Co-Evolutive Models for Firms Dynamics

Giulia Rotundo and Andrea Scozzari

Abstract This paper considers the Bak–Sneppen (B&S) Self-Organized Criticality model originally developed for species co-evolution. We focus both on the original application of the model on a lattice, and on scale-free networks. Stylized facts on firms size distribution are also considered for the application of the model to the analysis of firms size dynamics. Thus, the B&S dynamics under the uniform, Normal, lognormal, Pareto, and Weibull distributions is studied. The original model is also extended by introducing weights on links connecting species, and examining the topology of the resulting Minimum Spanning Tree (MST) of the underlying network. In a system of firms a MST may evidence the template of the strongest interactions among firms. Conditions that lead to particular configurations of a MST are investigated.

1 Introduction

In the framework of Econometrics, the availability of large electronic databases has led to an increasing number of statistical analysis of raw data. Numerical studies on the size of firms often are concerned with the detection of the probability distribution that best fits the data sets. It is remarkable that same probability distributions for the size of firms are validated on large data sets encompassing several different industry sectors and long time extension. There is now wide empirical evidence indicating that the distribution of the degrees of the nodes in many networks

G. Rotundo
Faculty of Economics, University of Tuscia, Viterbo, Italy
giulia.rotundo@uniroma1.it

A. Scozzari
Faculty of Economics, University of Rome "La Sapienza", Rome, Italy
andrea.scozzari@uniroma1.it

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representing system of firms follows a power law. Other probability distributions are validated only on constrained subsets of data, but they result invariant on quite long time intervals [2, 3, 11, 17]. Although each single firm can experience fluctuations in its size, at a first analysis the collective evolution of firms, traced through their sizes, can be represented by a stationary process. Hence, the contribution of many single units to observables at the macrolevel has opened the way to complex systems approaches mainly based on stochastic models in discrete time and space state. The emergence of such common behavior from several data sets was explained by the fact that most firms share the same kind of hierarchical managing structure and internal organization [2, 3]. Furthermore, the collective behavior of groups of agents has been often interpreted as a mechanism for social contacts, as the basis for opinion switching relevant for financial applications [20, 29, 45], as well as in many other fields [6, 31–33]. These studies point out the relevance of closeness of agents in term of their similarity in the preferential direction for information spreading. Moreover, spatial closeness plays the major role in neighbors definitions in crowd behavior [43], thus validating models that mainly consider local dynamics driven by short range interactions. In particular, percolation models are well suitable for the modelling of such phenomena.

Bak and Sneppen (B&S) have introduced a simple model in the class of percolation models, addressing to each unit (agent) the role of species in an evolutionary context [10]. The B&S model is of intrinsic interest, since it is one of the simplest models giving rise to Self-Organizing Criticality (SOC) behavior. Species co-evolve to a stationary state and exhibit "intermittent dynamics", that is, species undergo long periods of little changes, called stasis, which are punctuated by sudden bursts of activities called avalanches (which are correlated with extinction events). The original formulation of the model on d-dimensional lattices with the usual boundary conditions has been already extensively studied [10, 39]. However, as argued by Watts and Strogatz [49], most real-life networks are neither perfectly ordered, nor completely disordered, but fall under the category of "small-world" networks, which interpolate smoothly between the two extremes [1, 49]. Such networks are characterized by a high degree of local order, yet appear disordered on a large-scale because of the presence of shortcuts in the networks. Because of their wide applicability, there have recently been numerous papers characterizing the properties of such networks. Scale-free networks provide a good example of small-world networks. Extension of the B&S co-evolutive dynamics on scale-free networks has evidenced different stasis dynamics of the involved species depending on their number of first neighbors [34, 35]. In this paper, we consider the application of the B&S model to firms size co-evolution by referring to probability distributions of the firms size drawn from empirical literature. We study this dynamics on lattice and scale-free networks.

The dynamics of the B&S model, at its stationary phase, seems to be well suitable for firms size modelling, since it accomplishes the permanence of the same firm's size probability distribution on long time intervals. We study the B&S dynamics by considering the same probability distributions that were already validated in the literature for the description of the firms size. In particular, the B&S dynamics seems to be suitable for modelling the extinction process of the firms, whose sizes are well fitted by a Weibull distribution.

In the original B&S model applied to lattice networks, links only serve to drive the dynamics and to define the neighborhood of each species. A link between two species represents a dependence between them, whose straightforward interpretation in natural evolution models represents a prey-predator relationship.

In this paper we introduce weights on links connecting the nodes, representing species (e.g., firms), of the network. Weights are intended to give a measure of the closeness among nodes, and their natural meaning range, for instance, from the amount of commercial interchange among firms, to the closeness of management teams, to the intensity of technological innovation, and so forth [3, 4, 42].

