5

The Time Arrow of Spacetime Geometry

In the framework of general relativity, gravity is a consequence of spacetime curvature. Its dynamical laws (Einstein's field equations) are again symmetric under time reversal. However, if their *actual* global solution, that is, the observed spacetime, is asymmetric (such as a forever expanding universe), this must affect the dynamics of all matter. While this was well known, it came as a surprise during the early 1970s that strongly gravitating systems possess thermodynamical properties, thus indicating an intimate connection between two seemingly very different fields of physics.

Gravitating systems are already thermodynamically peculiar in Newton's theory, since they possess negative heat capacity, resulting from the universal attractivity of this force. In particular, attractive forces which depend homogeneously on the minus second power of distance, such as gravity and Coulomb forces, lead according to the virial theorem to the relation

$$\overline{E_{\mathrm{kin}}} = -\frac{1}{2}\overline{E_{\mathrm{pot}}} = -E \;, \tag{5.1}$$

between the mean values of kinetic and potential energies, and therefore between them and the total energy. This virial theorem is valid for mean values over a (quasi-)period of the motion, or approximately (in the case of semi-stable states) for mean values defined over sufficiently large intervals of time. In quantum theory, mean values have to be replaced by expectation values on proper (normalizable) energy eigenstates. The theorem can then be conveniently proved using Fock's *ansatz* $\psi(\lambda\boldsymbol{r}_1, \ldots, \lambda\boldsymbol{r}_N)$ and the homogeneity of T and V in a variational procedure, $\delta\big(\langle\psi|T+V|\psi\rangle/\langle\psi|\psi\rangle\big) = 0$, with respect to λ. So it must also hold for expectation values on density matrices whose non-diagonal elements can be neglected in the energy basis. (For relativistic generalizations of the virial theorem see Gourgoulkon and Bonazzola 1994.)

The anti-intuitive negative sign relating kinetic and total energy in (5.1) means, for example, that satellites are *accelerated* by friction when they enter the earth's atmosphere, and that stars *heat up* by radiating energy away. This second example is valid only as far as the quantum mechanical zero-point energy does not dominate $\overline{E_{\text{kin}}} = \text{Trace}\{\rho T\}$ – as it would in white dwarf stars or solid bodies. Early astrophysicists believed instead that stars always cool down in the course of time. The virial theorem also means that the heat flow from hot to cold objects which are governed by gravity causes a thermal inhomogeneity to *grow*.

To construct an example, first consider a monatomic ideal gas in two vessels under different conditions, but under exchange of energy (heat), $\delta U_1 = -\delta U_2$, and particles, $\delta N_1 = -\delta N_2$. Their partial entropies according to (3.14) are given by

$$S_i = kN_i\left(\frac{3}{2}\ln T_i - \ln\rho_i + C\right) , \tag{5.2}$$

with $i = 1, 2$ distinguishing the two vessels. Since the internal energy, $U = \overline{E_{\text{kin}}}$, is here $U = (3/2)NkT$, the total change of entropy becomes for fixed volumes V_i, or for fixed densities $\rho_i = N_i/V_i$,

$$\delta S_{\text{total}} = \delta S_1 + \delta S_2 = \left(\frac{1}{T_1} - \frac{1}{T_2}\right)\delta U_1 + k\left(\frac{3}{2}\ln\frac{T_1}{T_2} - \ln\frac{\rho_1}{\rho_2}\right)\delta N_1 . \tag{5.3}$$

This expression describes entropy changes δS_1 and δS_2 with opposite signs, which cancel only in thermodynamical equilibrium ($T_1 = T_2$ and $\rho_1 = \rho_2$). In this situation without gravity, an entropy increase in accordance with the Second Law requires a *reduction* of thermal and density inhomogeneities (except for the transient *thermo-mechanical effect*, that is, a thermally induced pressure difference that is caused by the temperature dependence of the second term).

However, the density of a gravitating star is not a free variable that can be kept fixed (as in the laboratory). A typical star, assumed for simplicity to be in thermal equilibrium, may to a very good approximation also be described as an ideal gas. Its temperature and volume are then related by means of the virial theorem according to

$$NT \propto U = \overline{E_{\text{kin}}} \propto -\overline{E_{\text{pot}}} \propto \frac{N^2}{R} \propto \frac{N^2}{V^{1/3}} , \tag{5.4}$$

that is, $V \propto N^3/T^3$. The entropy (5.2) of a star is therefore

$$\begin{aligned} S_{\text{star}} &= kN\left(\frac{3}{2}\ln T - \ln N + \ln V + C\right) \\ &= kN\left(-\frac{3}{2}\ln T + 2\ln N + C'\right) . \end{aligned} \tag{5.5}$$

In the second line, the signs of $\ln T$ and $\ln N$ are reversed. The total entropy change of a star embedded in an interstellar gas, $\delta S_{\text{star}} + \delta S_{\text{gas}}$, becomes after again using the virial theorem in the form $E_{\text{star}} = -U_{\text{star}}$,

$$\delta S_{\mathrm{total}} = \left(\frac{1}{T_{\mathrm{star}}} - \frac{1}{T_{\mathrm{gas}}} \right) \delta E_{\mathrm{star}} + k \ln \left[\frac{C'' N_{\mathrm{star}}^2 \rho_{\mathrm{gas}}}{(T_{\mathrm{star}} T_{\mathrm{gas}})^{3/2}} \right] \delta N_{\mathrm{star}} \, . \tag{5.6}$$

While heat must still flow from the hot star into cold interstellar space in order to comply with the Second Law, this leads now to a further increase of the star's temperature, and the accretion of matter – provided the 'star' is already sufficiently massive. Thermal and density inhomogeneities thus *grow* in the generic astrophysical situation, although there are also 'pathological' objects with non-periodic motion, such as gravitationally collapsing spherical matter shells or pressure-free dust spheres, for which the virial theorem does not hold.

These arguments show that the evolution of normal stars is dynamically controlled by thermodynamics rather than by gravity itself. If the thermodynamical arrow of time did change direction in a recontracting universe (as suggested by Gold 1962 – see Sect. 5.3), stars and other gravitating objects would have to re-expand by means of advanced incoming radiation in spite of their attractive forces.

A homogeneous universe must therefore describe an unstable state of very low entropy (though a 'simple' state in the sense of Sect. 3.5). So one may ask whether the evolution of matter into inhomogeneous clumps under gravitational forces represents an entropy capacity that is sufficient to explain the observed global thermodynamical arrow of time. The apparently required *Kaltgeburt* of the Universe might then be replaced by a *homogeneous birth*, since inhomogeneous local contraction leads to the formation of strong temperature and density gradients.

In order to estimate the improbability (negentropy) of a homogeneous universe, one has to know the maximum entropy that can be gained by gravitational contraction. Conceivable limits of contraction are:

- *Quantum degeneracy* (primarily of electrons) is essential for the stability of solid gravitating bodies and white dwarf stars. By emitting heat, these objects cool down rather than further heating up.
- *Repulsive short range forces* are important in neutron stars, for example.
- *Gravitation* itself may lead to black holes even in Newton's theory. Any radiation with bounded velocity cannot escape from the surface of a sufficiently dense and massive object. If this velocity bound is as universal as gravity (as in the theory of relativity), the further fate of matter inside this critical surface remains completely *irrelevant* to an external observer. This surface defines an *event horizon* for him. Matter disappearing behind the horizon is irreversibly lost except for its long range forces, such as gravity itself. In particular, it can no longer participate in the thermodynamics of the Universe.

Such *non-relativistic black holes* were discussed by Laplace as early as 1795, and before him by J. Mitchel at Cambridge. In general relativity, black holes are described by specific spacetime structures. This leads to the further consequence that neither of the first two mentioned limits to gravitational contraction may prevent an object of sufficiently large mass (that could always be reached by further accretion of matter) from collapsing into a black hole. Repulsive forces would give rise to a positive potential energy, that must eventually dominate as a source of gravity, while the increasing zero point pressure of a degenerate Fermi gas would force the fermions into effective bosons that may form a further contracting condensate.

Therefore, only black holes define a realistic upper limit for entropy production by gravitational contraction of matter from the point of view of an external observer. But what is the value of the entropy of a black hole? This question cannot be answered by investigating relativistic stars, that is, equilibrium systems, since the essential stages of the collapse proceed irreversibly. However, a unique and finite answer is obtained from a quantum aspect of black holes, viz., their Hawking radiation (Sect. 5.1).

Since in general relativity the spatial curvature represents a dynamical state (see Sect. 5.4), it may itself carry entropy. Its dynamics is described by Einstein's field equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \ , \tag{5.7}$$

in units with $G = c = 1$, where $T_{\mu\nu}$ is the energy–momentum tensor of matter. They define an initial (or final) value problem, since they are essentially of hyperbolic type (see Sect. 2.1). The Einstein tensor $G_{\mu\nu}$ is a linear combination of the components of the Ricci tensor $R_{\mu\nu} := R^{\lambda}{}_{\mu\lambda\nu}$, that is, the trace of the Riemann curvature tensor. Forming this trace is analogous to forming the d'Alembertian in the wave equation (2.1) for the electromagnetic potential from its matrix of second derivatives $\partial_\nu\partial_\lambda A^\mu$. Aside from nonlinearities (that are responsible for the self-interaction of gravity), the Riemann curvature tensor is similarly defined by the second derivatives of the metric $g_{\mu\nu}$, which thus assumes the role of the gravitational potential (analogous to A^μ in electrodynamics). In both cases, the trace of the tensor of derivatives is determined locally by the sources, while its trace-free parts represent the degrees of freedom of the vector or tensor field, respectively, which can therefore be freely *chosen* initially (as an incoming field).

Penrose (1969, 1981) used this freedom to conjecture that the trace-free part of the curvature tensor (the *Weyl tensor*) vanished when the Universe began. This situation describes a 'vacuum state of gravity', that is, a state of minimum gravitational entropy, and a space as flat as is compatible with the sources. It is analogous to the cosmic initial condition $A^\mu_{\mathrm{in}} = 0$ for the electromagnetic field discussed in Sect. 2.2 (with Gauss's law as a similar constraint). Gravity would then represent a retarded field, requiring 'causes' in the form of advanced sources. Since Penrose intends to explain the thermodynamical arrow, too, from this initial condition (see Sect. 5.3), his conjecture revives Ritz's position in his controversy with Einstein (see Chap. 2) by applying it to gravity rather than to electrodynamics.

In the big bang scenario, the beginning of the Universe is characterized by a past time-like curvature singularity (where time itself began). Penrose used this fact to postulate his Weyl tensor hypothesis on all past singularities, since this would allow only *one* of them: a uniform big bang. In the absence of an *absolute* direction of time, the past would then be distinguished from the future precisely and solely by this asymmetric boundary condition and its consequences (again introducing a 'double standard'). If the Weyl tensor condition could be derived from some other assumptions that did *not* arbitrarily select a time direction, it would have to exclude inhomogeneous future singularities as well. This may again lead to dynamical consistency problems, but it would not rule out collapsing objects to *appear* as black holes to external observers (see Sects. 5.1 and 6.2.3).

5.1 Thermodynamics of Black Holes

In order to discuss the spacetime geometry of black holes, it is convenient to consider the static and spherically symmetric vacuum solution, discovered by Schwarzschild and originally expected to represent a point mass. In terms of spherical spatial coordinates, this solution is described by the metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\;. \quad (5.8)$$

Here, r measures the size of a two-dimensional sphere – though *not* the distance from $r = 0$. This metric form is singular at $r = 0$ and $r = 2M$, but the second singularity, at the *Schwarzschild radius* $r = 2M$, is merely the result of an inappropriate choice of these coordinates. The condition $r = 2M$ describes a surface of fixed area $A = 4\pi(2M)^2$ (using Planck units $G = c = \hbar = k_\mathrm{B} = 1$) in spite of moving outwards at speed of light. In its interior (that is, for $r < 2M$) one has $g_{tt} = 2M/r - 1 > 0$ and $g_{rr} = (1 - 2M/r)^{-1} < 0$. Therefore, r and t interchange their physical meaning as spatial and temporal coordinates. This internal solution is *not* static, while the genuine singularity at $r = 0$ represents a time-like singular boundary rather than the space point expected by Schwarzschild.

