is

$$\mu(t,r,x,dy)=\rho(t,r,x,y)dy, \tag{21.29}$$

where $\rho(t,r,x,y)=G(t,r,x,y)$. It is called the transition probability density for the process $X_s^{t,x}$. (It is exactly the density of the Markov transition function, which is defined in the next section for the time-homogeneous case.) In order to prove (21.29), take any $f\in C_b(\mathbb{R}^d)$ and observe that

$$\int_{\mathbb{R}^d} f(y)\mu(t,r,x,dy)=\int_{\mathbb{R}^d} f(y)G(t,r,x,y)dy,$$

since both sides are equal to the solution to the same backward parabolic PDE evaluated at the point (t,x) due to Theorem 21.14 and Definition 21.16. Therefore, the measures $\mu(t,r,x,dy)$ and $G(t,r,x,y)dy$ coincide (see Problem 4, Chap. 8). We formalize the above discussion in the following lemma.

Lemma 21.17. *Suppose that* $\sigma_{ij}(t,x)$ *and* $v_i(t,x)$, $1\le i\le d$, $1\le j\le r$, $(t,x)\in\mathbb{R}^{1+d}$, *are continuous, uniformly Lipschitz continuous in the space variables, and do not grow faster than linearly.*

Then, the family of processes $X_s^{t,x}$ *defined by (21.25) has transition probability density* $\rho(t,r,x,y)$*, which for fixed* r *and* y *satisfies equation (21.26) (backward Kolmogorov equation). If, in addition, the partial derivatives in (21.28) are continuous, uniformly Lipschitz continuous in the space variables, and do not grow faster than linearly, then, for fixed* t *and* x*, the function* $\rho(t,r,x,y)$ *satisfies equation (21.27) (forward Kolmogorov equation).*

Now consider a process whose initial distribution is not necessarily concentrated in a single point.

Lemma 21.18. *Assume that the distribution of a square-integrable* $\mathbb{R}^d$*-valued random variable* ξ *is equal to* μ*, where* μ *is a measure with continuous density* p_0*. Assume that the coefficients* v_i *and* σ_{ij} *and their partial derivatives in (21.28) are continuous, uniformly Lipschitz continuous in the space variables, and do not grow faster than linearly. Let* X_t^μ *be the solution to (21.16) with initial condition* $X_0^\mu = \xi$*.*

Then the distribution of X_t^μ*, for fixed* t*, has a density* $p(t,x)$ *which belongs to* $C^{1,2}((0,\infty)\times\mathbb{R}^d)\cap C_b([0,\infty)\times\mathbb{R}^d)$ *and is the solution of the forward Kolmogorov equation*

$$(-\frac{\partial}{\partial t} + L^*)p(t,x) = 0$$

with initial condition $p(0,x) = p_0(x)$*.*

Sketch of the Proof. Let $\widetilde{\mu}_t$ be the measure induced by the process at time t, that is, $\widetilde{\mu}_t(A) = \mathrm{P}(X_t^\mu \in A)$ for $A \in \mathcal{B}(\mathbb{R}^d)$. We can view $\widetilde{\mu}$ as a generalized function (element of $\mathcal{S}'(\mathbb{R}^{1+d})$), which acts on functions $f \in \mathcal{S}(\mathbb{R}^{1+d})$ according to the formula

$$(\widetilde{\mu}, f) = \int_0^\infty \int_{\mathbb{R}^d} f(t,x)d\widetilde{\mu}_t(x)dt.$$

Now let $f \in \mathcal{S}(\mathbb{R}^{1+d})$, and apply Ito's formula to $f(t, X_t^\mu)$. After taking expectation on both sides,

$$\mathrm{E}f(t,X_t^\mu) = \mathrm{E}f(0,X_0^\mu) + \int_0^t \mathrm{E}(\frac{\partial f}{\partial s} + Lf)(s, X_s^\mu)ds.$$