Furthermore, weights on links allow to devise subnetworks (or spanning subnetworks) pointing out the strongest relationships between species (e.g., firms). Among different classes of spanning subnetworks, we consider here the Spanning Trees, and, in particular, we are interested in finding a Minimum weight Spanning Tree (MST) of the underlying network. This is due to by both the existence of efficient algorithms for finding a MST even on huge networks, and the ease of interpreting its *geometry* from an economic viewpoint. The geometry of an MST is important in the *strong disordered limit* [19], and it remains unaltered on random graphs even if the distribution of disorder is made very broad [19]. The main reason can be addressed to the ordering of links weights, and any probability distribution that does not alter the order of the weights gives rise to the same MST configuration. Numerical studies on the geometry of Minimum Spanning Trees, when random uncorrelated weights are assigned to the links or edges of a network, have been provided for square and cubic lattices [19] and for scale-free networks [28, 36, 47]. The latter ones show universality of the spanning trees geometry highlighted in the scale-free structure of the MST itself. In fact, on non-sparse scale-free networks, the MST nodes degree distribution follows a power law with a degree exponent close to the one of the original network, and independent from the weight distribution. The B&S nodes co-evolution can be extended to links weights co-evolution. When applying the B&S dynamics on networks representing system of firms, we obtain not only a co-evolution of the firms size but also a evolution of the MSTs describing the evolution of the strongest relationships between firms. Furthermore, we are also able to investigate the conditions that give rise to different MST structures, with the aim to provide support for policies decisions.

We are aware of the fact that the B&S model is a toy model, but it has the advantage to hold some mathematical tractability, it can serve as a first approximation of collective behavior of market agents, and of the comprehension of properties and limits of the simple interactions between agents. Therefore, it can constitute a starting point for the development of more complex models.

The paper is organized as follows. The next section reviews the econometric literature about the distributions of the size of the firms. Section 3 shows the B&S model and its extensions, and finally Sect. 4 reports some results on our MST approach.

2 Empirical Studies on the Size of Firms

Several studies have been made on the detection of skew distributions of firms sizes and on the validity of Gibrat's law of proportionate effect for the growth rate in order to explain the empirically observed distribution of the firms size. This law states that the expected increment of a firm size in each period is proportional to the current size of the firm, and that the growth rate of each firm is independent from its size (Gibrat's law in weak form). Therefore, under the hypothesis that the growth rates are identically independent distributed, the distribution of the log of a firm's size tends to the lognormal distribution for $T \to \infty$, i.e. on sufficiently large time interval [16].

Mainstream econometric literature on firms size is aimed at showing the limits of the Gibrat model, and new growth rates and firms size distributions are proposed for fitting data. The studies about size and growth rate of firms differ for the hypotheses tested and for the data sets that were used. In the literature, data from Census and COMPUSTAT data bases are mainly analyzed. Census data give information about small firms, that are crucial for understanding the impact of social dynamics at the individual level. The volume of sales is used as a proxy for a firm size, and in some studies other fundamental variables like the total assets, sales and the number of employees are used as a complementary variables in order to check the validity and robustness of the results. Literature focuses mostly on the Pareto distribution as well as on the lognormal distribution as an alternative for the size of firms. The discussion is not purely an academic exercise. Right skewness implies that most firms have a size just below the average, and that there are few huge firms and some others very small. The detection of the proper distribution allows to explain differences in the reaction of the market to external shocks, like natural catastrophes, or the impact on some economy of exogenous economic factors. Computer aided simulations of economic systems show that, in the case of lognormally distributed data, shocks are absorbed, while in the case of Pareto distribution, correlations internal to the system can amplify the external shocks leading to strong oscillations of the entire system and to run the risk of a collapse [14]. These studies can help both in driving the best policies for economy development and in detecting the maximum charge of bad events (taxes, wars, natural catastrophes etc.) that can be supported without a complete crash. The next three subsections review the literature presenting some other studies on the distribution of the size of the firms supporting different probability distributions [11].

2.1 Econometric Analysis Supporting the Gibrat's Law

The distributions of the firms sizes in industrial countries are highly skewed, that is, a small number of large firms coexists along with a large number of small firms. The presence of right skewness supports both the Gibrat's law and the Pareto distribution.

Some studies [24, 25] inquiry the independence between growth rate and size. In [24] it is shown that the lognormal distribution hypothesis holds for UK firms larger than eight employers. Later, the same authors report that the size of the distribution of the UK companies is close to the lognormal, although the hypothesis of lognormality can be statistically rejected [25]. The test of the suitability of other distributions shows that the Pareto distribution performs the fit worse than the lognormal one at the upper tail. Other studies report that the fit of the lognormal distribution to size data is quite close to the mean, but it performs less on the tails. Families of functions, that include the lognormal one as a particular function, and that take into account a power law decay of tails, have also been developed. The goodness of statistical fit allows for some compromise. The weak form of Gibrat's law has been shown to be compatible with power law under further hypotheses. As an example, the first model is the Simon's model [16] where the Gibrat's law is combined with an entry process to obtain a Levy distribution for the firms size. Particular assumptions like the validity of the detailed balance, that states the time-reversal symmetry for the growth rate, show that Gibrat's law and Pareto–Zipf's law hold for firms larger than a fixed threshold [21]. This property is not valid in general [27], but the behavior of the largest companies is important because it influences the entire economy. Therefore, such analysis is useful for driving economic policies at the Country level. On the other hand, districts constitute small worlds with a prevalence of small sized industries, so that policies for district developments will be different from those based on the common behavior of big firms, and need a finer analysis.