Physical (time-like or light-like) world lines, that is, curves with $\mathrm{d}s^2 \leq 0$, hence with $(\mathrm{d}r/\mathrm{d}t)^2 \leq (1 - 2M/r)^2 \to 0$ for $r \to 2M$, can only approach the Schwarzschild radius parallel to the t-axis (see Fig. 5.1). Therefore, the interior region $r < 2M$ is physically accessible only via $t \to +\infty$ or $t \to -\infty$, albeit within finite proper time. These world lines can be extended regularly into the interior when t goes beyond $\pm\infty$. Their proper times continue into the physically finite future (for $t > +\infty$) or past (for $t < -\infty$) with the new time coordinate $r < 2M$. There are therefore *two* internal regions (II and IV in the figure), with their own singularities at $r = 0$ (at a finite distance in proper times). These internal regions must in turn each have access to a new *external*

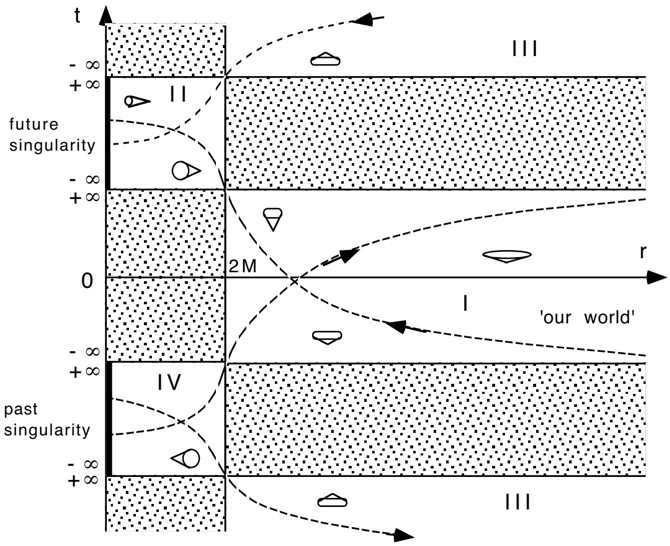

Fig. 5.1. Extension of the Schwarzschild solution from ‘our world’ beyond the *two* coordinate singularities at $r = 2M$, $t = \pm\infty$. Each point in the diagram represents a 2-sphere of size $4\pi r^2$. A consistent orientation of forward light cones (required from the continuation of physical orbits, such as those represented by *dashed lines*) is indicated in the different regions. There are also *two* genuine curvature singularities with coordinate values $r = 0$

region, also in their past or future, respectively, via different Schwarzschild surfaces at $r = 2M$, but with opposite signs of $t = \pm\infty$. There, proper times have to *decrease* with growing t. These two new external regions may then be identified with one another in the simplest possible topology (region III appearing twice in the figure).

This complete Schwarzschild geometry may be described by means of the regular *Kruskal–Szekeres coordinates* u and v, which eliminate the coordinate singularity at $r = 2M$. In the external region I they are related to the Schwarzschild coordinates r and t by

$$u = \sqrt{\frac{r}{2M} - 1}\; \mathrm{e}^{r/4M} \cosh\left(\frac{t}{4M}\right) , \tag{5.9a}$$

$$v = \sqrt{\frac{r}{2M} - 1}\; \mathrm{e}^{r/4M} \sinh\left(\frac{t}{4M}\right) . \tag{5.9b}$$

The Schwarzschild metric in terms of these new coordinates reads

$$\mathrm{d}s^2 = \frac{32M^2}{r}\mathrm{e}^{-r/2M}(-\mathrm{d}v^2 + \mathrm{d}u^2) + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) , \tag{5.10}$$

where $r = r(u, v)$ is determined by inverting (5.9a) and (5.9b). It is evidently regular for $r \to 2M$ and $t \to \pm\infty$, where u and v may remain finite. The

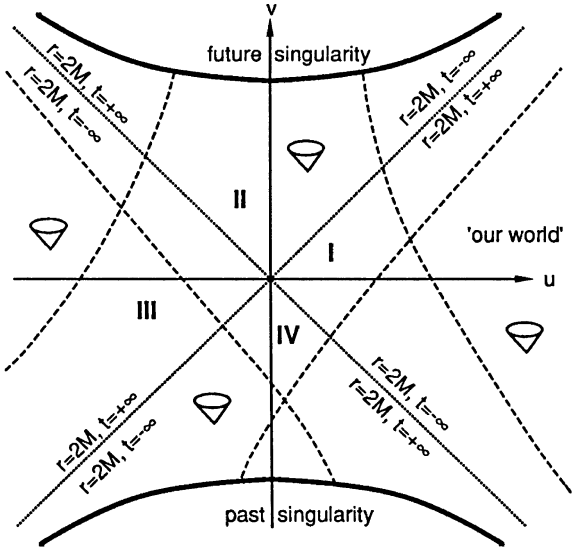

Fig. 5.2. Completed Schwarzschild solution represented in terms of Kruskal coordinates. Forward light cones now appear everywhere with a 45° opening angle around the $+v$-direction. Horizons are indicated by *dense-dotted lines*, possible orbits as *dashed lines*. Although the future horizon, say, moves in the outward direction with the speed of light from an inertial point of view, it does not increase in size. The center of the Kruskal diagram defines an 'instantaneous sphere' as a symmetry center, even though it does not specify a specific *external* time t_0

Kruskal coordinates are thus chosen in such a way that future light cones everywhere form an angle of 45° around the $+v$-direction (see Fig. 5.2). Sector I is again the external region outside the Schwarzschild radius ('our world'). One also recognizes the two distinct internal regions II and IV (connected only through the 'asymptotic sphere' at $t = \pm\infty$ that corresponds to the origin of the figure) with their two separate singularities $r = 0$. Both Schwarzschild surfaces are light-like, and thus represent one-way passages for physical orbits. Their interpretation as past and future horizons is now evident. Sector III represents the second asymptotically flat 'universe'. (It is *not* connected with the original one by a rotation in space, since u is not restricted to positive values like a radial coordinate.)

This vacuum solution of the Einstein equations is clearly T-symmetric, that is, symmetric under reflection at the hyperplane $v = 0$ (or any other hyperplane $t =$ constant). Therefore, it does not yet represent a black hole, and it would not be compatible with the Weyl tensor hypothesis. In the absence of gravitational sources, the Ricci tensor must vanish according to the Einstein equations (5.7), while a non-zero or even singular curvature tensor can then only be due to the Weyl tensor.

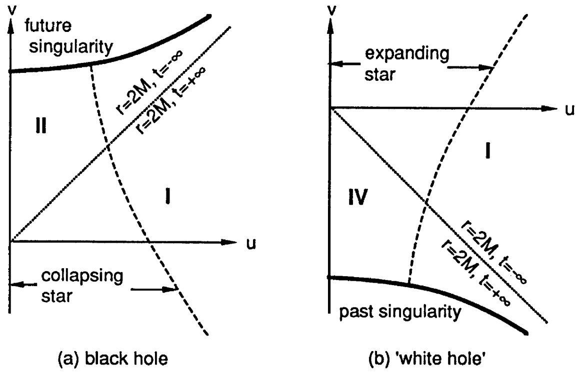

Fig. 5.3. Geometry of a Schwarzschild *black hole* (**a**) which forms by the gravitational collapse of a spherically symmetric mass, and its time-reverse (**b**) – usually called a *white hole*

A black hole is instead defined as an asymmetric spacetime structure that *arises* dynamically by the gravitational collapse of matter from a regular initial state. For example, if the in-falling geodetic sphere indicated by the dashed line passing through sectors I and II of Fig. 5.2 represents the collapsing surface of a spherically symmetric star, the vacuum solution is valid only outside it. The coordinates u and v can then be extended into the interior only with a different interpretation (see Fig. 5.3a, where $u = 0$ is chosen as the center of the collapsing star). This black hole is drastically asymmetric under time reversal, as it contains only a future horizon and a future singularity.

Because of the symmetry of the Einstein equations, a time-reversed black hole – not very appropriately called a *white hole* (Fig. 5.3b) – must also represent a solution. However, its existence in Nature would be excluded by the Weyl tensor hypothesis. If it were the precise mirror image of a black hole, the white hole could describe a star (perhaps with planets carrying time-reversed life) emerging from a past horizon. This would be inconsistent with an arrow of time that is valid everywhere in the external region. If a white hole were allowed to exist, we could receive light from its singularity, although this light would be able to carry *retarded information* about the vicinity of the singularity only if our arrow of time remained valid in this region. This seems to be required for thermodynamical consistency, but might be in conflict with such a local initial singularity (see Sect. 5.3).

Similar to past singularities, space-like singularities – so-called *naked singularities* – could also be visible to us. They, too, were assumed to be absent by Penrose. However, this 'cosmic censorship' assumption cannot generally be imposed directly as an initial condition. Rather, it has to be understood as a conjecture about the nature of singularities which may *form dynamically* during a collapse from generic initial value data which comply with the Weyl tensor hypothesis. Although counterexamples (in which naked singularities form during a gravitational collapse from appropriate initial conditions) have been explicitly constructed, they seem either to form sets of measure zero (which could be enforced by imposing exact symmetries that may be thermodynamically unstable in the presence of quantum matter fields), or to remain hidden behind black hole horizons (see Wald 1997, Brady, Moss and Myers 1998). The first possibility is similar to pathological solutions in mechanical systems that have been shown to exist as singular counterexamples to ergodic behavior. This similarity may already indicate a relationship between these aspects of general relativity and statistical thermodynamics.

The Schwarzschild–Kruskal metric may be generalized as a *Kerr–Newman metric*, which describes axially symmetric black holes with non-vanishing angular momentum J and charge Q. This solution is of fundamental importance, since its external region characterizes the final stage of *any* gravitationally collapsing object. For $t \to +\infty$ (although very soon in excellent approximation during a stellar collapse) every black hole may be completely described by the three parameters M, J and Q, up to translations and Lorentz transformations.

This result is known as the *no-hair theorem*. It means that black holes cannot maintain any external structure ('no hair'), since the collapsing star must radiate away all higher multipoles of energy and charge, while conserved quantities connected with short-range forces, such as lepton or baryon number, disappear behind the horizon. A *white hole* would therefore require coherently incoming (advanced) radiation in order to 'grow hair'. For this reason, *white holes seem to be incompatible with the radiation arrow of our world*. A general correlation between the time arrow of horizons and that of radiation has been derived in the form of a 'consistency condition' for certain de Sitter-type universes by Gott and Li (1997). Their model (though not representative for our world) is remarkable in possessing *different* arrows of time in different spacetime regions separated by an event horizon.

If the internal region of a black hole is regarded as *irrelevant* for external observers, the gravitational collapse effectively violates baryon and lepton number conservation. *Even the entropy* carried by collapsing matter would disappear from this point of view – in violation of the Second Law. Conservation laws would eventually have to be violated objectively at the future singularity if all physical properties were assumed to disappear from existence there. According to rather moderate assumptions, such a singularity must always arise behind any future horizon that comes into being (see Hawking and Ellis 1973).

Spacetime singularities would have particularly dramatic consequences in quantum theory because of the latter's kinematical nonlocality (see Sect. 4.2). Consider a global quantum state, propagating on space-like hypersurfaces ('simultaneities'), which define an arbitrary foliation of spacetime, that is, a time coordinate t. If these hypersurfaces met a singularity somewhere, not only the state of matter *on* this singularity, but also its entanglement with the rest of the Universe would be lost. While classical correlations occur only in statistical ensembles, quantum states would objectively cease to exist also on the non-singular part of the Universe unless the global state evolved 'just in time' into the factorizing form $\psi = \psi_{\text{singularity}}\psi_{\text{elsewhere}}$ whenever it approached a singularity. This would represent a very strong final condition. Therefore, several authors have concluded that quantum gravity must violate CPT invariance and unitarity, while this suggestion has led to a number of proposals for a gravity-based dynamical collapse of the wave function (Wald 1980, Penrose 1986, Károlyházy, Frenkel and Lukácz 1986, Diósi 1987, Ellis, Mohanty and Nanopoulos 1989, Percival 1997, Hawking and Ross 1997). According to these proposals, the existence of future singularities would explain the first step of (4.54).

However, this dynamical indeterminism of global quantum states would not only be inconsistent with canonical quantum gravity (see Sect. 6.2). It may also be avoided in quantum field theory on a classical spacetime if the foliation defining a time coordinate were chosen never to encounter a singularity. For example, the Schwarzschild–Kruskal metric could be foliated according to Schwarzschild time t in the external region, and according to the new time coordinate $r < 2M$ for $t > \infty$ (see Fig. 5.1). This choice, which leaves the entire black hole interior in the 'global future' of external observers *for all times*, is facilitated by the fact that this interior never enters their past, and therefore cannot be regarded as *causing* anything on them – no matter how long they wait (Zeh 2005a).