If f is equal to zero for all sufficiently large t, we obtain

$$0 = \int_{\mathbb{R}^d} f(0,x)d\mu(x) + (\widetilde{\mu}, \frac{\partial f}{\partial t} + Lf),$$

or, equivalently,

$$((-\frac{\partial}{\partial t} + L^*)\widetilde{\mu}, f) + \int_{\mathbb{R}^d} f(0,x)d\mu(x) = 0. \tag{21.30}$$

A generalized function $\widetilde{\mu}$, such that (21.30) is valid for any infinitely smooth function with compact support, is called a generalized solution to the equation

$$(-\frac{\partial}{\partial t} + L^*)\widetilde{\mu} = 0$$

with initial data μ. Since the partial derivatives in (21.28) are continuous, uniformly Lipschitz continuous in the space variables, and do not grow faster than linearly, and μ has a continuous density $p_0(x)$, the equation

$$(-\frac{\partial}{\partial t} + L^*)p(t,x) = 0$$

with initial condition $p(0,x) = p_0(x)$ has a unique solution in $C^{1,2}((0,\infty) \times \mathbb{R}^d) \cap C_b([0,\infty) \times \mathbb{R}^d)$. Since $\widetilde{\mu}_t$ is a finite measure for each t, it can be shown that the generalized solution $\widetilde{\mu}$ coincides with the classical solution $p(t,x)$. Then it can be shown that for t fixed, $p(t,x)$ is the density of the distribution of X_t^μ. □

21.4 Markov Property of Solutions to SDE's

In this section we prove that solutions to stochastic differential equations form Markov families.

Theorem 21.19. *Let X_t^x be the family of strong solutions to the stochastic differential equation (21.16) with the initial conditions $X_0^x = x$. Let L be the infinitesimal generator for this family of processes. If the coefficients v_i and σ_{ij} are Lipschitz continuous and do not grow faster than linearly, and L is uniformly elliptic in $\mathbb{R}^d$, then X_t^x is a Markov family.*

Proof. Let us show that $p(t,x,\Gamma) = \mathrm{P}(X_t^x \in \Gamma)$ is Borel-measurable as a function of $x \in \mathbb{R}^d$ for any $t \geq 0$ and any Borel set $\Gamma \subseteq \mathbb{R}^d$. When $t = 0$, $\mathrm{P}(X_0^x \in \Gamma) = \chi_\Gamma(x)$, so it is sufficient to consider the case $t > 0$. First assume that Γ is closed. In this case, we can find a sequence of bounded continuous functions $f_n \in C_b(\mathbb{R}^d)$ such that $f_n(y)$ converge to $\chi_\Gamma(y)$ monotonically from above. By the Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(y)p(t,x,dy) = \int_{\mathbb{R}^d} \chi_\Gamma(y)p(t,x,dy) = p(t,x,\Gamma).$$

By Theorem 21.14, the integral $\int_{\mathbb{R}^d} f_n(y)p(t,x,dy)$ is equal to $u(0,x)$, where u is the solution of the equation

$$(\frac{\partial}{\partial t} + L)u = 0 \qquad (21.31)$$

with the terminal condition $u(t,x) = f(x)$. Since the solution is a smooth (and therefore measurable) function of x, $p(t,x,\Gamma)$ is a limit of measurable functions, and therefore measurable. Closed sets form a π-system, while the collection of sets Γ for which $p(t,x,\Gamma)$ is measurable is a Dynkin system. Therefore, $p(t,x,\Gamma)$ is measurable for all Borel sets Γ by Lemma 4.13. The second condition of Definition 19.2 is clear.

To verify the third condition of Definition 19.2, it suffices to show that

$$\mathrm{E}(f(X^x_{s+t})|\mathcal{F}_s) = \int_{\mathbb{R}^d} f(y)p(t,X^x_s,dy) \tag{21.32}$$

for any $f \in C_b(\mathbb{R}^d)$. Indeed, we can approximate χ_Γ by a monotonically non-increasing sequence of functions from $C_b(\mathbb{R}^d)$, and, if (21.32) is true, by the Conditional Dominated Convergence Theorem,

$$\mathrm{P}(X^x_{s+t} \in \Gamma|\mathcal{F}_s) = p(t,X^x_s,\Gamma) \quad \text{almost surely.}$$