2.2 Econometric Analysis Supporting the Pareto Distribution

Histograms of companies sizes exhibit skewness. In some data set, skewness has been shown to be robust over time [8]. It even lived through large-scale demographic transitions in the work forces and widespread technological changes. Finer analysis have shown that skewness grows during growing phases of the economy and decreases during recessions [22], thus being an indicator of such economic cycles. A characteristic that emerges is that, although the position of individual firms in a size distribution does depend on the definition of size, the shape of the distribution does not. The main concern is to select the best fit to data histograms. Although in older studies [24, 25] the lognormal hypothesis received great attention, in recent papers the main results indicate power law for firm size and Laplace law for firms growth rates [16]. The two results are strictly connected. In fact, it can be shown, under proper hypotheses, that the logarithm of a Pareto random variable follows an exponential distribution, and that the difference of two exponential random variables results in a Laplace distribution [40].

The power law behavior seems to be valid also for parameters that are common in most developed Countries. The results reported in [15] can be interpreted as the existence of a significant range of the world GDP distribution where countries share a common size-independent average growth rate. Further particular hypotheses like entry and exit of companies from the market, provide results that contradict the Gibrat's law. As an example in [2, 3, 44] the exponential distribution for the growth rate of firms has been found to hold for the 20 years 1974–1993 of COMPUSTAT publicly-traded United States manufacturing firms, whilst the variance of the growth rate should grow with the size of the firm. A model is also proposed which offers a possible explanation for the power law relationship between firms size and the variance of growth rate [46], showing a power law dependence of the variance of the growth rate conditioned to the size of the firms. It has been shown that such kind of dependence may rise from a hierarchical management organization provided with a disobedience probability.

2.3 Econometric Analysis Supporting the Weibull Distribution

A comparison with the distribution of the extinction rate of species introduces immediately the comparison between the B&S model for species co-evolution and the studies presented in Di Guilmi et al. [17]. In the former case the species, corresponding to nodes of a lattice in the B&S model, are the extinguished ones. The changing in the nodes' value in the B&S dynamics, indicates the extinction of a specie (firm) that will be replaced by a new specie represented by that node. The classification of firms by size is best fitted by a Weibull function. A best fit Weibull parameters table, reported in [17], considers data about the extinction rate of firms in eight OECD countries, and divides data into six classes by number of employees. In our application, we perform simulations with the B&S model referring to each of the six classes.

3 The Dynamic Model

In its original formulation, at each time $t = 0,\dots,T$, the d-dimensional B&S dynamic model considers L^d species organized in a simple regular lattice with the usual boundary conditions in dimension d. Each species, represented by each node of the lattice, is fully described at time t by its fitness, $f_i^d(t)$, $i = 1,\dots,L^d$, drawn at time 0 from a uniform distribution in $[0,1]$. Therefore, we are considering a network where to each node is assigned a value (fitness) while each link or edge of the network simply represents the connection between two nodes. In financial applications the values $f_i^d(t) \in [0,1)$ can be chosen for representing firm fitness [5, 7], as well as prices or opinions [12, 41, 50]. At each time step, the B&S model selects the node with the minimum fitness and changes its fitness and those of its $2d$ adjacent nodes by randomly generating new values from a uniform distribution in $[0,1)$.

One of the key problems related to the B&S evolution model is to compute the limit distribution for the values of the nodes at a stationary *regime*, as the time of the system grows to infinity. Computer simulations show that for a time long enough, under the B&S dynamics the maximum of minima of fitnesses are above a critical threshold f_c, apart from some periods, called avalanches, where they fall below f_c [9, 10, 23, 26]. In the one-dimensional case (i.e., $d = 1$) the limit (marginal) distribution is uniform on $(f_c, 1)$, with $f_c \sim 0.667$. These results were confirmed theoretically through the mean field approximation [18, 39, 48]. Many other properties of the B&S model were obtained after a change of distributions: from uniform to exponential.

3.1 The B&S-Exponential Model

The B&S-exponential model is defined to be the model obtained from the B&S-uniform one by substituting the hypothesis $f_i^d(t) \sim U[0,1)$, $i = 1, \ldots, L^d$, by $f_i^d(t) \sim D[0,\infty)$, $i = 1, \ldots, L^d$, where $D[0,\infty)$ is the exponential distribution in $[0,\infty)$.

Intermediate results in the proof assessing the existence of the critical threshold f_c and the behavior of the joint distribution of $f_i^d(t)$, $i = 1, \ldots, L^d$, are provided in [37, 38] using the exponential setup. The results were reported to the original B&S model through the following lemma based on some remarks in [37, 38].