A *general* singularity-free foliation is given by *York time*, which is defined by hypersurfaces of uniform extrinsic spatial curvature scalar K (see Qadir and Wheeler 1985). A foliation that excludes singularities also appears appropriate because consequences of an elusive unified theory are expected to become relevant close to them. Many hypothetical theories have been proposed, which replace the singular big bang by other scenarios. Among them are oscillating universes or inflationary 'bubble universes' in an eternal inhomogeneous superuniverse. It is questionable, though, whether the traditional concept of time can be maintained in a situation where (quasi-)classical general relativity breaks down (see Sect. 6.2).

The conceivable salvation of *global* unitarity by excluding future singularities is quite irrelevant for local observers who remain outside the event horizon, since the reality accessible to them can be completely described by a reduced density matrix ρ_{ext} in the sense of a Zwanzig projection $\hat{P}_{\text{sub}}$ – see (4.28) – regardless of how their local reference frame is globally extrapolated to form a complete foliation. The non-unitary dynamics of these reduced density matrices has the same origin as it did for quantum mechanical subsystems of Sect. 4.3: nonlocal entanglement. Thereby, the horizon appears as a maximal boundary separating subsystems of interest. One may therefore appropriately describe the phenomenological properties of black holes (including their Hawking radiation – see below) without referring to the singularity or the precise nature of quantum gravity. This 'effective non-unitarity' of black holes reflects the usual time arrow of decoherence (see Chap. 4 and Sect. 6.2.3).

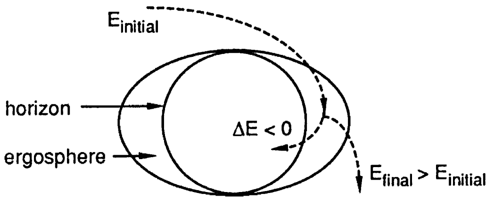

Fig. 5.4. Extraction of rotational energy from a black hole by means of the *Penrose mechanism*, using a booster in the *ergosphere* close to the horizon

From the point of view of an external observer, the information about matter collapsing under the influence of gravity becomes irreversibly irrelevant, except for the conserved quantities M, J and Q. However, the mass of a Kerr–Newman black hole is not completely lost (even if Hawking radiation is neglected). Its rotational and electromagnetic contributions can be recovered by means of a process discovered by Penrose (1969) – see Fig. 5.4. It requires boosting a rocket in the 'ergosphere', that is, in a region between the Kerr–Newman horizon, $r_+ := M + \sqrt{M^2 - Q^2 - (J/M)^2}$, and the 'static limit', $r_0(\theta) := M + \sqrt{M^2 - Q^2 - (J/M)^2 \cos^2\theta}$. In this ergosphere, the cyclic coordinate ϕ is time-like ($g_{\phi\phi} < 0$) as a consequence of extreme relativistic frame dragging. Because of the properties of this metric, ejecta from the booster which fall into the horizon may possess negative energy with respect to an asymptotic frame (even though this energy is locally positive). The mass of the black hole may thus be reduced by reducing its angular momentum. Similar arguments hold with respect to electric charge if the ejecta carry charged particles with an appropriate sign.

The efficiency of this process for extracting energy from a black hole is limited – precisely as it is for a heat engine. According to a geometro-dynamical theorem (Hawking and Ellis 1973), the area A of a future horizon (or the sum of several such horizon areas) may never decrease. For all known processes which involve black holes, this can be formulated in analogy to thermodynamics as (Christodoulou 1970)

$$\mathrm{d}M = \mathrm{d}M_{\mathrm{irrev}} + \Omega\, \mathrm{d}J + \Phi\, \mathrm{d}Q \,, \tag{5.11}$$

where the 'irreversible mass change' $\mathrm{d}M_{\mathrm{irrev}} \geq 0$ is defined by the change of total horizon area, $\mathrm{d}M_{\mathrm{irrev}} = (\kappa/8\pi)\mathrm{d}A$ – in analogy to $T\mathrm{d}S$. Here, κ is the *surface gravity*, which turns out to be constant on each horizon, while Φ is the electrostatic potential at the horizon, and Ω the angular velocity defined by the dragging of inertial frames at the horizon. The last two terms in (5.11) describe work done reversibly on the black hole by adding angular momentum or charge. All quantities are defined relative to an asymptotic rest frame, where they remain regular even when they diverge locally on the horizon. For a Schwarzschild metric, the surface gravity is $\kappa = 1/4M$. The quantities Φ and Ω are also constant on the horizon, in analogy to other thermodynamical equilibrium parameters, such as pressure and chemical potential, which appear in the expression for the work done on a thermodynamical system in the form $\mu \mathrm{d}N - p\mathrm{d}V$.

These similarities led to the proposal of the following *Laws of Black Hole Dynamics*, which form an analogy to the Laws of Thermodynamics (see Bekenstein 1973, Bardeen, Carter and Hawking 1973, Israel 1986):

0. The surface gravity of a black hole must approach a uniform equilibrium value $\kappa(M, Q, J)$ on a black hole horizon for $t \to \infty$.
1. The total energy of black holes and external matter, measured from asymptotically flat infinity, is conserved.
2. The total horizon area, $A := \sum_i A(M_i, Q_i, J_i)$, never decreases:

$$\frac{\mathrm{d}A}{\mathrm{d}t} \geq 0 \,. \tag{5.12}$$

3. It is impossible to reduce the surface gravity to zero by a finite number of physical operations.

Other versions of the Third Law of thermodynamics may not possess a direct analog in black holes because of the latters' negative heat capacity. In particular, the surface area A does not vanish with vanishing surface gravity in a similar way as the entropy does with vanishing temperature.

Bekenstein conjectured that these analogies are not just formal, but indicate genuine thermodynamical properties of black holes. He proposed not only a complete *equivalence of thermodynamical and spacetime-geometrical laws and concepts*, but even their *unification*. In particular, in order to 'legalize' the transformation of thermodynamical entropy into black hole entropy A (when dropping hot matter into a black hole), he required that instead of the two separate Second Laws, $\mathrm{d}S/\mathrm{d}t \geq 0$ and $\mathrm{d}A/\mathrm{d}t \geq 0$, there is only one *Unified Second Law*

$$\frac{\mathrm{d}(S + \alpha A)}{\mathrm{d}t} \geq 0 \,, \tag{5.13}$$

with an appropriate constant α (in units of $k_{\mathrm{B}}c^3/\hbar G$). Its value remains undetermined from the analogy, since the term $(\kappa/8\pi)\mathrm{d}A$, equivalent to $T\mathrm{d}S$, may equally well be written as $(\kappa/8\pi\alpha)\mathrm{d}(\alpha A)$. The *black hole temperature* $T_{\mathrm{bh}} := \kappa/8\pi\alpha$ is classically expected to vanish, since the black hole would otherwise have to emit heat radiation proportional to AT_{bh}^4 according to Stefan and Boltzmann's law. The constant α should therefore be infinite, and so should the *black hole entropy* $S_{\mathrm{bh}} := \alpha A$.

Nonetheless, Bekenstein suggested a finite value for α (of the order of unity in Planck units). This was confirmed by means of quantum field theory by Hawking's (1975) prediction of *black hole radiation*. His calculation revealed that black holes must emit thermal radiation according to the value $\alpha = 1/4$. This process may be described by means of 'virtual particles' with negative energy tunnelling from a virtual ergosphere into the singularity (York

1983), while their entangled partners with positive energy may then propagate towards infinity. (Again, all energy values refer to an asymptotic frame of reference.) The probabilities for these processes lead precisely to a black body radiation with temperature

$$T_{\rm bh} = \frac{\kappa}{2\pi} \;, \tag{5.14}$$

with κ in units of $\hbar/ck_B$, and therefore to the black hole entropy[1]

$$S_{\rm bh} = \frac{A}{4} \;. \tag{5.15}$$

The mean wavelength of the emitted radiation is of order $\sqrt{A}$.

A black hole not coupled to any quantum fields ($\alpha = \infty$) would possess zero temperature and infinite entropy, corresponding to an ideal absorber in the sense of Sect. 2.2. This result would also be obtained for *classical black body radiation*, that is, for classical electromagnetic waves in thermal equilibrium – reflecting the historically important infrared catastrophe for classical fields (Gould 1987).

According to (5.14), a black hole of solar mass would possess a temperature of no more than $T_{\rm bh} \approx 10^{-6}$ K. In the presence of a cosmic background radiation of 2.7 K, it would therefore absorb far more energy than it emits (even in the complete absence of interstellar dust). Only black holes with mass below 3×10^{-7} solar masses could effectively *lose* mass under the present conditions of the Universe (Hawking 1976). Black holes that have formed by gravitational collapse (that is, with a mass above 1.4 solar masses) require a further expansion and cooling of the Universe by a factor of almost 10^7 or more in order to be able to disappear by radiation. 'Black-and-white holes' in equilibrium with a heat bath would not possess any horizon, but according to classical general relativity require a *spatial* singularity at $r = 0$, which corresponds to a negative singular mass – signalling the need for quantum gravity (Zurek and Page 1984).

In vacuo (at $T = 0$), a black hole would eventually completely decay into thermal radiation. The resulting entropy can be estimated to be somewhat larger than that of the black hole (Zurek 1982b). Since the future horizon and the singularity would thereby also disappear, this process seems to represent a *genuine global indeterminism* – known as the 'information loss paradox'. It is remarkable, though, that no conservation laws would be violated in a Schwarzschild foliation. The diverging time dilation close to the horizon prevents all matter from ever reaching the horizon on these simultaneities, which define a global history that covers the complete external world.

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[1] It is important to realize that this result is quite independent of the nature of existing fields. Therefore, it cannot be used to support any *specific* theory, such as M-theory, by its explicit confirmation.

Various 'absolute' resolutions of the information loss paradox have been proposed in the context of quantum gravity (see also Sect. 6.2.3). Conventional unitary quantum theory requires that the entropy of the radiation is the consequence of a Zwanzig projection which regards entanglement between decay fragments as irrelevant. In a complete nonlocal description, photons, gravitons, neutrinos and other radiation fields would all have to be entangled to form a pure total state (Page 1980), while the latter can for all practical purposes be assumed to be a thermal mixture. However, similar quantum correlations between the constituents of incoming (advanced) radiation would be dynamically relevant for a white hole to 'grow hair'.

The description of black holes by a probabilistic 'super-scattering matrix' $\$$ (as suggested by Hawking 1976) can thus be explained by means of decoherence in a similar manner as the apparent collapse of the wave function (see Demers and Kiefer 1996, Kiefer 2004). However, an S-matrix of any kind is *not* a realistic tool for describing black holes – just as it would not be appropriate for describing any macroscopic objects, since they can never approach an asymptotic state of perfect isolation. Unitarity would then imply the superposition of many different Everett branches (in quantum gravity including different spacetimes), while symmetries and conservation laws may be broken within individual branches. In contrast, *microscopic* 'holes' – if they exist – would not possess the classical properties 'black' or 'white', which are formally analogous to the chirality of molecules (Sect. 4.3.2), but would instead have to be described by their T-symmetric or antisymmetric superpositions.

General Literature: Bekenstein 1980, Unruh and Wald 1982.

5.2 Thermodynamics of Acceleration

While the time arrow of black holes is defined by their (quasi-)classical spacetime structure, Hawking radiation requires *quantum* fields on them. It is a consequence of quantum nonlocality, facilitated by the presence of an event horizon as a separator between different 'subsystems'. However, the relevance (or even existence) of this horizon depends on the worldlines of observers or detectors, which define comoving local frames of reference. A black hole horizon is relevant for observers in its flat asymptotic spacetime, or for those staying at a fixed distance, while it would not exist for observers freely falling in. Would their detectors then register any Hawking radiation?

