Games of Coalition and Network Formation: A Survey

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Abstract This paper presents some recent developments in the theory of coalition and network formation. For this purpose, a few major equilibrium concepts recently introduced to model the formation of coalition structures and networks among players are briefly reviewed and discussed. Some economic applications are also illustrated to give a flavour of the type of predictions such models are able to provide.

1 Introduction

Very often in social life individuals take decisions within groups (households, friendships, firms, trade unions, local jurisdictions, etc.). Since von Neumann and Morgenstern's [45] seminal work on game theory, the problem of the formation of coalitions has been a highly debated topic among game theorists. However, during this seminal stage and for a long period afterward, the study of coalition formation was almost entirely conducted within the framework of games in characteristic form (cooperative games) which proved not entirely suited in games with externalities, i.e. virtually all games with genuine interaction among players. Only in recent years, a widespread literature on what is currently known as noncooperative coalition formation or endogenous coalition formation has come into the scene with the explicit purpose to represent the process of formation of coalitions of agents and hence modelling a number of relevant economic and social phenomena.[1] Moreover, following this theoretical and applied literature on coalitions, the recent paper by

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[1] Extensive surveys of the coalition formation literature are contained in Greenberg [23], Bloch [4, 5], Yi [46, 46] and Ray and Vohra [41].

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Jackson and Wolinsky [34] opened the door to a new stream of contributions using networks (graphs) to model the formation of links among individuals.[2]

Throughout these brief notes, I survey non exhaustively some relevant contributions of this wide literature, with the main aim to provide an overview of some modelling tools for economic applications. For this purpose, some basic guidelines to the application of coalition formation in economics are presented using as primitives the games in strategic form. As far as economic applications are concerned, most of the examples presented here mainly focus, for convenience, on a restricted number of I.O. topics, as cartel formation, horizontal merger and R&D alliances.

2 Coalitions

2.1 Cooperatives Games with Externalities

Since von Neumann and Morgenstern [45], a wide number of papers have developed solution concepts specific to games with coalitions of players. This literature, known as cooperative games literature, made initially a predominant use of the characteristic function as a way to represent the worth of a coalition of players.

Definition 1. A cooperative game with transferable utility (TU cooperative game) can be defined as a pair (N, v), where $N = \{1, 2, ..i, ..N\}$ is a finite set of players and $v : \mathbb{N} \to R_+$ is a mapping (characteristic function) assigning a value or worth to every feasible coalition, i.e. every nonempty subset of players $S \subset N$ belonging to $\mathbb{N}$, the family of nonempty coalitions $2^N \backslash \{\varnothing\}$.[3]

The value $v(S)$ can be interpreted as the maximal aggregate amount of utility members of coalition S can achieve by coordinating their strategies. In strategic environments, players' payoffs are defined on the strategies of all players and the worth of a group of players S depends on their expectations about the strategies played by the remaining players $N \backslash S$. Hence, to obtain $v(S)$ from a strategic situation, we need first to define an underlying strategic form game.

Definition 2. A strategic form game is a triple $G = \{N, (X_i; u_i)_{i \in N}\}$, in which for each $i \in N$, X_i is the set of strategies with generic element x_i, and $u_i : X_1 \times \cdots \times X_n \to R_+$ is every player's payoff function.

Moreover, henceforth we restrict the action space of each coalition $S \subset N$ to $X_S \equiv \prod_{i \in S} X_i$. Let, also, $v(S) = \sum_{i \in S} u_i(x)$, for $x \in X_N \equiv \prod_{i \in N} X_i$.[4]

[2] Myerson [36] and Aumann and Myerson [2] were among the first authors to use graphs to model cooperation between individuals. Excellent surveys of the network literature are contained in Dutta and Jackson [17] and in Jackson [28–31].

[3] Here we mainly deal with games with transferable utility. In games without transferable utility, the worth of a coalition associates with each coalition a players' utility frontier (a set of vectors of utilities).

[4] See Sect. 2.3 for an interpretation of these restrictions.

Example 1. Two-player prisoner's dilemma.

	A	B
A	*3,3*	*1,4*
B	*4,1*	*2,2*

Therefore, $v(N) = 6$ and $v(\{i\}) = \begin{cases} 4, & \text{if } x_j = A \\ 2, & \text{if } x_j = B \end{cases}$ for $j \neq i$.

The cooperative allocation (3; 3) can be considered stable only if every player is expected to react with strategy B to a deviation of the other player from the cooperative strategy A.

The above example shows that in order to define the worth of a coalition of players, a specific assumption on the behaviour of the remaining players is required.

2.1.1 α- and β-Characteristic Functions

The concepts of core, formally studied by Aumann [1], are based on von Neumann and Morgenstern's [45] early proposal of representing the worth of a coalition as the minmax or maxmin aggregate payoff that it can guarantee its members in the underlying strategic form game. Accordingly, the characteristic function $v(S)$ in games with externalities can be obtained assuming that outside players act to minimize the payoff of every deviating coalition $S \subset N$. In this minimax formulation, if members of S move second, the obtained characteristic function,

$$v_\beta(S) = \min_{x_{N\setminus S}} \max_{x_S} \sum\nolimits_{i \in S} u_i(x_S, x_{N\setminus S}), \tag{1}$$

denoted β-characteristic function, represents what members in S *cannot be prevented from getting*. Alternatively, if members of S move first, we have

$$v_\alpha(S) = \max_{x_S} \min_{x_{N\setminus S}} \sum\nolimits_{i \in S} u_i(x_S, x_{N\setminus S}) \tag{2}$$

denoted α-characteristic function, which represents what members in S *can guarantee themselves*, when they expect a retaliatory behaviour from the complement coalition $N\setminus S$.[5]

When the underlying strategic form game G is zero-sum, (1) and (2) coincide. In non-zero sum games they can differ and, usually, $v_\alpha(S) < v_\beta(S)$ for all $S \subset N$.

However, α- and β-characteristic functions express an irrational behaviour of coalitions of players, acting as if they expected their rivals to minimize their payoff. Although appealing because immune from any *ad hoc* assumption on the reaction of the outside players (indeed, their minimizing behavior is here not meant to represent the expectation of S but rather as a mathematical way to determine the lower

[5] Note that here players outside S are treated as one coalition, so the implicit assumption is that players in $N\setminus S$ stick together after S departure from the grand coalition N.

bound of S's aggregate payoff), still this approach has important drawbacks: deviating coalitions are too heavily penalized, while outside players often end up bearing an extremely high cost in their attempt to hurt deviators. Moreover,the little profitability of coalitional objections yield very large set of solutions (e.g. large cores).

2.1.2 Nash Behaviour Among Coalitions

Another way to define the characteristic function in games with externalities is to assume that in the event of a deviation from N, a coalition S plays *à la* Nash with remaining players.[6]

Although appealing, such a modelling strategy requires some specific assumptions on the coalition structure formed by remaining players $N \backslash S$ once a coalition S has deviated from N.

Following the Hart and Kurtz's [25] coalition formation game, two extreme predictions can be assumed on the behaviour of remaining players. Under the so called γ-assumption,[7] when a coalition deviates from N, the remaining players split up in singletons; under the δ-assumption, players in $N \backslash S$ stick together as a unique coalition.[8]

Therefore, the obtained characteristic functions can be defined as follows:

$$v_\gamma(S) = \sum_{i \in S} u_i \left(\overline{x}_S, \left\{ \overline{x}_j \right\}_{j \in N \backslash S} \right), \tag{3}$$

where $\overline{x}$ is a strategy profile such that, for all $S \subset N$, $\overline{x}_S \in X_S$ and $\forall j \in N \backslash S$, $\overline{x}_j \in X_j$

$$\overline{x}_S = \arg\max_{x_S \in X_S} \sum_{i \in S} u_i \left(x_S, \left\{ \overline{x}_j \right\}_{j \in N \backslash S} \right)$$

$$\overline{x}_j = \arg\max_{x_j \in X_j} \; u_j \left(\overline{x}_S, \left\{ x_k \right\}_{k \in (N \backslash S) \backslash \{j\}}, x_j \right).$$

Moreover,

$$v_\delta(S) = \sum_{i \in S} u_i \left(\overline{x}_S, \overline{x}_{N \backslash S} \right) \; ,$$

where,

$$\overline{x}_S = \arg\max_{x_S \in X_S} \sum_{i \in S} u_i \left(x_S, \overline{x}_{N \backslash S} \right)$$

$$\overline{x}_j = \arg \max_{x_{N \backslash S} \in X_{N \backslash S}} \sum_{j \in N \backslash S} u_j \left(\overline{x}_S, x_{N \backslash S} \right).$$

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[6] The idea that coalitions in a given coalition structure can play noncooperatively among them was firstly explored by Ichiishi [26].

[7] Hurt and Kurz's [25] Γ- game is indeed a strategic coalition formation game with fixed payoff division, in which the strategies consist of the choice of a coalition. Despite the different nature of the two games, there is an analogy concerning the coalition structure induced by a deviation from the grand coalition.

[8] See Chander and Tulkens [14] for applications of this approach.

In both cases, for (3) and (4) to be well defined, the Nash equilibrium of the strategic form game played among coalitions must be unique. Moreover, usually, $v_\alpha(S) < v_\beta(S) < v_\delta(S)$ for all $S \subset N$.