Lemma 1. *The B&S-exponential model has the same dynamics properties of the B&S-uniform one.*

Proof. We follow the rules for random numbers generation. Let x be a random number sampled by a random variable uniformly distributed in $[0,1)$. The function $q : [0,1) \to [0,\infty)$, such that $q(x) := -ln(1-x)$ transforms x into a random number $y = q(x)$ sampled by a random variable exponentially distributed in $[0,\infty)$. If at times $t = 0, \ldots, T$, $q(\cdot)$ is applied to the $f_i^d(t)$, $i = 1, \ldots, L^d$, then it provides a one-to-one mapping between the fitnesses of the B&S-uniform and the B&S-exponential model. Moreover, the dynamics of the evolution of the fitnesses is still based on the minimum value of the fitnesses. Actually, $q(\cdot)$ is a monotone function, therefore at time t, the transformation applied on the $f_i^d(t)$, $i = 1, \ldots, L^d$, maintains the ordering of the values of the nodes, and the evolution rule selects the same node and its neighbors both in the uniform and in the exponential setup. $\square$

After Lemma 1, the function $q(\cdot)$ changes the values of the fitness of each node, but it does not change the dynamics of the B&S-uniform model. Furthermore, the results on the threshold in the B&S-exponential model can be easily suited to the original B&S model by the following remark.

Remark 1. If the B&S-exponential model has threshold f_c, then the B&S-uniform model has threshold $q(f_c) = 1 - e^{-f_c}$.

3.2 Extension of B&S Model to Other Probability Distributions

Let us consider the B&S model where the uniform distribution has been substituted by using a random variable X, defined on an arbitrary interval I, with cumulative probability distribution $F_X(x)$.

Theorem 1. *Let f_c be the threshold for the B&S-uniform model. Consider now the B&S model where the uniform distribution has been replaced by a random variable X with cumulative probability distribution $F_X(x)$. Then, the resulting B&S model has threshold $F_X(f_c)$.*

Proof. Given a random variable U uniformly distributed over the interval $(0,1)$, $X = F_X^{-1}(U)$ (provided F is invertible). Hence, $F_X(X)$ is uniformly distributed in $(0,1)$. Since $F_X(X)$ is continuous, monotone and non decreasing it is order preserving. Therefore, $F_X(X)$ maps the B&S model with any probability distribution to the B&S model with uniform distribution and $F_X(f_c)$ holds. □

Theorem 2. *If the B&S model has limit distribution given by the product of uniform distributions above f_c, then the B&S model where the uniform distribution is replaced by a random variable X exponentially distributed has limit distribution given by the product of exponential distributions above $F_X(f_c)$.*

Proof. The proof follows from Theorem 1 by considering the transformation $F_X(\cdot)$ on the fitnesses, where $F_X(\cdot)$ is the cumulative distribution function of a random variable X exponentially distributed. □

Firstly, in this paper we are interested in studying the B&S evolution model by referring to the distributions mostly used for describing the size of the firms. Therefore, here we consider the uniform, Normal, lognormal, Pareto, log-Pareto, Exponential, and Weibull distributions. Let us consider the B&S model in which the uniform distribution is substituted by one of the above distributions. The following remarks hold.

Remark 2. If the B&S-uniform model has threshold f_c, then the B&S model using Normal, lognormal, Pareto, log-Pareto, Exponential, Weibull, distributions, has thresholds $F_X(f_c)$, where $F_X(x)$ is the cumulative distribution function of a random variable X following the Normal, lognormal, Pareto, log-Pareto, Exponential, Weibull distribution, respectively.

Remark 3. If the B&S-uniform model has limit distribution of the fitnesses given by the product of uniform distributions, then the B&S model using the lognormal, Pareto, log-Pareto, Weibull distributions has limit distribution given by the product of the lognormal, Pareto, log-Pareto, Weibull distribution, respectively.

3.3 Extension of the B&S Model: The Case of Scale-Free Networks

The B&S model on lattices can be generalized by referring to arbitrary finite connected networks, like, for instance, small world networks [30] and scale-free (SF) networks. SF networks have been recognized to describe several real growing networks, and, at the same time, have proved to show very peculiar properties for diffusion properties. In a system of firms this serves for modelling the impact of external factors, as the spreading of innovation, the external modification of demand and supply and so forth. In particular, the diffusion properties corresponding to the fault tolerance property can provide the maximal amount of changes that a system can bear before having a deep drastic change in the firms organization. It is then natural to ask whether and to what extent the topology of these complex networks would affect the results obtained in classical evolution models like the B&S one. It results that the critical thresholds continue to exist only on a subset of SF networks [34].