Homogeneous gravitational fields are known to be equivalent to uniformly accelerated frames of reference. They do not require any spacetime curvature, but can be transformed away by means of accelerated (curved) spacetime coordinates. A massive plane, for example, is equivalent to a discontinuity of inertial frames, separating the half-spaces on both sides of the plane by a uniform relative acceleration in the direction orthogonal to the plane. Must an accelerated detector in vacuo therefore be expected to register thermal

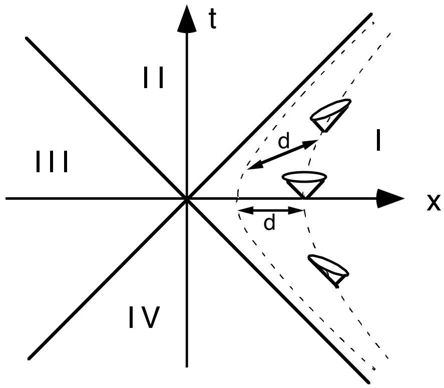

Fig. 5.5. Horizons in Minkowski spacetime are defined for uniformly accelerated local observers (*dashed* world lines) by the asymptotes of their hyperbolic world lines, which are described by the equation $\rho := (x^2 - t^2)/4 =$ constant. Proper acceleration depends on the specific world line. Distances d between two parallel observers remain constant in their comoving rest frames, thus defining global rigid frames in regions I and III

radiation 'equivalent' to (5.14)? For a uniformly accelerated observer in flat Minkowski spacetime, there would indeed be a past and a future horizon, represented by the asymptotes of his hyperbolic relativistic world line (see Fig. 5.5). He shares these horizons with a whole family of 'parallelly accelerated' observers (who require different accelerations in order to remain on parallel hyperbolae – *equivalent* to two observers at different fixed distances from a black hole). These observers also share their comoving rest frames, and thus define an accelerated global rigid frame in keeping fixed distances d in spite (or rather because) of their different accelerations. The same kinematical situation had to be discussed for uniformly accelerated charges and detectors of classical electromagnetic waves in Sect. 2.3.

The two-dimensional Minkowski diagram of Fig. 5.5 appears similar to the Kruskal–Szekeres diagram (Fig. 5.2), although it is singularity-free, as each point in Fig. 5.5 represents a flat $\mathbb{R}^2$ (with coordinates y, z) rather than a 2-sphere. Therefore, points in regions I and III are now related by a π-rotation around the t-axis. If the acceleration had *begun* at a certain finite time ($t = 0$, say), no past horizon would exist (in analogy to a black hole – see Fig. 5.3a). The world lines of this family of local observers can be used to define a new spatial coordinate $\rho(x, t)$ that is constant for each of them, and may be conveniently scaled by $\rho(x, 0) = x^2/4$. Together with a new time coordinate $\phi(x, t)$ that is related to proper times τ along the world lines according to $\mathrm{d}\tau = \sqrt{\rho}\,\mathrm{d}\phi$, and the coordinates y and z, it defines the *Rindler coordinates* of flat spacetime. In region I of Fig. 5.5, they are related to the Minkowski coordinates by

$$x = 2\sqrt{\rho}\cosh\frac{\phi}{2} \quad \text{and} \quad t = 2\sqrt{\rho}\sinh\frac{\phi}{2} \, . \tag{5.16}$$

The proper accelerations $a(\rho)$ along $\rho =$ constant are given by $a = (2\sqrt{\rho})^{-1}$, while the resulting non-Minkowskian representation of the Lorentz metric,

$$\mathrm{d}s^2 = -\rho\,\mathrm{d}\phi^2 + \rho^{-1}\,\mathrm{d}\rho^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \, , \tag{5.17}$$

describes a coordinate singularity at $\rho = 0$ that is analogous to $r - 2M = 0$ for the Schwarzschild solution. The Minkowski coordinates can therefore be compared with the Kruskal coordinates u and v of Fig. 5.2, while the Rindler coordinates are analogous to the Schwarzschild coordinates.

The Rindler coordinates are also useful for describing the uniformly accelerated point charge of Sect. 2.3 and its relation to a co-accelerated detector. The radiation propagating along the forward light cone of an event on the accelerated world line of the charge must somewhere hit the latter's future horizon (see Fig. 5.5), and asymptotically completely enter region II. However, from the point of view of a co-accelerated (uniformly Lorentz-rotated) observer with the same comoving simultaneities $\phi =$ constant, which all intersect the horizon at the origin, the radiation would *never* reach the horizon at $\phi = \infty$.

This explains why the accelerated charge radiates from the point of view of an inertial observer, but not for a co-accelerated one (Boulware 1980). While Dirac's *invariant* radiation reaction (2.25) vanishes for uniform acceleration, the definition of radiation is based on the distinction between near fields and far fields by their dependence on different powers of distance according to (2.14), and therefore depends on the acceleration of the reference frames. Even though global inertial frames are *absolutely* defined in special relativity, it is the *relative* acceleration between source and detector that is relevant for the resulting effects. For a *uniformly* accelerated charge in region I, time reversal symmetry has the consequence that its total retarded field is identical with its advanced field in this whole sector (except on the horizons), while elsewhere one has either just the retarded outgoing fields in region II, or just the advanced incoming fields in region IV (or a superposition of these two cases) – depending on the boundary conditions.

Unruh (1976) was able to show that an accelerated detector in the inertial vacuum of a quantum field must register an isotropic thermal radiation of *all* existing fields, corresponding to the temperature

$$T_\mathrm{U} := \frac{a}{2\pi} = \frac{a\hbar}{2\pi c k_\mathrm{B}} \, . \tag{5.18}$$

This is precisely what had to be expected from (5.14) according to the principle of equivalence, and from the analogy with Mould's (1964) result for classical radiation. For a generalization of (5.18) to other trajectories see Louko and Satz (2006). However, the response of a detector appears as an objective fact. It cannot just be a matter of spacetime perspective or definition (such as the distinction between near field and far field in different coordinate systems): wave functions live on configuration space.

The result (5.18) can be understood when representing the inertial *Minkowski vacuum* $|0_{\mathrm{M}}\rangle$ in terms of 'Rindler modes', that is, wave modes which factorize in the Rindler coordinates (with frequencies Ω with respect to the time coordinate ϕ). If Minkowski plane wave modes $\mathrm{e}^{\mathrm{i}(kx-\omega t)}$ are expanded in terms of such Rindler modes, this leads to a *Bogoljubow transformation* for the corresponding 'particle' creation operators:

$$a_k^+ \longrightarrow b_{\Omega s}^+ := \sum_k (\alpha_{\Omega s,k} a_k^+ + \beta_{\Omega s,k} a_k) \; .$$

Here, the index $s = \mathrm{I}$ or III specifies two Rindler modes (both with time dependence $\mathrm{e}^{-\mathrm{i}\Omega\phi}$) which vanish in the regions III or I of Fig. 5.5, respectively. On flat simultaneities through the origin ($\phi = \text{const.}$), they are complete on the corresponding half-spaces with $x > 0$ or $x < 0$, respectively. These Bogoljubow transformations combine creation and annihilation operators, since the non-linear coordinate transformations (5.16) do not preserve the sign of frequencies. These signs distinguish particle and antiparticle modes in the usual interpretation, such that the two terms of the Fourier representation of field operators, $\Phi(\boldsymbol{r},t) \propto \int \{ \exp\left[\mathrm{i}(kx+\omega t)\right] a_k + \exp\left[\mathrm{i}(kx-\omega t)\right] a_k^+ \} \mathrm{d}k$, are not separately transformed. (Recall that 'particle creation' operators are just raising operators for harmonic oscillator quantum numbers characterizing quantum states of field modes.)

In terms of the Rindler modes, the Minkowski vacuum becomes an *entangled* state in the form of a BCS ground state of superconductivity (Bardeen, Cooper and Schrieffer 1957):

$$|0_{\mathrm{M}}\rangle = \prod_{\Omega} \left(\sqrt{1-\mathrm{e}^{-4\pi\Omega}} \sum_n \mathrm{e}^{-2\pi\Omega n} |n\rangle_{\Omega,\mathrm{I}} |n\rangle_{\Omega,\mathrm{III}} \right) \; , \tag{5.19}$$

where $|n\rangle_{\Omega,s} = (n!)^{-1/2}(b_{\Omega s}^+)^n|0_{\mathrm{R}}\rangle$ are the Rindler particle occupation number eigenstates (see also Gerlach 1988). The *Rindler vacuum* $|0_{\mathrm{R}}\rangle$, defined by $b_{\Omega s}|0_{\mathrm{R}}\rangle = 0$ for all Ω and s, is therefore different from the Minkowski vacuum. It must also be a *pure* state in the Minkowski representation, while its reduced density matrix on the half spaces $x > 0$ or $x < 0$ describes a thermal mixture. This demonstrates that the concepts of quantum 'particles' and their vacua are not invariant under non-Lorentzian transformations. While the *actual quantum state* may be regarded as absolutely defined ('real'), its interpretation in terms of 'particles' depends on the local choice of simultaneities – conveniently identified with the comoving rest frames of a detector. For example, the Rindler basis characterizes detectors accelerated relative to inertial frames, while a specific 'vacuum' would represent an actual (physically meaningful) state. This distinction between physical states and their various representations is obscured in the Heisenberg picture.

Equation (5.19) is the *Schmidt canonical representation* (4.27) of nonlocal quantum correlations between the two sectors I and III of Fig. 5.5 (which together are spatially complete for hyperplanes intersecting the origin at $x = t = 0$). It illustrates the kinematical nonlocality of a relativistic Minkowski vacuum. The diagonal elements represent a canonical distribution with dimensionless formal temperature $1/4\pi$ – compatible with the dimensionless time coordinate ϕ. Since proper times along the world lines $\rho, y, z =$ constant are given by $\mathrm{d}\tau = \sqrt{\rho}\,\mathrm{d}\phi = (2a)^{-1}\,\mathrm{d}\phi$, energies are given by $2an\Omega$. The (ρ-dependent) temperature is therefore $T = a/2\pi$ – in accordance with (5.18). Disregarding quantum correlations with the other half-space thus leads to the *apparent ensemble* of states representing a heat bath. As one needs measurement times Δt larger than $(a\Omega)^{-1}$ to measure a frequency Ω, the acceleration has to remain approximately uniform for more than this interval of time in order to mimic the presence of an event horizon for this mode.

While the result (5.18) might have been expected from the principle of equivalence, it is more general than (5.14), since it is independent of gravity (spacetime curvature). In general, the equivalence principle holds only locally. Its exceptional global applicability is a consequence of the specific field of uniform accelerations depicted in Fig. 5.5 (see also Sect. 2.3). Therefore, Unruh radiation cannot in general be *globally* equivalent to Hawking radiation. While the whole future light cone of an event on the world line of a uniformly accelerated object must asymptotically intersect the latter's horizon for an inertial observer (as discussed above), only *part* of the future light cone of an event in the external region of a black hole will ever enter its internal region. For an observer approaching a black hole, the horizon will eventually cover his whole celestial sphere because of the bending of light rays. (He would have to speed towards the remaining 'hole in the sky' in order not to be swallowed.) Such spacetime-geometric aspects of boundary conditions also determine the specific 'actual vacuum' (Unruh 1976). Only in the immediate neighborhood of the horizon can the freely falling observer be completely equivalent to the inertial one in flat spacetime, and thus precisely experience a vacuum. While the Unruh radiation is isotropic and T-symmetric, the Hawking radiation observed by a non-inertial detector at a fixed distance from a black hole specifies a direction in space as well as in time by its non-vanishing energy flux coming from the black hole.

A real (and in principle observable) accelerated QED vacuum could be produced by a uniformly accelerated ideal mirror (Davies and Fulling 1977). A mirror at rest, representing a plane boundary condition to the field, leads to the removal of an infinite number of field modes (those not matching the boundary condition). This in turn leads to an infinite energy renormalization (defining a 'dressed mirror') by subtracting their zero point energies. This dressing would not be additive for two or more parallel mirrors at fixed distances, while the adiabatic variation of their distances defines the finite and observable *Casimir effect* (a force between them). An *accelerated* mirror, acting as an accelerated boundary, produces a quantum field state that would be experienced as a vacuum by a co-accelerated detector, but as a thermal bath by an inertial one. A uniformly accelerated mirror would completely determine this QED state on the concave side of its spacetime hyperbola in Fig. 5.5, while the convex side offers the freedom of additional boundary conditions in regions II or IV (similar to the classical field of a uniformly accelerated charge). According to the equivalence principle, an 'ideal graviton mirror' would even redefine (completely 'drag') inertial frames.