2.1.3 Timing and the Characteristic Function

It is also conceivable to modify the γ- or δ-assumption reintroducing the temporal structure typical of the α and β-assumptions.[9]

When a deviating coalition S moves first under the γ-assumption, the members of S choose a coordinated strategy as leaders, thus anticipating the reaction of the players in $N \backslash S$, who simultaneously choose their best response as singletons. The strategy profile associated to the deviation of a coalition S is the Stackelberg equilibrium of the game in which S is the leader and players in $N \backslash S$ are, individually, the followers. We can indicate this strategy profile as a $\widetilde{x}(S) = (\widetilde{x}_S, x_j(\widetilde{x}_S))$ such that

$$\widetilde{x}_S = \arg\max_{x_S \in X_S} \sum_{i \in S} u_i \left(x_S, \left\{ x_j(x_S) \right\}_{j \in N \backslash S} \right) \tag{4}$$

and, for every $j \in N \backslash S$,

$$x_j(x_S) = \arg\max_{x_j \in X_j} \; u_j \left(\widetilde{x}_S, \{ x_k(\widetilde{x}_S) \}_{k \in (N \backslash S) \backslash \{j\}}, x_j \right). \tag{5}$$

Sufficient condition for the existence of a profile $\widetilde{x}(S)$ can be provided. Assume that $G(N \backslash S, x_S)$, the restriction of the game G to the set of players $N \backslash S$ given the fixed profile x_S, possesses a unique Nash Equilibrium for every $S \subset N$ and $x_S \in X_S$, where X_S is assumed compact. Let also each player's payoff be continuous in each player's strategy. Thus, by the closedness of the Nash equilibrium correspondence (see, for instance, [20]), members of S maximize a continuous function over a compact set and, by Weiestrass Theorem, a maximum exists. As a consequence, for every $S \subset N$, there exists a Stackelberg equilibrium $\widetilde{x}(S)$. We can thus define the characteristic function $v_\lambda(S)$ as follows:

$$v_\lambda(S) = \sum_{i \in S} u_i \left(\widetilde{x}_S, \left\{ \overline{x}_j(\widetilde{x}_S) \right\}_{j \in N \backslash S} \right).$$

Obviously, $v_\lambda(S) \geq v_\gamma(S)$. Inverting the timing of deviations and reactions, the γ-assumption can be modified by assuming that a deviating coalition S plays as follower against all remaining players in $N \backslash S$ acting as singleton leaders. Obviously, the same can be done under the δ-assumption.

2.1.4 The Core in Games with Externalities

We can test the various conversions of $v(S)$ introduced above by examining the different predictions obtained using the *core* of (N, v).

[9] See Currarini and Marini [15] for more details.

We first define an imputation for (N,v) as a vector $\mathbf{z} \in R^n_+$ such that $\sum_{i\in N} z_i \leq v(N)$ (feasibility) and $z_i \geq v(i)$ (individual rationality) for all $i \in N$.

Definition 3. The core of a TU cooperative game (N,v) is the set of all imputations $\mathbf{z} \in R^n_+$ such that $\sum_{i\in S} z_i \geq v(S)$ for all $S \subseteq N$.

Given that coalitional payoffs are obtained from an underlying strategic form game, the core can also be defined in terms of strategies, as follows.

Definition 4. The joint strategy $\mathbf{x} \in X_N$ is core-stable if there is no coalition $S \subset N$ such that $v(S) > \sum_{i\in S} u_i(\mathbf{x})$.

Example 2 (Merger in a linear Cournot oligopoly). Consider three firms $N = \{1,2,3\}$ with linear technology competing à la Cournot in a linear demand market. Let the demand parameters a and b and the marginal cost c, be selected in such a way that interior Nash equilibria for all coalition structures exist. The set of all feasible coalitions of the N players is

$$\mathbb{N} = (\{1,2,3\},\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}).$$

Note that if all firms merge, they obtain the monopoly payoff $v(\{1,2,3\} = \frac{A}{4}$, where $A = (a-c)^2/b$, independently of the assumptions made on the characteristic function. These assumptions matters for the worth of intermediate coalitions. Under the α- and β-assumptions, if either one single firm or two firms leave the grand coalition N, remaining firms can play a minimizing strategy in such a way that, for every $S \subset N$, $v_\alpha(S) = v_\beta(S) = 0$. In this case, the core coincides with all individually rational Pareto efficient payoff, i.e. all points weakly included in the set of coordinates, $\mathbf{Z} = \left[\left(\frac{A}{8},\frac{A}{16},\frac{A}{16}\right),\left(\frac{A}{16},\frac{A}{8},\frac{A}{16}\right),\left(\frac{A}{16},\frac{A}{16},\frac{A}{8}\right)\right]$. Under the γ-assumption, we know that when, say firms 1 and 2, jointly leave the merger, a simultaneous duopoly game is played between the coalition $\{1,2\}$ and firm $\{3\}$. Hence, $v_\gamma(\{1,2\}) = \frac{A}{9}$. Similarly for all other couples of firms deviating from N. When instead a single firm i leaves the grand coalition N, a triopoly game is played, with symmetric payoffs $v_\gamma(\{i\}) = \frac{A}{16}$ (all these payoffs are obtained from the general expression $v(S) = \frac{A}{(n-s+2)^2}$ expressing firms' profits in a n-firm oligopoly). In this case, since intermediate coalitions made of two players do not give each firm more than their individually rational payoff, the core under the γ-assumption coincides with the core under the α- and β-assumptions. We know from Salant et al. [42] model of merger in oligopoly, that $v_\gamma(S) > \sum_{i\in S} v_\gamma(\{i\})$ only for $|S| > 0;8\ |N|$. This means that in the merger game the core under the γ-assumption shrinks with respect to the core under the α- and β-assumptions only for $n > 5$. Under the δ-assumption, when a single firm leaves N, a simultaneous duopoly game is played between the firm $\{i\}$ and the remaining firms $N\backslash\{i\}$ acting as a single coalition. As a result, $v(\{i\}) = \frac{A}{9}$, which is greater than $\frac{A}{12}$, the maximum payoff at least one firm will obtain in the grand coalition. Therefore, under the δ-assumption, the core is empty. Finally, note that since under the λ-assumption every single firm playing as leader obtains $v(\{i\}) = \frac{A}{12}$, in such a case the core is unique and contains only the equal split imputation $\mathbf{z} = (\frac{A}{12},\frac{A}{12},\frac{A}{12})$ [see Figs. 1 and 2].

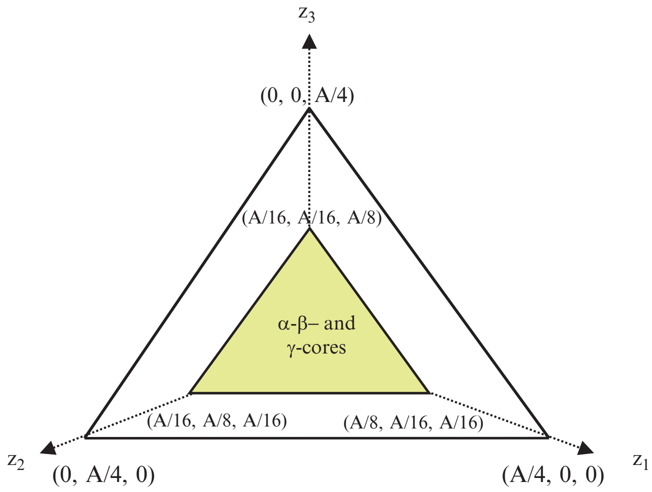

Fig. 1 Merger game: α, β and γ-cores

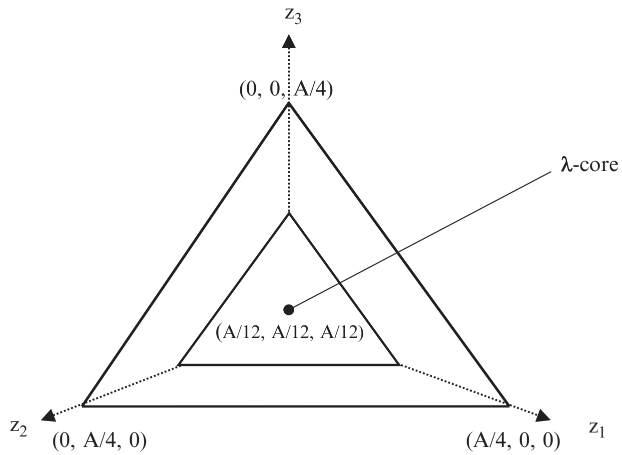

Fig. 2 Merger game: λ-core

2.2 Noncooperative Games of Coalition Formation

Most recent approaches have looked at the process of coalition formation as a strategy in a well defined game of coalition formation (see [7, 8, 47] for surveys). Within this stream of literature, usually indicated as *noncooperative theory of coalition formation* (or *endogenous coalition formation*), the work by Hurt and Kurz [25] represents the main seminal contribution. Most recent contributions along these lines include Bloch [4, 5], Ray and Vohra [40, 41] and Yi [46]. In all these works, cooperation is modelled as a two stage process: at the first stage players form coalitions, while at the second stage formed coalitions interact in a well defined strategic setting. This process is formally described by a *coalition formation game*, in which a given rule of coalition formation maps players' announcements of coalitions into a well defined coalition structure, which in turns determines the equilibrium strategies chosen by players at the second stage. A basic difference among the various models lies on the timing assumed for the coalition formation game, which can either be simultaneous (Hurt and Kurz [25], Ray and Vohra [40], Yi [46]) or sequential ([5], Ray and Vohra [41]).

