Some Topics in Graph Theory

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Abstract In this short introductory course to graph theory, possibly one of the most propulsive areas of contemporary mathematics, some of the basic graph-theoretic concepts together with some open problems in this scientific field are presented.

1 Some Basic Concepts

A *simple graph* X is an ordered pair of sets $X = (V, E)$. Elements of V are called *vertices* of X and elements of E are called *edges* of X. An edge joins two vertices, called its endvertices. Formally, we can think of the elements of E as subsets of V of size 2. A simple graph is thus an undirected graph with no loops or multiple edges.

If $u \neq v$ are vertices of a simple graph X and $\{u, v\}$ (sometimes shortened to uv) is an edge of X, then this edge is said to be *incident to u and v*. Equivalently, u and v are said to be *adjacent* or *neighbors*, and we write $u \sim v$. Phrases like, *"an edge joins u and v"* and *"the edge between u and v"* are also commonly used.

Graphs can be nicely represented with diagrams consisting of dots standing for vertices and lines standing for edges (see Fig. 1).

In a simple graph there is at most one edge joining a pair of vertices. In a *multi-graph*, multiple edges are permitted between pairs of vertices. There may also be edges, called *loops*, that connect a vertex to itself (see Fig. 2).

As opposed to a simple graph where edges are undirected, a *directed graph* (in short, *digraph*) is an ordered pair of sets (V, E) where V is a set of vertices and E is

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A.K. Naimzada et al. (eds.), *Networks, Topology and Dynamics*,
Lecture Notes in Economics and Mathematical Systems 613,

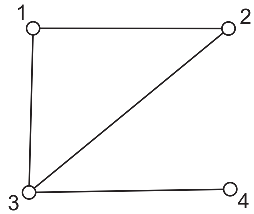

Fig. 1 A diagram of a graph $X = (V, E)$ where $V = \{1,2,3,4\}$ and $E = \{\{1,2\},\{1,3\},\{2,3\},\{3,4\}\}$

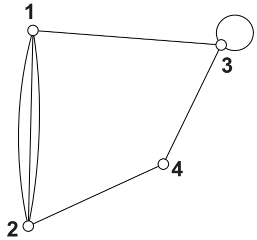

Fig. 2 An example of a multigraph with three edges between vertices 1 and 2 and loop on vertex 3

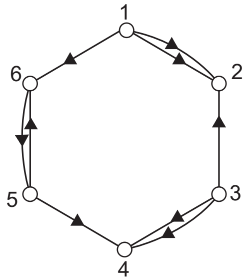

Fig. 3 A directed graph

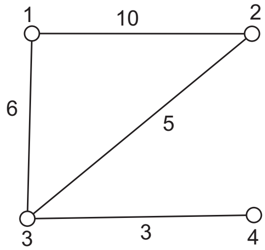

Fig. 4 A weighted graph

a subset of ordered pairs of vertices from V. Now the edges may be thought of as arrows going from a *tail (vertex)* to a *head (vertex)* (see Fig. 3).

Sometimes it is useful to associate a number, often called its *weight*, with each edge in a graph. Such graphs are called *edge-weighted* or simply *weighted graphs*; they may be simple, directed, etc. (see Fig. 4).

From now on by a graph we shall mean a simple and, unless otherwise specified, finite, undirected and connected graph. Let $X = (V, E)$ be a graph. The *degree* $d(v)$ of a vertex $v \in V$ is the number of edges with which it is incident. The set of

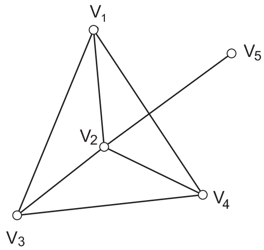

Fig. 5 $|N(v_1)| = |N(v_3)| = |N(v_4)| = 3$, $|N(v_2)| = 4$ and $|N(v_5)| = 1$

neighbors $N(v)$ of a vertex v is the set of vertices adjacent to v. Hence, $d(v) = |N(v)|$ (see Fig. 5). In the case of a directed graph $X = (V, E)$ the *indegree* and the *outdegree* of a vertex $v \in V$ is the number of edges having v as a head vertex and tail vertex, respectively. For example the vertex 5 in the graph shown in Fig. 3 has indegree 1 and outdegree 2.

When people at a party shake hands the total number of hands shaken is equal to twice the number of handshakes. Representing the party by a graph (with each person represented by a vertex and a handshake between two people represented by an edge between the corresponding vertices), the above fact, translated into graph-theoretic terminology, reads as follows.

Proposition 1 (Handshaking lemma). *The sum of degrees of all vertices in a simple graph equals twice the number of edges.*

Proof. Every edge contributes two to the sum of the degrees, one for each of its endvertices. □

As a corollary we have the following result.