In order to prove (21.32), we can assume that $s,t>0$, since otherwise the statement is obviously true. Let u be the solution to (21.31) with the terminal condition $u(s+t,x)=f(x)$. By Theorem 21.14, the right-hand side of (21.32) is equal to $u(s,X^x_s)$ almost surely. By the Ito formula,

$$u(s+t,X^x_{s+t}) = u(0,x) + \sum_{i=1}^d \sum_{j=1}^r \int_0^{s+t} \frac{\partial u}{\partial x_i}(X^x_u)\sigma_{ij}(X^x_u)dW^j_u.$$

After taking conditional expectation on both sides,

$$\begin{aligned}\mathrm{E}(f(X^x_{s+t})|\mathcal{F}_s) &= u(0,x) + \sum_{i=1}^d \sum_{j=1}^r \int_0^{s} \frac{\partial u}{\partial x_i}(X^x_u)\sigma_{ij}(X^x_u)dW^j_u \\ &= u(s,X^x_s) = \int_{\mathbb{R}^d} f(y)p(t,X^x_s,dy).\end{aligned}$$

□

Remark 21.20. Since $p(t,X^x_s,\Gamma)$ is $\sigma(X^x_s)$-measurable, it follows from the third property of Definition 19.2 that

$$\mathrm{P}(X^x_{s+t} \in \Gamma|\mathcal{F}_s) = \mathrm{P}(X^x_{s+t} \in \Gamma|X^x_s).$$

Thus, Theorem 21.19 implies that X^x_t is a Markov process for each fixed x.

We state the following theorem without a proof.

Theorem 21.21. *Under the conditions of Theorem 21.19, the family of processes X^x_t is a strong Markov family.*

Given a Markov family of processes X^x_t, we can define two families of Markov transition operators. The first family, denoted by P_t, acts on bounded measurable functions. It is defined by

$$(P_t f)(x) = \mathrm{E} f(X_t^x) = \int_{\mathbb{R}^d} f(y) p(t, x, dy),$$

where p is the Markov transition function. From the definition of the Markov property, we see that $P_t f$ is again a bounded measurable function.

The second family of operators, denoted by P_t^*, acts on probability measures. It is defined by

$$(P_t^* \mu)(C) = \int_{\mathbb{R}^d} \mathrm{P}(X_t^x \in C) d\mu(x) = \int_{\mathbb{R}^d} p(t, x, C) d\mu(x).$$

It is clear that the image of a probability measure μ under P_t^* is again a probability measure. The operators P_t and P_t^* are adjoint. Namely, if f is a bounded measurable function and μ is a probability measure, then

$$\int_{\mathbb{R}^d} (P_t f)(x) d\mu(x) = \int_{\mathbb{R}^d} f(x) d(P_t^* \mu)(x). \tag{21.33}$$

Indeed, by the definitions of P_t and P_t^*, this formula is true if f is an indicator function of a measurable set. Therefore, it is true for finite linear combinations of indicator functions. An arbitrary bounded measurable function can, in turn, be uniformly approximated by finite linear combinations of indicator functions, which justifies (21.33).

Definition 21.22. *A measure μ is said to be invariant for a Markov family X_t^x if $P_t^* \mu = \mu$ for all $t \geq 0$.*

Let us answer the following question: when is a measure μ invariant for the family of diffusion processes X_t^x that solve (21.16) with initial conditions $X_0^x = x$? Let the coefficients of the generator L satisfy the conditions stated in Lemma 21.19. Assume that μ is an invariant measure. Then the right-hand side of (21.33) does not depend on t, and therefore neither does the left hand side. In particular,

$$\int_{\mathbb{R}^d} (P_t f - f)(x) d\mu(x) = 0.$$

Let f belong to the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. In this case,

$$\begin{aligned} \int_{\mathbb{R}^d} Lf(x) d\mu(x) &= \int_{\mathbb{R}^d} \lim_{t \downarrow 0} \frac{(P_t f - f)(x)}{t} d\mu(x) \\ &= \lim_{t \downarrow 0} \int_{\mathbb{R}^d} \frac{(P_t f - f)(x)}{t} d\mu(x) = 0, \end{aligned}$$

where the first equality is due to (21.18) and the second one to the Dominated Convergence Theorem. Note that we can apply the Dominated Convergence Theorem, since $(P_t f - f)/t$ is uniformly bounded for $t > 0$ if $f \in \mathcal{S}(\mathbb{R}^d)$, as is clear from the discussion following (21.18).