4 Minimum Spanning Tree

We extend the B&S model by assigning weights to each link of the underlying network (lattice or scale-free networks). In our application, each node-firm value (fitness) represents the size of a firm, while a link provides the connection between two firms. Depending on the application, a weight associated to a link or edge may represent the quote of participation of a firm into another [42], the intensity of technological innovation [4], or the tightness of management structure [3]. In networks or graphs applications, given a graph, a customary problem is to find out the relevant relationships between nodes. This is often accomplished by finding certain spanning subgraphs of the given graph. Different classes of subgraphs can be considered, each providing different properties about the closeness of the nodes. In this paper we refer to trees. A spanning subtree of a network gives the minimum way of connecting all the nodes of the graph. Among all the possible spanning subtrees we will look for the one (or the ones since there may be more that one) that minimizes the sum of the weights. A Minimum weight Spanning Tree (MST) of a given network points out the strongest relationships between the nodes. Searching for a spanning subtree is also preferred compared with other spanning subgraphs since both the availability of efficient algorithms for finding a spanning tree even on huge networks, and the ease of interpreting its geometry. Thus, along with evaluating the nodes dynamics with the classical B&S model, we are able to evaluate the changes in the relationships between firms over times by observing the changes in the topology of a MST. Actually, MST topologies and particular kind of spanning subtrees may represent structures relevant for Economics, like, for instance, the raise of oligopolies. In the next subsections we show the conditions that give rise to very different shapes of a MST, leaving the most proper definition of weights to econometric work. In order to evaluate this dynamics we considered different probability distributions for assigning weights to the nodes and edges of the underlying network, and again we refer to the uniform, Normal, lognormal, Pareto, log-Pareto and Weibull distributions.

4.1 Weights not Correlated with the Degrees of the Nodes

Numerical results on MST shapes when random edge weights uncorrelated with nodes weights are considered, appear in Dobrin et al. [19] for square and cubic lattices. For scale-free networks, in Szabo et al. [47] is shown that if the weights of the links incident to a given node are independent from the node's degree (i.e., the degree of connectivity of the node) then the geometry of a MST depends only on the topology of the network. Therefore, the probability distribution of the nodes values (fitnesses) is not relevant for the MST topology, and the B&S dynamics changes the MST, but not its topology. As an example, Fig. 1 reports on the result in the case of a Weibull distribution on a two-dimensional square lattice. At time t, the edge connecting nodes i and j has weight $w_{ij} = \mid f_i^d(t) - f_j^d(t) \mid$. This is a first raw measure of distance between i and j, and it is independent of the degree of connectivity of both i and j. The degree distribution of the nodes is stable under the B&S dynamics and for the different probability distributions considered.

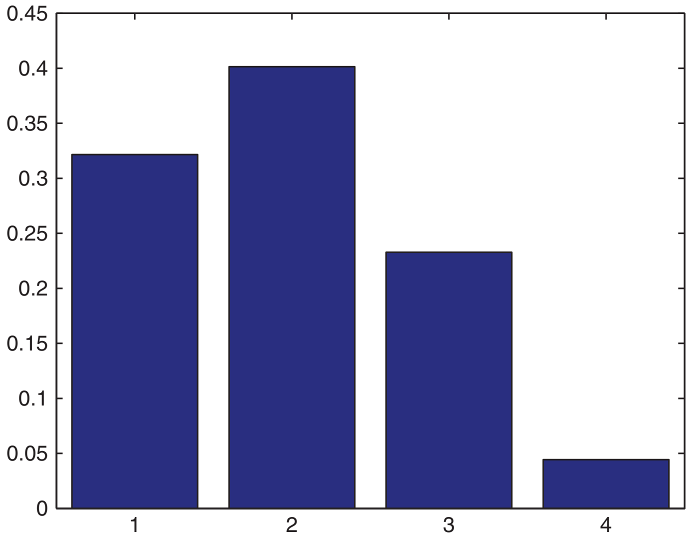

Fig. 1 Distribution of the degrees of the nodes in the MSTs on a lattice network and with a Weibull probability distribution

4.2 Weights Correlated with the Degrees of the Nodes

In order to observe other MST shapes, it is necessary to assign weights to the links depending on the degrees of the nodes. We leave the definition and interpretation of these weights, and their exact meaning problem-oriented, to empirical econometric papers addressing real-world problems.

In real-world applications, weights should be correlated to both the fitnesses and the degree of the nodes. Although the values of the fitnesses may change during the dynamics, only the nodes degree may change the topology of the MST. Referring to SF networks, due to their practical relevance, an immediate implication of the above remark is that any rewiring procedure mapping the network into another network having the same SF property will lead to the same MST topology. Therefore, it is worth examining the effects of the correlation function between edge weights and connectivity degree for the general class of SF networks, without adding any rewiring dynamics.

In [36] it is shown that solely by changing the nature of the correlations between wights and network topology, the structure of a MST may change from SF to exponential. In particular, they explore the MST behavior considering weights w_{ij} from node i to node j defined as being directly proportional to $(k_i k_j)\vartheta$, $\vartheta > 0$, $max(k_i, k_j)$, $min(k_i, k_j)$, $1/(k_i k_j)$, $1/min(k_i, k_j)$, $1/max(k_i, k_j)$.

Their numerical results indicate that in the presence of correlations, two classes of MSTs exist for scale-free networks, having either a power law or an exponential degree distribution. Correlated weights choices $w_{ij} \propto k_i k_j$, or $w_{ij} \propto max(k_i, k_j)$ give rise to MSTs with exponential degree distributions, while the other choices result in MSTs with power law distributions.

The exponential nature of the first two weights choices is due to the tendency of a MST algorithm to avoid links with large weights, so that, for this weight selection, a MST algorithm effectively shuns the highly connected nodes by using, when possible, links connecting low degree nodes. The remaining weights choices give rise to power law degree distribution of a MST [28, 36, 47].