Corollary 1. *In every graph, there is an even number of vertices of odd degree.*

Proof. Partitioning the vertices into those of even degree and those of odd degree, we know

$$\sum_{v \in V} d(v) = \sum_{d(v)\, is\, odd} d(v) + \sum_{d(v)\, is\, even} d(v)$$

The value of the left-handside of this equation is even, and the second summand on the right-handside is even since it is entirely a sum of even values. So the first summand on the right-handside must also be even. But since it is entirely a sum of odd values, it must contain an even number of terms. In short, there must be an even number of vertices with odd degree. □

A graph X is *d-regular* if all vertices have the same degree d. A 3-regular simple graph is usually called a *cubic graph*.

Corollary 2. *Every cubic graph has an even number of vertices.*

A *subgraph* of a graph X is a graph having all of its vertices and edges in X (see Fig. 6). A *spanning subgraph* of X is a subgraph containing all vertices of X.

The *complement* $\overline{X}$ of a graph X has $V(X)$ as its vertex set, and two vertices are adjacent in $\overline{X}$ if and only if they are not adjacent in X (see Fig. 7).

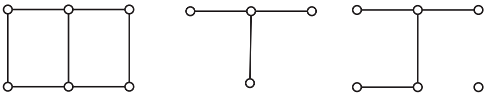

Fig. 6 A graph and two of its subgraphs

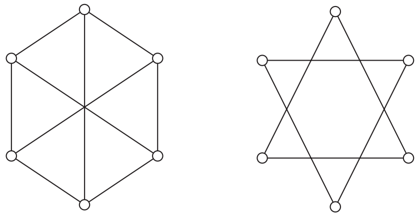

Fig. 7 A graph and its complement

Exercise 1. (a) Rewrite the proof of Corollary 1 more carefully as an inductive proof on the number of edges in a simple graph. (b) Extend Corollary 1 to multigraphs. (c) Extend Corollary 1 to digraphs.

Exercise 2. Show that a connected graph with n vertices and degree d, where $2 \leq d \leq (n-1)/2$, contains an induced 3-path. *(An induced subgraph is a subset of the vertices of a graph together with any edges whose endpoints are both in this subset. The m-path is a connected graph with two vertices of degree* 1*, and the other* $m-2$ *vertices of degree* 2*.)*

2 Traversability

A *walk* (or v_0v_k*-walk*) of a graph X is a sequence of vertices

$$v_0, v_1, v_2, \ldots, v_k$$

such that v_i is adjacent to v_{i+1} for every $i \in \{0,1,2,\ldots,k-1\}$. It is *closed* if $v_0 = v_k$, and it is *open* otherwise. It is a *path* if all the vertices are distinct. It is a *trail* if all the edges are distinct. A graph is *connected* if there is a path connecting any two distinct vertices, and it is *disconnected* otherwise. A *cycle* (also called a *circuit*) in a graph is a closed path $uv_1,\ldots,v_{k-1}u$ in which all u, v_i, $i \in \{1,\ldots,k-1\}$, are distinct. A *tree* is a connected graph with no cycles.

Let us give a few examples of some well known graphs. A *cycle* C_n is a connected graph with n vertices in which each vertex is of degree 2. A *complete graph* K_n is a graph with n vertices where any two vertices are adjacent. A *bipartite graph* is a graph whose vertex set V can be partitioned into two subsets Y and Y' in such a way that each edge has one end in Y and the other in Y'. Such a partition (Y,Y') is called a *bipartition* of the graph. A *complete bipartite graph* is a simple bipartite graph with bipartition (Y,Y') in which each vertex of Y is joined to each vertex of Y'. If $|Y| = m$ and $|Y'| = n$, such a graph is denoted by $K_{m,n}$ (see Fig. 8).

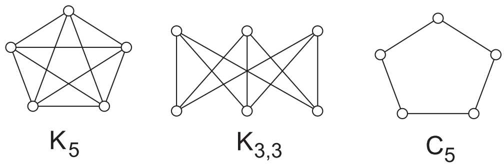

Fig. 8 Examples

Proposition 2. *For a simple graph $X = (V, E)$, the following are equivalent:*

(1) X is connected and $|E| = |V| - 1$.
(2) X is connected and acyclic (X is a tree).
(3) X is connected, but removing any edge from X leaves a disconnected graph.
(4) There is a unique simple path between any two distinct vertices of X.

2.1 Graphs and Matrices

Let X be a graph. The *adjacency matrix* of X relative to the vertex labeling $v_1, v_2, \ldots, v_n$ is the $n \times n$ matrix $A(X)$ whose entries a_{ij} are given by the rule

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases} .$$

For example, the graph K_4 has adjacency matrix

$$A(K_4) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} .$$

Note that the adjacency matrix of an undirected graph is symmetric. The following result gives one important use of powers of the adjacency matrix of a graph (see for example [12, 40]).

Proposition 3. *If A is the adjacency matrix of a graph X of order n relative to the vertex labeling $v_1, v_2, \ldots, v_n$, the (i, j)-entry of A^r represents the number of distinct r-walks from vertex v_i to vertex v_j in the graph.*

Taking the square of the matrix $A(K_4)$ above gives

$$A^2 = A(K_4)^2 = \left(\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \right)^2 = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix} .$$

The resulting matrix gives us the number of different paths using two edges between the vertices of the graph K_4.

