From Graph Theory to Models of Economic Networks. A Tutorial

Michael D. König and Stefano Battiston

Abstract Networks play an important role in a wide range of economic phenomena. Despite this fact, standard economic theory rarely considers economic networks explicitly in its analysis. However, a major innovation in economic theory has been the use of methods stemming from graph theory to describe and study relations between economic agents in networks. This recent development has lead to a fast increase in theoretical research on economic networks. In this tutorial, we introduce the reader to some basic concepts used in a wide range of models of economic networks.

1 Introduction

Networks are ubiquitous in social and economic phenomena. The use of methods from graph theory has allowed economic network theory to improve our understanding of those economic phenomena in which the embeddedness of individuals in their social inter-relations cannot be neglected. In this tutorial will give a brief overview of network models, starting from simple network constructions to more complex models that allow for the strategic formation of links.

When discussing these models we try to introduce the reader to the most important concepts of economic networks. However, the literature that is discussed in this tutorial is far from being exhaustive. For a more detailed introduction to economic network theory we recommend the books Vega-Redondo [84], Jackson [52] and Goyal [42] as well as the lecture notes by Calvó-Armengol [15], Zenou [93]. A more mathematical treatment of complex networks can be found in Chung and Lu [20] and Durrett [30]. Standard references for graph theory are Bollobas [8], Diestel [28], West [91].

M.D. König and S. Battiston
Chair of Systems Design, ETH Zurich, Kreuzplatz 5, 8032 Zurich, Switzerland
mkoenig@ethz.ch

A.K. Naimzada et al. (eds.), *Networks, Topology and Dynamics*,
Lecture Notes in Economics and Mathematical Systems 613,

This tutorial is organized as follows. First, we will argue in Section 2 that standard economic theory is in the need of incorporating networks in its analysis. In Section 3 we will mention several applications of economic network theory and we will introduce the basic terminology used to describe networks in Section 4. We will proceed by discussing several prominent network models with an increasing degree of complexity ranging from Poisson random networks in Section 5.1, its generalization in Section 5.2, growing random networks in Section 6 to models of strategic network formation in Section 7.

2 Why Networks in Economics?

Gallegati and Kirman [36], Kirman [60] propose that the aggregate behavior of an economy cannot be investigated in terms of the behavior of isolated individuals, as it is usually done in standard economic theory. Firms interact only with a few other firms, out of all firms present in the economy. Moreover, there are different ways in which firms interact, and they may learn over time to adapt their interactions, meaning that they strengthen profitable ones while they cut costly ones. All this is based on their previous experience. We may then view the economy as an evolving network.

Viewing the economy as an evolving network is different from what a standard neoclassical model[1] of the economy would look like. In such a model it is assumed that anonymous and autonomous individuals take decisions independently and interact only through the price system which they cannot influence at all. This situation refers to a market with perfect competition. However, competition easily becomes imperfect because, if agents have only minimal market power, they will anticipate the consequences of their actions and anticipate the actions of others. In order to overcome this deficiency, game theorists have tried to integrate strategically interacting firms into a general equilibrium [2] framework. But still they leave two questions unanswered. First, it is assumed that the behavior is fully optimizing considering all possible actions as well as all possible actions of others. This leads to agents with extremely sophisticated information processing capabilities. Such ability of passing these enormous amounts of information in short times cannot be found in any realistic setting of human interaction. Advances in weakening that assumption are referred to as "bounded rationality" [40]. Second, the problem of coordination of activities is not addressed in the standard equilibrium model of the economy. Instead

[1] A standard neoclassical model includes the following assumptions [35]: (1) perfect competition, (2) perfect information, (3) rational behavior, (4) all prices are flexible (all markets are in equilibrium). The resulting market equilibrium (allocation of goods) is then efficient. See [49] for a discussion of these assumptions.

[2] The individual decision making process is represented as maximizing a utility function. A utility function is a way of assigning a number to every possible choice such that more-preferred choices have a higher number than less-preferred ones [82]. The gradients of the utility function are imagined to be like forces driving people to trade, and from which economic equilibria emerge as a kind of force balance [32].

it is assumed that every agent can interact and trade with every other agent, which becomes quite unrealistic for large systems. One has to specify the framework within the individual agents take price decisions and thus limit the environment within which they operate and reason. An obvious way is to view the economy as a network in which agents interact only with their neighbors. In the case of technological innovation, neighbors might be similar firms within the same industry, but these firms will then be linked either through customers or suppliers with firms in other industries. Through these connections innovations will diffuse throughout the network. The rate and extent of this diffusion then depends on the structure and connectivity of the network.

Finally, the evolution of the network itself should be made endogenous. In this case the evolution of the link structure is dependent on the agents' experience from using the links (respectively contacts) available to them. Individuals learn and adapt their behavior and this in turn leads to an evolution of the network structure which then feeds back into the incentives of agents to form or sever links. We will briefly discuss this coupled dynamic interaction between individuals' incentives and the network dynamics in Section 4.4.

3 Examples of Networks in Economics

In this section we point to several applications of network models in economics. We have restricted ourselves to a few applications but this list could of course be greatly expanded [see e.g. also 84, p. 10].

Corporate Ownership and Boars of Directors

Ownership relations between firms, as well as members in common in the boards of directors, give raise to intricate networks. On one hand ownership relations are instruments to exert corporate control and several works have studied indirect ownership relations [13] and patterns such as the so-called pyramids and cross-shareholdings [19], as well as business groups [33, 45]. Other works have also studied the financial architecture of corporations in national or global economies [5, 21, 37, 61].

On the other hand, interlocked directors among firms are known to convey information and power [4, 27]. The spread of corporate practices through the director network and the role of inter-organizational imitation of managers has been studied by Davis and Greve [26]. Moreover, it has been shown that the structure of the intrlock network has implications for the decision making process [3].

Labor markets

A wide range of empirical studies of labor markets have shown that a significant fraction of all jobs are found through social networks. The role of informal

social networks in labor markets has been emphasized first by Granovetter [44]. He found that over 50% of jobs were found through personal contacts. In a recent paper, Jackson and Calvo-Armengol [53] introduce a network model of job information transmission. The model reproduces the empirically stylized fact that the employment situation of individuals that are connected, either directly or indirectly, is correlated. Further, they show that the topology of the network influences the length and correlation of unemployment among individuals. Finally, with this model the authors can explain the pervasive inequalities in wages, employment and drop-out rates.

Diffusion in Networks

In economics diffusion is usually related to the spread of a technology through a society or industry. A new technology or idea might be generated by an innovator and then be subsequently adopted by others over time. The literature on technological diffusion focuses on alternative explanations of the dominant stylized fact: that the usage of new technologies over time typically follows an S-curve. Geroski [39] gives an excellent survey on models of technological diffusion [see also 24, 79, 85]. Most models assume that there are no restrictions on the interactions between agents and the path along which knowledge can flow. This assumption is clearly not supported by the restrictions and limited contacts firms realistically maintain [47, 74]. In particular, if knowledge diffuses through social contacts or personal interrelations then the diffusion of a technology critically depends on the underlying network structure. Thus, a proper understanding of the diffusion of innovations needs to be grounded in economic network theory.

Formal and Informal Organizations

The central question in the theory of organizations is how a complex decision problem can be efficiently decomposed into distinct tasks, distributed among the different units of an organization. A network can represent the paths along which these tasks are distributed in an organization [see 80, 81, for a general discussion of network forms of organization].

One can distinguish between formal or informal networks in an organization. Formal network usually refer to the hierarchical structure of an organizational chart. On the other hand, informal organizational networks are usually referred to “communities of practice”. They can serve as a complement to the formal organizational structure [11, 14, 65]. Beyond the formal working relationships institutionalized in the organizational chart, informal working relationships may coexists or may even play a predominant role [16]. In principle. a hierarchical formal organization assumes that a central coordinator can distribute tasks efficiently among the members of an organization. However, central coordination may not be feasible when the number of agents in the organization is large, the problems the organization has to

solve are highly complex and their nature varies considerably such that they cannot be decomposed and distributed.

The existing literature has mainly focussed on the formal organizational structure whereas recent works try to incorporate both the formal as well as the informal communication networks [51] among individuals in an organization. A recent example is the work by Dodds et al. [29]. The authors find a particular organizational network structure that enhances the robustness of the organization and reduces the possibility of a communication overload among its members.

R&D Collaborations

There exist many theoretical works in the literature on industrial organization trying to explain the effects and incentives of R&D collaborations between competing firms [see e.g. 59] and Veugelers [86, for a review]. However, these works do not address the heterogeneity of inter-firm collaborations that have been observed in empirical studies [e.g. 72]. In recent works by Goyal and Moraga-Gonzalez [43] and Vega-Redondo [83] R&D collaborations are investigated in a network setup in which these collaborations are not exogeneously given but the endogeneous outcome of the incentives of firms to collaborate. In this way, heterogeneous interaction profiles are possible. Their equilibrium analysis, however, leads to simple network structures. These simple networks are in contradiction to the empirical literature that shows that R&D networks can have complex network topologies, in general characterized by high clustering, sparseness and a heterogeneous degree distribution. A recent example of a model that tries to incorporate these empirical stylized facts can be found in König et al. [62], König et al. [63] and we will give a brief overview of this model in Section 7.3.

4 Characterization of Networks

If the links in a network do not change over time (we have a static network) we can associate a state variable to the nodes based on their position in the network. If, for example, an agent has many neighbors which in turn have many neighbors, for instance, she may have much better opportunities to gather information from others compared to an agent that maintains only a few connections to other loosely connected agents. We can assign this agent a high centrality in the network (see Section 4.3.3). But a highly central agent may also be much more frequently exposed to any threat propagating through the network, e.g. viruses or avalanches of insolvencies.

In this section, we start with some concepts of graph theory that deals with the properties of static networks. We then review some of the measures that are used to characterize networks, discussing briefly their meaning in economic systems and point to some relevant literature. Finally, we introduce a possible classification scheme for models of networks in economics.

4.1 Elements of Graph Theory

In this section we follow closely West [91] to which we refer the reader for further details. In the following, we will use the terms graph and network as synonyms. The same holds for nodes and nodes as well as links and edges.

A *graph* G is a pair, $G = (V,E)$, consisting of a node set $V(G)$ and an edge set $E(G)$. The edge set $E(G)$ induces a symmetric binary relation on $V(G)$ that is called the adjacency relation of G. Nodes i and j are adjacent if $e_{ij} \in E(G)$.

The *degree*, d_i, of a node i is the number of edges incident to it. A graph can either be *undirected* or *directed*, where in the latter case one has to distinguish between *indegree*, d_i^-, and *outdegree*, d_i^+, of node i. In the case of an undirected graph, the (first-order) neighborhood of a node i in G is $N_i = \{w \in V(G) : e_{wi} \in E(G)\}$. The degree of a node i is then $d_i = |N_i|$. The second-order neighborhood of node i is $\bigcup_{u \in N_i} N_u \backslash \{i \cup N_i\}$. Similarly, higher order neighborhoods can be defined (as well as neighborhoods for directed graphs). A graph G is *regular* if all nodes have the same degree. A graph G is $k-$*regular* if every node has degree k.

The *adjacency matrix*, $\mathbf{A}(G)$, of G, is the $n \times n$ matrix in which the entry a_{ij} is 1 if the edge $e_{ij} \in E(G)$, otherwise a_{ij} is 0. For an undirected graph $\mathbf{A}$ is symmetric, i.e. $a_{ij} = a_{ji}\ \forall i, j \in V(G)$. An example of a simple directed graph on four nodes and its associated adjacency matrix $\mathbf{A}$ is given in Figure (1).

The *eigenvalues* of the adjacency matrix $\mathbf{A}$ are the numbers λ such that $\mathbf{Ax} = \lambda\mathbf{x}$ has a nonzero solution vector, which is an *eigenvector* associated with λ. The term λ_{PF} denotes the *largest real eigenvalue* of $\mathbf{A}$ [the *Perron-Frobenius eigenvalue*, cf. 50, 75], i.e. all eigenvalues λ of $\mathbf{A}(\mathbf{G})$ satisfy $|\lambda| \leq \lambda_{\mathrm{PF}}$ and there exists an associated nonnegative eigenvector $\mathbf{v} \geq 0$ such that $\mathbf{Av} = \lambda_{\mathrm{PF}}\mathbf{v}$. For a connected graph G the adjacency matrix $\mathbf{A}(G)$ has a unique largest real eigenvalue λ_{PF} and a positive associated eigenvector $\mathbf{v} > 0$.