Of course, intermediate degree of correlation between edge weights and nodes degree may give rise to very different shapes of a MST.

4.3 Numerical Results

We examined the B&S dynamics under different probability distributions. We considered two-dimensional square lattices with the usual boundary conditions and scale-free networks as underlying networks. Since the B&S dynamics shows a transient phase, we considered 10^9 iterations of the B&S dynamics before starting our analysis, and statistics were drawn on the next 10^7 iterations. On a two-dimensional square lattice we considered $L^d = 10,000$ evolving species. In order to provide statistics about the MST evolution, at each time $t = 0, \ldots, T$, we used the distance measure between nodes i and j defined by $w_{ij} = \mid f_i^d(t) - f_j^d(t) \mid$. The more the values of the nodes are close to each other, the lower is the distance w_{ij}. Mutations following the B&S dynamics randomly change the $f_i^d(t)$, $i = 1,\ldots,L^d$, and therefore the edge weights w_{ij}, for each edge (i, j). The sampling of the fitnesses $f_i^d(t)$, $i = 1,\ldots,L^d$, was performed according to the uniform, Normal, lognormal, Pareto, log-Pareto and Weibull distributions. Figure 1 shows the histogram representing the distribution of the degrees of the nodes in the MSTs in the case of a Weibull distribution.

The analysis on the edge weights in a MST, presented in [19, 36], does not consider any evolution w.r.t. the time. On the other hand, we provide a synthetical analysis that is obtained by considering the behavior of the weights of the MSTs obtained during the dynamics. We notice that the $f_i^d(t)$, $i = 1,\ldots,L^d$, and w_{ij} are not co-monotone, thus allowing the social interpretation of the possibility of disagreement among agents as they change their opinion. In the stationary state we have that all the $f_i^d(t) \in (f_c, 1)$, $i = 1,\ldots,L^d$ apart from avalanches. Thus, most of the weights w_{ij} belong to $[0, 1 - f_c)$, and exhibit avalanches as a consequence of fitnesses avalanches. Following Dobrin et al. [19], we also provide the probability that a link with a given weight lies on the MST (see Fig. 2).

We also note that for $d = 2$ and for lattice networks, a changing of the fitness of a node only implies a change of the weights of its $2d = 4$ adjacent nodes. This corresponds to modify the weights of 16 edges at a time, thus resulting in a fast updating procedure for the nodes and the edges values. The number of links changed at each step of the dynamics is centered at 5 over the 16 links changed. The MST constructing algorithm is based on the Kruskal procedure.

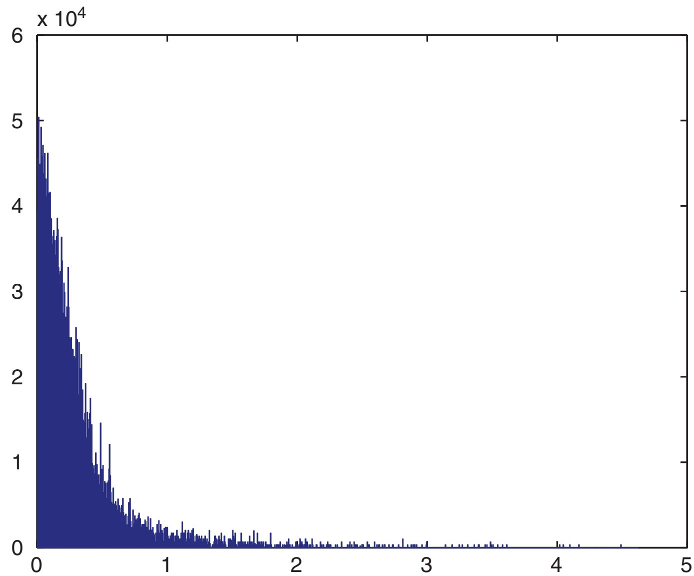

Fig. 2 Probability that a link with a given weight lies on the MST in a lattice and with a log-Pareto distribution

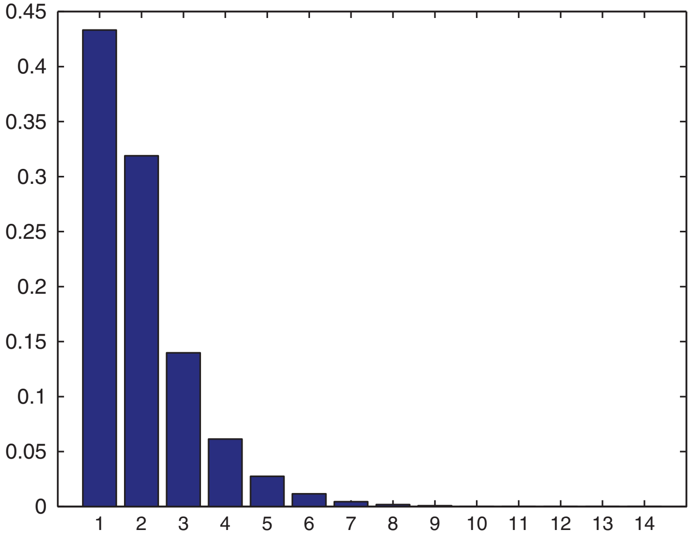

Fig. 3 Power law property for the distribution of the degrees of the nodes in a MST by using the Weibull distribution on scale-free networks