We end this subsection with the following simple exercise.

Exercise 3. How many different paths using three edges between two vertices of the graph K_5 exist?

2.2 Eulerian Graphs

A bit of history: Koenigsberg was a city in Prussia situated on the Pregel River, which served as the residence of the dukes of Prussia in the sixteenth century. Today, the city is named Kaliningrad, and is a major industrial and commercial center of western Russia. The river Pregel flowed through the town, dividing it into four regions. In the eighteenth century, seven bridges connected the four regions. Koenigsberg people used to take long walks through town on Sundays. They wondered whether it was possible to start at one location in the town, travel across all the bridges without crossing any bridge twice and return to the starting point. This problem was first solved by the prolific Swiss mathematician Leonhard Euler, who, as a consequence of his solution invented the branch of mathematics now known as graph theory. Euler's solution consisted of representing the problem by a graph with the four regions represented by four vertices and the seven bridges by seven edges as shown in Fig. 9. Stated as a general graph theory problem, the problem is to construct a closed walk of the graph that traverses every edge exactly once. By Proposition 4 it follows that the trail of Koenigsberg bridges is not possible.

An *Eulerian trail* of an undirected graph X is a closed walk that traverses every edge of X exactly once.

Proposition 4. *If an undirected graph X has Eulerian trail, then X is connected and every vertex has even degree.*

Proof. The Eulerian trail naturally orients every edge. We enter a vertex as many times as we leave it. So every vertex must have indegree equal to outdegree. In particular, the degrees are all even. □

Proposition 5. *If X is a connected graph and every vertex has even degree, then X has an Eulerian trail.*

Proof. If X has only one vertex then it has an empty (1-vertex) Eulerian trail. So assume that in X each vertex has (even) degree at least 2. The result is clearly true if all vertices are of degree 2, as then X is a cycle. By induction, assume the result is true for all graphs with "average" degree smaller than that of X. As X is not a tree we can find a cycle C in X. Then by induction, $X - C$ has an Eulerian trail which together with C gives us the desired Eulerian trail in X. □

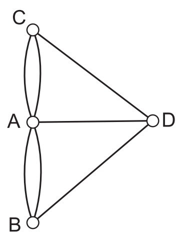

Fig. 9 Koenigsberg bridges in a graph

Exercise 4. Convince yourself that the following graphs are Eulerian: the cycle C_n for every n; the complete graph K_n if and only if n is odd; the bipartite graph $K_{n,m}$ if and only if both n and m are even.

2.3 Hamiltonian Graphs

A simple cycle that traverses every vertex exactly once is called a *Hamiltonian cycle* (*Hamiltonian circuit*). Similarly, a simple path that traverses every vertex exactly once is a *Hamiltonian path*. A *Hamiltonian graph* is a graph that possesses a Hamiltonian cycle.

Proposition 6 (Dirac theorem [31]). *Any graph with n vertices, $n \geq 3$, in which the minimum degree of each vertex is at least $n/2$, has a Hamiltonian cycle.*

Proof. Suppose by contradiction, that some non-Hamiltonian graph has n vertices, $n \geq 3$, and that the minimum degree of each vertex is at least $n/2$. If we add edges to this graph one at a time, we eventually end up with a complete graph, which does have a Hamiltonian cycle. Somewhere along this process we get a graph, call it X, that does not have a Hamiltonian cycle, but adding an edge uv yields a Hamiltonian graph, call it X'. We now get a contradiction for the graph X.

Since X' is Hamiltonian, X must have a Hamiltonian path $u = u_1, \ldots, u_n = v$ joining u and v. By definition this path includes all the vertices of X. Now let us play with this path and turn it into a cycle, and thus get a contradiction. We use the fact that u and v each have degree at least $n/2$ to produce two edges to replace an edge (u_i, u_{i+1}) on this path.

Let us count: of the $n-2$ intermediate vertices on the path $u_2, \ldots, u_{n-1}$ we know that at least $n/2$ are neighbors of u, and at least $n/2$ are neighbors of v. Consequently, by the Pigeon hole principle, there are two adjacent vertices, u_i and u_{i+1}, where u_i is a neighbor of v and u_{i+1} is a neighbor of u. Namely, let $S = \{i \mid u_{i+1} \sim u\}$ and $T = \{i \mid u_i \sim v\}$. Then each of S and T has at least $n/2$ elements. Since there are only $n-2$ possible values of i, some i must be in both sets. That is, u_i is a neighbor of v and u_{i+1} is a neighbor of u. To produce a Hamiltonian cycle in X is now an easy exercise, giving us the desired contradiction. □

To wrap up this subsection we give Hamilton-flavored exercises.