We can rewrite the equality $\int_{\mathbb{R}^d} Lf(x)d\mu(x) = 0$ as $(L^*\mu, f) = 0$, where $L^*\mu$ is the following generalized function

$$L^*\mu = \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{\partial^2}{\partial x_i \partial x_j}[a_{ij}(x)\mu(x)] - \sum_{i=1}^{d}\frac{\partial}{\partial x_i}[v_i(x)\mu(x)].$$

Here, $a_{ij}(x)\mu(x)$ and $v_i(x)\mu(x)$ are the generalized functions corresponding to the signed measures whose densities with respect to μ are equal to $a_{ij}(x)$ and $v_i(x)$, respectively. The partial derivatives here are understood in the sense of generalized functions. Since $f \in \mathcal{S}(\mathbb{R}^d)$ was arbitrary, we conclude that $L^*\mu = 0$.

The converse is also true: if $L^*\mu = 0$, then μ is an invariant measure for the family of diffusion processes X_t^x. We leave this statement as an exercise for the reader.

Example. Let X_t^x be the family of solutions to the stochastic differential equation

$$dX_t^x = dW_t - X_t^x dt$$

with the initial data $X_t^x = x$. (See Sect. 21.1, in which we discussed the Ornstein-Uhlenbeck process.) The generator for this family of processes and the adjoint operator are given by

$$Lf(x) = \frac{1}{2}f''(x) - xf'(x) \quad \text{and} \quad L^*\mu(x) = \frac{1}{2}\mu''(x) + (x\mu(x))'.$$

It is not difficult to see that the only probability measure that satisfies $L^*\mu = 0$ is that whose density with respect to the Lebesgue measure is equal to $p(x) = \frac{1}{\sqrt{\pi}}\exp(-x^2)$. Thus, the invariant measure for the family of Ornstein-Uhlenbeck processes is $\mu(dx) = \frac{1}{\sqrt{\pi}}\exp(-x^2)\lambda(dx)$, where λ is the Lebesgue measure.

21.5 A Problem in Homogenization

Given a parabolic partial differential equation with variable (e.g. periodic) coefficients, it is often possible to describe asymptotic properties of its solutions (as $t \to \infty$) in terms of solutions to a simpler equation with constant coefficients. Similarly, for large t, solutions to a stochastic differential equation with variable coefficients may exhibit similar properties to those for an SDE with constant coefficients.

In order to state one such homogenization result, let us consider the $\mathbb{R}^d$-valued process X_t which satisfies the following stochastic differential equation

$$dX_t = v(X_t)dt + dW_t \tag{21.34}$$

with initial condition $X_0 = \xi$, where ξ is a bounded random variable, $v(x) = (v_1(x), \ldots, v_d(x))$ is a vector field on $\mathbb{R}^d$, and $W_t = (W_t^1, \ldots, W_t^d)$ is a d-dimensional Brownian motion. We assume that the vector field v is smooth, periodic ($v(x+z) = v(x)$ for $z \in \mathbb{Z}^d$) and incompressible ($\text{div} v = 0$). Let T^d be the unit cube in $\mathbb{R}^d$,

$$T^d = \{x \in \mathbb{R}^d : 0 \le x_i < 1,\ i = 1 \ldots, d\}$$

(we may glue the opposite sides to make it into a torus). Let us assume that $\int_{T^d} v_i(x)dx = 0$, $1 \le i \le d$, that is the "net drift" of the vector field is equal to zero. Notice that we can consider X_t as a process with values on the torus.