All these thermodynamic consequences of acceleration or curvature are too small to be confirmed with presently available techniques. However, they were drawn by combining two well established theories (general relativity and quantum field theory), and they appear necessary for consistency (see Unruh and Wald 1982). So they can hardly be regarded as merely hypothetical.

General Literature: Birrell and Davies 1983.

5.3 Expansion of the Universe

Since Hubble's discovery of 1923, we have known that the Universe is expanding. This is often regarded as a confirmation of general relativity, since it can be described by Friedmann's solutions of the Einstein equations of 1922. However, a very similar dynamical universe could have been derived in Newton's theory, although this nonrelativistic model would have to specify an inertial center. Evidently, applying the laws of mechanics and gravity to the whole Universe met even stronger reservations than applying them to the celestial objects a few hundred years earlier, when Kepler and his contemporaries were surprised to discover that planets can 'fly like the birds' rather than being guided by the crystal spheres.

Since a static universe would not be stable under gravity, Einstein quite artificially introduced his 'cosmological constant' in order to make his theory compatible with what he believed to be empirically correct. A similar novel kind of repulsive global force would have been required in Newton's theory for this purpose. In an *open* Newtonian universe these consequences might at most be obscured, but not avoided (Bondi 1961). Without such a repulsive force, Newton's theory, too, would have required the Universe to expand or to contract (depending on the initial conditions), and this would have led to a big bang or a big crunch, respectively, or both. However, in contrast to general relativity, a singularity could then be avoided by appropriate repulsive forces that become relevant at very high densities.

In Einstein's theory, a homogeneous and isotropic universe is described by the Friedmann–Robertson–Walker (FRW) metric,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\left\{\mathrm{d}\chi^2 + \Sigma^2(\chi)\left[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right]\right\}, \tag{5.20}$$

with $\Sigma(\chi) = \sin\chi$, $\sinh\chi$, or χ, depending on the sign of the spatial curvature, $k = +1$, -1 or 0, respectively. The Friedmann time coordinate t in (5.20) describes the proper time for objects which are at rest in these coordinates ('comoving clocks'). This metric may remain valid close to the big bang (for $a = 0$) in accordance with the Weyl tensor hypothesis. It can be generalized by means of a multipole expansion on the Friedmann sphere (see Halliwell and Hawking 1985, and Sect. 5.4). This general-relativistic form has the advantages of not requiring a special 'center at rest', and of allowing a finite universe without a boundary (for positive curvature).

The exact FRW metric (5.20) depends only on the expansion parameter $a(t)$. The latter's dynamics, derived from the Einstein equations (5.7) with an additional cosmological constant, assumes the form of an 'energy integral' with a *fixed vanishing* value of the energy:

$$\frac{1}{2}\left(\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 = \frac{1}{2}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right)^2 = -V(\alpha) \ . \qquad (5.21)$$

The logarithm of spatial extension, $\alpha = \ln a$, which formally sends the big bang to minus infinity, will prove convenient on several occasions. The Friedmann potential $V(\alpha)$ is given by the energy density of matter $\rho(a)$, the cosmological constant Λ, and the spatial curvature k/a^2, in the form

$$V(\alpha) = -\frac{4\pi\rho(\mathrm{e}^{\alpha})}{3} - \frac{\Lambda}{3} + k\mathrm{e}^{-2\alpha} \ . \qquad (5.22)$$

One would have obtained essentially the same equation (without curvature term and cosmological constant, but with variable energy) from Newton's dynamics for the radius of a gravitating homogeneous sphere of matter.

The energy density ρ may depend on a in various ways. In the matter-dominated epoch it is proportional to the inverse density, a^{-3}. During the radiation era – less than 10^{-4} of the present age of the universe – it decreased according to a^{-4}, since all wavelengths expand with a. Much earlier (for extremely high matter density), quite novel phenomena must be expected to have affected the relativistic equation of state, here described by $\rho(a)$. According to some theories, for example, the vacuum state of matter passed through one or several phase transitions (see Sect. 6.1). Similar to a condensation process, this situation may be characterized by a constant function of state, $\rho(a) = \rho_0$. The matter term in the potential V would then *simulate* a cosmological constant – albeit only for a limited time (see Fig. 5.6). In the 'Planck era', that is, for values of a of order unity, quantum gravity must become essential (see Sect. 6.2).

Different eras, described by such analytic equations of state $\rho(a)$, possess different solutions $a(t)$. For example, a dominating (fundamental or simulated) cosmological constant would lead to a 'de Sitter era' with $a(t) = c\mathrm{e}^{\pm Ht}$ and a 'Hubble constant' $H = \dot{a}/a = \dot{\alpha}$. For a matter- or radiation-dominated universe, one has $a(t) = c't^{2/3}$ or $a(t) = c''t^{1/2}$, respectively, while for low matter densities the curvature term may dominate. Recent observations indicate that our Universe is approximately flat (negligible curvature term), while an effec-

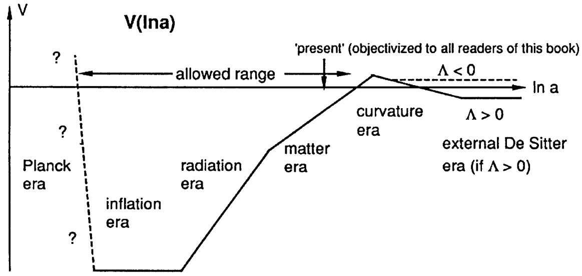

Fig. 5.6. Schematic behavior of the 'potential energy' for the dynamics of $\ln a$ in the case of positive spatial curvature. Since only regions with positive kinetic energy $E - V = -V > 0$ are allowed, turning points of the cosmic expansion would arise at values of $V = 0$. An upper turning point would lead to recontraction, while a lower turning point describes a 'bouncing' universe (without big bang or big crunch singularities)

tive positive cosmological constant of unknown origin ('dark energy') already contributes two thirds of the potential $V(a)$.

5.3.1 Instability of Homogeneity

While the Friedmann model is an exact solution of the Einstein equations, and apparently a reasonable approximation to the very large scale behavior of the real Universe, it is not stable against density fluctuations (as discussed in the introduction to this chapter and in Sect. 5.1). This local instability cannot be compensated by a *global* force, such as a cosmological constant. It is in fact successfully used to explain the formation of stars, galaxies, galaxy clusters, possibly larger structures, and eventually black holes in the present Universe. Thereby, the *assumed* initial symmetries of the Friedmann universe must be dynamically broken. In classical physics, density fluctuations would be microscopically determined (Sect. 3.4). In quantum theory they may also result from an indeterministic (genuine or apparent) collapse of the wave function, induced by decoherence (see Calzetta and Hu 1995, Kiefer, Polarski and Starobinsky 1998, and Sect. 6.1). A similar quantum effect is known to limit the retardation of symmetry-breaking phase transitions (their hysteresis). The onset of these primordial structures of the Universe is now believed to be observed in the cosmic background radiation.

The arrow of time characterizing these irreversible processes is thus again based on an improbable (but 'simple') cosmic initial condition: homogeneity. When Boltzmann (1896) discussed the origin of the Second Law in the context of an infinite and eternal universe, he had to conclude that we, here and now, are living in the aftermath of a gigantic cosmic fluctuation. Its maximum (that is, a state of very low entropy) must have occurred in the distant past in order to explain the existence of fossils and other documents in terms of causal history and evolution (see Sect. 3.5).

How improbable is the novel initial condition of homogeneity that Boltzmann did not even recognize as an essential assumption? We may calculate its probability by means of Einstein's relation (3.56) if we know the entropy of the most probable state. The entropy of a non-degenerate homogeneous physical state in local equilibrium is proportional to the number of particles, N. All other parameters enter this expression only logarithmically – as exemplified for the ideal gas in (3.14). In the present Universe, the number of photons contained in the 2.7 K background radiation exceeds that of massive particles by a factor 10^8. The entropy of a finite 'standard universe' of 10^{80} baryons (now often regarded as no more than a 'bubble' in a much larger or infinite universe) would therefore possess an entropy of order 10^{88} plus a small but important contribution resulting from gravitating objects. Most of this entropy must therefore have been produced in the early Universe by the creation of photons and other particles, which are strongly entangled in a chaotic way.

However, the present entropy is far from its maximum that would be achieved by the production of black holes. In Planck units, the horizon area of a neutral and spherical black hole of mass M is given by $A = 4\pi(2M)^2$. Its entropy according to (5.15) thus grows with the *square* of its mass,

$$S_{\mathrm{bh}} = 4\pi M^2 \ . \tag{5.23}$$

Merging black holes will therefore produce an enormous amount of entropy. If the standard universe of 10^{80} baryons consisted of 10^{23} solar mass black holes (since $M_{\mathrm{sun}} \approx 10^{57} m_{\mathrm{baryon}}$), it would already possess a total entropy of order 10^{100}, that is, 10^{12} times its present value. If most of the matter eventually formed a single black hole, this value would increase by another factor of 10^{23}. The probability for the present, almost homogeneous universe is therefore a mere

$$p_{\mathrm{hom}} \approx \frac{\exp(10^{88})}{\exp(10^{123})} = \exp(10^{88} - 10^{123}) \approx \exp(-10^{123}) \tag{5.24}$$

(Penrose 1981), indistinguishable in this approximation from the much smaller probability at the big bang. Gravitational contraction thus offers an enormous further entropy capacity to assist the formation of structure and complexity.

This improbable initial condition of homogeneity as an origin of thermodynamical time asymmetry is different from attempts (see Gold 1962) to derive this arrow from a homogeneous expansion of the Universe in a causal manner (see Price 1996 and Schulman 1997 for critical discussions). While it is true that non-adiabatic expansion of an equilibrium system may lead to a retarded non-equilibrium, this would equally apply to non-adiabatic *contraction* in our causal world. The growing space (and thus phase space, representing increasing entropy capacity) cannot form the master arrow of time, since it is insufficient to explain causality (the absence of any advanced correlations). Non-adiabatic *compression* of a vessel would lead to retarded pressure waves emitted from the walls, but not to a reversal of the thermodynamical arrow. The entropy capacity of gravitational contraction is far more important than homogeneous expansion, but probably not very relevant for the very early stages of the Universe.

There are other examples of *using* causality in thermodynamical arguments rather than *deriving* it in this cosmic scenario. For example, Gal-Or (1974) discussed retarded equilibration due to the slow nuclear reactions in stars. Even though nuclear fusion controls the time scale and energy production during most stages of stellar contraction, it presumes a strong initial non-equilibrium.

5.3.2 Inflation and Causal Regions

The finite age of an expanding universe that starts from an initial singularity (a big bang) leads to the consequence that the backward light cones of two events may not overlap. These events would then not be causally connected. A sphere formed by the light front originating in a point-like event at the big bang, where $a(0) = 0$, is therefore called a *causality horizon*. Its radius $s(t)$ at Friedmann time t is given by

$$s(t) = \int_0^t \frac{a(t)}{a(t')} \mathrm{d}t' \ . \tag{5.25}$$

In a matter- or radiation-dominated universe, this integral would converge for $t' \to 0$, and thus define a finite horizon size. Only *parts* of the Universe may then be causally connected – excluding even readily observable distant pairs of objects that strongly indicate a simultaneous origin.

In particular, the homogeneity of the universe on the large scale would thereby remain causally unexplained. This *horizon problem* was the major motivation for postulating a phase transition of the vacuum or another mechanism of quantum fields that would lead to a transient cosmological constant, and thus to an early de Sitter era. In an exponentially expanding universe, the big bang singularity could in principle be shifted arbitrarily far into the past – depending on the duration of this era. However, in an extremely short time span (of the order of 10^{-33} s), the universe, and with it all causality horizons, would have been *inflated* by a huge factor that was sufficient for the sources of the whole now observable cosmic background radiation to be causally connected (Linde 1979). On the other hand, since causality horizons started with zero radius, this would explain the initial absence of nonlocal correlations and entanglement, provided they were assumed to *require* a causal origin.

Measurements of the cosmic background radiation indicate that an inflation era did in fact occur. Since the corresponding repulsive force counteracts gravity, it has also been conjectured to have driven the universe into a state of homogeneity in a causal manner. This *cosmological no-hair conjecture* is supported by a theorem of Hawking and Moss (1982). However, this theorem remains insufficient for the required purpose, since the global effect of a cosmological constant cannot generally force *local* gravitating systems, in particular black holes, to expand into a state of homogeneity. Proofs of the cosmic no-hair theorem had therefore to exclude positive spatial curvature. (Expanding white holes would require acausally incoming advanced radiation, as explained in Sect. 5.1.)