As introduced before, we also examined the results of the co-evolutive model when applied on scale-free networks, that are supposed to be the most spread form of contacts organizations. We used the free downloadable Barabasi–Albert software for scale-free networks generation. We run simulations on 1,005 nodes, and 20,000 edges. Then we run the B&S dynamics again considering the uniform, Normal, lognormal, Pareto, log-Pareto, and the Weibull distribution. As shown in Fig. 3, the distribution of the degrees of the nodes of the MSTs follows a power law, and this holds also for all the other probability distributions considered. All the programs implementing the B&S dynamics as well as the Kruskal algorithms for finding the MSTs are written in C code.

5 Conclusions

The paper goes beyond the B&S model features introducing weights and analyzing the co-evolution of nodes and their relationship with the MST evolution of a given underlying network. Actually, given a network representing the relationships between firms (nodes of the networks), a Minimum weight Spanning Tree points out the strongest relationships between the firms. In this paper, we are interested in evaluating the evolution of an MST and, in particular, the evolution of the geometry of an MST since different MST shapes may reveal structures relevant, for instance, in an economic system, like the raise of oligopolies. We notice that this paper differs from other papers in the literature (see e.g., [4]), where the values assigned to each node and edge of a network are constant. In fact, while their dynamics concerns activation/disactivation of links, thus considering the so called *rewiring problem*, we are interested in studying the evolution of a spanning subgraph of the network that summarizes the relationships between the nodes of the network. Indeed, in social models, a MST may represent the most strong social interactions between agents, that can be transferred to management links when considering the evolution of firms.

MST dynamics and its evolution strictly depends on the underlying network as well as on the sampling distribution of the values assigned to its nodes and edges. Our toy model provides a dynamical co-evolutive model that considers functional properties observed in empirical studies, and can constitute a starting point for the construction of more realistic models. Finer econometric analysis evidenced fluctuations of the parameters of probability distribution of the size of the firms, depending on the selection of the time windows [13]. The models presented here can be easily adapted to accomplish the arise and the change of the shape and scale parameters of probability distribution during cycles of recession and expansion of economies. The introduction of edge weights correlated to the degrees of nodes in place of weights that are proportional to them, and the rewiring mechanism for changing network topologies are the main tools for calibrating the models. We leave this for future research as well as further investigations introducing asymmetry in the network's links.

References

1. Albert R, Barabasi AL (2002) Statistical mechanics of complex networks. Rev Mod Phys 74:47–97
2. Amaral LAN, Buldyrev SV, Havlin S, Leschhorn H, Maass P, Salinger MA, Stanley HE, Stanley MHR (1997) Scaling behavior in economics: I. Empirical results for company growth. J Phys I 7:621–633
3. Amaral LAN, Buldyrev SV, Havlin S, Leschhorn H, Maass P, Salinger MA, Stanley HE, Stanley MHR (1997) Scaling behavior in economics: II. Modeling of company growth. J Phys I 7:635–650
4. Andergassen R, Nardini F, Ricottilli M (2007) The emergence of paradigm setters through firms interaction and network formation (to appear)
5. Ausloos M, Pekalski A (2007) Model of wealth and goods dynamics in a closed market. Physica A 373:560–568
6. Ausloos M, Petroni F (2007) Statistical dynamics of religions and adherents. Europhys Lett 77:38002
7. Ausloos M, Clippe P, Pekalski A (2004) Evolution of economic entities under heterogeneous political/environmental conditions within a Bak–Sneppen-like dynamics. Physica A 332:394–402
8. Axtell RL (2001) Zipf distribution of u.s. firm sizes. Science 293:1818–1820
9. Bak P (1996) How nature works. Springer, New York
10. Bak P, Sneppen K (1993) Punctuated equilibrium and criticality in a simple model of evolution. Phys Rev Lett 71:4083–4086
11. Cerqueti R, Rotundo G (2007) Processi di rinnovamento nei cluster di imprese. In Garofalo G, Capitalismo distrettuale, localismi d'impresa, globalizzazione. Firenze University Press: 129–143