A *walk* is an alternating list, $\{v_0, e_{01}, v_1, ..., v_{k-1}, e_{k-1k}, v_k\}$, of nodes and edges. A *trail* is a walk with no repeated edge. A *path* is a walk with no repeated node. The shortest path between two nodes is also known as the *geodesic distance*. If the endpoints of a trail are the same (a closed trail) then we refer to it as a *circuit*. A circuit with no repeated node is called a *cycle*. In particular, C_n denotes the *cycle* on

$$A = \begin{pmatrix} 0\,1\,0\,0 \\ 0\,0\,1\,1 \\ 0\,0\,0\,1 \\ 1\,0\,0\,0 \end{pmatrix}$$

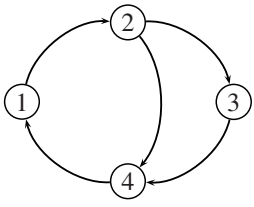


Fig. 1 (Right) a directed graph consisting of 4 nodes and 5 edges. (Left) the corresponding adjacency matrix A. For example, in the first row in $\mathbf{A}$ with elements, $a_{11} = 0, a_{12} = 1, a_{13} = 0, a_{14} = 0$, the element $a_{12} = 1$ indicates that there exist an edge from node 1 to node 2 while node 1 has not other outgoing links

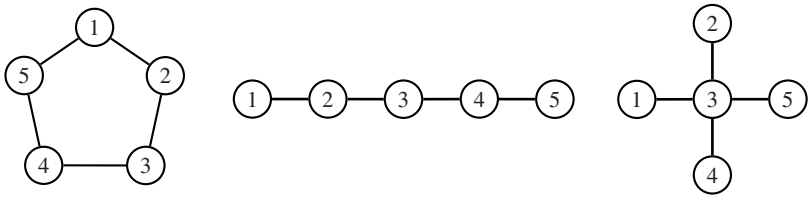


Fig. 2 A cycle C_5 (left), a path P_5 (middle) and the star $K_{1,4}$ (right). All graphs are undirected and contain 5 nodes

n nodes. Note that a cycle is also a circuit but a circuit is not necessarily a cycle. Examples of simple graphs are shown in Figure (2).

The k^{th} power of the adjacency matrix is related to walks of length k in the graph. In particular, $\left(\mathbf{A}^k\right)_{ij}$ gives the number of walks of length k from node i to node j [41].

A *subgraph*, G', of G is the graph of subsets of the nodes, $V(G') \subseteq V(G)$, and edges, $E(G') \subseteq E(G)$. A graph G is *connected*, if there is a path connecting every pair of nodes. Otherwise G is disconnected. The *components* of a graph G are the maximal connected subgraphs. A graph is said to be *complete* if every node is connected to every other node. K_n denotes the *complete graph* on n nodes.

4.2 Graphs and Matrices

We will state some useful facts about matrices and graphs in this section. The study of irreducible and primitive graphs is important in linear dynamic network models. We will present the theory here and discuss a particular application in Section 7.3. Next, we introduce bipartite graphs and show how they can be applied to study networks between members of boards of different companies.

Irreducible and Primitive Graphs

If a graph G is not connected then its adjacency matrix $\mathbf{A}(G)$ can be decomposed in blocks, each block correspond to a connected component. An $n \times n$ matrix $\mathbf{A}$ is said to be a reducible matrix if and only if for some permutation matrix $\mathbf{P}$, the matrix $\mathbf{P}^T\mathbf{A}\mathbf{P}$ is block upper triangular. If a square matrix is not reducible, it is said to be an irreducible matrix. If a graph is connected then there exists a path from every node to every other node in the graph. The adjacency matrix of a connected graph is irreducible [50] and in particular it cannot be decomposed in blocks. Irreducible matrices can be primitive or cyclic (imprimitive) [75]. This distinction is relevant for several results on the convergence of linear systems Boyd [12], Horn and Johnson [50] and we will apply it in Section 7.3.

A non-negative matrix $\mathbf{A}$ is primitive if $\mathbf{A}^k > 0$ for some positive integer $k \leq (n-1)n^n$. This means that, $\mathbf{A}$ is primitive if, for some k, there is a walk of length

k from every node to every other node. Notice that this definition is a much more restrictive than the one of irreducible (or connected) graphs in which it is required that there exits a walk from every node to every other node, but not necessarily of the same length. A graph G is said to be primitive if its associated adjacency matrix $\mathbf{A}(G)$ is primitive.

It is useful to look at an alternative but equivalent way to characterize a primitive graph. A graph G is primitive if and only if it is connected and the greatest common divisor of the set of length of all cycles in G is 1 [50, 92]. This means for instance that the connected graph consisting of two connected nodes is not primitive as the only cycle has length 2 (since the link is undirected a walk can go forward and backward along the link). Similarly, a chain or a tree is also not primitive, since all cycles have only even length. However, if we add one link in order to form a triangle, the graph becomes primitive. The same is true, if we add links in order to form any cycle of odd length. In general, if the graph of interaction between agents is connected, the presence of one cycle of odd length is a sufficient condition for the primitivity of the graph.

Bipartite Graphs

In a *bipartite* graph G, $V(G)$ is the union of two disjoint independent sets V_1 and V_2. In a bipartite graph, if $e_{12} \in E(G)$ then $v_1 \in V_1$ and $v_2 \in V_2$. In other words, the two endpoints of any edge must be in different sets. The *complete bipartite graph* with partitions of size $|V_1| = n_1$ and $|V_2| = n_2$ is denoted K_{n_1,n_2}. A special case is the *star* which is a complete bipartite graph with one partition having size $n_1 = 1$ and $n_2 = n - 1$, denoted as $K_{1,n-1}$ in Figure (2).

A bipartite graph can be 'projected' into two one-mode networks. For sake of clarity let us take the following example. Assume that in Figure (3) each node denoted with a number represents the board of directors of a company, while each node denoted with a letter represents a person. A link, say, between person B and board 1 represents the fact that person B serves in board 1. Notice that B serves also in board 2. The one-mode projection on the directors is a new graph in which there is a link between two persons if they serve together in one or more boards. In doing this projection some information is lost: consider for instance three directors connected in a triangle (not shown). The links do not specify whether each pair of directors sit in a different board or whether the three directors sit all in the same board. Denote $\mathbf{C}$ the adjacency matrix of our network of boards and persons,

$$C_{\alpha i} = \begin{cases} 1 \text{ if } \alpha \text{ sits in board } i \\ 0 \text{ otherwise.} \end{cases} \tag{1}$$

$\mathbf{C}$ is an $M \times N$ matrix, M being the number of persons, and N being the number of boards. This is a binary matrix, and in general it is neither square, nor symmetric. For the one-mode projection relative to the boards, we should take into account that the number of directors sitting in boards i and j, is equivalent to the number of paths of length 2 connecting i and j in the bipartite graph. Therefore, this number can be

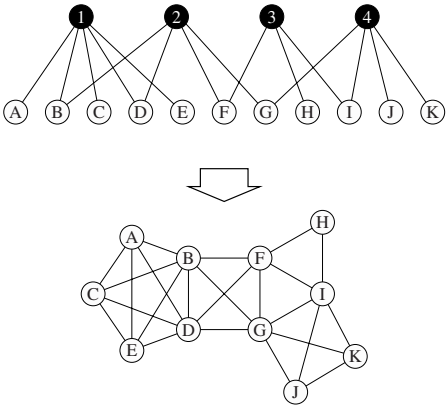


Fig. 3 Example of bipartite network (top). There are two classes of nodes and links are assigned only between nodes that do not belong to the same class. A one-mode projection is a new graph consisting only of nodes of one class in which a link between two nodes implies that, in the original bipartite graph, the two nodes where connected to a same third node

assigned as the weight of the connection between i and j, and result in a natural way from the follwong operation on the adjacency matrix. If we define the adjacency matrix of the board network as

$$B_{ij} = \begin{cases} w_{ij} \text{ if } i \text{ and } j \text{ are connected with weight } w_{ij} \\ 0 \text{ if } i \text{ and } j \text{ are not connected.} \end{cases} \tag{2}$$

then it holds that

$$B_{ij} = \sum \alpha C_{\alpha i} C_{\alpha j}. \tag{3}$$

In terms of matrix product this means $\mathbf{B} = \mathbf{C}^T\mathbf{C}$. In analogous way, the adjacency matrix of the director network is related to the initial board-person network as follows

$$D_{\alpha\beta} = \sum_i C_{\alpha i} C_{\beta i}. \tag{4}$$

which is equivalent to $\mathbf{D} = \mathbf{C}\mathbf{C}^T$. While the off-diagonal entries correspond to the edge weights, the diagonal entries, are, respectively, the size B_{ii} of board i (the number of directors serving on it), and the number $D_{\alpha\alpha}$ of boards which director α serves on.

4.3 Network Measures

This section covers only a few network measurements. For a more extensive survey see Costa et al. [22] and also Newman [67] as well as Wasserman and Faust [87]. The following definitions assume undirected graphs.

4.3.1 Average Path Length

The *average path length* $\mathcal{L}$ is the mean geodesic (i.e. shortest) distance between node pairs in a graph

$$\mathcal{L} = \frac{1}{\frac{1}{2}n(n-1)} \sum_{i \geq j}^{n} d_{ij}$$

where d_{ij} is the geodesic distance from node i to node j. The average path length is important for instance in networks in which agents benefit from the knowledge of the others (so called knowledge spillovers, see Section 7.2 and 7.3 for examples). The smaller is the average distance among agents the more intense is the knowledge exchange.

For Poisson random graphs (Section 5.1) we obtain $\mathcal{L} = \frac{\ln n}{lnz}$ where n denotes the number of nodes in the graph and z the average degree. For a regular graph the average path length is $\mathcal{L} = \frac{n}{2z}$. For a complete graph K_n it is trivially $\mathcal{L} = 1$. For a cycle C_n it is half the length of the cycle $\mathcal{L} = \frac{n}{2}$ and for scale free networks (see Section 6.3) it is $\mathcal{L} = \frac{\ln n}{\ln \ln n}$. [1].

4.3.2 Clustering

For each node i, the *local clustering coefficient*, $\mathcal{C}_l(i)$, is simply defined as the fraction of pairs of neighbors of i that are themselves neighbors. The number of possible links between the neighbors of node i is simply $d_i(d_i - 1)/2$. Thus we get

$$\mathcal{C}_l(i) = \frac{|\{e_{jk} \in E(G) : e_{ij} \in E(G) \wedge e_{ik} \in E(G)\}|}{d_i(d_i-1)/2}$$

The *global clustering coefficient* $\mathcal{C}_l$ is then given by $\mathcal{C}_l = \frac{1}{n}\sum_{i=1}^{n} \mathcal{C}_l(i)$.

A high clustering coefficient $\mathcal{C}_l$ means (in the language of social networks), that two of your friends are likely to be also friends of each other. It also indicates a high redundancy of the network. For a complete graph K_n it is trivially $\mathcal{C}_l = 1$. Let $\langle d \rangle$ denote the average degree then we get for a Poisson random graph $\mathcal{C}_l = \frac{\langle d \rangle}{n-1}$ and for a cycle $\mathcal{C}_l \sim \frac{3}{4}$ for large n [1].

4.3.3 Centrality

Centrality measures the importance of a node on the basis of its position in the network [9, 10, 34]. We can look at a simple example. Consider the star $K_{1,n-1}$ in Figure (2). the most central node is node 3 which has the highest centrality, and all other nodes have minimum centrality. Actually, the star is also the most centralized graph [87].

In the following paragraphs we will introduce different measures of centrality which incorporate different aspects of a nodes position in the network. Degree

centrality counts the number of links incident to a node. Closeness centrality measures how many steps it takes to reach any other node in the network. Betweenness centrality measures how many paths between any pair of nodes pass through a node. Finally, eigenvector centrality measures the importance of a node as a function of the importance of its neighbors. The different measures of centrality capture different aspects of the position of an agent in a network and therefore the choice of the right measure depends on the particular application under consideration.

Degree Centrality

The degree centrality of node i is just the number of links d_i . We have that $d_i = \sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ji}$ (since $\mathbf{A}$ is symmetric). If we consider the degree of an agent as a measure of centrality then her centrality depends on the size of the network (with maximum centrality given by $n-1$). In order to overcome this bias one can consider the normalized degree centrality that divides the degree by $n-1$, yielding a measure in $[0,1]$. There are several applications of degree centrality, for example the popularity in friendship networks, the diffusion of information and the spread of infections.

Closeness Centrality

The closeness $\mathcal{C}_C(i)$ of i is the reciprocal of the sum of geodesic distances to all other nodes in the graph, that is

$$\mathcal{C}_C(i) = \sum_{v \neq i} \frac{1}{d_{iv}}. \tag{5}$$

If an agent has high closeness centrality she can quickly interact with other agents and gather information from them since she has short communication paths to the others.

Betweenness Centrality

The betweenness centrality of i, denoted by $\mathcal{C}_B(i)$ is defined as follows.

$$\mathcal{C}_B(i) = \sum_{u,v \neq i} \frac{g_{uv}(i)}{g_{uv}}. \tag{6}$$

More precisely, if g_{uv} is the number of geodesic paths d_{uv} from u to v and $g_{uv}(i)$ is the number of paths from u to v that pass through i, then $\frac{g_{uv}(i)}{g_{uv}}$ is the fraction of geodesic paths from u to v that pass through i. Normalized betweenness divides simple betweenness by its maximum value. Agents who are not directly connected

might depend on another agent if she lies on a path connecting them. If an agent lies on many such path connecting different components in a network then she has a high betweenness centrality.