Since a cosmological constant that was simulated by a phase transition of the vacuum would depend on the local density, it may at least overcompensate the effect of gravity until strong inhomogeneities begin to form. This may *partly* explain the homogeneity of the observed part of our universe. It can be described by saying that the Weyl tensor 'cooled down' as a consequence of this spatial expansion – similar to the later red-shifting of the primordial electromagnetic radiation. While these direct implications of the expansion of the universe define reversible phenomena, equilibration during the radiation era or during the phase transition would be irreversible in the statistico-thermodynamical sense (based on microscopic causality).

This explanation of homogeneity is incomplete as it has to presume the absence of *strong* initial inhomogeneities (abundant initial black holes, in particular). In order to work in a deterministic theory, it would furthermore require the state that precedes inflation to be even less probable than the homogeneous state after inflation.

Similar inflationary scenarios have been discussed in various hypothetical models of quantum cosmology (see Caroll and Chen 2004, and Chap. 6).

5.3.3 Big Crunch and a Reversal of the Arrow

These questions may also be discussed by means of a conceivable recontracting universe. A consistent analysis of the arrow of time for this case is helpful regardless of what will happen to our own Universe. Would the thermodynamical arrow have to reverse direction when this universe starts recontracting towards the big crunch after having reached maximum extension? The answer would have to be 'yes' if the cosmic expansion represents the master arrow, but it is often claimed to be 'no' on the basis of causal arguments if they are continued into this region. For example, some authors argued that the background radiation would reversibly heat up during contraction (blueshifting), while the temperature gradient between interstellar space and the fixed stars would first have to be inverted in order to reverse stellar evolution long after the universe had reached its maximum extension. However, this argument presupposes the overall validity of the 'retarded causality' in question, that is, the absence of future-relevant correlations in the contraction phase. It would be justified if the relevant initial condition held at only one 'end' of this otherwise symmetric cosmic history. The absence or negligibility of any anti-causal events in our present epoch seems to indicate either that our Universe is thermodynamically asymmetric in time, or that it is still 'improbably young' in comparison to its total duration.

Paul Davies (1984) argued in a similar causal manner that there can be no reversed inflation leading to a homogeneous big crunch, since correlations which would be required for an inverse phase transition have to be excluded for being extremely improbable. Instead of a homogeneous big crunch one would either obtain locally re-expanding 'de Sitter bubbles' forming an inhomogeneous 'bounce', or inhomogeneous singularities at variance with a reversed Weyl tensor condition, or both. This probability argument fails, however, if the required correlations are *caused in the backward direction of time* by a final condition that was thermodynamically a mirror image in time of the initial one (see also Sect. 6.2.3). Similarly, if the big bang was replaced by a non-singular *homogeneous bounce* by means of some kind of 'Planck potential' (Fig. 5.6), entropy must have decreased prior to the bounce. In particular, decoherence would have to be replaced by recoherence in all contraction eras. In this case, an observer complying with the Second Law would always experience an expanding universe; the sign of the dynamical time parameter used in this description is merely formal (see Sect. 5.4).

On the other hand, a low entropy big bang *and* an equivalent big crunch may lead to severe consistency problems, since the general boundary value problem (Sect. 2.1) allows only one complete (initial or final) condition. Although the requirement of low entropy is not a complete boundary condition, statistically independent two-time conditions would lead to the square of the already very small probability of (5.24), that is,

$$p_{\text{two-time}} = p_{\text{hom}}^2 \approx \left[\exp(-10^{123})\right]^2 \approx \exp(-10^{123.301}) \, . \qquad (5.26)$$

The RHS appears as a small correction to (5.24) only because of this double-exponential form, although an element of phase space corresponding to (5.26) could now easily be much smaller than a Planck cell (see Zeh 2005b). A two-time boundary condition of homogeneity may thus be inconsistent with 'ergodic' quantum cosmology (that would have to include the repeated formation and decay of black holes, which contribute most of phase space).

The consistency of general two-time boundary conditions has been investigated for simple deterministic systems (see Cocke 1967 and Schulman 1997). Davies and Twamley (1993) discussed the more realistic situation of classical electromagnetic radiation in an expanding and recollapsing universe. According to their estimates, our Universe will remain essentially transparent all the way between the two opposite radiation eras (in spite of the reversible red- and blue-shifting over many orders of magnitude in between) – in contrast to ergodic assumptions used in (5.26). Following a suggestion by Gell-Mann and Hartle, they concluded that light emitted *causally* by all stars before the 'turning of the tide' propagates freely until it reaches the time-reversed radiation era – thus giving rise to an asymmetric history of this universe.

David Craig (1996) argued on this basis, but by *assuming* a thermodynamically time-symmetric universe, that the night sky at optical frequencies should contain an almost homogeneous component that represents the advanced radiation from stars existing during the contraction era. It should be observable as a non-Planckian high frequency tail in the isotropic background radiation with a total intensity at least equalling that of the light now observed from all stars and galaxies in our past – but probably much higher because of the advanced light corresponding to that which will have to be produced until the turning point is reached. However, since classical radiation would preserve all information about its origin, it is inconsistent with a time-reversed absorber (the opposite radiation era), that allows only its thermal radiation in *its* causal future (Sect. 2.2). Craig also concluded that the intensity of the thermal part of the background radiation would be doubled because of the two radiation eras, but this does not seem to be required, since the 'two' *thermal* components may be identical. (Retarded and advanced fields do not add – see Sect. 2.1 – but they must be consistent with one another.) Only in the non-thermal frequency range can retarded and advanced radiation be conceptually distinguished and thus carry information about their origin.

These conclusions have to be modified in an essential way when the quantum aspect of electromagnetic radiation is taken into account. The information content of radiation consisting of photons is limited, as first emphasized by Brillouin (1962). This consequence had also turned out to be important for Borel's argument of Sect. 3.1.2 – see footnote 4 of Chap. 3. Each photon, even if emitted into intergalactic space as a spherical wave, disappears from the whole quasi-classical universe as soon as it is absorbed *somewhere*. A reversal of this process would again require recoherence, that is, the superposition of many Everett branches. This argument requires consistent quantum cosmology (Chap. 6), where initial or final conditions can only affect the total, unitarily evolving Everett wave function. If the Schrödinger dynamics was instead modified by means of a collapse of the wave function (as implicitly assumed also for Gell-Mann and Hartle's 'histories'[2]), the corresponding new

[2] Gell-Mann and Hartle (1994) discussed quantum mechanical 'histories', which are defined in terms of time-ordered series of projections in Hilbert space. These *individual* histories are thus equivalent to successions of stochastic collapse events (global quantum jumps) – even though a collapse is not explicitly used. The authors nonetheless discussed the possibility of a thermodynamically time reversal-symmetric cosmic history by presuming a final condition that is similar to the initial one. This proposal is based on the equivalence of the upper and lower diagrams of Fig. 4.4, but neglects the asymmetric structure (4.56) of a collapse, which would have to include all retarded entanglement with 'information gaining systems'. Therefore, it leads to insurmountable problems as soon as one attempts to justify the probabilistic interpretation ('consistent histories') by an in practice irreversible decoherence process (see Fig. 4.5). Time reversal symmetry could be restored in the contraction era only by means of a complete process of recoherence. This would not only have to include those Everett components that have been disregarded by the Hilbert space projections which lead to individual measurement outcomes, and in this way define quasi-classical 'histories' as a *partial* quantum reality. It should also require components that have to be regarded as being retro-caused in the future.

dynamical law would have to be reversed, too, in order to save a thermodynamically time-symmetric (but now indeterministic) universe.

This problem of consistent cosmic two-time boundary conditions will assume a conceptually quite novel form in the context of quantum gravity, where any fundamental concept of time disappears from the description of a closed universe (Sect. 6.2).

5.4 Geometrodynamics and Intrinsic Time

In general relativity, the 'block universe picture' is traditionally preferred to a dynamical description, as its unified spacetime concept is then manifest. So it took almost half a century before its dynamical content was sufficiently understood, in particular by means of its Hamiltonian form, invented by Arnowitt, Deser and Misner (1962). This approach, which is essential for a quantization of the theory, has not always been welcomed, as it seems to destroy the beautiful relativistic spacetime concept by reintroducing a 3+1 (space and time) representation. However, only in this *form* can the dynamical content of general relativity be fully appreciated (see Chap. 21 of Misner, Thorne and Wheeler 1973). A similarly symmetry-violating form in spite of Lorentz invariance is known for the electromagnetic field when described in the Coulomb gauge by the vector potential $\boldsymbol{A}$ as the dynamical field configuration on a space-like hypersurface of Minkowski spacetime.

This dynamical reformulation requires the separation of unphysical gauge degrees of freedom (which in general relativity simply represent the choice of coordinates), and the skillful handling of boundary terms. The result of this technically demanding procedure turns out to have a simple interpretation. It describes the *dynamics of the spatial geometry* ('three-geometry') ${}^{(3)}G(t)$, that is, a propagation of the intrinsic curvature on space-like hypersurfaces with respect to a time coordinate t that labels a foliation of the spacetime arising dynamically in this way. This foliation has to be *chosen* simultaneously with the construction of the solution. The extrinsic curvature, which describes the embedding of the three-geometries into spacetime, is represented by the corresponding canonical momenta. The configuration space of three-geometries ${}^{(3)}G$ has been dubbed *superspace* by Wheeler, since the form of its kinetic energy defines a metric. Trajectories in this superspace define four-dimensional spacetime geometries ${}^{(4)}G$.

This 3+1 description may appear ugly not only as it hides Einstein's beautiful spacetime concept, but also since the foliation of a given ${}^{(4)}G$ by means of space-like hypersurfaces, on which ${}^{(3)}G(t)$ is defined, is quite arbitrary. Many trajectories ${}^{(3)}G(t)$ therefore represent the same spacetime ${}^{(4)}G$, which is absolutely defined. It is only in special situations – such as for the FRW metric (5.20) – that there may be a 'preferred choice' of coordinates, which then reflect their exceptional symmetry. The time coordinate t, characterizing a foliation, is just one of the four arbitrary (physically meaningless) spacetime coordinates. As a parameter labelling trajectories it could just as well be eliminated and replaced by one of the dynamical variables (a global 'clock' – see Chap. 1), such as the size (or scale) of an expanding universe. The abstract four-geometry defines all spacetime distances – including *all* proper times of real or imagined local clocks. Classically, spacetime may always be assumed to be filled with a 'dust of test clocks' of negligible mass (see Brown and Kuchař 1995). However, such clocks are not required to *define* proper times; in general relativity, time as a property of the metric is itself a dynamical variable (see below), while proper times assume the role of Newton's time as controllers of motion for all material clocks.

Einstein's equations (5.7) possess a similar hyperbolic structure as the wave equation (2.1). They may therefore be expected to determine the metric $g_{\mu\nu}(x,y,z,t)$ by means of two boundary conditions for $g_{\mu\nu}$ – at t_0 and t_1, say. (For $t_1 \to t_0$ this would correspond to $g_{\mu\nu}$ and its 'velocity' at t_0. This pair of variables would in general also define the extrinsic curvature.) Since the time coordinate is physically meaningless, its value on the boundaries is irrelevant: two metric functions on three-space, $g^{(0)}_{\mu\nu}(x,y,z)$ and $g^{(1)}_{\mu\nu}(x,y,z)$, without mentioning time coordinates, suffice to determine a solution and hence physical time. Not even their order is essential, since there is no *absolute* direction of light cones. Similarly, the t-derivative of $g_{\mu\nu}$, resulting in the limit $t_1 \to t_0$, is required only up to a scalar factor (that would specify a meaningless initial 'speed of three-geometry' in superspace).

If one also eliminates all *spatial* coordinates from the metric $g_{\mu\nu}(x,y,z)$, it describes precisely the coordinate-independent three-geometry ${}^{(3)}G$. One may therefore expect the coordinate-independent content of the Einstein equations to determine the complete four-dimensional spacetime geometry in-between (and possibly beyond) two spatial geometries ${}^{(3)}G^{(0)}$ and ${}^{(3)}G^{(1)}$. However, the existence and uniqueness of a solution for this boundary value problem has not yet been generally proved (Bartnik and Fodor 1993, Giulini 1998).

