17

Generalized Random Processes

17.1 Generalized Functions and Generalized Random Processes

We start this section by recalling the definitions of test functions and generalized functions.[1] Thenwe shall introduce the notion of generalized random processes and see that they play the same role, when compared to ordinary random processes, as the generalized functions, when compared to ordinary functions.

As the space of test functions we shall consider the particular example of infinitely differentiable functions whose derivatives decay faster than any power. To simplify the notation we shall define test functions and generalized functions over $\mathbb{R}$, although the definitions can be easily replicated in the case of $\mathbb{R}^n$.

Definition 17.1. *The space $\mathcal{S}$ of test functions consists of infinitely differentiable complex-valued functions φ such that for any non-negative integers r and q,*

$$\max_{0\le s\le r}\sup_{t\in\mathbb{R}}((1+t^2)^q|\varphi^{(s)}(t)|) = c_{q,r}(\varphi) < \infty.$$

Note that $c_{q,r}(\varphi)$ are norms on the space $\mathcal{S}$, so that together with the collection of norms $c_{q,r}$, $\mathcal{S}$ is a countably-normed linear space. It is, therefore, a linear topological space with the basis of neighborhoods of zero given by the collection of sets $U_{q,r,\varepsilon} = \{\varphi : c_{q,r}(\varphi) < \varepsilon\}$.

Let us now consider the linear continuous functionals on the space $\mathcal{S}$.

Definition 17.2. *The space $\mathcal{S}'$ of generalized functions consists of all the linear continuous functionals on the space $\mathcal{S}$.*

[1] This chapter can be omitted during the first reading.

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The action of a generalized function $f \in \mathcal{S}'$ on a test function φ will be denoted by $f(\varphi)$ or (f, φ). Our basic example of a generalized function is the following. Let $\mu(t)$ be a σ-finite measure on the real line such that the integral

$$\int_{-\infty}^{\infty} (1+t^2)^{-q} d\mu(t)$$

converges for some q. Then the integral

$$(f, \varphi) = \int_{-\infty}^{\infty} \varphi(t) d\mu(t)$$

is defined for any $\varphi(t) \in \mathcal{S}$ and is a continuous linear functional on the space of test functions. Similarly, if $g(t)$ is a continuous complex-valued function whose absolute value is bounded from above by a polynomial, then it defines a generalized function via

$$(f, \varphi) = \int_{-\infty}^{\infty} \varphi(t) \overline{g(t)} dt$$

(the complex conjugation is needed here if $g(t)$ is complex-valued). The space of generalized functions is closed under the operations of taking the derivative and Fourier transform. Namely, for $f \in \mathcal{S}'$, we can define

$$(f', \varphi) = -(f, \varphi') \quad \text{and} \quad (\widehat{f}, \varphi) = (f, \widetilde{\varphi}),$$

where $\widetilde{\varphi}$ stands for the inverse Fourier transform of the test function φ. Note that the right-hand sides of these equalities are linear continuous functionals on the space $\mathcal{S}$, and thus the functionals f' and $\widehat{f}$ belong to $\mathcal{S}'$.

Since all the elements of $\mathcal{S}$ are bounded continuous functions, they can be considered as elements of $\mathcal{S}'$, that is $\mathcal{S} \subset \mathcal{S}'$. The operations of taking derivative and Fourier transform introduced above are easily seen to coincide with the usual derivative and Fourier transform for the elements of the space $\mathcal{S}$.

Let us now introduce the notion of generalized random processes. From the physical point of view, the concept of a random process X_t is related to measurements of random quantities at certain moments of time, without taking the values at other moments of time into account. However, in many cases, it is impossible to localize the measurements to a single point of time. Instead, one considers the "average" measurements $\Phi(\varphi) = \int \varphi(t) X_t dt$, where φ is a test function. Such measurements should depend on φ linearly, and should not change much with a small change of φ.

This leads to the following definition of generalized random processes.