Eigenvector Centrality

Eigenvector centrality measures the importance of a node from the importance of its neighbors. Even if a node is only connected to a few others (thus having a low degree centrality) its neighbors may be important, and therefore the node is important too, giving it a high eigenvector centrality. Let's assume that the importance of a node i is measured by x_i. Then the eigenvector centrality of node i is proportional to the sum of the eigenvector centralities of all nodes which are connected to i [68].

$$x_i = \frac{1}{\lambda} \sum_{j \in N_i} x_j = \frac{1}{\lambda} \sum_{j=1}^{n} a_{ij} x_j, \tag{7}$$

where N_i is the set of nodes that are connected to node i, n is the total number of nodes and λ is a constant. In matrix-vector notation we can write $\mathbf{Ax} = \lambda \mathbf{x}$, which is the eigenvector equation. If the proportionality factor λ is given by the largest eigenvalue λ_{PF} (Section 4.1) of the adjacency matrix $\mathbf{A}$ then all the elements in the eigenvector must be positive [50] and we get a proper measure of centrality.

4.4 Dynamics of State Variables and Network Evolution

In the following we introduce a classification of network models in four types. This classification has mainly a didactic value and it should help readers to find their way in the growing landscape of network models.

As mentioned in the beginning of this section, the agents $N = \{1, ..., n\}$ in an economic network G can be associated with a state variable x_i, representing agent i's wealth, firm i's output or, in the case of R&D collaborations, knowledge. The links between the agents i and j can be indicated by the elements a_{ij} of an adjacency matrix $\mathbf{A}$. It is important to distinguish between (1) the dynamics taking place on the state variables $\mathbf{x}(t)$ and (2) the evolution of the network $\mathbf{A}(t)$. In the first, the state variables are changed as a result of the interaction among connected nodes. In the latter, nodes or edges are added to/removed from the network by a specific mechanism. For example, the value of the assets of a firm depends on the value of the firms it holds shares in. Even if the links do not change the asset value may change. On the other hand, the links may change in time, depending or not on the asset value. Consequently, there are four types of dynamics that can be investigated in models of economic networks, as illustrated in Figure (4).

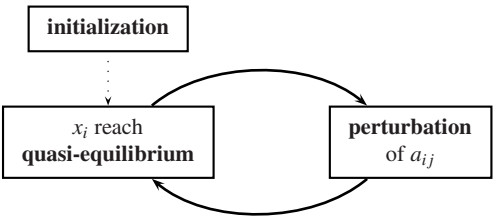
In socio-economic systems dynamics and evolution are often coupled. The utility of agents depends on their links to the other agents and agent modify their links

network \ state variables	static	dynamic
static	$\frac{dx_i}{dt}=0, \frac{da_{ij}}{dt}=0$	$\frac{dx_i}{dt}\neq 0, \frac{da_{ij}}{dt}=0$
dynamic	$\frac{dx_i}{dt}=0, \frac{da_{ij}}{dt}\neq 0$	$\frac{dx_i}{dt}\neq 0, \frac{da_{ij}}{dt}\neq 0$

Fig. 4 Possible combinations of static and dynamic state variables x_i associated with the nodes and fixed or changing links indicated by a_{ij} between the nodes for $i, j \in N$

over time depending on the utility they expect or they experience from a link. So, in principle, all systems should be studied with models in which the state variables and the network are dynamic, they co-evolve. However, evolution and dynamics do not necessarily have the same time scale.

Assume that agents have a certain inertia for creating new links and evaluating their existing ones. The rate at which links are formed is much slower than the rate at which the state variables change. In other words, there are two different time scales in our dynamical system: the fast dynamics of the state variables and the slow evolution of the network. The state variables immediately reach their quasi-equilibrium state, whereas the network remains unchanged during this short adaptation time. An illustration can be seen in Figure (4.4). One can say that the variables with the fast dynamics are "slaved" by the variables with the slow dynamics [48], [see also 46, for a review][3]. We will introduce such an approach in Section 7.3 when studying the evolution of R&D networks.



Another example for the coupling of a dynamic network with dynamic state variables are credit relations among firms. The links may represent credit relations among firms, established through contracts. Many financial variables (such as total asset value or solvency ration) of a firm are affected when financial variables change in the connected firms. Despite that, some relations maybe fixed until the expiration of the contract. Therefore, while links may be modified on a time scale of, say, several months, financial variables may vary on a time scale of days.

In the following sections we will discuss several models of networks. According to the classification we have introduced in this section, the models in Section 5 and Section 6 do not consider a state variable attached to the nodes. These models

[3] This principle has been used e.g by [57, 58] in the context of evolutionary biology and by [64] in order to explain the sustainability of informal knowledge exchange in innovation networks.

consider different ways how networks can be constructed in a stochastic network formation process. The process can be viewed as a network evolution. Since these models do not consider a dynamic state variable, they are easier to analyze and so we take them as a starting point before moving on to more complex network models. More complex models follow in the next sections. Both models in Section 7 introduce a state variable attached to the nodes. The nodes are interpreted as agents and the state variable is their utility. The model in Section 7.2 considers the case of a dynamic network but does not assume any dynamics on the state variables (even though the state variables depend on the network). Finally, in Section 7.3 we discuss a model that includes both a dynamic state variable and a dynamic network and it assumes a time-scale separation between the two.

5 Random Network Constructions

In this section we present some basic models of networks. In this discussion we follow Newman [67] as well as Vega-Redondo [84]. For a more detailed mathematical treatment see Chung and Lu [20] and Durrett [30]. The network construction algorithms introduced in this section can be simulated with the Java package "econnet" available upon request to the authors[4]. The algorithms used there serve for educational purposes only and we refer to Batagelj and Brandes [2] for an efficient implementation.

5.1 Poisson Random Graphs

We denote the Poisson random graph by $G(n,p)$ with n nodes and in which every edge is present with probability p. The expected degree is $z = \frac{2p}{n}\binom{n}{2} = p(n-1)$ where $\binom{n}{2}$ is the number of edges in the complete graph K_n. The degree distribution of $G(n,p)$ is given by

$$p_k = \binom{n-1}{k} p^k (1-p)^{n-1-k}, \tag{8}$$

where p_k is the probability that a randomly chosen node has degree k. We have that

$$\lim_{n\to\infty} p_k = \frac{z^k e^{-z}}{k!} = \mathrm{Pois}(z;k). \tag{9}$$

Many results on the topological properties and phase transitions can be derived for Poisson random graphs. We refer to West [91], Chung and Lu [20], Durrett [30] and Bollobas [7] for the interested reader.

[4] Mail to mkoenig@ethz.ch.

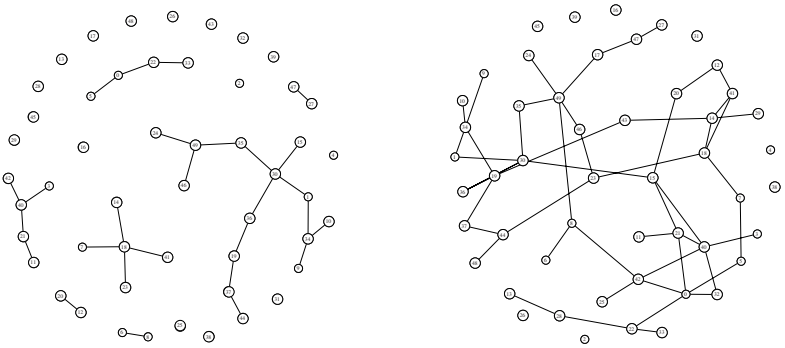


Fig. 5 Poisson random graph $G(n,p)$ 50% below the phase transition $p = \frac{1}{n-1}$ (left) and at the phase transition (right). The graph was generated with the Java package "econnet" and the ARF layout algorithm [38]

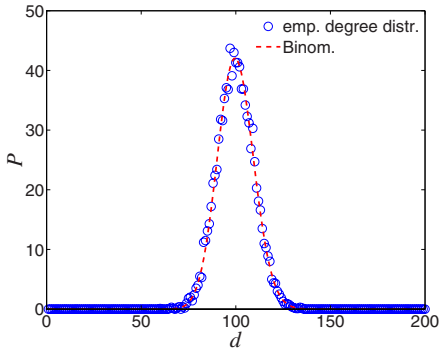


Fig. 6 Degree distribution of the Poisson random graph $G(n,p)$ with $p = 0.1$, $n = 1000$ and averaged over 10 realizations

5.2 Generalized Random Graphs

In the following we give a short introduction to random graphs with arbitrary degree distributions. For a detailed discussion (including all the material presented here) see Newman et al. [69, 70].

5.2.1 Random Graph Construction

Consider a set of nodes $N = \{1,...,n\}$. A degree sequence of a graph is a list of node degrees $d_1 \geq d_2 \geq ... \geq d_n$ with the property that $\sum_{k=1}^{n} d_k$ must be even. We construct the random graph G by creating d_i half-edges attached to node i, and then pair the half-edges at random. The resulting graph may have loops and multiple edges.

5.2.2 Neighborhood Size, Diameter, Phase Transition and Clustering

The probability of a randomly chosen node having degree k is given by

$$p_k = \frac{1}{n}|\{i \in N : d_i = k\}|. \tag{10}$$

Its generating function is defined by [31]

$$G_0(x) = \sum_{k=0}^{\infty} p_k x^k. \tag{11}$$

p_k is the probability that a randomly chosen node has degree k. The distribution p_k is assumed to be correctly normalized, so that

$$G_0(1) = \sum_{k=0}^{\infty} p_k = 1. \tag{12}$$

$G_0(x)$ is finite for all $|x| \leq 1$. If the distribution is Poisson, $p_k = z^k e^{-z}/k!$, then the generating function is

$$G_0(x) = \sum_k \frac{1}{k!} e^{-z} z^k x^k = e^{-z} \sum_k \frac{(zx)^k}{k!} = e^{z(x-1)}. \tag{13}$$

The probability p_k is given by the k^{th} derivative of G_0 according to

$$p_k = \left. \frac{1}{k!} \frac{d^k G_0}{dx^k} \right|_{x=0}. \tag{14}$$

Thus, the function $G_0(x)$ encapsulates all the information of the discrete probability distribution p_k.

The mean (first-order moment), e.g. the average degree z of a node, is given by

$$z = \langle k \rangle = \sum_{k=0}^{\infty} k p_k = G_0'(1). \tag{15}$$

Higher order moments of the distribution can be calculated from higher derivatives. In general we have

$$\langle k^n \rangle = \sum_{k=0}^{\infty} k^n p_k = \left. \left(x \frac{d}{dx} \right)^n G_0(x) \right|_{x=1}. \tag{16}$$

For the first two moments of the Poisson distribution we obtain

$$x \frac{d}{dx} e^{z(x-1)} = z \tag{17}$$

$$\left(x \frac{d}{dx} \right)^2 e^{z(x-1)} = z(1+z). \tag{18}$$

If we select a node i then the number of neighbors has distribution p. However, the distribution of the first neighbors of a node is not the same as the degree distribution of nodes on the graph as a whole. Because high-degree nodes have more edges connected to it, there is a higher probability that a randomly chosen edge is incident to it, in proportion to the node degree. The number of nodes with degree k is np_k. The number of edges incident to nodes with degree k is given by knp_k. This is equal the number of possibilities to select an edge which is incident to a node with degree k. Thus, the probability that a node incident to a randomly chosen edge has degree k is proportional to kp_k and not just p_k. Through normalization we get that the probability distribution of the degree among neighbors of a randomly selected node i is given by [67]

$$q_k = \frac{kp_k}{\sum_s sp_s}. \tag{19}$$

The average degree of a neighboring node is then

$$\sum_k kq_k = \frac{\sum_k k^2 p_k}{\sum_s sp_s} = \frac{\langle k^2 \rangle}{\langle k \rangle}. \tag{20}$$

The corresponding generating function is

$$\begin{aligned} \sum_k q_k x^k &= \frac{\sum_k kp_k x^k}{\sum_s sp_s} \\ &= \frac{1}{\langle k \rangle} x \underbrace{\sum_k p_k k x^{k-1}}_{G'_p(x)} \\ &= \frac{xG'_p(x)}{G'_p(1)}. \end{aligned} \tag{21}$$

If we are interested in the (excess) distribution p^*_k of links of a node that can be reached along a randomly chosen edge, other then the one we arrived along, $p^* = q_{k+1} \propto (k+1)p_{k+1}$, then its generating function is

$$\begin{aligned} G_{p^*}(x) &= \sum_k \frac{(k+1)p_{k+1}}{\sum_s sp_s} x^k \\ &= \frac{1}{G'_p(1)} \sum_k kp_k x^{k-1} \\ &= \frac{G'_p(x)}{G'_p(1)}. \end{aligned} \tag{22}$$

In order to compute the expected number of second neighbors we have to exclude node i from the degree count of its neighboring node and obtain

$$q_{k-1} = \frac{kp_k}{\sum_s p_s}, \tag{23}$$

or equivalently

$$q_k = \frac{(k+1)p_{k+1}}{\sum_s s p_s}. \tag{24}$$

The average (excess) degree of such a node is then

$$\begin{aligned} \sum_k k q_k &= \frac{\sum_k (k+1)p_{k+1}}{\sum_s s p_s} \\ &= \frac{\sum_k (k-1)k p_k}{\sum_s s p_s} \\ &= \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle}. \end{aligned} \tag{25}$$

The average total number of second neighbors of a node is given by the average degree of the node times the excess degree of the first neighbours:

$$z_2 = \langle k \rangle \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} = \langle k^2 \rangle - \langle k \rangle. \tag{26}$$

The average number of second neighbors is then equal to the difference between the second- and first-order moments of the degree distribution p. The expectation of the first neighbors is $z_1 = G_p'(1)$ and for the second neighbors one derives $z_2 = G_p''(1)$. Note that in general the number of r^{th} neighbors is not simply the r^{th} derivative of the generating function.