2.2.1 Hurt and Kurz's Games of Coalition Formation

Hurt and Kurz [25] were among the first to study games of coalition formation with a valuation in order to identify stable coalition structures.[10] As valuation, Hurt and Kurz adopt a general version of Owen value for TU games [38], i.e. a Shapley value with prior coalition structures, that they call Coalitional Shapley value, assigning to every coalition structure a payoff vector $\varphi_i(\pi)$ in $\mathbf{R}^N$, such that (by the efficiency axiom) $\sum_{i\in N}\varphi_i(\pi) = v(N)$. Given this valuation, the game of coalition formation is modelled as a game in which each player $i \in N$ announces a coalition $S \ni i$ to which he would like to belong; for each profile $\sigma = (S_1, S_2, \dots, S_n)$ of announcements, a partition $\pi(\sigma)$ of N is assumed to be induced on the system. The rule according to which $\pi(\sigma)$ originates from σ is obviously a crucial issue for the prediction of which coalitions will emerge in equilibrium. Hurt and Kurz's game Γ predicts that a coalition emerges if and only if all its members have declared it (from which the name of "unanimity rule" also used to describe this game).

Formally:

$$\pi^{\gamma}(\sigma) = \{S_i(\sigma) : i \in N\} \ ,$$

where

$$S_i(\sigma) = \begin{cases} S_i \text{ if } \ S_i = S_j \text{ for all } j \in S_i \\ \quad \{i\} \text{ otherwise.} \end{cases}$$

Their game Δ predicts instead that a coalition emerges if and only if all its members have declare the same coalition S (which may, in general, differs from S). Formally:

$$\pi^{\delta}(\sigma) = \left\{S \subset N : i, j \in S \text{ if and only if } S_i = S_j\right\}.$$

It can be seen that the two rules generate different partitions after a deviation by a coalition: in the Γ-game, remaining players split up in singletons; in the Δ-game, they stick together.

Example 3. $N = \{1,2,3\}$, $\sigma_1 = \{1,2,3\}$; $\sigma_2 = \{1,2,3\}$; $\sigma_3 = \{3\}$

$$\pi^{\gamma}(\sigma) = (\{1\},\{2\},\{3\}),$$

$$\pi^{\delta}(\sigma) = (\{1,2\},\{3\}).$$

[10] Another seminal contribution is Shenoy [43].

Note that the two rules of formation of coalitions are "exclusive" in the sense that each player of a forming coalition has announced a list of its members. Moreover, in the gamma-game this list has to be approved unanimously by all coalition members.

Once introduced these two games of coalition formation, a stable coalition structure for the game Γ (Δ) can be defined as a partition induced by a Strong Nash Equilibrium strategy profile of these games.

Definition 5. The partition π is a γ-stable (δ-stable) coalition structure if $\pi = \pi^{\gamma}(\sigma^*)$ (or $\pi = \pi^{\delta}(\sigma^*)$) for some σ^* with the following property: there exists no $S \subset N$ and $\sigma_S \in \Sigma_S$ such that

$$u_i(\sigma_S, \sigma^*_{N\setminus S}) \geq u_i(\sigma^*) \text{ for all } i \in S$$

and

$$u_h(\sigma_S, \sigma^*_{N\setminus S}) > u_h(\sigma^*) \text{ for at least one } h \in S.$$

In the recent literature on endogenous coalition formation, the coalition formation game by Hurt and Kurz is usually modelled as a first stage of a game in which, at the second stage formed coalitions interact in some underlying strategic setting. The coalition formation rules are used to derive a valuation mapping from the set of all players' announcements Σ into the set of real numbers. These payoff functions are obtained by associating with each partition $\pi = \{S_1, S_2, \ldots, S_m\}$ a game in strategic form played by coalitions

$$G(\pi) = (\{1,2,\ldots,m\}, (X_{S_1}, X_{S_2}, \ldots, X_{S_m}), (U_{S_1}, U_{S_2}, \ldots, U_{S_m})),$$

in which X_{S_k} is the strategy set of coalition S_k and $U_{S_k} : \Pi^m_{k=1} X_{S_k} \to R_+$ is the payoff function of coalition S_k, for all $k = 1,2,\ldots,m$. The game $G(\pi)$ describes the interaction of coalitions after π has formed as a result of players announcements in Γ.or Δ-coalition formation games.

The Nash equilibrium of the game $G(\pi)$ (assumed unique) gives the payoff of each coalition in π; within coalitions, a fix distribution rule yields the payoffs of individual members.

Following our previous assumptions (see Sect. 1.2) we can derived the game $G(\pi)$ from the the strategic form game G by assuming that $X_{S_k} = \prod_{i \in S_k} X_i$ and $U_{S_k} = \sum_{i \in S_k} u_i$, for every coalition $S_k \in \pi$. We can also assume $u_i = \frac{U_{S_k}}{|S_k|}$ as the per capita payoff function of members of S_k. Therefore, using Example 1, for the Γ-game , $u_i(x^*(\{1,2,3\}) = \frac{A}{12}$, for $i = 1,2,3$, $u_i(x^*(\{i,j\},\{k\}) = u_j(x^*(\{i,j\},\{k\}) = \frac{A}{18}$, $u_k(x^*(\{i,j\},\{k\}) = \frac{A}{9}$ and $u_i(x^*(\{i\},\{j\},\{k\}) = \frac{A}{16}$, for $i = 1,2,3$. Therefore, the grand coalition is the only stable coalition structure of the Γ-game of coalition formation. For the Δ-game, there are no stable coalition structures.

If we extend the merger game to n firms, we know that the payoff of each firm $i \in S \subset N$ when all remaining firms split up in singletons, is given by:

$$u_i\left(x^*\left(\pi^{\gamma}\left(\sigma'\right)\right)\right) = \frac{(a-c)^2}{s(n-s+2)^2},$$

where $n \equiv |N|$, $s \equiv |S|$ and $\sigma' = \left(\{S\}_{i \in S}, \{N\}_{i \in N \setminus S}\right)$. The grand coalition, induced by the profile $\sigma^* = \left(\{N\}_{i \in N}\right)$, is a stable coalition structure in the Γ-game of coalition formation, if, for every $i \in N$,

$$u_i\left(x\left(\pi^{\gamma}\left(\sigma^*\right)\right)\right) = \frac{(a-c)^2}{4n} \geq u_i\left(x\left(\pi^{\gamma}\left(\sigma'\right)\right)\right) = \frac{(a-c)^2}{s(n-s+2)^2}.$$

The condition above is usually verified for every $s \leq n$. Therefore, the stability of the grand coalition for the Γ-merger game holds also for a n-firm oligopoly.

2.2.2 Timing in Games of Coalition Formation

Following the literature on endogenous timing (for instance, Hamilton and Slutsky's [24]) we can add a preplay stage to the basic strategic setting (denoted *basic game*) in which players declare independently both their intention to coordinate their action with the other players as well as the timing they want to play the basic game. More specifically, every player $i \in N$ is assumed to play an extensive form game in which at stage t_0 (*coalition timing game*) announces an 2-tuple of strategies $a_i = (S, \tau) \in \mathbb{N} \times \{t_1, t_2\}$, where $\tau = \{t_1, t_2\}$ represents the time (stage 1 or 2) she intends to play the *basic game* jointly with the selected coalition $S \in \mathbb{N}$. Given the profile of announcements of the N players $a = (a_1, a_2, \ldots, a_n)$, a coalition structure $P(a) = (S_1^{\tau}, S_2^{\tau}, \ldots, S_m^{\tau})$ endowed with a sequence of play of the basic game is induced, for instance, via the Hart and Kurz's unanimity rule: when a coalition of players announces both the same coalition S and the same timing, they will play the basic game of strategies simultaneously and coordinately as a coalition of players; otherwise, they will play as singletons with the timing prescribed by their own announcement. As the following example shows, the coalition formation timing rule constitutes a one-to-one mapping between the set of players' announcements and the set of feasible partitions of N.

Example 4 (Two-player). For every $i = 1, 2$ with $j \neq i$, each player's announcement set is:

$$A_i = [(\{i, j\}, t_1), (\{i, j\}, t_2), (\{i\}, t_1), (\{i\}, t_2)].$$

In this case the set of feasible partitions induced by the vector of announcement $a \in A_1 \times A_2$ includes the following six partitions:

$$\left(\{1,2\}^{t_1}\right), \left(\{1,2\}^{t_2}\right), \left(\{1\}^{t_1}, \{2\}^{t_1}\right), \left(\{1\}^{t_2}, \{2\}^{t_2}\right), \left(\{1\}^{t_1}, \{2\}^{t_2}\right), \left(\{1\}^{t_2}, \{2\}^{t_1}\right).$$

The existence of a Strong Nash equilibrium of the *coalition timing game* can be investigated. It can be shown [35] that for a symmetric strategic setting with no discount, the strategy for players of acting all together at period one constitutes an equilibrium when players' actions are strategic substitutes (in the sense of Bulow et al. [12]). Conversely, acting together at period two constitutes an equilibrium when players' actions are strategic complements.