Exercise 5. Prove that the Petersen graph (the graph shown in Fig. 10) has no Hamiltonian cycle.

Exercise 6. Given any two vertices u and v of the dodecahedron, the graph shown in Fig. 11. Is there a Hamiltonian path between these two vertices? *(This problem is known as Hamilton's Icosian game proposed by W. R. Hamilton in T. P. Kirkman in 1857.)*

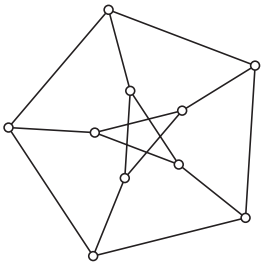

Fig. 10 The Petersen graph

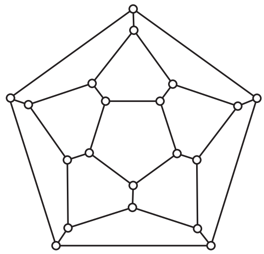

Fig. 11 The dodecahedron

Exercise 7. Prove that a graph with n vertices ($n > 3$) has a Hamiltonian cycle if for each pair of non-adjacent vertices, the sum of their degrees is n or greater. *(This problem is known as Ore theorem [62].)*

3 Factorizations, Colorings and Tournaments

A *factor* of a graph X is a spanning subgraph of X which is not totally disconnected. (A *totally disconnected* graph of order n is the complement $nK_1 = \bar{K}_n$ of the complete graph K_n.) We say that X is the *sum* of factors X_i if it is their edge-disjoint union, and such a union is called a *factorization* of X. An *n-factor* is a factor which is an n-regular graph. If X is the sum of n-factors, their union is called an *n-factorization* and X itself is *n-factorable*.

When $X = (V,E)$ has a 1-factor it is clear that $|V|$ is even and the edges of this 1-factor are vertex disjoint. In particular, K_{2n+1} cannot have a 1-factor, but K_{2n} certainly can as is shown in the proposition below.

Proposition 7. *The complete graph K_{2n} is* 1*-factorable.*

Proof. We need only display a partition of the set E of edges of K_{2n} into $(2n-1)$ 1-factors. Denote the vertices of K_{2n} by $v_1, v_2, \ldots, v_{2n}$. Define, for $i \in \{1,2,\ldots,n-1\}$, the sets of edges

$$E_i = \{v_i v_{2n}\} \cup \{v_{i-j} v_{i+j} \mid j \in \{1,2,\ldots,n-1\}\},$$

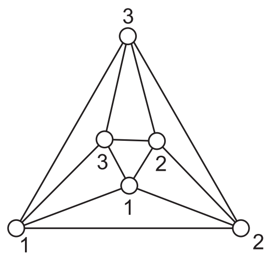

Fig. 12 Not a good scheduling

where each of the subscripts $i-j$ and $i+j$ is expressed as one of the numbers $1,2,\ldots,(2n-1)$ modulo $(2n-1)$. The collection $\{E_i\}$ is easily seen to give an appropriate partition of E, and the sum of the subgraphs X_i induced by E_i is a 1-factorization of K_{2n}. □

An example where the vertex colorings may be used is the exam scheduling problem. The Schedules Office needs to assign a time slot for each final exam. This is not easy, because some students are taking several classes with finals, and a student can take only one test during a particular time slot. The Schedules Office wants to avoid all conflicts, but wants to make the exam period as short as possible. Let each vertex represent a course for which final exams are taken. Put an edge between two vertices if there is some student taking both courses. Identify each possible time slot with a color. For example, Monday 9–12 is color 1, Monday 1–4 is color 2, Tuesday 9–12 is color 3, etc. If there is an edge between two vertices with the same color, then a conflict exam will have to be scheduled because there is a student who has to take exams for the courses represented by the vertices, but the exams are scheduled at the same time (see Fig. 12). Everyone wants to avoid conflict exams. So the registrar would like to color each vertex of the graph so that no adjacent vertices have the same color. To keep exam period as short as possible, the registrar would like to use the minimum possible number of colors.

The minimum number of colors needed to color the vertices of a graph X so that no two adjacent vertices are of the same color is called the *chromatic number* and is denoted by $\chi(X)$. For example: $\chi(C_n)=3$ if n is odd and $\chi(C_n)=2$ if n is even.

The following propositions gives the upper bound on the chromatic number.

Proposition 8. *For any graph X, $\chi(X)\leq 1+\Delta$ where Δ is the maximum degree of X.*

Proof. We use induction on the order n of the graph X. The only graph of order $n=1$ is the complete graph K_1 in which case $\Delta=1$ and $\chi(K_1)=1$. Now suppose that the result holds for all graphs of order less than or equal to $n-1$, and let X be a graph of order n and maximal degree Δ. Let Y be the graph obtained from X by deleting a vertex $v\in V(X)$ and all the edges having v as an endvertex. Since the order of Y is less than n and its maximal degree is less than or equal to Δ, by induction, $\chi(Y)\leq 1+\Delta$. Now a vertex coloring of X is obtained by coloring the vertex v with the color that is not the color of the neighboring vertices of v. Since v has at most Δ neighbors X is colored with at most $1+\Delta$ colors. □

Proposition 9 (Brook theorem [17]). *For a connected graph X different from a complete graph or an odd cycle, $\chi(X) \leq \Delta$ where Δ is the maximum degree of X.*

Now consider the task of coloring a political map. What is the minimum number of colors needed, with the obvious restriction that neighboring countries should have different colors? This is related to the so-called edge colorings of a graph.