The average number of edges leaving from a second neighbor is given by Equation (25). This also holds for any distance m away from a randomly chosen node. Thus, the average number of neighbors at distance m is

$$\begin{aligned} z_m &= \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} z_{m-1} \\ &= \frac{z_2}{z_1} z_{m-1} \\ &= \left(\frac{z_2}{z_1} \right)^{m-1} z_1, \end{aligned} \tag{27}$$

where $z_1 = \langle k \rangle$ and z_2 is given by Equation (26). Depending on whether z_2 is greater than z_1 or not, this expression will either diverge or converge exponentially as m becomes large so that the average number of neighbors of a node is either finite or infinite for $n \to \infty$. We call this abrupt change a phase transition at $z_1 = z_2$. This condition can be written as $\langle k^2 \rangle - 2\langle k \rangle = 0$ or

$$\sum_k k(k-2)p_k = 0. \tag{28}$$

In the above sum isolated nodes and nodes with degree one do not contribute since they can be removed from a graph without changing its connectivity.

We assume that $z_2 \gg z_1$ so that there exists a giant component essentially including all the nodes and most of the nodes are far from each other, at around distance D, the diameter of the graph. This means that

$$n \sim z_D = \left(\frac{z_2}{z_1}\right)^{D-1} z_1, \tag{29}$$

which leads to

$$\begin{aligned} \ln \frac{n}{z_1} &\sim (D-1) \ln \frac{z_2}{z_1} \\ D &\sim \frac{\ln \frac{n}{z_1}}{\ln \frac{z_2}{z_1}} + 1. \end{aligned} \tag{30}$$

For the special case of a Poisson network with $z_1 = z$ and $z_2 = z^2$ we obtain for large n

$$D \sim \frac{\ln \frac{n}{z}}{\ln z} + 1 = \frac{\ln n}{\ln z} \tag{31}$$

In the following we study the clustering coefficient $\mathcal{C}_l$ of a random graph. For this, we consider a particular node i. The j^{th} neighbor of i has k_j links emanating from it other than the edge e_{ij} and k_j is distributed according to the distribution q. The probability that node j is connected to another neighbor s is $\frac{k_j k_s}{nz}$, where k_s is distributed according to q. The average of this probability is precisely the clustering coefficient

$$\begin{aligned} \mathcal{C}_l &= \frac{\langle k_j k_s \rangle}{nz} \\ &= \frac{1}{nz} \left(\sum_k k q_k \right)^2 \\ &= \frac{z}{n} \left(\frac{\langle k^2 \rangle \langle k \rangle}{\langle k \rangle^2} \right)^2 \\ &= \frac{z}{n} \left(c_v^2 + 1 - \frac{1}{\langle k \rangle} \right)^2, \end{aligned} \tag{32}$$

where $c_v = \frac{\langle (k - \langle k \rangle) \rangle}{\langle k \rangle^2}$ is the coefficient of variation of the degree distribution - the ratio of the standard deviation to the mean. For Poisson networks we get $z_2 = \langle k^2 \rangle - \langle k \rangle = \langle k \rangle^2 = z^2$ and the clustering coefficient is $\mathcal{C}_l = \frac{z}{n}$. For arbitrary degree distributions we still have that $\lim_{n \to \infty} \mathcal{C}_l = 0$ but the leading term in (32) may be higher.

5.2.3 Average Component Size Below the Phase Transition

With similar methods one can compute the average size of the connected component a node belongs to. Here we closely follow the discussion in Baumann and Stiller [6]. The computation is valid under following assumptions:

(i) The network contains no cycles. One can show that this assumption is a good approximation for big, sparse random networks.
(ii) For any edge e_{uv} of a node u the degree of v is distributed independently of u's neighbors and independently of the degree of u.

We then choose an edge e uniform at random among the edges in $E(G)$. We select one of the incident nodes of e at random, say v. Let p^0 denote the distribution of the size of the component of v in the graph of $E(G)\setminus e$. Further, let p^* be the distribution of the degree of v in $E(G)\setminus e$. Then $p^0(1) = p^*(0)$. If the degree of v is k then we denote the neighbors of v in $E(G)\setminus e$ as $n_1,...,n_k$. We define the following probability: $P_k(s-1)$ is the probability that the size of the components of the k nodes $n_1,...,n_k$ in $E(G)\setminus\{e,e_{vn_1},...,e_{vn_k}\}$ sum up to $s-1$. Then we can write

$$p_s^0 = \sum_k p_k^* P_k(s-1). \tag{33}$$

Now let S denote a random variable that is the sum of m independent random variables $X_1,...,X_m$, that is

$$S = X_1 + ... + X_m, \tag{34}$$

then the generating function of S is given by

$$G_S(x) = G_{X_1}(x)G_{X_2}(x)\cdots G_{X_m}(x). \tag{35}$$

Consider the distribution of the sum of the degrees of two nodes when p_k is the distribution of a single node. Then the sum of the degrees has a generating function $G_0(x)^m$. For two nodes we get

$$\begin{aligned} G_0(x)^2 &= \left(\sum_k p_k x^k\right)^2 \\ &= \left(\sum_k p_k x^k\right)\left(\sum_j p_j x^j\right) \\ &= \sum_{j,k} p_j p_k x^{j+k} \\ &= p_0p_0x^0 + (p_0p_1 + p_1p_0)x^1 + (p_2p_0 + p_1p_1 + p_0p_2)x^2 + \cdots. \end{aligned} \tag{36}$$

The coefficients of the powers of x^n are clearly the sum of all products $p_j p_k$ such that $j+k=n$ and hence it gives the probability that the sum of the degrees of the two nodes will be n. We can use a similar argument to prove that higher order powers of generating functions can be computed in the same way.

Following our assumptions, the edges e_{vn_i} are chose independently and uniform at random among all edges in $E(G)$. Therefore, P_k is distributed as the sum of k random variables, which are in turn distributed according to p^0. Using the powers of

generating functions we have that $G_{P_k} = G_{p^0}(x)^k$. Moreover, the generating function of p^0 is

$$\begin{aligned} G_{p^0}(x) &= \sum_s p^0 x^s \\ &= \sum_s x^s \sum_k p_k^* P_k(s-1) \\ &= x \sum_k p_k^* \underbrace{\sum_s x^{s-1} P_k(s-1)}_{G_{P_k}(x) = G_{p^0}(x)^k} \\ &= x \sum_k p_k^* G_{p^0}(x)^k \\ &= x G_{p^*}\left(G_{p^0}(x)\right). \end{aligned} \tag{37}$$

The quantity we are actually interested in is the distribution of the size of the component a randomly chosen node belongs to. The number of edges emanating from such a node is distributed according to the degree distribution p_k. Each such edge leads to a component whose size is drawn from the distribution generated by the function $G_{p^0}(x)$. In a similar way to the derivation of Equation (37), one can show that the size of the component to which a randomly selected node belongs is generated by

$$\begin{aligned} G_{\tilde{p}}(x) &= x \sum_k p_k G_{p^0}(x)^k \\ &= x G_p\left(G_{p^0}(x)\right). \end{aligned} \tag{38}$$

The expected component size of a randomly selected node can be computed directly from above. The expectation of a distribution is the derivative of its generating function evaluated at point 1. Therefore the mean component size $\langle s \rangle$ is given by

$$\langle s \rangle = G'_{\tilde{p}}(1) = \underbrace{G_p\left(G_{p^0}(1)\right)}_{=1} + G'_p\left(G_{p^0}(1)\right) G'_{p^0}(1). \tag{39}$$

where we used the normalization of the generating function. From (37) we know that

$$\begin{aligned} G'_{p^0}(1) &= G_{p^*}\left(G_{p^0}(1)\right) + G'_{p^*}\left(G_{p^0}(1)\right) G'_{p^0}(1) \\ &= 1 + G'_{p^*}(1) G'_{p^0}(1), \end{aligned} \tag{40}$$

and thus $G'_{p^0}(1) = \frac{1}{1 - G'_{p^*}(1)}$. Inserting this equation into Equation (39) yields

$$\langle s \rangle = 1 + \frac{G'_p(1)}{1 - G'_{p^*}(1)}. \tag{41}$$

We further have that

$$\begin{aligned} G'_p(1) &= \sum_k k p_k = \langle k \rangle = z_1 \\ G'_{p^*}(1) &= \frac{\sum_k k(k-1)p_k}{\sum_l l p_l} \\ &= \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} \\ &= \frac{z_2}{z_1}. \end{aligned} \tag{42}$$

Therefore, the average component size below the transition is

$$\langle s \rangle = 1 + \frac{z_1^2}{z_1 - z_2}. \tag{43}$$

The above expression diverges for $z_1 = z_2$ which signifies the formation of the giant component. We can also write the condition for the phase transition as $G'_{p^*}(1) = 1$. We see that for $p = 0$ $\langle s \rangle = 1$ (an empty graph contains only isolated nodes). For the Poisson random graph $z_1 = z = p(n-1)$, $z_2 = z^2$ and thus we get $\langle s \rangle = 1 + \frac{p(n-1)}{1-p(n-1)}$.

5.3 The Watts-Strogatz "Small-World" Model

The model draws inspiration from social systems in which most people have friends among their immediate neighbors, but everybody has one or two friends who are a far away - people in other countries, old acquaintances, which are represented by the long-range edges obtained by rewiring. Empirically, in social networks the average distance turns out to be "small": the fact that any two persons in the US are separated on average by only six acquaintances is the so called "Small-World" phenomenon discovered by Milgram [66]. Watts and Strogatz [89] introduced a "Small-World" network model which has triggered an avalanche of works in the field. Their model generates a one-parameter family of networks laying in between an ordered lattice and a random graph. We will explain how such a "Small-World" network can be constructed in the next section.

5.3.1 "Small-World" Network Construction

The initial network is a one-dimensional ring of n nodes (if each node has only two neighbors it is a cycle) as shown in Figure (7), with periodic boundary conditions, each node being connected to its z nearest neighbors. The nodes are then visited one after the other: each link connecting a node to one of its $\frac{z}{2}$ neighbors in the clockwise order is left in place with probability $1-p$, and with probability p is reconnected to a randomly chosen node. With varying p the system exhibits a transition between order ($p = 0$) and randomness ($p = 1$).

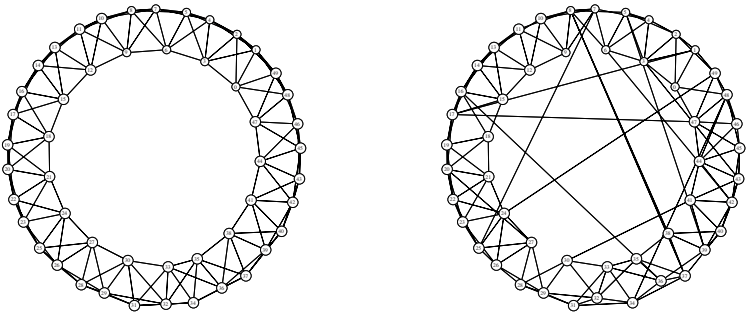


Fig. 7 Regular (lattice) graph with $n = 50$ nodes and neighborhood size $z = 6$ (left). Small World graph with $n = 50$ nodes, neighborhood size $z = 6$ of the underlying lattice and rewiring probability $p = 0.1$ (right). The graph was generated with the Java package "econnet" and the ARF layout algorithm [38]

5.3.2 Degree Distribution

For $p = 0$, each node has the same degree, z. On the other hand, a non-zero value of p introduces disorder in the network, in the form of a non-uniform degree distribution, while maintaining a fixed average degree $\langle X \rangle = z$. Let us denote $P(X = k)$ the probability of the degree of a node being equal k.