2.3 Some Guidelines to Coalition Formation in Economic Applications

In order to compare and interpret the main predictions that endogenous coalition formation theories obtain in some classical economic problems, it can be useful to use a very simple setup in which the equal sharing rule within each coalition is not assumed but it is obtained through some symmetry assumptions imposed on the strategic form game describing the economic problem at hand. Once some basic assumptions are imposed on the strategic form games underlying the games of coalition formation, the main economic applications can be divided in a few categories: 1) games with positive or negative players-externalities; 2) games with actions that are strategic complements or substitutes; 3) games with or without coalition-synergies. According to these three features, we may have a clear picture of some of the results which can be expected from the different concepts of coalitional stability illustrated above and, in particular, of the stability of the grand coalition.[11]

We start imposing some symmetry requirements on the strategic form game G.
Assumption 1. ***(Symmetric Players)*:** $X_i = X \subset R$ for all $i \in N$. Moreover, for all $x \in X_N$ and all pairwise permutations $p : N \to N$:

$$u_{p(i)}\left(x_{p(1)}, \ldots, x_{p(n)}\right) = u_i\left(x_1, \ldots, x_n\right).$$

Assumption 2. ***(Monotone Externalities)*:** One of the following two cases must hold for $u_i(x) : X_N \to R$ assumed quasiconcave:

1. Positive externalities: $u_i(x)$ strictly increasing in $x_{N\setminus i}$ for all i and all $x \in X_N$;
2. Negative externalities: $u_i(x)$ strictly decreasing in $x_{N\setminus i}$ for all i and all $x \in X_N$.

Assumption 1 requires that all players have the same strategy set, and that players payoff functions are symmetric, by this meaning that any switch of strategies between players induces a corresponding switch of payoffs. Assumption 2 requires that the cross effect on payoffs of a change of strategy have the same sign for all players and for all strategy profiles.

Lemma 1. *For all* $S \subseteq N$, $\widetilde{x}_S \in \arg\max_{x_S \in X_S} \sum_{i \in S} u_i(x_S, x_{N\setminus S})$ *implies* $\widetilde{x}_i = \widetilde{x}_j$ *for all* $i, j \in S$ *and for all* $x_{N\setminus S} \in X_{N\setminus S}$.

Proof. See Appendix.

An important implication of Lemma 1 is that all players belonging to a given coalition $S \subseteq N$ will play the same maximizing strategy and then will obtain the same payoff. We can thus obtain a game in valuation form from a game in partition function form without imposing a fixed allocation rule.

The next lemma expresses the fact that in every feasible coalition structure π, at the Nash equilibrium played by coalitions, when players-externalities are positive

[11] Some of the results presented here comes from Currarini and Marini [16].

(negative), being a member of bigger rather than a smaller coalition is convenient only when each member of S plays a strategy that is lower (higher) than that played by each member of a smaller coalition.

Lemma 2. *Let Assumptions 1 and 2 hold. Then for every S and $T \in \pi$, with $|T| \geq |S|$:*
(1) Under Positive Externalities, $u_s(x^*(\pi)) \geq u_t(x^*(\pi))$ if and only if $x_s \leq x_t$;
(2) Under Negative Externalities, $u_s(x^*(\pi)) \geq u_t(x^*(\pi))$ if and only if $x_s \geq x_t$.

Proof. See Appendix.

Finally, we can use a well known classification of all economic models in two classes: (1) games in which players' actions are *strategic complements*; (2) games in which players' actions are *strategic substitutes*.[12]

Definition 6. The payoff function u_i exhibits increasing differences on X_N if for all S, $x_S \in X_S$, $x'_S \in X_S$, $x_{N\setminus S} \in X_{N\setminus S}$ and $x'_{N\setminus S} \in X_{N\setminus S}$ such that $x'_S > x_S$ and $x'_{N\setminus S} > x_{N\setminus S}$ we have

$$u_i\left(x'_S, x'_{N\setminus S}\right) - u_i\left(x_S, x'_{N\setminus S}\right) \geq u_i\left(x'_S, x_{N\setminus S}\right) - u_i\left(x_S, x_{N\setminus S}\right).$$

This feature is typical of games, as price oligopoly models with differentiated goods, for which players' best-replies are upward-sloping. For these games, we can prove the following.

Lemma 3. *Let assumptions 1–2 hold, and let u_i have increasing differences on X_N, for all $i \in N$. Then for every S and $T \in \pi$, with $|T| \geq |S|$:*
(1) Positive Externalities imply $x_s \leq x_t$; (2) Negative Externalities imply $x_s \geq x_t$.

Proof. See Appendix.

Suppose now to have a game with actions that are strategic substitutes. This is the case of Cournot oligopoly and many other economic models. Suppose also that a boundary on the slope of the reaction mapping $f_S : R_{N\setminus S} \rightarrow R_S$ is imposed by the following contraction assumption.
Assumption 3. *(contraction)* Let $S \in \pi$. Then, there exists a $c < 1$ such that for all $x_{N\setminus S}$ and $x'_{N\setminus S} \in X_{N\setminus S}$

$$\left\| f_S\left(x_{N\setminus S}\right) - f_S\left(x'_{N\setminus S}\right)\right\| \leq c\left\| x_{N\setminus S} - x'_{N\setminus S}\right\|,$$

where $\|.\|$ denotes the euclidean norm defined on the space R^{n-s}.

Lemma 4. *Let assumptions 1–3 hold. Then for every S and $T \in \pi$, with $|T| \geq |S|$: (1) Positive Externalities imply $x_s \leq x_t$; (2) Negative Externalities imply $x_s \geq x_t$.*

Proof. See Currarini and Marini [16].

[12] See, for this definition, Bulow et al. [12].

Using all lemmata presented above we are now able to compare the valuation of players belonging to different coalitions in a given coalition structure and then, to a certain extent, the profitability of deviations. However, the above analysis is limited to games in which forming a coalition does not enlarge the set of strategy available to its members and does not modify the way payoffs within a coalition originate from the strategies chosen by players in N. In fact, as assumed at the beginning of the paper, the action space of each coalition $S \subset N$ is restricted to $X_S \equiv \prod_{i \in S} X_i$. Moreover $U_S = \sum_{i \in S} u_i(x(\pi))$. The only advantage for players to form coalitions is to coordinate their strategies in order to obtain a coalitional efficient outcome. This approach encompasses many well known games *without synergies*, such as Cournot and Bertrand merger or cartel formation and public good and environmental games, but rules out an important driving force of coalition formation, i.e. the exploitation of synergies, typically arising for instance in R&D alliances or mergers among firms yielding some sort of economies of scales. Within this framework, we can present the following result.

Proposition 1. *Let assumptions 1–2 hold, and let u_i possess increasing differences on X_N, for all $i \in N$. Then the grand coalition N is a stable coalition structure in the game of coalition formation Γ derived from the game in strategic form G.*

Proof. By Lemma 3, positive externalities imply that for all π, at $x(\pi)$ larger coalitions choose larger strategies than smaller coalitions, while the opposite holds under negative externalities, and then $\frac{U_S(x^*(\pi^\gamma))}{|S|} \geq \frac{U_S(x^*(\pi^\gamma))}{|T|}$ for all $S, T \in \pi^\gamma$ with $|T| \geq |S|$. This directly implies the stability of the grand coalition in Γ. To provide a sketch of this proof, we note that any coalitional deviation from the strategy profile σ^* yielding the grand coalition induces a coalition structure in which all members outside the deviating coalitions appear as singleton. Since these players are weakly better off than any of the deviating members, and since all players were receiving the same payoff at σ^*, a strict improvement of the deviating coalition would contradict the efficiency of the outcome induced by the grand coalition. □

In games with increasing differences, players strategies are strategic complements, and best replies are therefore positively sloped. The stability of the efficient coalition structure $\pi^* = \{N\}$ in this class of games can be intuitively explained as follows. In games with positive externalities, a deviation of a coalition $S \subset N$ will typically be associated with a lower level of S's members' strategies with respect to the efficient profile $x(\pi^*)$, and with a higher level in games with negative externalities (see lemma 3 and 4 above). If strategies are the quantity of produced public good or prices (positive player-externalities), S will try to free ride on non members by reducing its production or its price; if strategies are emissions of pollutant or quantities (negative player-externalities), S will try to emit or produce more and take advantage of non members' lower emissions or quantities. The extent to which these deviations will be profitable ultimately depend on the reaction of non members. In the case of positive externalities, S will benefit from an increase of non members' production levels or prices; however, strategic complementarity implies that the decrease of S's production levels or prices will be followed by a decrease of the produced levels or prices of non members. Similarly, the increase of S's pollutant emissions or quantities will induce higher pollution or quantity levels by non members. Free riding is therefore little profitable in these games. From the above discussion, it is clear that deviations can be profitable only if best reply functions are negatively sloped, that is, strategies must be substitutes in G. However, the above discussion suggests that some "degree" of substitutability may still be compatible with stability. Indeed, if S's decrease in the production of public good is followed by a moderate increase in the produced level of non members, S may still not find it profitable to deviate from the efficient profile. Therefore, if the absolute value of the slope of the reaction maps is bounded above by 1, the stability result of proposition 1 extends to games with strategic substitutes.