Although the solution to (21.34) cannot be written out explicitly, we can describe the asymptotic behavior of X_t for large t. Namely, consider the $\mathbb{R}^d$-valued process Y_t defined by

$$Y_t^i = \sum_{1 \le j \le d} \sigma_{ij} W_t^j, \quad 1 \le i, j \le d,$$

with some coefficients σ_{ij}. Due to the scaling property of Brownian motion, for any positive ε, the distribution of the process $Y_t^\varepsilon = \sqrt{\varepsilon} Y_{t/\varepsilon}$ is the same as that of the original process Y_t. Let us now apply the same scaling transformation to the process X_t. Thus we define

$$X_t^\varepsilon = \sqrt{\varepsilon} X_{t/\varepsilon}.$$

Let P_X^ε be the measure on $C([0,\infty))$ induced by the process X_t^ε, and P_Y the measure induced by the process Y_t. It turns out that for an appropriate choice of the coefficients σ_{ij}, the measures P_X^ε converge weakly to P_Y when $\varepsilon \to 0$. In particular, for t fixed, X_t^ε converges in distribution to a Gaussian random variable with covariance matrix $a_{ij} = (\sigma\sigma^*)_{ij}$.

We shall not prove this statement in full generality, but instead study only the behavior of the covariance matrix of the process X_t^ε as $\varepsilon \to 0$ (or, equivalently, of the process X_t as $t \to \infty$). We shall show that $\mathrm{E}(X_t^i X_t^j)$ grows linearly, and identify the limit of $\mathrm{E}(X_t^i X_t^j)/t$ as $t \to \infty$. An additional simplifying assumption will concern the distribution of ξ.

Let L be the generator of the process X_t which acts on functions $u \in C^2(T^d)$ (the class of smooth periodic functions) according to the formula

$$Lu(x) = \frac{1}{2}\Delta u(x) + (v, \nabla u)(x).$$

If u is periodic, then so is Lu, and therefore we can consider L as an operator on $C^2(T^d)$ with values in $C(T^d)$. Consider the following partial differential equations for unknown periodic functions u_i, $1 \le i \le d$,

$$L(u_i(x) + x_i) = 0, \tag{21.35}$$

where x_i is the i-th coordinate of the vector x. These equations can be rewritten as

$$Lu_i(x) = -v_i(x).$$

Note that the right-hand side is a periodic function. It is well-known in the general theory of elliptic PDE's that this equation has a solution in $C^2(T^d)$ (which is then unique up to an additive constant) if and only if the right-hand side is orthogonal to the kernel of the adjoint operator (see "Partial Differential Equations" by A. Friedman). In other words, to establish the existence of a solution we need to check that

$$\int_{T^d} -v_i(x)g(x)dx = 0 \text{ if } g \in C^2(T^d) \text{ and } L^*g(x) = \frac{1}{2}\Delta g(x) - \mathrm{div}(gv)(x) = 0.$$

It is easy to see that the only $C^2(T^d)$ solutions to the equation $L^*g = 0$ are constants, and thus the existence of solutions to (21.35) follows from $\int_{T^d} v_i(x)dx = 0$. Since we can add an arbitrary constant to the solution, we can define $u_i(x)$ to be the solution to (21.35) for which $\int_{T^d} u_i(x)dx = 0$.

Now let us apply Ito's formula to the function $u_i(x) + x_i$ of the process X_t:

$$u_i(X_t) + X_t^i - u_i(X_0) - X_0^i = \int_0^t \sum_{k=1}^d \frac{\partial(u_i + x_i)}{\partial x_k}(X_s)dW_s^k$$

$$+ \int_0^t L(u_i + x_i)(X_s)ds = \int_0^t \sum_{k=1}^d \frac{\partial(u_i + x_i)}{\partial x_k}(X_s)dW_s^k,$$

since the ordinary integral vanishes due to (21.35). Let $g_t^i = u_i(X_t) - u_i(X_0) - X_0^i$. Thus,

$$X_t^i + g_t^i = \int_0^t \sum_{k=1}^d \frac{\partial(u_i + x_i)}{\partial x_k}(X_s)dW_s^k.$$