Since $\frac{z}{2}$ of the original z edges are not rewired by the above procedure, the degree of node i can be written as [1].

$$X = \frac{z}{2} + n_0 + n_+ \tag{44}$$

with $n_0 + n_+ \geq 0$. n_0 denotes the number of links that have been left in place during the rewiring procedure (with probability $1 - p$) and n_0 denotes the number of links that have been rewired to node i from other nodes (with probability $p/(n-1)$, since there are $n - 1$ other nodes). This sequence of independent events (the links left in place as well as the rewired links) is actually a Bernoulli process. Thus, the probabilities are given by Binomial distributions

$$P(n_0 = s) = \binom{\frac{z}{2}}{s} (1-p)^s p^{\frac{z}{2}-s}, \tag{45}$$

with $0 \leq s \leq \frac{z}{2}$ and

$$P(n^+ = s) = \binom{(n-1)\frac{z}{2}}{s} \left(\frac{p}{n-1}\right)^s \left(1 - \frac{p}{n-1}\right)^{(n-1)\frac{z}{2}-s}, \tag{46}$$

where $0 \leq s \leq (n-1)\frac{z}{2}$ and $n - 1$ is the number of other nodes, $\frac{z}{2}$ the maximum number of edges that can be rewired by other nodes. If we define

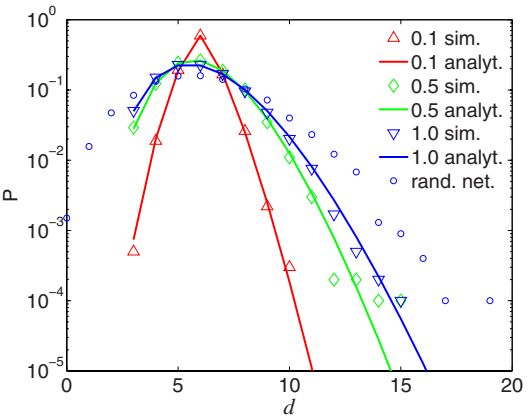


Fig. 8 Empirical and theoretical degree distributions of the "Small-World" network for $n = 500$, neighborhood size $z = 6$ and and different values of the rewiring probability $p \in \{0.1, 0.5, 1.0\}$

$$\begin{aligned} N &= (n-1)\tfrac{z}{2} \\ q &= \tfrac{p}{n-1} \\ \lambda &= Nq = \tfrac{z}{2}(n-1)\tfrac{p}{n-1} = nq, \end{aligned} \tag{47}$$

we get the standard form of the Binomial distribution

$$P(n^+ = s) = \binom{N}{s} q^s (1-q)^{N-s}. \tag{48}$$

For $N \to \infty$ respectively $n \to \infty$ we obtain the Poisson distribution

$$P(n^+ = s) = \frac{\lambda^s e^{-\lambda}}{s!} = \frac{\left(\frac{pz}{2}\right)^s e^{-\left(\frac{pz}{2}\right)}}{s!}. \tag{49}$$

Thus, we get for $k \geq \frac{z}{2}$ (k links remain unchanged by construction)

$$P(X = k) = \sum_{i=0}^{\min\{k-\frac{z}{2}, \frac{z}{2}\}} \binom{\frac{z}{2}}{2} (1-p)^i p^{\frac{z}{2}-i} \frac{\left(\frac{pz}{2}\right)^{k-\frac{z}{2}-i}}{\left(k-\frac{z}{2}-i\right)!} e^{-\frac{pz}{2}}. \tag{50}$$

The upper bound in the sum above guarantees that $n_0 \leq \frac{z}{2}$. Since any degree $k > \frac{z}{2}$ must come from new edges. Figure (8) shows the degree distribution for different values of p.

5.3.3 Average Path Length and Clustering Coefficient

For a cycle ($p = 0$) we have a linear chain of nodes and we find for the average path length (defined in Section 4.3) for large n [1]

$$L(p=0) = \frac{n(n+z-2)}{2z(n-1)} \sim n/2z \gg 1. \tag{51}$$

Moreover, for $p = 0$ each node has z neighbors and the number of links between these neighbors is $\frac{3z(z/2-1)}{4}$ and it follows that for large n [1]

$$\mathcal{C}_l(p=0) = \frac{3(z/2-1)}{2(z-1)} \sim 3/4. \tag{52}$$

Thus, $\mathcal{L}$ scales linearly with the system size, and the clustering coefficient is large and independent of n. On the other hand, for $p \to 1$ the model converges to a random graph for which $\mathcal{L}(p=1) \sim \ln(n)/\ln(z)$ and $\mathcal{C}_l(p=1) \sim z/n$ when n is large, thus $\mathcal{L}$ scales logarithmically with n and the clustering coefficient decreases with n. Based on these scaling relationships, one could expect that a large (small) value of $\mathcal{C}_l$ is always associated with a large (small) value of $\mathcal{L}$. Unexpectedly, it turns out that there is a broad range of values of p in which $\mathcal{L}(p<1)$ is close to $\mathcal{L}(p=1)$ and yet $\mathcal{C}_l(p<1) \gg \mathcal{C}_l(p=1)$. The coexistence of small $\mathcal{L}$ and large $\mathcal{C}_l$ means that the network is a "Small-world" like a random graph *and* has high clustering like a lattice. Interestingly, this feature is found in many real networks. In a regular lattice ($p = 0$) the clustering coefficient $\mathcal{C}_l$ does not depend on the system size but only on its topology. As the edges of the network are randomized, the clustering coefficient remains close to $\mathcal{C}_l(p=0)$ up to relatively large values of p, while the average path length $\mathcal{L}$ drops quite rapidly. This is the reason of the onset of the small world regime. We show examples for the clustering coefficient and the average path length in the "Small-World" network in Figure (9).

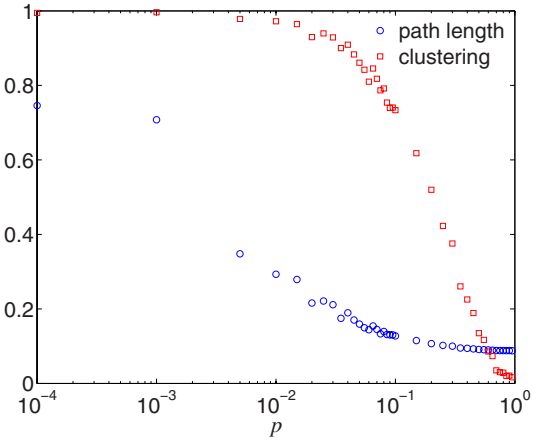


Fig. 9 Clustering coefficient $\mathcal{C}_l$ and average path length $\mathcal{L}$ of the "Small-World" network for with $n = 500$, neighborhood size $z = 6$ and and different values of the rewiring probability p. The average path length $\mathcal{L}$ is normalized to the corresponding value of the lattice. For $p = 1$ the normalized path length (proportional to $\ln n/n$) converges to zero for large n

6 Growing Random Networks

In the next sections we derive the degree distributions for two types of networks, the uniform and the preferential attachment network, illustrated in Figure (10) and their corresponding degree distributions in Figure (11). Both networks are generated by continuously adding nodes to the existing network. The difference is the following: in the uniform attachment network new nodes form links uniformly to the existing nodes and in the preferential attachment network new nodes form links more likely to existing nodes with higher degree. In the derivation of the degree distribution we follow closely Vega-Redondo [84].

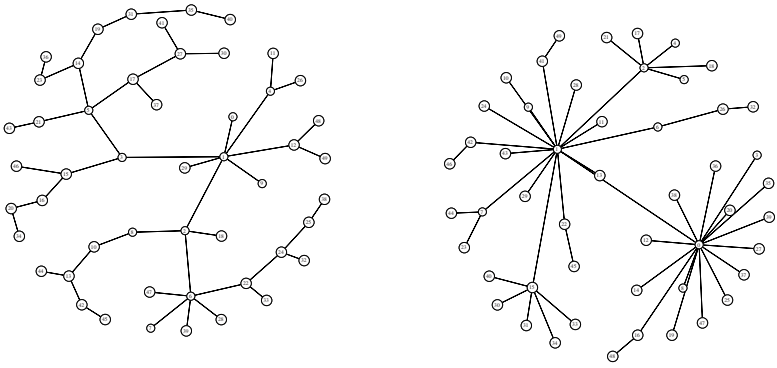


Fig. 10 Uniform attachment (left) and preferential attachment (right) networks with $n = 50$ nodes. The graph was generated with the Java package "econnet" and the ARF layout algorithm [38]

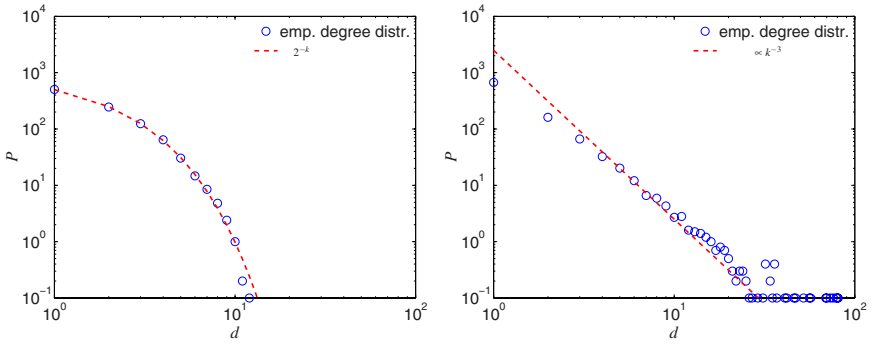


Fig. 11 Degree distribution of the uniform (left) and preferential attachment (right) networks for $n = 1000$ averaged over 10 realizations

6.1 Uniform Attachment Network Construction

The network is constructed as follows. Times is measured at countable dates $t \leq 0$. A node that enters the network at time t is attached the label t. We initialize nodes $1,2$ and the edge 12. Then, at every step $t > 2$ we add a new node t and create the edge e_{ts}, where node s is selected uniformly at random from the set $\{1,...,t-1\}$ of already existing nodes in the network.

6.2 Degree Distribution

In the following we derive the degree distribution if edges are attached to existing nodes with uniform probability. Denote by $q_t(s,k)$ the probability that a particular node s has degree k at time t where $s \leq t$. Any existing node s enjoys degree $k \geq 1$ at time $t+1$ if, and only if, one of the following events occurs: (i) Node s had degree $k-1$ at time t (with probability $q_t(s,k-1)$) and is chosen to be linked by the entering node at time t (with probability $\frac{1}{t+1}$), or (ii) node s already had degree k at time t (with probability $q_t(s,k)$) and is not chosen by the new node (with probability $1-\frac{1}{t+1}$).

Thus we get the following master equation [73, 90] and Vega-Redondo [84, p. 272]

$$q_{t+1}(s,k) = \frac{1}{t+1} q_t(s,k-1) + \left(1 - \frac{1}{t+1}\right) q_t(s,k), \tag{53}$$

with the boundary conditions[5]

$$\begin{aligned} &q_1(0,k) = q_1(1,k) = \delta_{k,1} \\ &q_t(t,k) = \delta_{k,1}. \end{aligned} \tag{54}$$

Denote $p_t(k)$ the probability that a randomly selected node has any given degree k at time t. $p_t(k)$ is the degree distribution at time t. Assuming that the selection of nodes is a sequence of stochastically independent events, it follows that

$$p_t(k) = \frac{1}{t+1} \sum_{s=0}^{t} q_t(s,k) \tag{55}$$

Summation over all nodes $s = 0,...,t$ in Equation (53) yields

$$\sum_{s=0}^{t} q_{t+1}(s,k) = \frac{1}{t+1} \sum_{s=0}^{t} q_t(s,k-1) + \left(1 - \frac{1}{t+1}\right) \sum_{s=0}^{t} q_t(s,k), \tag{56}$$

and further adding the term $q_{t+1}(t+1,k)$ on both sides gives

[5] The Kronecker-Delta is defined as $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

$$\begin{aligned}\sum_{s=0}^{t+1} q_{t+1}(s,k) &= \frac{1}{t+1}\sum_{s=0}^{t} q_t(s,k-1) + \left(1-\frac{1}{t+1}\right)\sum_{s=0}^{t} q_t(s,k) + \delta_{k,1} \\ &= p_t(k-1) + t\,p_t(k) + \delta_{k,1},\end{aligned} \tag{57}$$

where we used the boundary condition $q_{t+1}(t+1,k) = \delta_{k,1}$. This reflects the fact that, in every period $t+1$, the entering node $t+1$ always represents a unit contribution to the set of nodes with degree 1 (and only these nodes). Then, with

$$(t+2)\frac{1}{t+2}\sum_{s=0}^{t+1} q_{t+1}(s,k) = (t+2)p_{t+1}(k), \tag{58}$$

we may write Equation (57) as follows

$$(t+2)p_{t+1}(k) - t\,p_t(k) = p_t(k-1) + \delta_{k,1}, \tag{59}$$

which is the law of motion of the degree distribution. In the limit $t \to \infty$, $p_t(k)$ attains its stationary distribution $p(k)$.

$$2p(k) = p(k-1) + \delta_{k,1} \tag{60}$$

We can solve the above equation for $k > 1$ ($\delta_{k,1} = 0$):

$$p(k) = 2^{-k}. \tag{61}$$

Since there are no disconnected nodes in the network we have that $p(0) = 0$. For $k = 1$ we thus find that Equation (61) also solves Equation (60) for any $k = 1, 2, \ldots$. This means that the long run stationary degree distribution is geometric.