Proposition 2. *Let assumptions 1–3 hold. The grand coalition N is a stable coalition structure in the game of coalition formation Γ derived from the game in strategic form G.*

Moreover, we can extend the results of proposition 1 and 2 to games with negative *coalition-externalities*.[13]

Definition 7. A game $G(\pi)$ exhibits positive (negative) coalition-externalities if, for any feasible coalition structure π and coalition $S \in \pi$, for every player $i \in S$, $u_i(x^*(\pi')) > (<)\ u_i(x^*(\pi))$ where π' is obtained from π by merging coalitions in $\pi \backslash S$.

It is clear from the above definition, that under negative coalition-externalities, $u_i(x(\pi^\gamma(\sigma'))) < u_i\left(x\left(\pi^\delta(\sigma')\right)\right)$ where $\sigma' = \left(\{S\}_{i\in S}, \{N\}_{j\in N\backslash S}\right)$ just because $\pi^\gamma(\sigma') = \left(\{S\}, \{j\}_{j\in N\backslash S}\right)$ and $\pi^\delta(\sigma') = (\{S\}, \{N\backslash S\})$. The following propositions exploits this fact.

Proposition 3. *Let assumptions 1–2 hold, and let u_i possess increasing differences on X_N, for all $i \in N$. Let also the game $G(\pi)$ exhibits negative coalition-externalities. Then the grand coalition N is a stable coalition structure in the Δ-game of coalition formation derived from the game in strategic form G.*

Proposition 4. *Let assumptions 1–3 hold. Let also the game $G(\pi)$ exhibits negative coalition-externalities. Then the grand coalition N is a stable coalition structure in the Δ-game of coalition formation derived from the game in strategic form G.*

[13] See Bloch [6] or Yi [47] for such a definition. There is not a clear relationship between games with positive (or negative) player-externalities and games with positive (or negative) coalition-externalities. However, for most well known games without synergies, both positive-player externalities (PPE) plus strategic complement actions (SC) as well as negative-player externalities (NPE) plus strategic substitute actions (SS) yield games with positive coalition-externalities. These are the cases of merger or cartel games in quantity oligopolies (NPE+SS), merger or cartel games in price oligopolies (PPE+SC) and public goods (PPE+SS) or environmental games (NPE+SS). Similarly, we can obtain Negative Coalition-Externalities in a game by associating NPE and SC as in a cartel game in which goods are complements and then the game exhibits SC.

A comparison of the above results, obtained for Hurt and Kurz's (1985) games of coalition formation, with the other solution concepts can be mentioned. It can be shown (see [46]) that for all games without synergies in which - as in the merger example - players prefer to stay as singletons to free-ride on a forming coalition – Bloch's [5] sequential game of coalition formation gives rise to equilibrium coalition structures formed by one coalition and a fringe of coalition acting as singletons. Moreover, even in a linear oligopoly merger game, Ray and Vohra's [40] *Equilibrium Binding Agreement* may or may not support the grand coalition as a stable coalition structure, depending on the number of firms in the market. When the game G is a game with synergies, a classification of the possible results. becomes even more complex. To give an illustration, we can introduce a simple form of synergy by assuming, as in Bloch's [4] and Yi's [46] R&D alliance models, that when firms coordinate their action and create a R&D alliance, they pool their research assets in such a way to reduce the cost of each firm in proportion to the number of firms cooperating in the project.[14] Let the producing cost of firms participating to a R&D alliance of s firms be $c(x_i, s_i) = (c+1-s_i)x_i$, where s_i is the cardinality of the alliance containing firm i: Let also $a > c \geq n$. As shown by Yi [46], at the unique Nash equilibrium associated with every coalition structure , the profit of each firm in a coalition of size s_i is given by

$$u_i^\gamma \left(x\left(\pi^\gamma\right)\right) = \frac{\left(a-(n+1)(c+1-s_i)+\sum_{j=1}^{k} s_j (c+1-s_j)\right)^2}{(n+1)^2},$$

When $\pi = \pi(\sigma')$, symmetry can be used to reduce the above expression to

$$u_i^\gamma \left(\pi^\gamma \left(\sigma'\right)\right) = \frac{\left(a-(n-s_i+1)(c+1-s_i)+(n-s_i)c\right)^2}{(n+1)^2}.$$

Straightforward manipulations show that the deviation of a coalition S_i from the grand coalition in the game Γ is always profitable whenever:

$$s_i > -\frac{1}{2}n + c - \frac{1}{2}\sqrt{(n^2 - 4\left(nc - c^2\right) - 8(a-c-1)}.$$

For example, for $n = 8$, a deviation by a group of six firms ($s_i = 6$) induces a per firm payoff of $v_i^\gamma(\pi^\gamma(\sigma')) = \frac{(a-c+15)^2}{81}$ higher than the every firm's payoff in the grand coalition $v_i(\pi^\gamma(\sigma^*)) = \frac{(a-c+7)^2}{81}$. Therefore, it becomes more difficult to predict the stable coalition structures in Hurt and Kurz's Γ and Δ-games. In the sequential games of coalition formation [5, 41] for a linear Cournot oligopoly in which firms can form reducing-cost alliances, and each firm's $i \in S$ bears a marginal cost

$$c_i = \gamma - \theta s\,,$$

[14] This is usually classified as a game with negative coalition-externalities (see [46, 47]).

where s is the size of the alliance to which firm's i belongs, the equilibrium profit of each firm $i \in S$ is:

$$v_i(\pi) = \frac{1-\gamma}{n+1} + \theta s_i - \frac{\sum_{j\neq i} s_i^2}{n+1}.$$

Therefore, the formation of alliances induces negative externalities on outsiders, just because an alliance reduces marginal costs of participants and make them more aggressive in the market. Moreover, members of larger alliance have higher profits and then, if membership is open, all firms wants to belong to the association ([6], Bloch 2005). In the game of sequential coalition formation, anticipating that remaining players will form an association of size $(n-s)$, the first s players optimally decide to admit $s^* = (3n+1)/4$ and the unique equilibrium coalition structure results in the formation of two associations of unequal size $\pi^* = \left(\left\{\frac{3n+1}{4}\right\}, \left\{\frac{n-1}{4}\right\}\right)$.

3 Networks

3.1 Notation

We follow here the standard notation applied to networks.[15] A nondirected network (N,g) describes a system of reciprocal relationships between individuals in a set $N = \{1,2,\ldots,n\}$, as friendships, information flows and many others. Individuals are nodes in the graph g and links represent bilateral relationship between individuals.[16] It is common to refer directly to g as a network (omitting the set of players). The notation $ij \in g$ indicates that i and j are linked in network g. Therefore, a network g is just a list of which pairs of individuals are linked to each other. The set of all possible links between the players in N is denoted by $g^N = \{ij| i,j \in N, i \neq j\}$. Thus $G = \left\{g \subset g^N\right\}$ is the set of all possible networks on N, and g^N is denoted as the *complete network*. To give an example, for $N = \{1,2,3\}$, $g = \{12,13\}$ is the network with links between individuals 1 and 2 and 1 and 3, but with no link between player 2 and 3. The complete network is $g^N = \{12,23,13\}$. The network obtained by adding link ij to a network g is denoted by $g+ij$, while the network obtained by deleting a link ij from a network g is denoted $g-ij$. A *path* in g between individuals i and j is a sequence of players $i = i_1, i_2, \ldots, i_K = j$ with $K \geq 2$ such that $i_k i_{k+1} \in g$ for each $k \in \{1,2,\ldots,K-1\}$. Individuals who are not connected by a path are in different components C of g; those who are connected by a path are in the same component. Therefore, the components of a network are the distinct connected subgraphs of a network. The set of all component can be indicated as $C(g)$. Therefore, $g = \bigcup_{g' \in C(g)} g'$. Let also indicate with $N(g)$ the players who have at least one link in network g.

[15] See, for instance, Jakcson and Wolinski [34], Jackson [28] and van den Nouweland [44].

[16] Here both individuals engadged in a relationship have to give their consent for the link to form. If the relationship is unilateral (as in advertising) the appropriate model is a directed network. Also, here the intensity of a link is assumed constant.

3.2 Value Functions and Allocation Rules

It is possible to define a value function assigning to each network a worth.

Definition 8. A value function for a network is a function $v : G \to R$.

Let V be the set of all possible value functions. In some applications $v(g) = \sum_i u_i(g)$, where $u_i : G \to R$. A network $g \in G$ is defined (strongly) efficient if $v(g) \geq v(g')$ for all $g' \in G$. If the value is transferable across players, this coincides with Pareto-efficiency.[17]

Since the network is finite, it always exists an efficient network. Another relevant modelling feature is the way in which the value of a network is distributed among the individuals forming the network.

Definition 9. An allocation rule is a function $Y : G \times V \to R^N$.