Definition 17.3. *Let $\Phi(\varphi)$ be a collection of complex-valued random variables on a common probability space $(\Omega, \mathcal{F}, \mathrm{P})$ indexed by the elements of the space of test functions $\varphi \in \mathcal{S}$ with the following properties:*

1. *Linearity: $\Phi(a_1\varphi_1 + a_2\varphi_2) = a_1\Phi(\varphi_1) + a_2\Phi(\varphi_2)$ almost surely, for $a_1, a_2 \in \mathbb{C}$ and $\varphi_1, \varphi_2 \in \mathcal{S}$.*

2. *Continuity: If* $\psi_k^n \to \varphi_k$ *in* $\mathcal{S}$ *as* $n \to \infty$ *for* $k = 1, \ldots, m$, *then the vector-valued random variables* $(\Phi(\psi_1^n), \ldots, \Phi(\psi_m^n))$ *converge in distribution to* $(\Phi(\varphi_1), \ldots, \Phi(\varphi_m))$ *as* $n \to \infty$.

Then $\Phi(\varphi)$ *is called a generalized random process (over the space* $\mathcal{S}$ *of test functions).*

Note that if $X_t(\omega)$ is an ordinary random process such that $X_t(\omega)$ is continuous in t for almost every ω, and $|X_t(\omega)| \leq p_\omega(t)$ for some polynomial $p_\omega(t)$, then $\Phi(\varphi) = \int \varphi(t)\overline{X_t}dt$ is a generalized random process. Alternatively, we could require that $X_t(\omega)$ be an ordinary random process continuous in t as a function from $\mathbb{R}$ to $L^2(\Omega, \mathcal{F}, \mathrm{P})$ and such that $||X_t||_{L^2} \leq p(t)$ for some polynomial $p(t)$.

As with generalized functions, we can define the derivative and Fourier transform of a generalized random process via

$$\Phi'(\varphi) = -\Phi(\varphi'), \quad \widehat{\Phi}(\varphi) = \Phi(\widetilde{\varphi}).$$

A generalized random process Φ is called strictly stationary if, for any $\varphi_1, \ldots, \varphi_n \in \mathcal{S}$ and any $h \in \mathbb{R}$, the random vector $(\Phi(\varphi_1(t+h)), \ldots, \Phi(\varphi_n(t+h)))$ has the same distribution as the vector $(\Phi(\varphi_1(t)), \ldots, \Phi(\varphi_n(t)))$.

We can consider the expectation and the covariance functional of the generalized random process. Namely, assuming that the right-hand side is a continuous functional, we define

$$m(\varphi) = \mathrm{E}\Phi(\varphi).$$

Assuming that the right-hand side is a continuous functional of each of the variables, we define

$$B(\varphi, \psi) = \mathrm{E}\Phi(\varphi)\overline{\Phi(\psi)}.$$

Clearly, the expectation and the covariance functional are linear and hermitian functionals respectively on the space $\mathcal{S}$ (hermitian meaning linear in the first argument and anti-linear in the second). The covariance functional is non-negative definite, that is $B(\varphi, \varphi) \geq 0$ for any φ. A generalized process is called wide-sense stationary if

$$m(\varphi(t)) = m(\varphi(t+h)), \quad B(\varphi(t), \psi(t)) = B(\varphi(t+h), \psi(t+h))$$

for any $h \in \mathbb{R}$. If an ordinary random process is strictly stationary or wide-sense stationary, then so too is the corresponding generalized random process. It is easily seen that the only linear continuous functionals on the space $\mathcal{S}$, which are invariant with respect to translations, are those of the form

$$m(\varphi) = a \int_{-\infty}^{\infty} \varphi(t)dt,$$

where a is a constant. The number a can also be referred to as the expectation of the wide-sense stationary generalized process.

The notions of spectral measure and random spectral measure can be extended to the case of generalized random processes which are wide-sense stationary. Consider a generalized random process with zero expectation. In order to define the notion of spectral measure, we need the following lemma, which we provide here without a proof. (See "Generalized Functions", Volume 4, by I.M. Gelfand and N.Y. Vilenkin.)

Lemma 17.4. *Let $B(\varphi, \psi)$ be a hermitian functional on $\mathcal{S}$, which is continuous in each of the arguments, translation-invariant, and non-negative definite (that is $B(\varphi, \varphi) \geq 0$ for all $\varphi \in \mathcal{S}$). Then there is a unique σ-finite measure ρ on the real line such that the integral*

$$\int_{-\infty}^{\infty} (1 + t^2)^{-q} d\rho(t)$$

converges for some $q \geq 0$, and

$$B(\varphi, \psi) = \int_{-\infty}^{\infty} \widehat{\varphi}(\lambda)\overline{\widehat{\psi}(\lambda)} d\rho(\lambda). \tag{17.1}$$

Note that the covariance functional satisfies all the requirements of the lemma. We can thus define the spectral measure as the measure ρ for which (17.1) holds, where B on the left-hand side is the covariance functional.