6.3 Preferential Attachment Network Construction

The network is constructed in a similar way as in the uniform attachment network formation process. We initialize nodes $1, 2$ and edge 12, setting $t = 3$. Let $k_t(s)$ denote the degree of node s at time t. Then, at every step t we add a node t and create the edge e_{ts} with probability $k_t(s)/\sum_{r=0}^{t-1} k_t(r)$.

6.4 Degree Distribution

The master equation for the probabilities $q_t(s,k)$ that any node s has degree $k \geq 1$ at time t, $s \leq t$ is given by

$$q_{t+1}(s,k) = \frac{k-1}{2t} q_t(s,k-1) + \left(1-\frac{k}{2t}\right) q_t(s,k). \tag{62}$$

There are two exclusive events that may lead node s to have degree k in time step $t+1$: (i) Node s had degree $k-1$ at time t and the new node $t+1$ establishes a link to s, or (ii) node s had degree k at time t and the new node $t+1$ does not form a link to it.

The probability of event (i) is given by $q_t(s,k-1)$ multiplied by the ratio of the degree, $k-1$, to the sum of the degrees, that is $2t$. The probability of the event (ii) is the complement of the probability that the new node establishes a link to s with degree k, that is $1-\frac{k}{2t}$ times $q_t(s,k)$. Summing over all nodes $s \leq t+1$ in Equation (62) and adding the term $q_{t+1}(t+1,k)$ on both sides, we arrive at the law of motion for the degree distribution

$$\sum_{s=0}^{t+1} q_{t+1}(s,k) = \frac{k-1}{2t}\sum_{s=0}^{t} q_t(s,k-1) + \left(1-\frac{k}{2t}\right)\sum_{s=0}^{t} q_t(s,k) + \delta_{k,1}. \tag{63}$$

We have that

$$\begin{aligned}\sum_{s=0}^{t+1} q_{t+1}(s,k) &= \frac{1}{2}\frac{t+1}{t}\left[(k-1)\frac{1}{t+1}\sum_{s=0}^{t} q_t(s,k-1) \; -k\frac{1}{t+1}\sum_{s=0}^{t} q_t(s,k)\right] \\ &\quad + (t+1)\frac{1}{t+1}\sum_{s=0}^{t} q_t(s,k) + \delta_{k,1} \\ &= \frac{1}{2}\frac{t+1}{t}\left((k-1p_t(k-1) - kp_t(k))\right) + (t+1)p_t(k) + \delta_{k,1}.\end{aligned} \tag{64}$$

Using the fact that

$$\begin{aligned}\sum_{s=0}^{t+1} q_{t+1}(s,k) &= (t+2)\frac{1}{t+2}\sum_{s=0}^{t+1} q_{t+1}(s,k) \\ &= (t+2)p_{t+1}(k),\end{aligned} \tag{65}$$

we get

$$(t+2)p_t(k) = \frac{1}{2}\frac{t+1}{t}\left((k-1)p_t(k-1) - kp_t(k)\right) + (t+1)p_t(k) + \delta_{k,1}. \tag{66}$$

In the limit, as $t \to \infty$, and each $p_t(k)$ converges to its stationary distribution $p(k)$,we obtain

$$p(k) = \frac{1}{2}\left((k-1)p(k-1) - kp(k)\right) + \delta_{k,1}, \tag{67}$$

since $p_{t+1}(k) = p_t(k)$ in the stationary state and for large t, $t+2 \sim t+1 \sim t$. The solution for $k > 1$ of Equation (67) is given by

$$p(k) = \frac{4}{k(k+1)(k+2)}. \tag{68}$$

One can write Equation (67) in the form

$$\begin{aligned} p(k) &= \frac{1}{2}\left(k\left[p(k-1)-p(k)\right]-p(k-1)\right)+\delta_{k,1} \\ &= -\frac{1}{2}\left(k\frac{p(k)-p(k-\Delta k)}{\Delta k}+p(k-\Delta k)\right)+\Delta_k, \end{aligned} \tag{69}$$

where $\Delta k = 1$. Taking the limit $\Delta k \to 0$ one obtains the continuous form of (67)

$$\begin{aligned} p(k) &= -\frac{1}{2}\left(k\frac{dp}{dk}+p(k)\right) \\ &= -\frac{1}{2}\frac{d}{dk}\left(kp(k)\right). \end{aligned} \tag{70}$$

The solution of this equation is given by

$$p(k) = 2k^{-3}, \tag{71}$$

where the factor 2 comes from the normalization condition $\int_1^\infty p(k)dk = 1$. We find, therefore, that the degree distribution satisfies a power law of the form $p(k) \propto k^{-\gamma}$. If the frequency of nodes with a degree k is proportional to $k^{-\gamma}$, then the distribution is scale-free.

7 Strategic Network Formation

In the preceding sections we have studied the formation of networks under different stochastic processes governing the way in which links are formed between nodes. However, in social and economic settings the choice of forming a link or not is governed by individual incentives and the potential benefits versus costs that arise from the establishment or withdrawal from a relationship. Strategic network formation[6] thus constitute strategic settings in which the payoffs of agents are interdependent and this interdependency is rooted in a network structure.

7.1 Efficiency and Pairwise Stability

If we want to model network formation based on individual incentives then we first need to introduce a utility function that describes the net benefits an agent enjoys from being part of the network. This can formally be done via a utility function $u_i : G \to \mathbb{R}$ that assigns each agent $i \in N = \{1,...,n\}$ a utility from the network G.

[6] We restrict our discussion in this tutorial to non-cooperative games on networks [see also 42, 55, 84, 93, for an excellent introduction]. Cooperative games on networks have been treated in [76]. For algorithmic issues we refer to Nisan et al. [71].

Based on a properly defined utility function we can address the question of how efficient or stable certain network structures are. We treat both of these issues in the next paragraphs.

A measure of the global performance of the network is introduced by its efficiency. The total utility of a network is defined by $U(G) = \sum_{i=1}^{n} u_i(G)$. A network is considered efficient if it maximizes the total utility of the network $U(G)$ among all possible networks, G with n nodes [56].

Definition 1. Denote the set of networks with n nodes by $\mathcal{G}(n)$. A network G is *efficient* if $U(G) = \sum_{i=1}^{n} u_i(G) \geq U(G') = \sum_{i=1}^{n} u_i(G')$ for all $G' \in \mathcal{G}(n)$.

The evolution of the network is the result of strategic interactions between agents when they decide to create or delete links. In the following we consider a particularly simple network formation process. At every time step a pair of agents is chosen at random and tries to establish a new link between them or delete an already existing one. If a link is added, then the two agents involved must both agree to its addition, with at least one of them strictly benefiting (in terms of a higher utility) from its formation. Similarly a deletion of a link can only take place in a mutual agreement. The subsequent addition and deletion of links creates a sequence of networks. If no new links are accepted nor old ones are deleted then the network reaches an equilibrium. An equilibrium under the above described network formation process leads us to the notion of pairwise stability, introduced by Jackson and Wolinsky [56].

Definition 2. A network G is *pairwise stable* if and only if

(i) for all $e_{ij} \in E(G)$, $u_i(G) \geq u_i(E \backslash e_{ij})$ and $u_j(G) \geq u_j(E \backslash e_{ij})$,
(ii) for all $e_{ij} \notin E(G)$, if $u_i(G) < u_i(E \cup e_{ij})$ then $u_i(G) > u_j(E \cup e_{ij})$.

A network is pairwise stable if and only if (i) removing any link does not increase the utility of any agent, and (ii) adding a link between any two agents, either does not increase the utility of any of the two agents, or if it does increase one of the two agents' utility then it decreases the other agent's utility.

The point here is that establishing a new link with an agent requires the consensus, that is, an increase in utility, of both of them. The notion of pairwise stability can be distinguished from the one of Nash equilibrium[7] which is appropriate when each agent can establish or remove unilaterally a connection with another agent.

In Section 7.2 and in Section 7.3 we will give specific examples for different utility functions. As we will show, the particular choice of the utility function significantly shapes individual incentives to form or severe links. As a result, different incentive structures translate into network outcomes that can vary considerably in terms of efficiency and stability.

[7] Considering two agents playing a game (e.g. trading of knowledge) and each adopting a certain strategy. A Nash equilibrium is characterized by a set of strategies where each strategy is the optimal response to all the others.

7.2 The Connections Model

In the Connections Model introduced in Jackson and Wolinsky [56] agents receive information from others to whom they are connected to. Through these links they also receive information from those agents that they are indirectly connected to, that is, trough the neighbors of their neighbors, their neighbors, and so on[8].

The utility, $u_i(G)$, agent i receives from network G with n agents is a function $u_i : G \to \mathbb{R}$ with

$$u_i(G) = \sum_{j=1}^{n} \delta^{d_{ij}} - \sum_{j \in N_i} c, \tag{72}$$

where d_{ij} is the number of edges in the shortest path between agent i and agent j. $d_{ij} = \infty$ if there is no path between i and j. $0 < \delta < 1$ is a parameter that takes into account the decrease of the utility as the path between agent i and agent j increases. $N(i)$ is the set of nodes in the neighborhood of agent i.

There exists a tension between stability and efficiency in the connections model. This will become clear, after we state the following two propositions.

Proposition 1. *The unique efficient network in the symmetric Connections Model is*

(i) the complete graph K_n if $c < \delta - \delta^2$,
(ii) a star encompassing everyone if $\delta - \delta^2 < c < \delta + \frac{n-2}{2}\delta^2$,
(iii) the empty graph (no links) if $\delta + \frac{n-2}{2}\delta^2 < c$.

Proof. (i) We assume that $\delta^2 < \delta - c$. Any pair of agents that is not directly connected can increase its utility (the net benefit for creating a link is $\delta - c - \delta^2 > 0$) and thus the total utility, by forming a link. Since every pair of agents has an incentive to form a link, we will end up in the complete graph K_n, where all possible links have been created and no additional links can be created any more.

(ii) Consider a component of the graph G containing m agents, say G'. The number of links in the component G' is denoted by k, where $k \geq m-1$, otherwise the component would not be connected. E.g. a path containing all agents would have $m-1$ links. The total utility of the direct links in the component is given by $k(s\delta - 2c)$. There are at most $\frac{m(m-1)}{2} - k$ left over links in the component, that are not created yet. The utility of each of these left over links is at most $2\delta^2$ (it has the highest utility if it is in the second order neighborhood). Therefor the total utility of the component is at most

$$k2(\delta - c) + \left(\frac{m(m-1)}{2} - k\right) 2\delta^2. \tag{73}$$

Consider a star $K_{1,m-1}$ with m agents. The star has $m-1$ agents which are not in the center of the star. An example of a star with 4 agents is given in Figure (12). The utility of any direct link is $2\delta - 2c$ and of any indirect link $(m-2)\delta^2$, since

[8] Here only the shortest paths are taken into account.

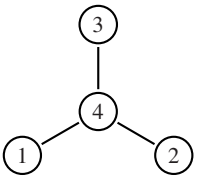


Fig. 12 A star encompassing 4 agents

any agent is 2 links away from any other agent (except the center of the star). Thus the total utility of the star is

$$\underbrace{(m-1)(2\delta-2c)}_{\text{direct connections}}+\underbrace{(m-1)(m-2)\delta^2}_{\text{indirect connections}}. \tag{74}$$

The difference in total utility of the (general) component and the star is just $2(k-(m-1))(\delta-c-\delta^2)$. This is at most 0, since $k \geq m-1$ and $c > \delta-\delta^2$, and less than 0 if $k > m-1$. Thus, the value of the component can equal the value of the star only if $k = m-1$. Any graph with $k = m-1$ edges, which is not a star, must have an indirect connection with a distance longer than 2, and getting a total utility less than $2\delta^2$. Therefore the total utility from indirect connections of the indirect links will be below $(m-1)(m-2)\delta^2$ (which is the total utility from indirect connections of the star). If $c < \delta-\delta^2$, then any component of a strongly efficient network must be a star.

Similarly it can be shown [56] that a single star of $m+n$ agents has a higher total utility than two separate stars with m and n agents. Accordingly, if an efficient network is non-empty, it must be a star.

(iii) A star encompassing every agent has a positive value only if $\delta+\frac{n-2}{2}\delta^2 > c$. This is an upper bound for the total achievable utility of any component of the network. Thus, if $\delta+\frac{n-2}{2}\delta^2 < c$ the empty graph is the unique strongly efficient network. □

Moreover, Jackson and Wolinsky [56] also determine the stable networks in the Connections Model.

Proposition 2. *Consider the Connections Model in which the utility of each agent is given by Equation (72).*

(i) A pairwise stable network has at most one (non-empty) component.
(ii) For $c < \delta-\delta^2$, the unique pairwise stable network is the complete graph K_n.
(iii) For $\delta-\delta^2 < c < \delta$ a star encompassing every agent is pairwise stable, but not necessarily the unique pairwise stable graph.
(iv) For $\delta < c$, any pairwise stable network that is non-empty is such that each agent has at least two links (and thus is inefficient).