12. Cuniberti G, Valleriani A, Vega JL (2001) Effects of regulation on a self-organized market. Quantitative Finance 1:332–338
13. Delli Gatti D, Di Guilmi C, Gaffeo E, Giulioni G, Gallegati M, Palestrini A (2004) Business cycle fluctuations and firms' size distribution dynamics. Adv Complex Syst 7(2):1–18
14. Delli Gatti D, Di Guilmi C, Gaffeo E, Giulioni G, Gallegati M, Palestrini A (2005) A new approach to business fluctuations: heterogeneous interacting agents, scaling laws and financial fragility. J Econ Behav Organ 56:489–512
15. Di Guilmi C, Gaffeo E, Gallegati M (2003) Power law scaling in the world income distribution. Econ Bull 15(6):1–7
16. Di Guilmi C, Gaffeo E, Gallegati M (2004) Empirical results on the size distribution of business cycle phases. Physica A 333:325–334
17. Di Guilmi C, Gallegati M, Ormerod P (2004) Scaling invariant distributions of firms exit in OECD countries. Physica A 334:267–273
18. Dickman R, Muñoz MA, Vespignani A, Zapperi S (2000) Paths to self-organized criticality. Bras J Phys 30:27–41
19. Dobrin R, Duxbury PM (2001) Minimum spanning trees on random networks. Phys Rev Lett 86(22):5076–5079
20. Föllmer H, Horst U, Kirman A (2005) Equilibria in financial markets with etherogeneous agents: a prbobabilistic perspective. J Math Econ 41:123–125
21. Fujiwara Y, Di Guilmi C, Aoyama H, Gallegati M, Souma W (2000) Do Pareto–Zipf and Gibrat laws hold true? An analysis with European firms. Physica A Stat Mech Appl 335 (1–2):197–216
22. Gaffeo E, Gallegati M, Palestrini A (2003) On the size distribution of firms: additional evidence from the G7 countries. Physica A Stat Mech Appl 324(1–2):117–123
23. Grassberger P (1995) The Bak–Sneppen model for punctuated evolution. Phys Lett A 200:277–282
24. Hart PE, Oulton N (1996) Growth and size of firms. Econ J 106:1242–1252
25. Hart PE, Oulton N (1997) Zipf and the size distribution of firms. Appl Econ Lett 4:205–206
26. Jensen HJ (1998) Self-organized criticality. Cambridge Lecture Notes in Physics
27. Kertesz J, Kullmann L, Zawadowski AG, Karadi R, Kaski K (2003) Correlations and response: absence of detailed balance on the stock market. Physica A 324(1):74–80
28. Kim DH, Noh JD, Jeong H (2004) Scale-free trees: skeleton of complex networks. Phys Rev E 70:046126
29. Kirman A, Teyssiere G (2002) Microeconomic models for long-memory in the volatility of financial time series. Studies in Noninear Dynamics and Econometrics 5:281–302
30. Kulkarni RV, Almaas E, Stroud D (1999) Evolutionary dynamics in the Bak–Sneppen model on small-world networks. arXiv:cond-mat/9905066 v1 6
31. Lambiotte R, Ausloos M (2005) Uncovering collective listening habits and music genres in bipartite networks. Phys Rev E 72:066107
32. Lambiotte R, Ausloos M (2006) Modeling the evolution of coupled networks. In: First World congress on Social Simulation (WCSS 2006), e-Proceedings 1:375–381
33. Lambiotte R, Ausloos M, Holyst JA (2007) Majority model on a network with communities. Phys Rev E 75
34. Lee S, Kim Y (2005) Coevolutionary dynamics on scale-free networks. Phys Rev E 71:057102
35. Lee KE, Hong BH, Lee JW (2005) Universality class of Bak–Sneppen model on scale-free network. arXiv:cond-mat/0510067
36. Macdonald PJ, Almaas E, Barabási AL (2005) Minimum spanning trees of weighted scale-free networks. Europhys Lett. 72:308314
37. Maslov S (1996) Infinite hierarchy of exact equations in the Bak–Sneppen model. Phys Rev Lett 77:1182
38. Meester R, Znamenski D (2004) Critical thresholds and the limit distribution in the Bak–Sneppen model. Commun Math Phys 246
39. Paczuski M, Maslov S, Bak P (1996) Avalanche dynamics in evolution, growth, and depinning models. Phys Rev E 53:414

40. Palestrini A (2007) Analysis of industrial dynamics: a note on the relationship between firms' size and growth rate. Econ Lett 94(3):367–371
41. Rotundo G, Ausloos M (2007) Microeconomic co-evolution model for financial technical analysis signals. Physica A 373:569–585
42. Salvemini MT, Simeone B, Succi R (1995) Analisi del possesso integrato nei gruppi di imprese mediante grafi. L'Industria XVI:641–662
43. Sornette D (2002) Why the stock markets crash. Princeton University Press, Princeton
44. Stanley MHR, Amaral LAN, Buldyrev SV, Havlin S, Leschhorn H, Maass P, Salinger MA, Stanley HE (1996) Scaling behaviour in the growth of companies. Lett Nature 379:804–806
45. Stauffer D, Sornette D (1999) Self-organized percolation model for stock market fluctuations. Physica A 271:496
46. Sutton J (2003) The variance of corporate growth rates. Physica A 324(1):45–48
47. Szabó GJ, Alava M, Kertesz J (2003) Geometry of minimum spanning trees on scale-free networks. Physica A 330:31–36
48. Vergeles M (1995) Self-prganization at nonzero temperatures. Phys Rev Lett 75
49. Watts DJ, Strogatz SH (1998) Collective dynamics of 'small-world' networks. Nature 393:4
50. Yamano T (2001) Regulation effects on market with Bak–Sneppen model in high dimenzions. Int J Mod Phys C 12:13291333