Thus, $Y_i(g,v)$ is the payoff obtained by every player $i \in N(g)$ under the value function v. Some important properties of the value functions v and of the allocation rules Y can be defined.[18]

When compared to the characteristic function of cooperative games (see Sect. 1.1), here a value function v is sensitive not only to the number of players connected (in a component of g) but also to the specific architecture in which they are connected. However, v can be restricted to depend only on the number of players connected in a coalition. In a seminal contribution, Myerson [36] starts with a TU cooperative game (N,v) and overlaps a communication network g to such a framework. Myerson [36] associates a "graph-restricted value" $v^g : 2^N \to R$, assigning to each coalition S a value equal to the sum of worth generated by the connected components of players in S. Formally, players in S have links in $g(S) = \{ij \in g | i \in S,\ j \in S\}$ and this induces a partition of S into subsets of players $S(g)$ that are connected in S by g. Thus, $v^g(S) = \sum_{g' \in C^S(g)} v(g')$ for every $S \subset N$, where $C^S(g)$ indicates the set of components induced by g involving players belonging to coalition S. This value assumes that players in S can coordinate their action only within their own components.[19] Two assumptions underline this value: (1) there are no externalities between different components of a network; (2) what matters for the worth v^g is only the worth of the coalition of players which are in a component, not the type of connections existing within the coalition. Within this framework, Myerson characterizes a specific allocation rule (known as *Myerson value*) distributing the payoffs among individuals, and shows that under two axioms - fairness and component additivity - the unique allocation rule satisfying these properties is the Shapley value of the graph-restricted game (N,v^g):

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[17] A network g is Pareto efficient (PE) with respect to a value v and an allocation rule Y if there not exists any $g' \in G$ such that $Y_i(g',v) \geq Y_i(g,v)$ with strict inequality for some i Note that if a network is PE with respect to v and Y for all possible allocation rules Y; it is (strong) efficient (see [28]).

[18] See Jackson and Wolinsky [34] and Jackson [29] for details.

[19] This implies a component balanced allocation rule Y.

$$Y_i(g, v^g) = \sum_{S \subset N \setminus \{i\}} \frac{|S|!(|N|-1-|S|)!}{|N|!} \left(v^g\left(S \cup \{i\}\right) - v^g\left(S\right)\right).$$

3.3 Networks Formation Games

3.3.1 Networks Formation in Extensive Form

Aumann and Myerson [2] propose an extensive form game to model the endogenous formation of cooperation structures. In their approach, which involves a sequential formation of links among players, bilateral negotiations take place in some predetermined order. Firstly, an exogenous rule determines the sequential order in which pairs of players negotiate to form a link. A link is formed if and only if both players agree and, once formed, cannot be broken. The game is one of perfect information and each player knows the entire history of links accepted or rejected at any time of the game. Once all links between pairs of players have formed, single players can still form links. Once all players have decided, the process stops and the network g forms and the payoff is assigned according to the Myerson value, i.e. the Shapley value of the restricted game (N, v^g). Stable cooperative structure are considered only those associated with subgame perfect equilibria of the game.

Example 5.[20] Suppose a TU majority game with $N = \{1,2,3\}$ and $v(S) = 1$ if $|S| \geq 2$ and $v(S) = 0$ otherwise. If the exogenous rule specifies the following order of pairs: $\{1,2\}$, $\{1,3\}$, $\{2,3\}$. The structure $\{1,2\}$ is the only cooperation structure supported by a subgame perfect equilibrium of the game. Neither player 1 nor player 2 have an interest to form a link with player 3, provided that the other player has not formed a link with 3. So, using backward induction, if at the final stage $\{2,3\}$ has formed, at stage 2 also $\{1,3\}$ forms and player 1 obtains a lower payoff than in a coalition with only player 2. Thus, at stage 1 player 1 forms a link with player 2 and the latter accepts. No other links are formed at the following stages.

It is possible that a subgame Nash equilibrium of the Aumann and Myerson's network formation game in extensive form does not support the formation of the complete network even for superadditive games. Moreover, no general results are known for the existence of stable complete networks even for symmetric convex games.[21]

3.3.2 Networks Formation in Strategic Form

Myerson [37] suggests a noncooperative game of network formation in strategic form.[22]

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[20] This example is taken from Dutta et al. [19].

[21] See, for a survey of this approach, van den Nouweland [45].

[22] This game is also analyzed by Quin [39] and Dutta et al. [19].

For each player $i \in N$ a strategy $\sigma_i \in \Sigma_i$ is given by the set of players with whom she want to form a link, i.e. $\Sigma_i = (S | S \subseteq N \setminus \{i\})$. Given a n-tuple of strategies $\sigma \in \Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_n$ a link ij is formed if and only if $j \in \sigma_i$ and $i \in \sigma_j$. Denoting the formed (undirected) network $g(\sigma)$, the payoff of each player is given by $Y_i(v, g(\sigma))$ for every $\sigma \in \Sigma_N$. A strategy profile is a Nash equilibrium of the Myerson's linking game if and only if, for all player $i$and all strategies $\sigma' \in \Sigma_i$

$$Y_i(v, g(\sigma)) \geq Y_i(v, g(\sigma'_i, \sigma_{-i})).$$

We can also define a network g *Nash stable* with respect to a value function v and an allocation rule Y, if there exists a pure strategy Nash equilibrium σ such that $g = g(\sigma)$.

The concept of Nash equilibrium applied to the network formation game appears a too weak notion of equilibrium, due to the bilateral nature of links. The empty network (a g with no links) is always Nash stable for any v and Y. Moreover, all networks in which there is a gain in forming additional links but no convenience to sever existing links are also Nash stable. Refinements of the Nash equilibrium concept for the network formation process have been proposed. The *pairwise stability* introduced by Jackson and Wolinsky [34] plays a prominent role in the recent developments of the analysis of networks formation.

3.3.3 Pairwise Stability

We should expect that in a stable network players do not benefit by altering the structure of the network. Accordingly, Jackson and Wolinsky [34] defines a notion of network stability denoted *pairwise stability*.

Definition 10. A network g is pairwise stable with respect to the allocation rule Y and value function v if

(1) For all $ij \in g$, $Y_i(v, g) \geq Y_i(g - ij, v)$ and $Y_j(v, g) \geq Y_j(g - ij, v)$, and

(2) For all $ij \notin g$, if $Y_i(g + ij, v) > Y_i(g, v)$ then $Y_j(g + ij, v) < Y_j(g, v)$.

As shown by Jackson and Watts [33], a network is pairwise stable if and only if it has no *improving path* emanating from it. An improving path is a sequence of networks $\{g_1, g, \ldots, g_K\}$, where each network g_k is defeated by a subsequent (adjacent) network g_{k+1}, i.e. $Yi(g_{k+1}, v) > Yi(g_k, v)$ for $g_{k+1} = g_k - ij$ or $Y_i(g_{k+1}, v) \geq Y_i(g_k, v)$ and $Y_j(g_{k+1}, v) \geq Y_j(g_k, v)$ for $g_{k+1} = g_k + ij$, with at least one inequality holding strictly. Thus, if there not exists any pairwise stable network, then it must exists at least one cycle, i.e. an improving path $\{g_1, g, \ldots, g_K\}$ with $g_1 = g_K$. Jackson and Wolinsky [34] show that the existence of pairwise stable networks is always ensured for certain allocation rules. They prove that under the egalitarian and the component-wise egalitarian rules,[23] pairwise stable networks always exists. In particular, under

[23] The *egalitarian allocation rule* Y^e is such that $Y_i^e(g; v) = \frac{v(n)}{n}$ for all i and g. The component-wise allocation rule Y^{ce} is an egalitarian rule respecting component balance, i.e. such that $Y_i^{ce}(g; v) = \frac{v(C)}{|N(C)|}$ when $N(C)$, the set of players in component C is non empty and $Y_i^{ce}(g; v) = 0$ otherwise. See Jackson and Wolinsky [34] and Jackson [28] for details.

the egalitarian rule, any efficient network is pairwise stable. Under the component-wise allocation rule, a pairwise stable network can always be found. This can be done for component additive v by finding components C that maximize the payoffs of its players, and then continuing this process for the remaining players $N \backslash N(C)$. The network formed by all these components is pairwise stable. Another allocation rule with strong existence properties is the Myerson value. As shown by Jackson and Wolinsky [34], under Myerson's allocation rule there always exists a pairwise network for every value function $v \in V$. Moreover, all improving paths emanating from any network lead to pairwise stable networks, i.e. there are no cycles under the Myerson value allocation rule.[24]

However, as it is shown by Jackson and Wolinsky and by Jackson [28], there exists a tension between efficiency and stability whenever the allocation rule Y is component balanced and anonymous, in the sense that there does not exists an allocation rule with such properties that for all $v \in V$ yields an efficient network that is pairwise stable.

3.3.4 Further Refinements of Network Stability Concepts

As in the case of coalition formation, equilibrium concepts immune to coordinated deviations by players are also conceivable for networks (see, [18, 19, 32]). By allowing every subset of players to coordinate their strategies in arbitrary ways yields a strong Nash equilibrium for network formation games. That is, a strategy profile $\sigma \in \Sigma_N$ is a *strong Nash equilibrium* of the network formation game if there not exist a coalition $S \subseteq N$ and a strategy profile $\sigma'_S \in \Sigma_S$ such that

$$Y_i(v, g(\sigma'_S, \sigma_{N \backslash S})) \geq Y_i(v; g(\sigma)),$$

with strict inequality for at least one $i \in S$. Hence, a network g is strongly stable with respect to a value function v and an allocation rule Y, if there exists a strong Nash equilibrium σ such that $g = g(\sigma)$.