Proof. (i) Lets assume, for the sake of contradiction, that G is pairwise stable and has more than one non-empty component. Let u^{ij} denote the utility of agent i

having a link with agent j. Then, $u^{ij} = u_i(G+e_{ij}) - u_i(G)$ if $e_{ij} \notin E(G)$ and $u^{ij} = u_i(G) - u_i(G-e_{ij})$ if $e_{ij} \in E(G)$. We consider now $e_{ij} \in E(G)$. Then $u^{ij} \geq 0$. Let e_{kl} belong to a different component. Since i is already in a component with j, but k is not, it follows that $u^{jk} > u^{ij} \geq 0$, because agent k will receive an additional utility of δ^2 from being indirectly connected to agent i. For similar reasons $u^{jk} > u^{lk} \geq 0$. This means that both agents in the separate component would have an incentive to form a link. This is a contradiction to the assumption of pairwise stability.

(ii) The net change in utility from creating a link is $\delta - \delta^2 - c$. Before creating the link, the geodesic distance between agent i and agent j is at least 2. When they create a link, they gain δ but they lose the previous utility from being indirectly connected by some path whose length is at least 2. So if $c < \delta - \delta^2$, the net gain from creating a link is always positive. Since any link creation is beneficial (increases the agents' utility), the only pairwise stable network is the complete graph, K_n.

(iii) We assume that $\delta - \delta^2 < c - \delta$ and show that the star is pairwise stable. The agent in the center of the star has a distance of 1 to all other agents and all other agents are separated by 2 links from each other. The center agent of the star cannot create a link, since she has already maximum degree. She has no incentive to delete a link either. If she deletes a link, the net gain is $c - \delta$, since there is no path leading to the then disconnected agent. By assumption, $\delta - \delta^2 < c < \delta$, $c - \delta < 0$ and the gain is negative, and the link will not be removed. We consider now an agent that is not the center of the star. She cannot create a link with the center, since they are both already connected. The net gain of creating a link to another agent is $\delta - \delta^2 - c$, which is strictly negative by assumption. So she will not create a link either. The star is pairwise stable.
Now consider the star encompassing all agents. Suppose an agent would not be connected to the star. If the center of the star would create a link to this isolated agent, the net gain would be $\delta - c > 0$ and the benefit of the isolated agent is again $\delta - c > 0$. So both will create the link.
The star is not the unique pairwise stable network. We will show that for 4 agents, the cycle, C_4 is also a pairwise stable network (see Figure (13)).
If agent 3 removes a link to agent 4, then her net gain is $c - \delta - \delta^3$. For the range of costs of $\delta - \delta^2 < c < \delta - \delta^3 < \delta$, she will never do it. If agent 3 adds a link to agent 1, Figure (13), the net gain is $\delta - \delta^2 < 0$. Thus, for $n = 4$ and

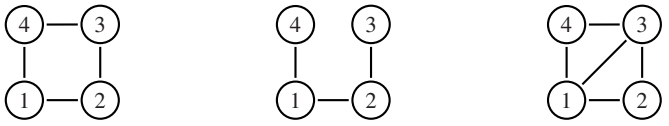


Fig. 13 A cycle of 4 agents (left) and the resulting graph after the deletion of a link from agent 3 to agent 4 (middle) and and the resulting graph after the creation of a link from agent 3 to agent 1 (right)

$\delta - \delta^2 < c < \delta - \delta^3$, then there are at least two pairwise stable networks: the star and the cycle.

(iv) For $\delta < c$ the star is not a pairwise stable network because the agent in the center of the star would gain $c - \delta$ from deleting a link. Moreover, it can be shown [56] that any connected agent has at least 2 links. □

One can see, from the two propositions described above, that a pairwise stable network is not necessarily efficient. For high cost $c > \delta$ there are non-empty pairwise stable networks but they are not efficient. Moreover, Watts [88] shows that if the benefit from maintaining an indirect link of length two is greater than the net benefit from maintaining a direct link ($\delta^2 > \delta - c > 0$) then the probability that the unique efficient network, the star $K_{1,n-1}$, is reached vanishes for large n.

The existence of inefficient equilibria is of interest because it indicates that the system, let alone to evolve, does not always reach an efficient configuration. In this respect, the result is important from the point of view of designing of policies that help the system to reach an efficient configuration.

Finally, we note that Jackson and Rogers [54] have proposed an extension of the Connections Model in which stable networks show the properties of a "Small-World" (see Section 5.3).

7.3 A Model of Dynamic Innovation Networks

We now briefly discuss a recent model of dynamic innovation networks in which agents compete for the most valuable knowledge for production, while knowledge can only be created through collaborations and knowledge exchange [62, 63]. By knowledge exchange we mean R&D partnerships (either informal or formal), characterized by bilateral interactions among agents. We characterize the emerging network topologies in terms of their efficiency (total knowledge growth) and in terms of the individual agents' knowledge growth.

Consider a set of agents, $N = \{1, ..., n\}$, represented as nodes of an undirected graph G, with an associated variable x_i representing the knowledge of agent i. A link e_{ij}, represents the transfer of knowledge between agent i and agent j. Knowledge is shared among an individual's direct and indirect acquaintances. The level of knowledge an agent holds is proportional to the knowledge levels of its neighbors. We assume that knowledge $\mathbf{x} = (x_1, ..., x_n)$ grows, starting from positive values, $x_i(0) > 0 \; \forall i \in N$, according to the following linear ordinary differential equation

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j, \tag{75}$$

where $a_{ij} \in \{0, 1\}$ are the elements of the adjacency matrix $\mathbf{A}$ of the graph G. In vector notation we have $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. Similar to Carayol and Roux [17, 18] we assume that the gross return of agent i is proportional to her knowledge growth rate, with

proportionality constant set to 1 for sake of simplicity[9]. We also assume that maintaining a link induces a constant marginal cost $c \geq 0$ for both agents connected via the link. Therefore the utility $u_i(t)$ of agent i is given by

$$u_i(t) = \frac{\dot{x}_i(t)}{x_i(t)} - cd_i, \tag{76}$$

where d_i denotes the degree of agent i. From Equation (75) one can show that [50] $\lim_{t\to\infty} \frac{\dot{x}_i(t)}{x_i(t)} = \lambda_{\mathrm{PF}}(G_i)$ where λ_{PF} is the largest real eigenvalue (Section 4.1) of the connected component G_i to which agent i belongs. Therefore, for large times t the utility function of agent i in a network G is given by

$$u_i = \lambda_{\mathrm{PF}}(G_i) - cd_i. \tag{77}$$

We can compare the utility functions of the Connections Model introduced in Equation (72) and the utility function from Equation (77). In both the utility depends on the position of an agent in the network. In Equation (77) and Equation (72) the cost term is identical. However, while the utility function in the Connections Model considers the length d_{ij} of the shortest path from node i to node j, the utility function in Equation (77) takes into account all paths of all lengths (in fact, λ_{PF} is proportional to the asymptotic growth rate of walks in a graph [25]). Indeed, it has been argued that knowledge gets transferred not only along the shortest path but also along all other paths in a network [78, 87]. Accordingly, all agents to which agent i is indirectly connected to along path of any length, contribute to the utility of agent i in this model.

Based on the definition of efficiency we can derive the efficient networks for certain values of the marginal cost. For a full characterization of efficient networks in this model we refer to König et al. [63].

Proposition 3. *The complete graph K_n is efficient for $c < \frac{1}{2}$. For costs $c \geq n$ the empty graph is efficient.*

Proof. Since for the complete graph it is $\lambda_{\mathrm{PF}} = n-1$ and $m = \frac{n(n-1)}{2}$, its aggregate utility is $U(K_n) = n(n-1) - 2\frac{n(n-1)}{2}c = n(n-1)(1-c)$.

On the other hand, the largest real eigenvalue λ_{PF} of a graph G with m edges is bounded from above so that $\lambda_{\mathrm{PF}} \leq \frac{1}{2}(\sqrt{8m+1}-1)$ [77]. For the aggregate utility of the network we then have

$$\begin{aligned} U(G) &= \sum_{i=1}^{n} \lambda_{\mathrm{PF}}(G_i) - 2mc \leq n \max_{1\leq i\leq n} \lambda_{\mathrm{PF}}(G_i) - 2mc \\ &\leq \frac{n}{2}(\sqrt{8m+1}-1) - 2cm := b(n,m,c), \end{aligned} \tag{78}$$

[9] The detailed derivation of the relation between an agents profits and her knowledge growth rate can be found in König et al. [63].

with $n \leq m \leq \binom{n}{2}$. For fixed cost c and number of nodes n, the number of edges maximizing Equation (78) is given by $m^* = \frac{n^2-c^2}{8c^2}$ if $\frac{n^2-c^2}{8c^2} < \binom{n}{2}$ and $m^* = \frac{n(n-1)}{2}$ if $\frac{n^2-c^2}{8c^2} > \binom{n}{2}$. The graph with the latter number of edges is the complete graph. Inserting m^* into Equation (78) yields

$$b(n,m^*,c) = \begin{cases} \frac{n}{2}\left(\sqrt{\frac{n^2-c^2}{c^2}+1}-1\right) - \frac{n^2-c^2}{4c} & c > \frac{n}{2n-1} \\ n(n-1)(1-c) = \Pi(K_n) & c < \frac{n}{2n-1}. \end{cases} \tag{79}$$

The bound for $c \leq \frac{n}{2n-1} \sim \frac{1}{2}$ coincides with the aggregate utility of the complete graph K_n for large n. K_n is therefore the efficient graph. If instead $c = n$ then $m^* = 0$. The efficient graph is the empty graph. This concludes the proof. □

In the following we make an important assumption. The network evolution process is assumed to be much slower than the knowledge growth of Equation (75), so that agents make decisions based on the asymptotic growth rate of knowledge and the utility function in Equation (77). Thus, we introduce a coupling of fast knowledge growth coupled with a slow network evolution, as illustrated in Figure (4.4) in Section 4.4.

Given the utility function in Equation (77) and the network evolution introduced in Section 7.1 one can show that the network evolution can lead to pairwise stable networks [63]. However, there exists a multiplicity of different equilibria, as the next proposition reveals (for the proof see König et al. [63]).

Proposition 4. *Consider costs c, $c' = \alpha c$, $\alpha \in [0,1]$ and the network G with n nodes such that*[10] $\lceil \frac{2}{c} \rceil \leq n \leq \lfloor \frac{1+c'^2(6+c'^2)}{4c'^2} \rfloor$. *If there exists an integer $k \leq n$, $\mod(n,k) = 0$ such that $\lceil \frac{1+c(1-c)}{c} \rceil \leq k \leq \lfloor \frac{2-c'(1-c')}{c'} \rfloor$ then G can be stable for at least two cases:*

(i) G consists of disconnected cliques $K_k^1, \ldots, K_k^d$, $n = kd$ or
(ii) G consists of a spanning star $K_{1,n-1}$.

There are at least two stable networks for the same level of marginal cost c (degenerate cost region).

The variety of the possible equilibria is not only restricted to cliques and stars of different sizes but it also includes networks with complex topologies which are characteristic of many real word networks. Examples of such equilibrium networks can be seen in Figure (14). Differently to the Connections Model this model is able to reproduce some stylized empirical facts of R&D networks, namely that networks are sparse, locally dense and show heterogeneous degree distributions [23, 72].

We have seen that in this model there exist multiple equilibrium networks, some of them being inefficient. Depending on the cost of interactions the system can get stuck in stable but inefficient structures. However, in König et al. [62] it is shown

[10] In the following, $\lceil x \rceil$, where x is a real valued number $x \in \mathbb{R}$, denotes the smallest integer larger or equal than x (the ceiling of x). Similarly, $\lfloor x \rfloor$ the largest integer smaller or equal than x (the floor of x).

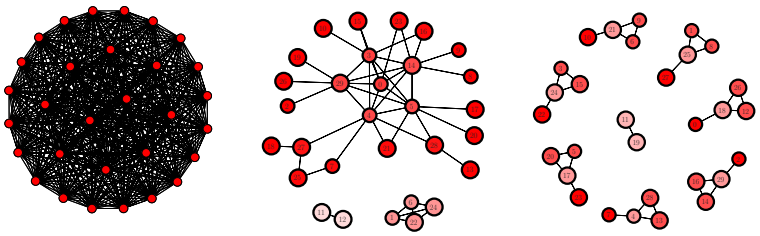


Fig. 14 Equilibrium networks for vanishing cost $c = 0$ (left), intermediate cost $c = 0.2$ (middle) and high cost $c = 0.5$ (right) for $n = 50$ agents (without link removal in the network formation process) adopted from König et al. [62]. Clearly, the higher the cost c, the sparser and the more clustered are the equilibrium networks

that if it is difficult to break up already existing collaborations and agents maintain R&D collaborations even when, in the short run, they may be unprofitable, then emerging networks are efficient.

7.4 Summary and Conclusion

In this tutorial we have given an introduction to models of economic networks and we have tried to show the wide applicability and importance of these models. Since the field of economic network theory is growing at an increasing pace, this tutorial is far from being exhaustive. However, the models presented here can serve as a starting point for interested students and prospective researchers in the field.