The minimum number of colors needed to color the edges of a graph X in such a way that no two incident edges are of the same color is called the *edge chromatic number* $\bar{\chi}(X)$. By the famous Vizing theorem, given below, there are only two possibilities for the edge chromatic number of a graph.

Proposition 10 (Vizing theorem [71]). *For any graph X, $\Delta \leq \bar{\chi}(X) \leq 1+\Delta$ where Δ is the maximum degree of X.*

By Proposition 10 the edge chromatic number in a cubic graph is equal either to 3 or to 4. A *snark* is a bridgeless cubic graph with edge chromatic number 4. (A *bridge* is an edge that disconnects a graph.) The smallest snark is the Petersen graph shown in Fig. 10. A search for new snarks is an active topic of research (see [16, 41, 47, 61, 66]).

A *tournament* is an oriented complete graph. Tournaments are named so because an n-vertex tournament corresponds to a tournament in which each member of a group of n players plays all other $n-1$ players. The players are represented by vertices. For each pair of vertices an arc is drawn from the winner to the loser. The score sequence for a given tournament is obtained from the set of outdegrees sorted in nondecreasing order (see Fig. 13).

The following result holds.

Proposition 11 (Landau [42]). *The sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ of positive integers is the score sequence of a tournament if and only if*

$$d_1 + d_2 + \cdots + d_t \geq \binom{t}{2}$$

for all $1 \leq t \leq n-1$, and $d_1 + d_2 + \cdots + d_n = \binom{n}{2}$.

Exercise 8. Give all possible tournaments on five vertices.

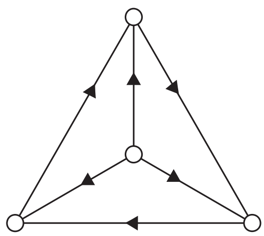

Fig. 13 An 4-vertex tournament

4 Graphs and Configurations

Suppose that in a presidential election with eight candidates a TV station wishes to organize eight TV debates with three candidates in each debate. Can candidates be arranged into eight groups of three such that no two meet more than once? This is typical problem that can be solved with the use of configurations.

An n_k *configuration* is a finite incidence structure with n points and n lines such that each line has k points and each point is on k lines. Also, two different lines intersect each other at most once and two different points are connected by at most one line. With each n_k configuration we can associate the so-called *Levi graph*. An *incidence graph* or *Levi graph* of an n_k configuration C is a bipartite k-regular graph of order $2n$, with n vertices representing the points of C and n vertices representing the lines of C, and with an edge joining two vertices if and only if the corresponding point and line are incident in C. In Figs. 14 and 15 the Moebius–Kantor configuration, the configuration solving the above election problem, and its Levi graph are shown. Since the Moebius–Kantor configuration is the only 8_3 configuration, this particular election problem has a unique solution.

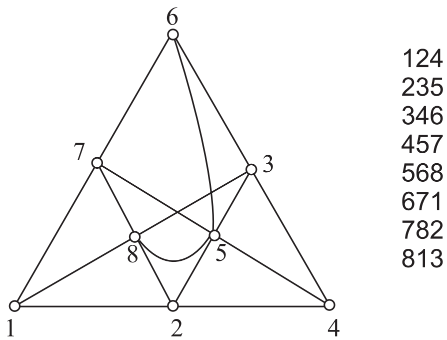

Fig. 14 The Moebius–Kantor configuration

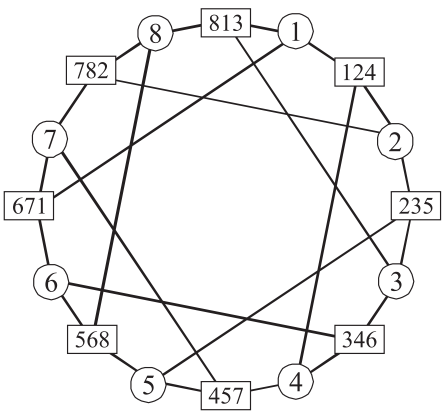

Fig. 15 The Levi graph of the Moebius–Kantor configuration

The property that two different points of a configuration are contained in at most one line implies that Levi graphs have girth (the length of the shortest cycle) at least 6. Conversely, each bipartite k-regular graph with girth at least 6 and with a chosen black-white coloring of the vertex set determines precisely one n_k configuration. If the coloring is not chosen in advance, such graphs determine a pair of dual configurations, that is, configurations with the role of their points and lines interchanged. Therefore the following result holds.