Furthermore, it can be shown that there exists a unique orthogonal random measure Z such that $\mathrm{E}|Z(\Delta)|^2 = \rho(\Delta)$, and

$$\Phi(\varphi) = \int_{-\infty}^{\infty} \widehat{\varphi} dZ(\lambda). \tag{17.2}$$

Let μ_ρ be the generalized function corresponding to the measure ρ. Let $F = \widetilde{\mu}_\rho$ be its inverse Fourier transform in the sense of generalized functions. We can then rewrite (17.1) as

$$B(\varphi, \psi) = (F, \varphi * \psi^*),$$

where the convolution of two test functions is defined as

$$\varphi * \psi(t) = \int_{-\infty}^{\infty} \varphi(s)\psi(t - s) ds,$$

and $\psi^*(t) = \overline{\psi(-t)}$. For generalized processes which are wide-sense stationary, the generalized function F is referred to as the covariance function.

Let us assume that X_t is a stationary ordinary process with zero expectation, which is continuous in the L^2 sense. As previously mentioned, we can also consider it as a generalized process, $\Phi(\varphi) = \int \varphi(t)\overline{X_t} dt$. We have two sets of definitions of the covariance function, spectral measure, and the random orthogonal measure (one for the ordinary process X_t, and the other for the generalized process Φ). It would be natural if the two sets of definitions led

to the same concepts of the covariance function, spectral measure, and the random orthogonal measure. This is indeed the case (we leave this statement as an exercise for the reader).

Finally, let us discuss the relationship between generalized random processes and measures on $\mathcal{S}'$. Given a Borel set $B \subseteq \mathbb{C}^n$ and n test functions $\varphi_1, \ldots, \varphi_n$, we define a cylindrical subset of $\mathcal{S}'$ as the set of elements $f \in \mathcal{S}'$ for which $(f(\varphi_1), \ldots, f(\varphi_n)) \in B$. The Borel σ-algebra $\mathcal{F}$ is defined as the minimal σ-algebra which contains all the cylindrical subsets of $\mathcal{S}'$. Any probability measure P on $\mathcal{F}$ defines a generalized process, since $f(\varphi)$ is a random variable on $(\mathcal{S}', \mathcal{F}, \mathrm{P})$ for any $\varphi \in \mathcal{S}$ and all the conditions of Definition 17.3 are satisfied. The converse statement is also true. We formulate it here as a theorem. The proof is non-trivial and we do not provide it here. (See "Generalized Functions", Volume 4, by I.M. Gelfand and N.Y. Vilenkin.)

Theorem 17.5. *Let $\Phi(\varphi)$ be a generalized random process on $\mathcal{S}$. Then there exists a unique probability measure* P *on $\mathcal{S}'$ such that for any n and any $\varphi_1, \ldots, \varphi_n \in \mathcal{S}$ the random vectors $(f(\varphi_1), \ldots, f(\varphi_n))$ and $(\Phi(\varphi_1), \ldots, \Phi(\varphi_n))$ have the same distributions.*

17.2 Gaussian Processes and White Noise

A generalized random process Φ is called Gaussian if for any test functions $\varphi_1, \ldots, \varphi_k$, the random vector $(\Phi(\varphi_1), \ldots, \Phi(\varphi_k))$ is Gaussian. To simplify the notation, let us consider Gaussian processes with zero expectation. We shall also assume that the process is real-valued, meaning that $\Phi(\varphi)$ is real, whenever φ is a real-valued element of $\mathcal{S}$.

The covariance matrix of the vector $(\Phi(\varphi_1), \ldots, \Phi(\varphi_k))$ is simply $B_{ij} = \mathrm{E}(\Phi(\varphi_i)\overline{\Phi}(\varphi_j)) = B(\varphi_i, \varphi_j)$. Therefore, all the finite-dimensional distributions with $\varphi_1, \ldots, \varphi_k$ real are determined by the covariance functional. We shall say that a hermitian form is real if $B(\varphi, \psi)$ is real whenever φ and ψ are real.