Similarly, using the index j instead of i, we can write

$$X_t^j + g_t^j = \int_0^t \sum_{k=1}^d \frac{\partial(u_j + x_j)}{\partial x_k}(X_s)dW_s^k.$$

Let us multiply the right-hand sides of these equalities, and take expectations. With the help of Lemma 20.17 we obtain

$$\mathrm{E}\Big(\int_0^t \sum_{k=1}^d \frac{\partial(u_i + x_i)}{\partial x_k}(X_s)dW_s^k \int_0^t \sum_{k=1}^d \frac{\partial(u_j + x_j)}{\partial x_k}(X_s)dW_s^k\Big)$$

$$= \int_0^t \mathrm{E}((\nabla u_i, \nabla u_j)(X_s) + \delta_{ij})ds,$$

where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$.

Notice that, since v is periodic, we can consider (21.34) as an equation for a process on the torus T^d. Let us assume that $X_0 = \xi$ is uniformly distributed on the unit cube (and, consequently, when we consider X_t as a process on the torus, X_0 is uniformly distributed on the unit torus). Let $p_0(x) \equiv 1$ be the density of this distribution. Since $L^* p_0(x) = 0$, the density of X_s on the torus is also equal to p_0. (Here we used Lemma 21.18, modified to allow for processes to take values on the torus.) Consequently,

$$\int_0^t \mathrm{E}((\nabla u_i, \nabla u_j)(X_s) + \delta_{ij}) ds = \int_0^t \int_{T^d} ((\nabla u_i, \nabla u_j)(x) + \delta_{ij}) dx ds$$
$$= t \int_{T^d} ((\nabla u_i, \nabla u_j)(x) + \delta_{ij}) dx.$$

Thus,

$$\mathrm{E}((X_t^i + g_t^i)(X_t^j + g_t^j))/t = \int_{T^d} ((\nabla u_i, \nabla u_j)(x) + \delta_{ij}) dx. \tag{21.36}$$

Lemma 21.23. *Under the above assumptions,*

$$\mathrm{E}(X_t^i X_t^j)/t \to \int_{T^d} ((\nabla u_i, \nabla u_j)(x) + \delta_{ij}) dx \quad \text{as} \quad t \to \infty. \tag{21.37}$$

Proof. The difference between (21.37) and (21.36) is the presence of the bounded processes g_t^i and g_t^j in expectation on the left-hand side of (21.36). The desired result follows from the following simple lemma.

Lemma 21.24. *Let f_t^i and h_t^i, $1 \le i \le d$, be two families of random processes. Suppose*

$$\mathrm{E}\left((f_t^i + h_t^i)(f_t^j + h_t^j)\right) = \phi^{ij} \ . \tag{21.38}$$

Also suppose there is a constant c such that

$$t\mathrm{E}(h_t^i)^2 \le c. \tag{21.39}$$

Then,

$$\lim_{t\to\infty} \mathrm{E}(f_t^i f_t^j) = \phi^{ij}.$$

Proof. By (21.38) with $i = j$,

$$\mathrm{E}(f_t^i)^2 = \phi^{ii} - \mathrm{E}(h_t^i)^2 - 2\mathrm{E}(f_t^i h_t^i) \ . \tag{21.40}$$

By (21.40) and (21.39), we conclude that there exists a constant c' such that

$$\mathrm{E}(f_t^i)^2 < c' \quad \text{for all} \quad t > 1 \ . \tag{21.41}$$

By (21.38),

$$\mathrm{E}(f_t^i f_t^j) - \phi^{ij} = -\mathrm{E}(h_t^i h_t^j) - \mathrm{E}(f_t^i h_t^j) - \mathrm{E}(f_t^j h_t^i) \ . \tag{21.42}$$

By the Schwartz Inequality, (21.39) and (21.41), the right-hand side of (21.42) tends to zero as $t \to \infty$. □

To complete the proof of Lemma 21.23 it suffices to take $f_t^i = X_t^i/\sqrt{t}$, $h_t^i = g_t^i/\sqrt{t}$, and apply Lemma 21.24. □