Proposition 12. *A k-regular graph of order $2n$ is a Levi graph of an n_k configuration if and only if it is bipartite and contains no 4-cycles.*

A slight modification of the above election problem may be posed: Can nine candidates be arranged into nine groups of size 3 such that no two meet more than once? The answer is again yes. But now there are three possible solutions (see Fig. 16). The upper 9_3 configuration on Fig. 16 allows no additional line, the lower allows one additional line and the middle allows three additional lines. This middle 9_3 configuration is known as the Pappus configuration. Among others this configuration solves the classical Orchard Planting Problem for $n = 9$, $k = 3$ and $r(n,k) = 10$ which is as follows: Can n trees be planted so as to produce $r(n,k)$ straight rows with k trees in each row.

More information on graphs and configurations can be found in [11, 29, 39].

Exercise 9. Show that the 8_3 configuration given in Fig. 15 is the only 8_3 configuration.

Exercise 10. Show that the three 9_3 configurations given in Fig. 16 are the only 9_3 configurations.

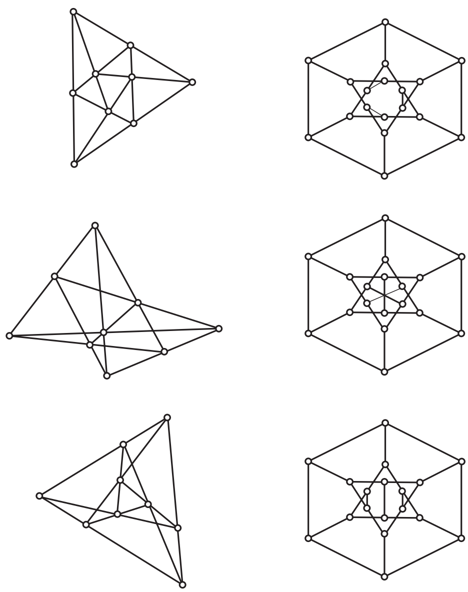

Fig. 16 The 9_3 configurations on the left handside pictures and the corresponding Levi graphs on the right handside pictures

5 Symmetries in Graphs

Digraphs $X_1 = (V_1, E_1)$ and $X_2 = (V_2, E_2)$ are *isomorphic* if and only if there is a bijection $f : V_1 \to V_2$ such that for all $u, v \in V_1$

$$(u,v) \in E_1 \Leftrightarrow (f(u), f(v)) \in E_2.$$

The bijection f is called an *isomorphism* between the graphs.

Example 1. Let X_1 be a simple graph whose vertices are integers $1, \ldots, 2n$ with an edge between two vertices if and only if they are of the same parity. Let X_2 be a simple graph whose vertices are integers $-1, \ldots, -2n$ with an edge between two vertices if and only if either both vertices are less than $-n$ or both are more than or equal to $-n$. Then the function $f : \{-1, \ldots, -2n\} \to \{1, \ldots, 2n\}$, where $f(k) = -2k$ if $-n \leq k \leq -1$ and $f(k) = 2(2n+k)+1$ if $-2n \leq k < -n$, is an isomorphism between X_1 and X_2.

A nonempty set G together with a binary operation $\cdot$ is a *group* if the following four axioms are satisfied: (1)*(Closure)* For all $g_1, g_2 \in G$, $g_1 \cdot g_2 \in G$. (2) *(Associativity)* For all $g_1, g_2, g_3 \in G$, $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$. (3) *(Identity)* There is an element e in G such that $e \cdot g = g \cdot e = g$ for all $g \in G$. (4) *(Inversion)* For each $g \in G$, there is an element denoted g^{-1} such that $g^{-1} \cdot g = g \cdot g^{-1} = e$.

An *automorphism* of a graph X is an isomorphism of X with itself. Thus each automorphism α of X is a permutation of the vertex set V which preserves adjacency. The set of automorphisms of a graph X is a group, called the *automorphism group* of X and denoted by $\mathrm{Aut}(X)$. For example, the automorphism group of the Petersen graph is isomorphic to the symmetric group $\mathrm{Aut}(GP(5,2)) = S_5$

Proposition 13. *A graph and its complement have the same automorphism group* $\mathrm{Aut}(X) = \mathrm{Aut}(\overline{X})$.

If G is a group and Ω is a set, then a *(right) group action* of G on Ω is a binary function $\Omega \times G \to \Omega$ with notation $(\omega, g) \mapsto \omega^g$ which satisfies the following two axioms:

(1) $\omega^1 = \omega$ for every $\omega \in \Omega$ (where 1 denotes the identity element of G).
(2) $(\omega^g)^h = \omega^{gh}$ for all $g, h \in G$ and $\omega \in \Omega$.

In a similar way we can define *(left) group action*.