Recall that the covariance functional of any generalized random process is a non-negative definite hermitian form which is continuous in each of the variables. We also have the converse statement.

Theorem 17.6. *Let $B(\varphi, \psi)$ be a real non-negative definite hermitian form which is continuous in each of the variables. Then there is a real-valued Gaussian generalized process with zero expectation with $B(\varphi, \psi)$ as its covariance functional.*

To prove this theorem we shall need the following important fact from the theory of countably normed spaces. We provide it here without a proof.

Lemma 17.7. *If a hermitian functional $B(\varphi, \psi)$ on the space $\mathcal{S}$ is continuous in each of the variables separately, then it is continuous in the pair of the variables, that is $\lim_{(\varphi,\psi)\to(\varphi_0,\psi_0)} B(\varphi, \psi) = B(\varphi_0, \psi_0)$ for any (φ_0, ψ_0).*

Proof of Theorem 17.6. Let $\mathcal{S}_r$ be the set of real-valued elements of $\mathcal{S}$. Let Ω be the space of all functions (not necessarily linear) defined on $\mathcal{S}_r$. Let $\mathcal{B}$ be the smallest σ-algebra containing all the cylindrical subsets of Ω, that is the sets of the form

$$\{\omega : (\omega(\varphi_1), \dots, \omega(\varphi_k)) \in A\},$$

where $\varphi_1, \dots, \varphi_k \in \mathcal{S}_r$ and A is a Borel subset of $\mathbb{R}^k$. Let $\mathcal{B}_{\varphi_1,\dots,\varphi_k}$ be the smallest σ-algebra which contains all such sets, where A is allowed to vary but $\varphi_1, \dots, \varphi_k$ are fixed. We define the measure $\mathrm{P}_{\varphi_1,\dots,\varphi_k}$ on $\mathcal{B}_{\varphi_1,\dots,\varphi_k}$ by

$$\mathrm{P}_{\varphi_1,\dots,\varphi_k}(\{\omega : (\omega(\varphi_1), \dots, \omega(\varphi_k)) \in A\}) = \eta(A),$$

where η is a Gaussian distribution with the covariance matrix $B_{ij} = B(\varphi_i, \varphi_j)$. The measures $\mathrm{P}_{\varphi_1,\dots,\varphi_k}$ clearly satisfy the assumptions of Kolmogorov's Consistency Theorem and, therefore, there exists a unique measure P on $\mathcal{B}$ whose restriction to each $\mathcal{B}_{\varphi_1,\dots,\varphi_k}$ coincides with $\mathrm{P}_{\varphi_1,\dots,\varphi_k}$.

We define $\Phi(\varphi)$, where $\varphi \in \mathcal{S}_r$ for now, simply by putting $\Phi(\varphi)(\omega) = \omega(\varphi)$. Let us show that $\Phi(\varphi)$ is the desired generalized process. By construction, $\mathrm{E}(\Phi(\varphi)\overline{\Phi}(\psi)) = B(\varphi, \psi)$. Next, let us show that $\Phi(a\varphi + b\psi) = a\Phi(\varphi) + b\Phi(\psi)$ almost surely with respect to the measure P, when $\varphi, \psi \in \mathcal{S}_r$ and $a, b \in \mathbb{R}$. Note that we defined Ω as the set of all functions on $\mathcal{S}_r$, not just the linear ones. To prove the linearity of Φ, note that the variance of $\Phi(a\varphi+b\psi)-a\Phi(\varphi)-b\Phi(\psi)$ is equal to zero. Therefore $\Phi(a\varphi + b\psi) = a\Phi(\varphi) + b\Phi(\psi)$ almost surely.