The procedure is made transparent by writing the metric with respect to a chosen foliation as

$$\begin{pmatrix} g_{00} & g_{0l} \\ g_{k0} & g_{kl} \end{pmatrix} = \begin{pmatrix} N^i N_i - N^2 & N_l \\ N_k & g_{kl} \end{pmatrix} . \qquad (5.27)$$

The submatrix $g_{kl}(x,y,z,t)$ (with $k,l = 1,2,3$) for $t =$ constant is now the spatial metric on a hypersurface, while the *lapse function* $N(x,y,z,t)$ and the three *shift functions* $N_i(x,y,z,t)$ define arbitrary increments of time and space coordinates, respectively, for an orthogonal transition to an infinitesimally close space-like hypersurface. These four 'gauge functions' have to be *chosen* for convenience when solving an initial value problem.

The six functions forming the remaining symmetric matrix $g_{kl}(x,y,z,t)$ still contain three gauge functions representing the spatial coordinates. Their initial choice is specified by the initial matrix $g^{(0)}_{kl}(x,y,z)$, while the free shift functions determine their change with time. The three remaining, geometrically meaningful functions may be physically understood as representing the two polarization components of gravitational waves and the 'many-fingered' (local) physical time that describes the increase of all proper times along world lines connecting two infinitesimally close space-like hypersurfaces. These three degrees of freedom are not always separable from one another in practice, but all three are gauge-free (physical) dynamical variables. In contrast, the lapse function $N(x, y, z, t)$, together with the shift functions, merely determines how a specific time *coordinate* is related to this many-fingered time.

Therefore, the three-geometry ${}^{(3)}G$, representing the *dynamical state* of general relativity, is itself the 'carrier of information on physical time' (Baierlein, Sharp and Wheeler 1962): it *contains* physical time rather than *depending* on it. By means of the Einstein equations, ${}^{(3)}G$ determines a *continuum of physical clocks*, that is, all time-like distances from an 'initial' ${}^{(3)}G_0$ (provided a solution of the corresponding boundary value problem does exist). Given yesterday's geometry, today's geometry could not be tomorrow's – an absolutely non-trivial statement, since ${}^{(3)}G_0$ by itself is not a complete initial condition that would determine the solution of (5.7) up to a gauge. A mechanical clock can meaningfully go 'wrong'; for a rotating planet one would have to know the initial angle *and* the initial rotation velocity in order to read time from motion. However, a *speed* of three-geometry (in contrast to the *direction* of its velocity in superspace) would be as tautological as a 'speed of time'.

In this sense, Mach's principle (here with respect to time)[3] is anchored in general relativity: time must be realized by dynamical objects (such as spatial geometry). Dynamical laws that do *not* implicitly presume an absolute time are characterized by their *reparametrization invariance*, that is, invariance under monotonic transformations, $t \to t' = f(t)$. In general relativity, the time parameter t labels trajectories in superspace by the values of an appropriate time coordinate. No specific choice may then 'simplify' the laws according to Poincaré's definition (see Chap. 1), and no distinction between active and passive reparametrizations remains meaningful (see Norton 1989). It is therefore amazing to observe ongoing attempts to re-establish an external concept of time – even by means of 'phantom fields' (Thiemann 2006). The latter attempt was inspired (though not justified) by the problematic distinction between coordinate transformations and 'active' diffeomorphisms (see also Sect. 6.2.2).

Newton's equations are *not* invariant under a reparametrization. His time t is not an arbitrary parameter, but a dynamically preferred one ('absolute' time). Its reparametrization would merely be 'Kretzschmann invariant', that is, invariant under a trivial substitution of the old coordinates by new ones – thereby allowing for a reformulation of the dynamical laws by means of Coriolis-type pseudo-forces. Newton's equations can be brought into a reparametrization-invariant *form* only by artificially parametrizing the time variable t itself, $t(\lambda)$, and treating it as an additional dynamical variable with respect to λ. If $L(q, \dot{q})$ is the original Lagrangean, this leads to the *new*

[3] See Barbour and Pfister (1995) for various interpretations of Mach's principle.

variational principle

$$\delta \int \tilde{L}\left(q, \frac{\mathrm{d}q}{\mathrm{d}\lambda}, \frac{\mathrm{d}t}{\mathrm{d}\lambda}\right) \mathrm{d}\lambda := \delta \int L\left(q, \frac{\mathrm{d}q}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right) \frac{\mathrm{d}t}{\mathrm{d}\lambda}\mathrm{d}\lambda = 0 , \tag{5.28}$$

where the absolute time $t(\lambda)$ has to be varied, too. This procedure also helps to understand the meaning of the 'Δ-variation' that often appears somewhat unmotivated in analytical mechanics (see Sect. 8.6 of Goldstein 1980). Evidently, (5.28) is invariant under the reparametrization $\lambda \rightarrow \lambda' = f(\lambda)$.

Eliminating the formal variable t from (5.28) then leads to *Jacobi's principle* (see below), which was partially motivated by the pragmatic requirements of astronomers who did not have better clocks than the objects they were dynamically describing. These clocks, which define *ephemeris time*, are given by stellar positions when compared with *tables of ephemeris* produced by colleague astronomers. Since all celestial motions must be more or less 'perturbed' by others, they do not offer any obvious way to define Newton's time operationally. Jacobi's principle allowed astronomers to solve the equations of motion without explicitly using Newton's time. Einstein's equations of general relativity, on the other hand, are invariant under reparametrization of their time coordinate, $t \rightarrow t' = f(t)$, without any further and artificial parametrization $t(\lambda)$. There is no longer any time beyond the many-fingered dynamical variable contained in ${}^{(3)}G$!

In (5.28), $\mathrm{d}t/\mathrm{d}\lambda =: N(\lambda)$ may be regarded as a Newtonian lapse function (the relation between absolute time and a time parameter). For a time-independent Lagrangean L, t then appears as a cyclic variable. Its canonical momentum, $p_t := \partial\tilde{L}/\partial N = L - \sum p_i \dot{q}_i = -H$, which is conserved, is remarkable only because its quantization leads to the time-dependent Schrödinger equation. However, the 'super-Hamiltonian' $\tilde{H}$ that describes the extended system which includes $t(\lambda)$ is trivial:

$$\tilde{H} := \sum p_i \frac{\mathrm{d}q_i}{\mathrm{d}\lambda} + p_t \frac{\mathrm{d}t}{\mathrm{d}\lambda} - \tilde{L} = N\left(\sum p_i \frac{\mathrm{d}q_i}{\mathrm{d}t} - H - L\right) \equiv 0 . \tag{5.29}$$

More dynamical content can be extracted from Dirac's procedure of treating $N(\lambda)$ rather than $t(\lambda)$ as a new variable. The corresponding momentum, $p_N := \partial\tilde{L}/\partial(\mathrm{d}N/\mathrm{d}\lambda) \equiv 0$, has to be regarded as a constraint, while the *new* super-Hamiltonian is

$$H_S := \sum p_i \frac{\mathrm{d}q_i}{\mathrm{d}\lambda} + p_N \frac{\mathrm{d}N}{\mathrm{d}\lambda} - \tilde{L} = NH . \tag{5.30}$$

Although $\mathrm{d}N/\mathrm{d}\lambda$ cannot be eliminated in the usual way here by inverting the definition of canonical momentum $p_N(N, \mathrm{d}N/\mathrm{d}\lambda, \dots)$, it drops out everywhere in the Hamiltonian equations except in the derivative $\partial H_S/\partial p_N$, since it occurs only as a factor multiplying the vanishing p_N. The two new Hamiltonian equations related to the variable $N(\lambda)$ are (1) $\mathrm{d}N/\mathrm{d}\lambda = \partial H_S/\partial p_N = \mathrm{d}N/\mathrm{d}\lambda$, which is an identity, and (2) $\mathrm{d}p_N/\mathrm{d}\lambda = -\partial H_S/\partial N = -H$. Because $p_N \equiv 0$, one obtains the (secondary) *Hamiltonian constraint* $H = 0$ (but not $\equiv 0$), characteristic of reparametrization invariant theories. This result is the origin of the vanishing energy in (5.21), and will turn out to be important for quantum gravity. In general relativity, there are also three *momentum constraints*, characterizing invariance under spatial coordinate transformations, and related to the shift functions when chosen as formal dynamical variables.

Hamilton's new principle (5.28) can be written in the form

$$\delta \int \left(\sum p_i \dot{q}_i - H \right) \frac{\mathrm{d}t}{\mathrm{d}\lambda} \mathrm{d}\lambda = 0 \, .$$

For fixed energy value, $H = E$, the second term would cancel under this variation because of the new boundary conditions $\delta t(\lambda) = 0$. For the usual quadratic form of the kinetic energy, $2T = \sum a_{ij} \dot{q}_i \dot{q}_j = \sum p_i \dot{q}_i = 2(E - V)$, the integrand can in this case be written homogeneously *linear* in $\mathrm{d}q_i/\mathrm{d}\lambda$:

$$\delta \int \sqrt{2(E - V) \sum a_{ij} \frac{\mathrm{d}q_i}{\mathrm{d}\lambda} \frac{\mathrm{d}q_j}{\mathrm{d}\lambda}} \mathrm{d}\lambda = 0 \, . \tag{5.31}$$

This is Jacobi's principle (see Lanczos 1970), useful for fixed energy. It is manifestly invariant under reparametrization of λ, and can thus describe only timeless orbits $q_i(\lambda)$. Even though these nonrelativistic equations of motion could be explicitly simplified by using Newton's time, (5.31) evidently does not depend on the choice of λ.

In Newton's theory, the energy E depends on absolute velocities $\mathrm{d}q_i/\mathrm{d}t$. Jacobi's principle would therefore describe a 'Machian' theory only if the fixed energy represented a *universal* constraint. Barbour and Bertotti (1982) were able to propose an illuminating nonrelativistic toy model for Machian mechanics by means of the action principle

$$\delta \int \sqrt{-VT} \mathrm{d}t = 0 \, , \tag{5.32}$$

inspired by (5.31). It is universally invariant under reparametrizations of t (just like general relativity). Nothing new could then be obtained from parametrizing t in order to vary $t(\lambda)$ as in (5.28). Barbour and Bertotti also eliminated absolute rotations from their configuration space. While this has other important consequences, it is irrelevant for the problem of time. In general relativity, this 'Leibniz group', consisting of time reparametrizations and spatial rotations, would have to be generalized to the whole group of *diffeomorphisms* (general coordinate transformations). In order to eliminate any absolute meaning of a time *coordinate* on spacetime, the Hamiltonian constraint has to be understood as a local condition on the Hamiltonian *density*, since in field theory spatial coordinates serve as 'indices' – not as variables.

Barbour (1999) refers to the absence of a physically meaningful function $t(\lambda)$ in general relativity as its *timelessness*. However, parametrizable trajectories still permit asymmetric boundary conditions, which would define a direction of intrinsic time. This is different in *quantum* cosmology, where the Hamiltonian constraint, combined with the time–energy uncertainty relation, leads to a complete elimination of time (Sect. 6.2). In classical general relativity, even a constrained Hamiltonian would define trajectories which represent cosmic histories in the form of spacetime foliations that can be parametrized, although no *external* time is required for this purpose. While the global states which form these histories depend on this arbitrary foliation, the resulting spacetime geometry does not. So it defines an invariant many-fingered time, that is, all *proper times*, for all local objects, such as 'test clocks' or observers, uniquely.

In the Friedmann model (5.20), where the shift function has been chosen as $N \equiv 1$, the increment of the time coordinate t is identical (up to a sign) with the increment of proper times τ of 'comoving' matter (being at rest in the Friedmann coordinates, which fulfill the condition $N_i \equiv 0$). Elimination of the global time parameter t would here merely reproduce the equation of state $\rho(a)$ as the corresponding 'trajectory', since ρ is not an independent dynamical variable. There is evidently no intrinsic distinction between expansion and contraction of this 'universe'. The single variable a would determine proper times τ for comoving matter up to this ambiguity, since $\dot{a}^2$ is given as a function of a by the energy constraint (5.21).