21.6 Problems

1. Prove the Gronwall Inequality (Lemma 21.4).

2. Let W_t be a one-dimensional Brownian motion. Prove that the process

$$X_t = (1-t)\int_0^t \frac{dW_s}{1-s}, \quad 0 \le t < 1,$$

is the solution of the stochastic differential equation

$$dX_t = \frac{X_t}{t-1}dt + dW_t, \quad 0 \le t < 1, \quad X_0 = 0.$$

3. For the process X_t defined in Problem 2, prove that there is the almost sure limit

$$\lim_{t \to 1-} X_t = 0.$$

Define $X_1 = 0$. Prove that the process X_t, $0 \le t \le 1$ is Gaussian, and find its correlation function. Prove that X_t is a Brownian Bridge (see Problem 13, Chap. 18).

4. Consider two European call options with the same strike price for the same stock (i.e., r, σ, P and S are the same for the two options). Assume that the risk-free interest rate γ is equal to zero. Is it true that the option with longer time till expiration is more valuable?

5. Let W_t be a one-dimensional Brownian motion, and $Y_t = e^{-t/2}W(e^t)$. Find a, σ and ξ such that Y_t has the same finite-dimensional distributions as the solution of (21.9).

6. Let W_t be a two-dimensional Brownian motion, and τ the first time when W_t hits the unit circle, $\tau = \inf(t : ||W_t|| = 1)$. Find $\mathrm{E}\tau$.

7. Prove that if a point satisfies the exterior cone condition, then it is regular.

8. Prove that regularity is a local condition. Namely, let D_1 and D_1 be two domains, and let $x \in \partial D_1 \cap \partial D_2$. Suppose that there is an open neighborhood U of x such that $U \bigcap \partial D_1 = U \bigcap \partial D_1$. Then x is a regular boundary point for D_1 if and only if it is a regular boundary point for D_2.

9. Let W_t be a two-dimensional Brownian motion. Prove that for any $x \in \mathbb{R}^2$, $||x|| > 0$, we have

$$\mathrm{P}(\text{there is } t \ge 0 \text{ such that } W_t = x) = 0.$$

Prove that for any $\delta > 0$

$$\mathrm{P}(\text{there is } t \ge 0 \text{ such that } ||W_t - x|| \le \delta) = 1.$$

10. Let W_t be a d-dimensional Brownian motion, where $d \ge 3$. Prove that

$$\lim_{t \to \infty} ||W_t|| = \infty$$

almost surely.

11. Let $W_t = (W_t^1, W_t^2)$ be a two-dimensional Brownian motion, and τ the first time when W_t hits the unit square centered at the origin, $\tau = \inf(t : \min(|W_t^1|, |W_t^2|) = 1/2)$. Find $\mathrm{E}\tau$.

12. Let D be the open unit disk in $\mathbb{R}^2$ and $u^\varepsilon \in C^2(D) \cap C(\overline{D})$ the solution of the following Dirichlet problem

$$\varepsilon \Delta u^\varepsilon + \frac{\partial u^\varepsilon}{\partial x_1} = 0,$$

$$u(x) = f(x) \quad \text{for } x \in \partial D,$$

where f is a continuous function on ∂D. Find the limit $\lim_{\varepsilon \downarrow 0} u^\varepsilon(x_1, x_2)$ for $(x_1, x_2) \in D$.

13. Let X_t be the strong solution to the stochastic differential equation

$$dX_t = v(X_t)dt + \sigma(X_t)dW_t$$

with the initial condition $X_0 = 1$, where v and σ are Lipschitz continuous functions on $\mathbb{R}$. Assume that $\sigma(x) \geq c > 0$ for some constant c and all $x \in \mathbb{R}$. Find a non-constant function f such that $f(X_t)$ is a local martingale.

14. Let X_t, v and σ be the same as in the previous problem. For which functions v and σ do we have

$$\mathrm{P}(\text{there is } t \in [0, \infty) \text{ such that } X_t = 0) = 1?$$