Similarly, an intermediate concept of stability, stronger than pairwise stability and weaker than strong Nash equilibrium, has been proposed [34] and denoted *pairwise Nash equilibrium*. This can be defined as a strategy profile $\sigma \in \Sigma_N$ such that, for all player i and all strategies $\sigma'_i \in \Sigma_i$,

$$Y_i(v, g(\sigma'_i, \sigma_{N \backslash \{i\}})) \geq Y_i(v, g(\sigma))$$

and there not exists a pair of agents (i, j) such that

$$Y_i(v, g(\sigma) + ij) \geq Y_i(v, g(\sigma))$$
$$Y_j(v, g(\sigma) + ij) \geq Y_j(v, g(\sigma))$$

[24] See Jackson [28] for details.

with strict inequality for at least one of the agents. Therefore, a network g is pairwise Nash stable with respect to a value function v and an allocation rule Y, if there exists a pairwise Nash equilibrium such that $g = g(\sigma)$.[25]

It can be shown that, given a value function v and an allocation rule Y, the set of strongly stable networks is weakly included in the set of pairwise Nash stable networks and that the latter set coincides with the intersection of pairwise stable networks and Nash stable networks.[26] Moreover, the set of pairwise stable networks and the set of Nash stable networks can be completely disjoint even though neither is empty.[27]

In the next section, I briefly illustrate some very simple applications of network formation games to classical I. O. models. These are taken from Bloch [9], Belleflamme and Bloch [3] as well as Goyal and Joshi [22].

3.4 Some Economic Applications

3.4.1 Collusive Networks

In Bloch [7] and in Belleflamme and Bloch [3] it is assumed that firms can sign bilateral market sharing agreements. Initially firms are present on different (geographical) markets. By signing bilateral agreement they commit not to enter each other's market.

If $ij \in g$, firm i withdraws from market j and firm j withdraws from market i. For every network g and given N firms, let $n_i(g)$ denote the number of firms in firm i' s market, with $n_i(g) = n - d_i(g)$ where $d_i(g)$ is the degree of vertex (firm) i in the network, i.e. the number of its links. If all firms are identical, firm i's total profit is

$$U_i(g) = u_i(n_i(g)) + \sum_{i,ij \notin g} u_i(n_j(g)).$$

With linear demand and zero marginal cost, under Cournot competition we obtain

$$U_i(g) = \frac{a^2}{[n_i(g)+1]^2} + \sum_{i,ij \notin g} \frac{a^2}{[n_j(g)+1]^2}.$$

If $n \geq 3$; there are exactly two pairwise stable networks, the empty network and the complete network. For $n = 2$, the complete network is the only stable network.

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[25] This equilibrium concept has been adopted in applications by Goyal and Joshi [22] and Belleflamme and Bloch [3] and formally studied by Calvo-Armengol and Ilkilic [13], Ilkilic [27] and Gilles and Sarangi [21].

[26] See, for instance, Jackson and van den Nouweland [32] and Bloch and Jackson [10].

[27] See Bloch and Jackson [10, 11], for an extensions of these equilibrium concepts to the case in which transfers among players are allowed.

Note that the empty network is stable since for every symmetric firm the benefit to form a link is

$$U_i(g+ij) - U_i(g) = \frac{a^2}{n^2} - 2\frac{a^2}{(n+1)^2}$$

that, for $n \geq 3$, is negative.

For every incomplete network, $U_i(g) - U_i(g-ij) \geq 0$, requires that

$$\frac{a^2}{[n_i(g)+1]^2} - \left[\frac{a^2}{[n_i(g)+2]^2} + \frac{a^2}{[n_j(g)+1]^2}\right] \geq 0$$

and this holds only for $n_i(g) = n_j(g) = 1$, i.e. when the network is complete.

In this case,

$$U_i(g^N) - U_i(g^N - ij) = \frac{a^2}{4} - \frac{2a^2}{9} > 0.$$

Therefore, it follows that the only nonempty network which is pairwise stable is the complete network.

3.4.2 Bilateral Collaboration Among Firms

Bloch [7] and Goyal and Joshi [22] consider the formation of bilateral alliances between firms that reduce their marginal cost, as

$$c_i = \gamma - \theta d_i(g)\ ,$$

where $d_i(g)$ denotes the degree of vertex i, i.e. the number of bilateral agreements signed by firm i.

Under Cournot competition with linear demand, we have each firm's profit is given by

$$U_i(g) = \left[\frac{a-\gamma}{n+1} + \theta d_i(g) - \frac{\theta \sum_j d_j(g)}{n+1}\right]^2.$$

For such a case, the only pairwise stable network turns out to be the complete network g^N (see [22]). This is because, by signing an agreement, each firm increases its quantity by $\Delta q_i = \dfrac{n\theta}{n+1}$, consequently, its profit. Moreover, when a large fixed cost to form a link is included in the model, Goyal and Joshi show that stable networks possess a specific form, with one complete component and a few singleton firms.

4 Concluding Remarks

This paper has attempted to provide a brief overview of the wide and increasing literature on games of coalition and network formation, paying a specific attention to the results which may be obtained by applying these games to some well known economic problems. It has been shown that, under reasonable assumptions mainly concerning the symmetry of players' payoffs, a number of general results can be obtained in games of coalition formation, which, in turn, can be easily applied to standard economic problems without synergies, as industry mergers and cartels, public goods games and many others. Network formation games appear as a natural extension of coalition formation games with, included, a detailed analysis of the effects of bilateral links among players. However, the issue of which network will form and which equilibrium concepts are suitable in a number of economic applications seems still largely unresolved, thus requiring further investigation. The future research agenda on the topic of network formation in social environments is certainly open to new exciting contributions.

Appendix

Lemma 1. *For all* $S \subseteq N$, $\widetilde{x}_S \in \arg\max_{x_S \in X_S} \sum_{i \in S} u_i(x_S, x_{N \setminus S})$ *implies* $\widetilde{x}_i = \widetilde{x}_j$ *for all* $i, j \in S$ *and for all* $x_{N \setminus S} \in X_{N \setminus S}$.

Proof. Suppose $\widetilde{x}_i \neq \widetilde{x}_j$ for some $i, j \in S$. By symmetry we can derive from $\widetilde{x}_S$ a new vector x'_S by permuting the strategies of players i and j such that

$$\sum_{i \in S} u_i(x'_S, x_{N \setminus S}) = \sum_{i \in S} u_i(\widetilde{x}_S, x_{N \setminus S}) \tag{6}$$

and hence, by the strict quasiconcavity of all $u_i(x)$, for all $\lambda \in (0,1)$ we have that:

$$\sum_{i \in S} u_i(\lambda x'_S + (1-\lambda)\widetilde{x}_S, x_{N \setminus S}) > \sum_{i \in S} u_i(\widetilde{x}_S, x_{N \setminus S}). \tag{7}$$

Since, by the convexity of X, the strategy vector $\left(\lambda x'_S + (1-\lambda)\widetilde{x}_S\right) \in X_S$, we obtain a contradiction. □

Lemma 2. *Let Assumptions 1 and 2 hold. Then for every* S *and* $T \in \pi$, *with* $|T| \geq |S|$: *(1) Under Positive Externalities,* $u_s(x(\pi)) \geq u_t(x(\pi))$ *if and only if* $x_s \leq x_t$; *(2) Under Negative Externalities,* $u_s(x(\pi)) \geq u_t(x(\pi))$ *if and only if* $x_s \geq x_t$.

Proof. We first prove the result for the case of positive externalities, starting with the "only if" part. By assumption 1, all members of T get the same payoff at $x(\pi)$. By definition of $x(\pi)$, the profile in which all members of T play x_t maximizes the utility of each member of T, so that

$$u_t((x_t, x_t)x_s) \geq u_t((x_s, x_s), x_s). \tag{8}$$

Suppose now that $x_s > x_t$. By assumption 1 and 2.1 we have

$$u_t((x_s, x_s), x_s) = u_{t_i}((x_s, x_s), x_s) = u_s((x_s, x_s), x_s) > u_s((x_t, x_t), x_s). \tag{9}$$

To prove the "if" part, consider coalitions T_1, T_2 and S which, as defined at the beginning of this section, are such that $|T_1| = |S|$ and such that $\{T_1, T_2\}$ forms a partition of T. By definition of $x(\pi)$, the utility of each member of S is maximized by the strategy profile x_S. Using the definition of u_s and of x_s we write:

$$u_s((x_t, x_t), x_s) \geq u_s((x_t, x_t), x_t). \tag{10}$$

By assumption 2.1, if $x_s \leq x_t$ then

$$u_s((x_t, x_t), x_t) \geq u_s((x_s, x_t), x_t). \tag{11}$$

Finally, by assumption 1 and the fact that $|T_1| = |S|$, we obtain

$$u_s((x_s, x_t), x_t) = u_{t_1}((x_t, x_t), x_s) = u_t((x_t, x_t), x_s), \tag{12}$$

implying, together with (11) and (12), that

$$u_s(x(\pi)) = u_s((x_t, x_t), x_s) \geq u_t((x_t, x_t), x_s) = u_t(x(\pi)). \tag{13}$$