We also need to demonstrate the continuity of $\Phi(\varphi)$. If $\psi_k^n \to \varphi_k$ in $\mathcal{S}_r$ as $n \to \infty$ for $k = 1, \dots, m$, then the covariance matrix of the vector $(\Phi(\psi_1^n), \dots, \Phi(\psi_m^n))$ is $B_{ij}^n = B(\psi_i^n, \psi_j^n)$, while the covariance matrix of the vector $(\Phi(\varphi_1), \dots, \Phi(\varphi_m))$ is equal to $B_{ij} = B(\varphi_i, \varphi_j)$. If $\psi_k^n \to \varphi_k$ in $\mathcal{S}_r$ as $n \to \infty$ for $k = 1, \dots, m$, then $\lim_{n\to\infty} B_{ij}^n = B_{ij}$ due to Lemma 17.7. Since the vectors are Gaussian, the convergence of covariance matrices implies the convergence in distribution.

Finally, for $\varphi = \varphi_1 + i\varphi_2$, where φ_1 and φ_2 are real, we define $\Phi(\varphi) = \Phi(\varphi_1) + i\Phi(\varphi_2)$. Clearly, $\Phi(\varphi)$ is the desired generalized random process. □

We shall say that a generalized function F is non-negative definite if $(F, \varphi * \varphi^*) \geq 0$ for any $\varphi \in \mathcal{S}$. There is a one-to-one correspondence between non-negative definite generalized functions and continuous translation-invariant non-negative definite hermitian forms. Namely, given a generalized function F, we can define the form $B(\varphi, \psi) = (F, \varphi * \psi^*)$. Conversely, the existence of the non-negative definite generalized function corresponding to a form is guaranteed by Lemma 17.4. Theorem 17.6 can now be applied in the translation-invariant case to obtain the following statement.

Lemma 17.8. *For any non-negative definite generalized function F, there is a real-valued stationary Gaussian generalized process with zero expectation for which F is the covariance function.*

Let us introduce an important example of a generalized process. Note that the delta-function (the generalized function defined as $(\delta, \varphi) = \varphi(0)$) is non-negative definite.

Definition 17.9. *A real-valued stationary Gaussian generalized process with zero expectation and covariance function equal to delta-function is called white noise.*

Let us examine what happens to the covariance functional of a generalized process when we take the derivative of the process. If B_Φ is the covariance functional of the process Φ and $B_{\Phi'}$ is the covariance functional of Φ', then

$$B_{\Phi'}(\varphi, \psi) = \mathrm{E}(\Phi'(\varphi)\overline{\Phi}'(\psi)) = \mathrm{E}(\Phi(\varphi')\overline{\Phi}(\psi')) = B_\Phi(\varphi', \psi').$$

If the process Φ is stationary, and F_Φ and $F_{\Phi'}$ are the covariance functions of Φ and Φ' respectively, we obtain

$$(F_{\Phi'}, \varphi * \psi^*) = (F_\Phi, \varphi' * (\psi')^*).$$

Since $\varphi' * (\psi')^* = -(\varphi * \psi^*)''$,

$$(F_{\Phi'}, \varphi * \psi^*) = (-F''_\Phi, \varphi * \psi^*).$$

Therefore, the generalized functions $F_{\Phi'}$ and $-F''_\Phi$ agree on all test functions of the form $\varphi * \psi^*$. It is not difficult to show that such test functions are dense in $\mathcal{S}$. Therefore, $F_{\Phi'} = -F''_\Phi$.

In Chap. 18 we shall study Brownian motion (also called Wiener process). It is a real Gaussian process, denoted by W_t, whose covariance functional is given by the formula

$$B_W(\varphi, \psi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(s,t)\varphi(s)\overline{\psi}(t)dsdt,$$

where

$$k(s,t) = \begin{cases} \min(|s|, |t|) & \text{if } s \text{ and } t \text{ have the same sign,} \\ 0 & \text{otherwise.} \end{cases}$$

Although the Wiener process itself is not stationary, its derivative is, as will be seen below. Indeed, by using integration by parts,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(s,t)\varphi'(s)\overline{\psi'}(t)dsdt = \int_{-\infty}^{\infty} \varphi(t)\overline{\psi}(t)dt.$$

Therefore, the covariance functional of the derivative of the Wiener process is equal to

$$B_{W'}(\varphi, \psi) = B_W(\varphi', \psi') = \int_{-\infty}^{\infty} \varphi(t)\overline{\psi}(t)dt = (\delta, \varphi * \psi^*).$$

Since the derivative of a Gaussian process is a (generalized) Gaussian process, and the distributions of a Gaussian process are uniquely determined by its covariance function, we see that the derivative of the Wiener process is a white noise.