Even for the exactly symmetric Friedmann universe, matter can be described *dynamically* by means of a homogeneous scalar field $\Phi(t)$. Its energy density may be chosen as

$$\rho = \frac{1}{2}(\dot{\Phi}^2 + m^2\Phi^2) \ . \tag{5.33}$$

The Hamiltonian of this simple 'quantum mechanical' model with respect to the variables $\alpha = \ln a$ and Φ, derived from (5.22) without cosmological constant, then reads

$$H = \frac{\mathrm{e}^{-3\alpha}}{2}\left(p_\alpha^2 - p_\Phi^2 + k\mathrm{e}^{4\alpha} - m^2\Phi^2\mathrm{e}^{6\alpha}\right) \ , \tag{5.34}$$

where the canonical momenta are $p_\alpha = \mathrm{e}^{3\alpha}\dot{\alpha}$ and $p_\Phi = -\mathrm{e}^{3\alpha}\dot{\Phi}$. A 'timeless orbit' for a closed universe ($k = 1$) in this model is depicted in Fig. 5.7. The freely chosen initial field $\Phi(a_0)$ at some small value a_0 first decays with increasing a, before it enters the 'matter-dominated' era, where it oscillates about the a-axis until it reaches a turning point in a as a consequence of the assumed positive curvature.

In the case of a Hamiltonian constraint, $H(p, q) = 0$, multiplying the Hamiltonian by a function $f(p, q)$, that is, $H \to H' = fH = 0$, would only induce an orbit-dependent reparametrization $t \to t'(t)$. This is given by $\mathrm{d}t'/\mathrm{d}t = f\big(p(t), q(t)\big)$, as can be seen by writing down the new Hamiltonian equations. For example, the choice $f \equiv -1$ would induce an inversion of the Hamiltonian time parameter for all trajectories. Therefore, the factor $\mathrm{e}^{-3\alpha}$ in (5.34) is irrelevant for the timeless orbits and can be omitted.

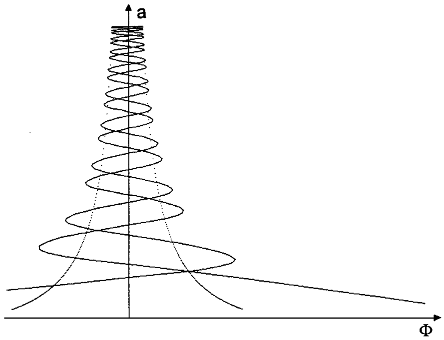

Fig. 5.7. Timeless classical orbit describing an expanding and recontracting dynamical Friedmann universe in terms of its expansion parameter a and a homogeneous massive scalar field Φ. *Dotted curves* represent vanishing Friedmann potential V as defined by (5.34). For slightly larger initial values $\Phi(a_0)$ than chosen in the figure, the 'inflation era', defined by the decaying initial field, would last over many orders of magnitude in a before the orbit entered the 'matter-dominated' era, where it performs a huge number of oscillations before reaching its turning point $a_{\max}$. (After Hawking and Wu 1985.) This dynamical description is very different in quantum gravity (see Fig. 6.3)

While this simple dynamical model cannot describe any thermodynamical aspects, it can be generalized by means of a multipole expansion on the Friedmann sphere,

$$\Phi(\chi, \theta, \phi, t) = \sum a_{nlm}(t) Q^n_{lm}(\chi, \theta, \phi) \; , \tag{5.35}$$

where $Q^n_{lm}(\chi, \theta, \phi)$ are spherical harmonics on a three-sphere (Halliwell and Hawking 1985). The variable $\Phi(t)$ in (5.34) represents the monopole component, $\Phi = a_{000}$, since $Q^0_{00} = 1$. A similar expansion of the metric tensor field g_{kl} requires vector and tensor harmonics in addition to the scalar harmonics Q^n_{lm}. Only the tensor harmonics turn out to represent physical (geometric) properties, while all others describe gauge degrees of freedom. In this 'perturbed Friedmann model', the time parameter t no longer automatically represents proper time on comoving world lines.

In (5.34) and its generalization to a multipole expansion, the kinetic energy of matter occurs with a negative sign (that is, with negative dynamical mass), since it entered the Hamiltonian as a source of gravity (representing negative potential energy). In Friedmann-type models, all gauge-free geometric degrees of freedom but the global expansion parameter a (or its logarithm) share this property (Giulini and Kiefer 1994, Giulini 1995), because gravitational waves imposed on a flat spacetime possess gravitating positive energy. The kinetic energy is thus not positive definite in cosmology, while the metric in infinite-dimensional superspace that it defines by its quadratic form is *super-Lorentzian* (with signature $+ - - - \dots$).[4]

This fact has important consequences. In the familiar case of mechanics, vanishing kinetic energy, $E - V = 0$, describes turning points of the motion. However, since there are no forbidden regions for indefinite kinetic energy, the boundary $V = V - E = 0$ does not force the trajectories to come to a halt and reverse direction here. Rather, this condition now describes a smooth transition between 'subluminal' and 'superluminal' directions in superspace (not in space!), as can be seen in Fig. 5.7. A trajectory would be reflected from an *infinite* potential 'barrier' only if this were either negative at a time-like boundary, or positive at a space-like one. Reversal of the cosmic expansion at $a_{\max}$ requires the vanishing of an appropriate $V_{\text{eff}}(\alpha)$ that includes the actual kinetic energy of the other degrees of freedom (similar to the effective radial potential in the Kepler problem). It is evident that this behavior must be important for a reversal of time and its arrow.

In the Friedmann model, a point on the trajectory in configuration space determines Friedmann time t (that could be read from comoving test clocks) – except where the curve intersects itself. In a mini-superspace with more than two degrees of freedom (adding a material clock, for example), physical time on a trajectory is generically *unique*, since intersections could occur only accidentally. This demonstrates that the essential requirement for the state to represent a carrier of information about time is reparametrization invariance of the dynamical laws – not its spacetime-geometric interpretation.

Although a time parameter is in general physically meaningless in these theories, it is often misused for an inappropriate interpretation. An example is Veneziano's (1991) string model, based on a dilaton field Φ. Its equations of motion lead to a time dependence of the form $f(t - t_0)$, with an integration constant t_0 that determines the value of the time parameter at the big bang (where $\alpha = -\infty$). A translation $t_0 \to t_0 + T$ would thus be meaningless (as already pointed out by Leibniz). The solution for $t < t_0$, where expansion *accelerates* exponentially in this model, has been interpreted as 'pre-big bang', while the absence of a smooth connection between pre- and post-big bang has been called a 'graceful exit problem' (Brustein and Veneziano 1994). However, this mathematical model has simply two different solutions, which could conceivably be related through an infinite parameter time, $t = \pm\infty$ – similar to Schwarzschild time at a horizon. Coordinate times $t < t_0$ would then represent physical times *later* than $t > t_0$, while a continuation through t_0 is merely formal (Dabrowski and Kiefer 1997).

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[4] There is also a *local*, 6-dimensional Lorentzian metric in superspace, corresponding to the 6 degrees of freedom of the submatrix g_{kl} at every space point, such that there seems to be an infinity of time-like variables (see Sect. 6.2.2). However, all but one of them are unphysical gauge degrees of freedom in a Friedmann type universe.

The shift functions N_i of (5.27) can be chosen to vanish even when spatial symmetries are absent. The secondary *momentum constraints* $H_i := \partial\tilde{H}/\partial N_i = 0$, which warrant conservation of vanishing canonical momenta p_{N_i}, and which are fulfilled automatically for the Friedmann solution because of its symmetry, then have to be solved explicitly. The lapse function $N(x, y, z, t)$ now determines genuine *many-fingered* time (as a spatial *field* on the dynamically evolving hypersurface) with respect to the coordinate t. If N is nonetheless chosen as a function of t alone, the foliation proceeds everywhere according to physical time (*normal* to the hypersurface, with fixed 'comoving' coordinates).

This may not always be a convenient choice. For example, observers coming very close to a black hole horizon would observe the stars moving very fast through a little hole that remains in the sky above the horizon because of their extreme time dilation. In a universe that is bound to recontract they could reach the contraction era within very short proper times. This renders the immediate vicinity of horizons very sensitive to a conceivable cosmic *final* condition, which may even exclude black hole horizons and singularities (see Zeh 1983, 2005a, and Sect. 6.2.3). In this case, a foliation according to *York time*, mentioned in Sect. 5.1, may be preferable, since it arrives 'simultaneously' at all final singularities. Note, however, that the external curvature scalar K, which defines York time, is *not* a function of state, $f({}^{(3)}G)$.

Among the simplest inhomogeneous models are the spherically symmetric ones, with a metric

$$ds^2 = -N(\chi, t)^2 dt^2 + L(\chi, t)^2 d\chi^2 + R(\chi, t)^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) . \qquad (5.36)$$

They contain one remaining spatial gauge function, that has to be eliminated by means of the momentum constraint $H_\chi = 0$. This is analogous to Gauß's law in electrodynamics, as it similarly refers to the radial coordinate.

Qadir (1988) proposed an illustrative toy model for such an inhomogeneous universe (Fig. 5.8). It forms a generalization of the Oppenheimer–Snyder model for the gravitational collapse of a homogeneous spherical dust cloud (see Misner, Thorne and Wheeler 1973, Chap. 32). The latter model pastes (or 'sutures') a comoving spherical surface surrounding part of a contracting closed Friedmann solution (representing the dust cloud) consistently to the external region a Schwarzschild–Kruskal solution. Qadir then pastes this Schwarzschild solution in turn to another (much larger) partial Friedmann solution with much smaller energy density (his universe proper). This pasting at two spatial boundaries, with Friedmann radial coordinate values χ_1 and χ_2, say, is consistent only if the total masses of the two partial Friedmann universes are identical, and can thus be identified with the Schwarzschild mass M characterizing the partial vacuum solution. The latter forms a strip from Fig. 5.2 between two non-intersecting geodesics that lead from the past to the future Kruskal singularity (big bang and big crunch).

In order to comply with the Weyl tensor hypothesis as much as possible, Qadir assumed the 'Schwarzschild corridor' to be absent at the big bang.

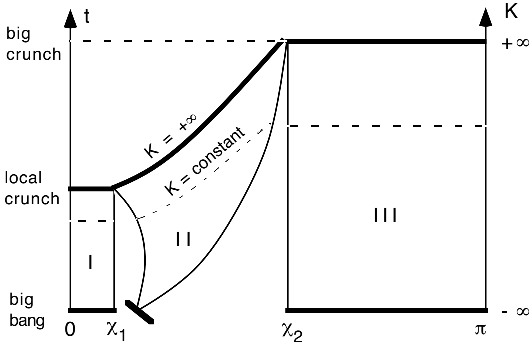

Fig. 5.8. Qadir's 'suture model' of a collapsing homogeneous dust cloud, I, as part of an expanding and recontracting Friedmann universe, III. The Friedmann spheres at χ_1 (*left*) and χ_2 (*right*) are initially identified. The Weyl tensor, representing the gravitational degrees of freedom, is thus chosen to vanish initially (except at the spatial boundary between the two regions I and III), but will grow by means of an emerging 'Schwarzschild–Kruskal corridor', II, (a strip from Fig. 5.2). The spatial boundaries of the three spacetime regions have to be identified (including proper times on them, all chosen to start at the big bang). According to a picture due to Penrose, the singularity inside the black hole (region I) together with its attached Kruskal singularity (in region II) appears as a 'stalactite' hanging from the 'ceiling' (which represents the big crunch singularity in region III). In contrast, there is only one (piecewise homogeneous) big bang singularity (a flat floor in Penrose's picture) at $K = -\infty$, that is chosen as the first slice of the foliation (corresponding to $t = 0$)

The density discontinuity then represents an initial inhomogeneity. Since the denser part of this toy universe feels stronger gravitational attraction than the less dense one, its expansion decelerates (or its contraction accelerates) faster. A vacuum corridor must then form and grow in size with increasing temporal distance from the big bang. As the energy–momentum tensor vanishes in the Schwarzschild–Kruskal region, the curvature is there entirely due to the Weyl tensor, while the latter vanishes inside the two partial Friedmann universes. The time arrow of this process of 'gravitational monopole radiation' (the formation of the corridor with its non-zero gravitational degrees of freedom) is once again a consequence of the special initial condition.

This model is certainly interesting as an illustration of the Weyl tensor hypothesis, but it does not describe statistical (entropic) aspects. For this purpose, many multipoles of (5.35) would have to be taken into account as radiation. Qadir's cosmic evolution process simply describes an example of motion away from the chosen initial state – similar to what is normally found in unbound mechanical systems regardless of any statistical considerations.

General Literature: Chap. 21 of Misner, Thorne and Wheeler 1973; Barbour 1999; Kiefer 2007.