Consider now the case of negative externalities (assumption 2.2). Condition (8) holds independently of the sign of the externality. Suppose therefore that $x_s < x_t$. By negative externalities and symmetry we have

$$u_t((x_s, x_s), x_s) = u_s((x_s, x_s), x_s) > u_s((x_t, x_t), x_s). \tag{14}$$

The "if" part is proved considering again coalitions T_1, T_2 and S. Again, Condition (10) holds independently of the sign of the externality. By negative externalities, if $x_s \geq x_t$ then

$$u_s((x_t, x_t), x_t) \geq u_s((x_s, x_t), x_t). \tag{15}$$

As before, we use assumption 1 and the fact that $|T_1| = |S|$ to obtain

$$u_s((x_s, x_t), x_t) = u_t((x_t, x_t), x_s), \tag{16}$$

and, therefore, that

$$u_s(x(\pi)) = u_s(x_t, x_s) \geq u_t(x_t, x_s) = u_t(x(\pi)). \tag{17}$$

□

Lemma 3. *Let assumptions 1–2 hold, and let u_i have increasing differences on X_N, for all $i \in N$. Then for every S and $T \in \pi$, with $|T| \geq |S|$: (1) Positive Externalities imply $x_s \leq x_t$; (2) Negative Externalities imply $x_s \geq x_t$.*

Proof. (1) Suppose that, contrary to our statement, positive externalities hold and $x_s > x_t$ By increasing differences of u_i *for all* $i \in N$ (and using the fact that the sum of functions with increasing difference has itself increasing differences), we obtain:

$$u_s((x_s, x_t), x_s) - u_s((x_s, x_t), x_t) \geq u_s((x_t, x_t), x_s) - u_s((x_t, x_t), x_t). \tag{18}$$

By definition of x_s we also have:

$$u_s((x_t,x_t),x_s) - u_s((x_t,x_t),x_t) \geq 0. \tag{19}$$

Conditions (18) and (19) directly imply:

$$u_s((x_s,x_t),x_s) - u_s((x_s,x_t),x_t) \geq 0. \tag{20}$$

Referring again to the partition of T into the disjoint coalitions T_1 and T_2, an application of the symmetry assumption 1 yields:

$$u_s((x_s,x_t),x_s) = u_{t_1}((x_s,x_t),x_s); \tag{21}$$
$$u_s((x_s,x_t),x_t) = u_{t_1}((x_t,x_t),x_s).$$

Conditions (20) and (21) imply

$$u_{t_1}((x_s,x_t),x_s) \geq u_{t_1}((x_t,x_t),x_s). \tag{22}$$

Positive externalities and the assumption that $x_s > x_t$ imply

$$u_{t_2}((x_s,x_t),x_s) > u_{t_2}((x_t,x_t),x_s). \tag{23}$$

Summing up conditions (22) and (23), and using the definition of T_1 and T_2, we obtain:

$$u_t((x_s,x_t),x_s) > u_t((x_t,x_t),x_s), \tag{24}$$

which contradicts the assumption that x_t maximizes the utility of T given x_s.

The case (2) of negative externalities is proved along similar lines. Suppose that $x_s < x_t$. Conditions (20) and (21), which are independent of the sign of the externalities, hold, so that (22) follows. Negative externalities also imply that if $x_s < x_t$ then (23) follows. We therefore again obtain condition (24). □

References

1. Aumann R (1967) A survey of games without side payments. In: Shubik M (ed) Essays in mathematical economics. Princeton University Press, Princeton, pp 3–27
2. Aumann R, Myerson R (1988) Endogenous formation of links between players and coalitions: an application of the shapley value. In: Roth A (ed) The shapley value. Cambridge University Press, Cambridge, pp 175–191.
3. Belleflamme P, Bloch F (2004) Market sharing agreements and stable collusive networks. Int Econ Rev 45:387–411.
4. Bloch F (1995) Endogenous structures of associations in oligopolies. Rand J Econ 26: 537–556.
5. Bloch F (1996) Sequential formation of coalitions with fixed payoff division. Games Econ Behav 14:90–123.

6. Bloch F (1997) Non cooperative models of coalition formation in games with spillovers. In: Carraro C, Siniscalco D (eds) New directions in the economic theory of the environment. Cambridge University Press, Cambridge
7. Bloch F (2002) Coalition and networks in industrial organization. The Manchester School 70:36–55
8. Bloch F (2003) Coalition formation in games with spillovers. In: Carraro C (ed) The endogenous formation of economic coalitions. Fondazione Eni Enrico Mattei Series on Economics and the Environment, Cheltenham, UK. Elgar, Northampton, MA
9. Bloch F (2004) Group and network formation in industrial organization. In: Demange G, Wooders M (eds) Group formation in economics: networks, clubs and coalitions. Cambridge University Press, Cambridge
10. Bloch F, Jackson MO (2006) Definitions of equilibrium in network formation games. Int J Game Theory 34:305–318
11. Bloch F, Jackson MO (2007) The formation of networks with transfers among players. J Econ Theory 133:83–110
12. Bulow J, Geanokoplos J, Klemperer P (1985) Multimarket oligopoly: strategic substitutes and complements. J Polit Econ 93:488–511
13. Calvó-Armengol A, Ilkilic R (2004) Pairwise stability and Nash equilibria in network formation. Universitat Autonoma de Barcelona (unpublished)
14. Chander P, Tulkens H (1997) The core of an economy with multilateral externalities. Int J Game Theory 26:379–401
15. Currarini S, Marini MA (2003) A sequential approach to the characteristic function and the core in games with externalities. In: Sertel M, Kara A (eds) Advances in economic design. Springer, Berlin
16. Currarini S, Marini MA (2006) Coalition formation in games without synergies. Int Game Theory Rev 8(1):111–126
17. Dutta B, Jackson MO (2003) On the formation of networks and groups. In: Dutta B, Jackson MO (eds) Networks and groups: models of strategic formation. Springer, Heidelberg
18. Dutta B, Mutuswami S (1997) Stable networks. J Econ Theory 76:322–344
19. Dutta B, van den Nouweland, Tijs AS (1998) Link formation in cooperative situations. Int J Game Theory 27:245–256
20. Fudenberg D, Tirole J (1991) Game theory. MIT, Cambridge, MA
21. Gilles RP, Sarangi S (2004) The role of trust in costly network formation. Virginia Tech (unpublished)
22. Goyal S, Joshi S (2003) Networks of collaboration in oligopoly. Games Econ Behav 43:57–85
23. Greenberg J (1994) Coalition structures. In: Aumann RJ, Hart S (eds) Handbook of game theory, vol 2. Elsevier, Amsterdam
24. Hamilton J, Slutsky S (1990) Endogenous timing in duopoly games: stackelberg or cournot equilibria. Games Econ Behav 2:29–46
25. Hart S, Kurz M (1983) Endogenous formation of coalitions. Econometrica 52:1047–1064
26. Ichiishi T (1983) A social coalitional equilibrium existence lemma. Econometrica 49:369–377
27. Ilkilic R (2004) Pairwise stability: externalities and existence. Universitat Autonoma de Barcelona (unpublished)
28. Jackson MO (2003) The stability and efficiency of economic and social networks. In: Koray S, Sertel M (eds) Advances in economic design. Springer, Heidelberg. Reprinted in Dutta B, Jackson MO (eds) Networks and groups: models of strategic formation. Springer, Heidelberg
29. Jackson MO (2005a) A survey of models of network formation: stability and efficiency. In: Demange G, Wooders M (eds) Group formation in economics: networks, clubs, and coalitions. Cambridge University Press, Cambridge
30. Jackson MO (2005b) The economics of social networks. In: Blundell R, Newey W, Persson T (eds) Proceedings of the 9th World Congress of the Econometric Society, Cambridge University Press, Cambridge, August 2005
31. Jackson MO (2007) Social and economic networks. Princeton University Press, Princeton

32. Jackson MO, van den Nouweland A (2005) Strongly stable networks. Games Econ Behav 51:420–444
33. Jackson MO, Watts A (2002) The evolution of social and economic networks. J Econ Theory 196(2):265–295
34. Jackson MO, Wolinsky A (1996) A strategic model of social and economic networks. J Econ Theory 71:44–74
35. Marini M (2007) Endogenous timing with coalitions of agents. CREI, Roma Tre (unpublished)
36. Myerson R (1977) Graphs and cooperation in games. Math Oper Res 2:225–229
37. Myerson R (1991) Game theory: analysis of conflict. Harvard University Press, Cambridge, MA
38. Owen R (1977) Value of games with a priori unions. In Hein R, Moeschlin O (eds) Essays in mathematical economics and game theory. Springer, New York, pp 76–88
39. Quin CZ (1996) Endogenous formation of cooperative structures. J Econ Theory 69:218–226
40. Ray D, Vohra R (1997) Equilibrium binding agreements. J Econ Theory 73:30–78
41. Ray D, Vohra R (1999) A theory of endogenous coalition structures. Games Econ Behav 26:286–336
42. Salant SW, Switzer R, Reynolds J (1983) Losses from horizontal merger: the effects of an exogenous change in industry structure on Cournot–Nash equilibrium. Q J Econ 98:185–99
43. Shenoy (1979) On coalition formation: a game theoretical approach. Int J Game Theory 8:133–164
44. van den Nouweland A (2005) Models of network formation in cooperative games. In: Demange G, Wooders M (eds) Group formation in economics: networks, clubs, and coalitions. Cambridge University Press, Cambridge
45. von Neumann J, Morgenstern O (1944) Theory of games and economic behaviour. Princeton University Press, Princeton
46. Yi S-S (1997) Stable coalition structure with externalities. Games Econ Behav 20:201–237
47. Yi S-S (2003) The endogenous formation of economic coalitions: the partition function approach. In: Carraro C (ed) The endogenous formation of economic coalitions. Fondazione Eni Enrico Mattei Series on Economics and the Environment, Cheltenham, UK. Elgar, Northampton, MA