For a group G acting on the set Ω the set

$$Orb_G(\omega) = \omega^G = \{\omega^g \mid g \in G\},$$

where $\omega \in \Omega$, is called a *G-orbit* (in short an *orbit* if the group G is clear from the contest) of the element ω with respect to the action of G. If the group orbit $Orb_G(\omega)$ is equal to the entire set Ω for some element ω in Ω, then G is *transitive*. For $\omega \in \Omega$ the set $G_\omega = \{g \in G \mid \omega^g = \omega\}$, the *stabilizer of the element* ω, is a subgroup of G. If $|G_\omega| = 1$ for every element $\omega \in \Omega$ then we say that G acts *semiregularly*. If

G acts on Ω transitively and $|G\omega| = 1$ for every element $\omega \in \Omega$ we say that G acts *regularly* (G is *regular*). For a more detail discussion on group actions we refer the reader to [64, 72].

A graph is said to be *vertex-transitive*, *edge-transitive*, and *arc-transitive* (also called *symmetric*) if its automorphism group acts vertex-transitively, edge-transitively and arc-transitively, respectively. Given a group G and a subset S of $G \setminus \{1\}$, the *Cayley graph* $X = Cay(G,S)$ has vertex set G and edges of the form $\{g, gs\}$ for all $g \in G$ and $s \in S$. Every Cayley graph is vertex-transitive but there exist vertex-transitive graphs that are not Cayley. Sabidussi [65] characterized Cayley graph in the following way: A graph is a Cayley graph of a group G if and only if its automorphism group contains a regular subgroup isomorphic to G.

Example 2. The graph shown in Fig. 5 is neither vertex-transitive nor edge-transitive. Graphs shown in Fig. 8 are all vertex-transitive, arc-transitive and Cayley. The Petersen graph and the dodecahedron (graphs shown in Figs. 10 and 11) are vertex-transitive and arc-transitive but they are not Cayley graphs.

Example 3. Let $n \geq 3$ be a positive integer, and let $k \in \{1, \ldots, n-1\} \setminus \{n/2\}$. The *generalized Petersen graph* $GP(n,k)$ is defined to have the following vertex set and edge set:

$$V(GP(n,k)) = \{u_i \mid i \in \mathbb{Z}_n\} \cup \{v_i \mid i \in \mathbb{Z}_n\},$$
$$E(GP(n,k)) = \{u_i u_{i+1} \mid i \in \mathbb{Z}_n\} \cup \{v_i v_{i+k} \mid i \in \mathbb{Z}_n\} \cup \{u_i v_i \mid i \in \mathbb{Z}_n\}.$$

There are infinitely many vertex-transitive but only 7 arc-transitive generalized Petersen graphs: $GP(4,1)$, $GP(5,2)$, $GP(8,3)$, $GP(10,2)$, $GP(10,3)$, $GP(12,5)$, $GP(24,5)$.

Clearly, a graph that is arc-transitive is also vertex-transitive and edge-transitive. But the converse is not true in general. In 1966 Tutte proved that the converse is true for graphs of odd valency.

Proposition 14 (Tutte [70]). *A vertex-transitive and edge-transitive graph of valence k, where k is odd, is also arc-transitive.*

Corollary 3. *Every cubic vertex-transitive and edge-transitive graph is arc-transitive.*

There exist graphs that are vertex-transitive and edge-transitive but not arc-transitive. These graphs are called *half-arc-transitive graphs*. By Proposition 14 the valency of such a graph is necessarily even. Moreover, in 1970 Bouwer [14] proved the existence of half-arc-transitive graphs of any given even valency. The smallest half-arc-transitive graph is the Doyle–Holt graph, the graph of order 27 shown in Fig. 17. The proof of the next proposition is straightforward.

Proposition 15. *Every graph that is edge-transitive but not vertex-transitive is bipartite.*

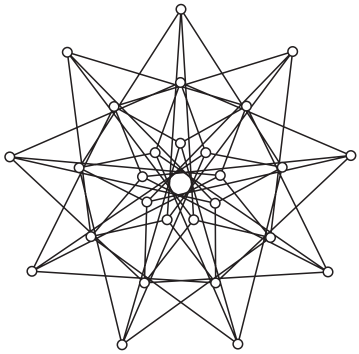

Fig. 17 The Doyle–Holt graph

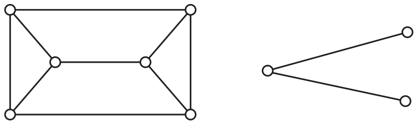

Fig. 18 The smallest graph that is vertex-transitive but not edge-transitive (the 3-prism $K_3\square K_2$) and the smallest graph that is edge-transitive but not vertex-transitive (the star $K_{1,2}$)

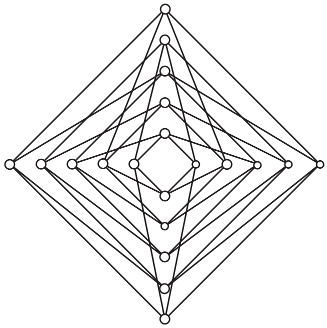

Fig. 19 The Folkman graph

Examples of edge-transitive but not vertex-transitive graphs are non-regular graphs $K_{m,n}$, $n \neq m$ (see also Fig. 18). A regular edge-transitive but not vertex-transitive graph is called *semisymmetric*. Research on these graphs has initiated and the foundations of the theory laid out by Jon Folkman [36]. The smallest such a graph has 20 vertices and is now known as the Folkman graph (see Fig. 19). The smallest cubic semisymmetric graph is the Gray graph shown in Fig. 20 (see also [24]).