References

1. Barrat, A. and Weigt, M. (2000). On the properties of small-world network models. *The European Physical Journal B*, (13):547–560.
2. Batagelj, V. and Brandes, U. (2005). Efficient generation of large random networks. *Physcal Review E*, 71.
3. Battiston, S., Bonabeau, E., and Weisbuch, G. (2003). Decision making dynamics in corporate boards. *Physica A: Statistical Mechanics and its Applications*, 322:567–582.
4. Battiston, S. and Catanzaro, M. (2004). Statistical properties of corporate board and director networks. *The European Physical Journal B-Condensed Matter*, 38(2):345–352.
5. Battiston, S., Rodrigues, J. F., and Zeytinoglu, H. (2007). The network of inter-regional direct investment stocks across europe. *Advances in Complex Systems (ACS)*, 10:29–51.
6. Baumann, N. and Stiller, S. (2005). *Network Analysis: Methodological Foundations*, chapter Network Models. Springer.
7. Bollobas, B. (1985). *Random Graphs*. Cambridge University Press, 2nd edition.
8. Bollobas, B. (1998). *Modern Graph Theory*. Graduate Texts in Mathematics. Springer.
9. Borgatti, S. (2005). Centrality and network flow. *Social Networks*, 27(1):55–71.

10. Borgatti, S. and Everett, R. (2006). A graph-theoretic perspective on centrality. *Social Networks*.
11. Bowles, S. and Gintis, H. (November 2002). Social capital and community governance. *The Economic Journal*, 112:419–436(18).
12. Boyd, S. (2006). Linear dynamical systems. Lecture Notes, Stanford University.
13. Brioschi, F., Buzzacchi, L., and Colombo, M. (1989). Risk capital financing and the separation of ownership and control in business groups. *Journal of Banking and Finance*, 13(1):747 – 772.
14. Brown, J. S. and Duguid, P. (1991). Organizational learning and communities-of-practice: Toward a unified view of working, learning, and innovation. *Organization Science*, 2(1):40–57. Special Issue: Organizational Learning: Papers in Honor of (and by) James G. March.
15. Calvó-Armengol, A. (2007). Networks in economics. lecture notes.
16. Calvó-Armengol, A. and Martí, J. d. (2007). On optimal communication networks. Working paper, Universitat Autonoma de Barcelona.
17. Carayol, N. and Roux, P. (2003). Self-organizing innovation networks: When do small worlds emerge? Working Papers of GRES - Cahiers du GRES 2003-8, Groupement de Recherches Economiques et Sociales.
18. Carayol, N. and Roux, P. (2005). *Nonlinear Dynamics and Heterogeneous Interacting Agents*, chapter Collective Innovation in a Model of Network Formation with Preferential Meeting, pages 139–153. Springer.
19. Chapelle, A. (2005). Separation of ownership and control: Where do we stand? *Corporate Ownership and Control*, 15(2).
20. Chung, F. and Lu, L. (2007). *Complex Graphs and Networks*. American Mathematical Society.
21. Corrado, R. and Zollo, M. (2006). Small worlds evolving: Governance reforms, privatizations, and ownership networks in italy. *Industrial and Corporate Change*, 15(2):319–352.
22. Costa, L. F., Rodrigues, F. A., Travieso, G., and Boas, P. V. (2007). Characterization of complex networks: A survey of measurements.
23. Cowan, R. (2004). Network models of innovation and knowledge diffusion. Research Memoranda 016, Maastricht : MERIT, Maastricht Economic Research Institute on Innovation and Technology.
24. Cowan, R. and Jonard, N. (2004). Network structure and the diffusion of knowledge. *Journal of Economic Dynamics and Control*, 28:1557–1575.
25. Cvetkovic, D., Doob, M., and Sachs, H. (1995). *Spectra of Graphs: Theory and Applications*. Johann Ambrosius Barth.
26. Davis, G. and Greve, H. (1997). Corporate Elite Networks and Governance Changes in the 1980s. *American Journal of Sociology*, 103(1):1–37.
27. Davis, G., Yoo, M., and Baker, W. (2003). The Small World of the American Corporate Elite, 1982-2001. *Strategic Organization*, 1(3):301.
28. Diestel, R. (2000). *Graphentheorie*. Springer, 2nd edition.
29. Dodds, P. S., Watts, D. J., and Sabel, C. F. (2003). Information exchange and robustness of organizational networks. *Working Paper Series, Center on Organizational Innovation, Columbia Univeristy*.
30. Durrett, R. (2007). *Random Graph Dynamics*. Cambridge University Press.
31. Durrett, R. A. (2004). *Probability. Theory and Examples*. Thomson Learning.
32. Farmer, J. D., Smith, E., and Shubik, M. (2005). Economics: The next physical science?
33. Flath, D. (1992). Indirect shareholding within japan's business groups. *Economics Letters*, 38(2):223–227.
34. Freeman, L. C. (1978). Centrality in social networks conceptual clarification. *Social Networks*, 1:215–239.
35. Gabszewicz, J. (2000). *Strategic Interaction and Markets*. Oxford University Press, Center for Operations Research and Econometrics (CORE), Catholic University of Louvain.
36. Gallegati, M. and Kirman, A. P., editors (1999). *Beyond the Representative Agent*. Edward Elgar Publishing.

37. Garlaschelli, D., Battiston, S., Castri, M., Servedio, V. D. P., and Caldarelli, G. (2005). The scale-free topology of market investments. *Physica*, A(350):491–499.
38. Geipel, M. (2007). Self-organization applied to dynamic network layout. *International Journal of Modern Physics C*. forthcoming.
39. Geroski, P. A. (2000). Models of technology diffusion. *Research Policy*, 29(4-5):603–625.
40. Gigerenzer, G. and Selten, R., editors (2002). *Bounded Rationality: The Adaptive Toolbox (Dahlem Workshop Reports)*. MIT Press.
41. Godsil, C. D. and Royle, G. F. (2001). *Algebraic Graph Theory*. Springer.
42. Goyal, S. (2007). *Connections: an introduction to the economics of networks*. Princeton University Press.
43. Goyal, S. and Moraga-Gonzalez, J. L. (2001). R&d networks. *RAND Journal of Economics*, 32:686–707.
44. Granovetter, M. (1995a). *Getting a Job: A Study of Contacts and Careers*. University of Chicago Press.
45. Granovetter, M. (1995b). *Industrial and Corporate Change*, chapter Coase Revisited: Business Groups in the Modern Economy. Oxford University Press.
46. Gross, T. and Blasius, G. (2007). Adaptive coevolutionary networks - a review. eprint arXiv: 0709.1858.
47. Growiec, J., Pammolli, F., Riccaboni, M., and Stanley, H. (2008). On the size distribution of business firms. *Economics Letters*, 98(2):207–212.
48. Haken, H. (1977). *Synergetics - An Introduction; Nonequilibrium Phase Transitions and Self-Organization in Physics, Chemistry and Biology*. Springer.
49. Hausman, D. M. (2003). *Inexact and Separate Science of Economics*. Cambridge University Press.
50. Horn, R. A. and Johnson, C. R. (1990). *Matrix Analysis*. Cambridge University Press.
51. Huberman, Bernardo, A. and Hogg, T. (1995). Communities of practice: Performance and evolution. *Computational and Mathematical Organization Theory*, 1(1):73–92.
52. Jackson, M. (2008). *Social and Economic Networks*. Princeton University Press.
53. Jackson, M. and Calvo-Armengol, T. (2007). Networks in labor markets: Wage and employment dynamics and inequality. *The Journal of Economic Theory*, 132(1):27–46.
54. Jackson, M. and Rogers, B. (2005). The Economics of Small Worlds. *Journal of the European Economic Association*, 3(2–3):617–627.
55. Jackson, M. O. (2006). *Advances in Economics and Econometrics: Theory and Applications, Ninth World Congress*, volume I, chapter The Economics of Social Networks. Cambridge University Press.
56. Jackson, M. O. and Wolinsky, A. (1996). A strategic model of social and economic networks. *Journal of Economic Theory*, 71(1):44–74.
57. Jain, S. and Krishna, S. (1998). Autocatalytic sets and the growth of complexity in an evolutionary model. *Physical Review Letters*, 81(25):5684–5687.
58. Jain, S. and Krishna, S. (2001). A model for the emergence of cooperation, interdependence, and structure in evolving networks. *Proceedings of the National Academy of Sciences*, 98(2):543–547.
59. Kamien, M. I., Muller, E., and Zang, I. (1992). Research joint ventures and R&D cartels. *The American Economic Review*, 82(5):1293–1306.
60. Kirman, A. (1997). The economy as an evolving network. *Journal of Evolutionary Economics*, 7(4):339–353.
61. Kogut, B. and Walker, G. (2001). The small world of germany and the durability of national networks. *American Sociological Review*, 66(3):317–335.
62. König, M. D., Battiston, S., Napoletano, M., and Schweitzer, F. (2008). On algebraic graph theory and the dynamics of innovation networks. *Networks and Heterogeneous Media*, 3(2):201–219.
63. König, M. D., Battiston, S., Napoletano, M., and Schweitzer, F. (2009). Efficiency and stability of dynamic innovation networks. forthcoming.

64. König, M. D., Battiston, S., and Schweitzer, F. (2008). *Innovation Networks - New Approaches in Modeling and Analyzing*, chapter Modeling Evolving Innovation Networks. Springer Complexity Series.
65. Lave, J. and Wenger, E. (1991). *Situated Learning: Legitimate Peripheral Participation*. Cambridge University Press.
66. Milgram, S. (1967). The small world problem. *Psychology Today*, 2(1):60–67.
67. Newman, M. (2003). The structure and function of complex networks. *SIAM review*, 45(2):167–256.
68. Newman, M. E. J. (2007). *The New Palgrave Encyclopedia of Economics*, chapter Mathematics of networks. Palgrave Macmillan, Basingstoke.
69. Newman, M. E. J., Strogatz, S. H., and Watts, D. J. (2001). Random graphs with arbitrary degree distributions and their applications. *Physcal Review E*, 64(2):026118.
70. Newman, M. E. J., Watts, D. J., and Strogatz, S. H. (2002). Random graph models of social networks. *Proceedings of the National Academy of Sciences*, 99(90001):2566–2572.
71. Nisan, N., Roughgarden, T., Tardos, E., and Vazirani, V. V., editors (2007). *Algorithmic Game Theory*. Cambridge University Press.
72. Powell, W. W., White, D. R., Koput, K. W., and Owen-Smith, J. (2005). Network dynamics and field evolution: The growth of interorganizational collaboration in the life sciences. *American Journal of Sociology*, 110:1132–1205.
73. Reichl, L. E. (2004). *A Modern Course in Statistical Physics*. Wiley-VCH, 2 edition.
74. Riccaboni, M. and Pammolli, F. (2002). On firm growth in networks. *Research Policy*, 31(8-9):1405–1416.
75. Seneta, E. (2006). *Non-negative Matrices And Markov Chains*. Springer.
76. Slikker, M. and van den Nouweland, A. (2001). *Social and Economic Networks in Cooperative Game Theory*. Springer.
77. Stanley, R. P. (1987). A bound on the spectral radius of graphs with e edges. *Linear Algebra and its Applications*, 87:267–269.
78. Stephenson, K. and Zelen, M. (1989). Rethinking centrality: Methods and examples. *Social Networks*, 11:1–37.
79. Stoneman, P. (2002). *The Economics of Technological Diffusion*. Blackwell Publishers.
80. Van Alstyne, M. (1997). The state of network organization: A survey in three frameworks. *Journal of Organizational Computing and Electronic Commerce*, 7(2):83–151.
81. Van Zandt, T. (1997). *Contemporary Economic Development Reviewed, Volume 4: The Enterprise and its Environment*, chapter Decentralized Information Processing in the Theory of Organizations, pages 125–160. MacMillan Press Ltd.
82. Varian, H. R. (1996). *Intermediate Microeconomics: A Modern Approach*. WW Norton.
83. Vega-Redondo, F. (2006). Diffusion and growth in an evolving network. *forthcoming in International Journal of Game Theory*.
84. Vega-Redondo, F. (2007). *Complex Social Networks*. Series: Econometric Society Monographs. Cambridge University Press.
85. Venkatesh, B. and Goyal, S. (1998). Learning from neighbors.
86. Veugelers, R. (October 1998). Collaboration in R&D: An assessment of theoretical and empirical findings. *De Economist*, 146:419–443(25).
87. Wasserman, S. and Faust, K. (1994). *Social Network Analysis: Methods and Applications*. Cambridge University Press.
88. Watts, A. (2001). A dynamic model of network formation. *Games and Economic Behavior*, 34(2):331–341.
89. Watts, D. J. and Strogatz, S. H. (1998). Collective dynamics of small-world networks. *Nature*, 393:440–442.
90. Weidlich, W. (2002). *Sociodynamics - a systematic approach to mathematical modelling in the social sciences*. Taylor & Francis.
91. West, Douglas, B. (2001). *Introduction to Graph Theory*. Prentice-Hall, 2nd edition.
92. Xu, J. (2003). *Theory and Application of Graphs*. Kluwer Academic Publishers.
93. Zenou, Y. (2006). Course on networks: Theory and applications. lecture notes.