Additional information on symmetries in graphs can be found in [38]. To wrap up this section we give the following two exercises.

Exercise 11. Prove that the Petersen graph is the smallest vertex-transitive graph that is not a Cayley graph.

Exercise 12. Prove Proposition 15.

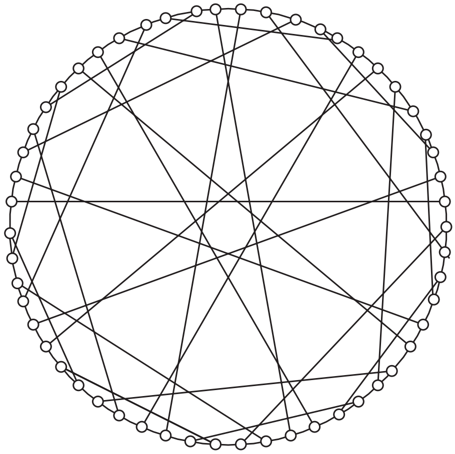

Fig. 20 The Gray graph

6 Some Well-Known Open Problems

The following unsolved problem proposed by Lovász can be thought of as the main motivation for the study of vertex-transitive graphs.

Open problem 1 (Lovász [46]) *Does every connected vertex-transitive graph have a Hamiltonian path?*

With the exception of K_2, only four connected vertex-transitive graphs that do not have a Hamiltonian cycle are known to exist. These four graphs are the Petersen graph, the Coxeter graph and the two graphs obtained from them by replacing each vertex by a triangle. The fact that none of these four graphs is a Cayley graph has led to a folklore conjecture that every Cayley graph is hamiltonian (see [7, 9, 10, 30, 37, 52] for the current status of this conjecture). Coming back to vertex-transitive graphs, it was shown in [32] that, with the exception of the Petersen graph, a connected vertex-transitive graph whose automorphism group contains a transitive subgroup with a cyclic commutator subgroup of prime-power order, is hamiltonian. Furthermore, it has been shown that connected vertex-transitive graphs of orders p, $2p$ (except for the Petersen graph), $3p$, $4p$, p^2, p^3, p^4 and $2p^2$, where p is a prime, are hamiltonian (see [1, 19, 44, 53–55, 68]). On the other hand, connected vertex-transitive graphs of orders $5p$ and $6p$ are only known to have Hamiltonian paths (see [45, 59]). Some other partial results related to this problem are known (see [2–6, 8]).

The following open problem was posed in [18, 51].

Open problem 2 ([18, 51]) *Does every vertex-transitive graph have a semiregular automorphism?*

An element of a permutation group is *semiregular*, more precisely (m,n) – semiregular, if it has m orbits of size n. It is known that every cubic vertex-transitive graph has a semiregular automorphism [60]. Recently this result have been extended to quartic vertex-transitive graphs [33], but the problem of existence of semiregular automorphisms in vertex-transitive graphs [18, 51] is still open for larger valencies. However, some results with restriction to certain orders of the graphs in question are also known (see [51]). As for arc-transitive graphs of valency pq, where p and q are primes, recently Xu [73] proved existence of semiregular automorphisms in the case when their automorphism groups have a nonabelian minimal normal subgroup with at least three orbits on the vertex set of the graph.

Open problem 3 *Classification and structural results for cubic symmetric graphs of different transitivity degrees ($s = 1, 2, 3, 4$ or 5).*

In 1947 Tutte [69] proved that in a cubic symmetric graph the order of a vertex stabilizer is 3×2^s where $s \leq 4$. All cubic symmetric graphs on up to $2,048$ vertices and some partial results related to the above problem are known (see [15, 20–22, 26–28]).

Open problem 4 *Classification and structural results for quartic half-arc-transitive graphs.*

The list of all quartic half-arc-transitive graphs on up to 500 vertices is close to being completed. Although quartic half-arc-transitive graphs are an active topic of research these days the above problem is still open (see [25, 34, 43, 48, 49, 56, 58, 67]).

Open problem 5 *Classification and structural results for semisymmetric graphs.*

The list of all cubic semisymmetric graphs on up to 768 vertices in given in [24]. There exist 43 cubic semisymmetric graphs on up to 768 vertices, 21 with solvable automorphism group and 22 with nonsolvable automorphism group. Partial results on the above problem were proven in [13, 23, 35, 50, 57, 63].

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