8

Theory of Small Oscillations

The study of the oscillations of a system in a neighbourhood of an equilibrium position or a periodic motion usually begins with linearization. The linearized system can be integrated. After this is done, the main properties of the oscillations in the original system can often be determined by using the theory of Poincaré–Birkhoff normal forms. This theory is an analogue of perturbation theory (§ 6.2). The linearized system plays the role of the unperturbed system with respect to the original one. In this chapter we describe the basic elements of this approach.

The central problem of theory of small oscillations is the study of stability of an equilibrium or a periodic motion. There is extensive literature devoted to stability theory (see the surveys [11, 12, 24]). We consider briefly only some results of this theory, which enable one to make conclusions on stability based on studying normal forms. We also describe results related to the problem of finding converses of Lagrange's theorem on the stability of an equilibrium in a conservative field.

8.1 Linearization

We consider a natural Lagrangian system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0, \qquad L = T - U(q), \qquad T = \frac{1}{2}(A(q)\dot{q}, \dot{q}). \tag{8.1}$$

The equilibrium positions of system (8.1) are critical points of the potential energy U. In order to linearize system (8.1) about the equilibrium position $q = 0$ it is sufficient to replace the kinetic energy T by its value T_2 at $q = 0$, and the potential energy U by its quadratic part U_2 in a neighbourhood of zero.

Example 8.1. For a one-dimensional system,

$$L = a(q)\dot{q}^2/2 - U(q), \qquad L_2 = T_2 - U_2 = \frac{1}{2}(a\dot{q}^2 - bq^2),$$
$$a = a(0), \qquad b = \left.\frac{\partial^2 U}{\partial q^2}\right|_{q=0},$$

and the linearized equation of motion is $a\ddot{q} + bq = 0$. △

We now consider a Hamiltonian system. Its equilibrium positions are critical points of the Hamiltonian. In order to linearize a Hamiltonian system near an equilibrium position it is sufficient to replace the Hamiltonian by its quadratic part in a neighbourhood of this equilibrium position.

The linearization of a Hamiltonian system near a periodic trajectory is considered in § 8.3.2.

8.2 Normal Forms of Linear Oscillations

8.2.1 Normal Form of a Linear Natural Lagrangian System

We consider a dynamical system with a quadratic Lagrange function $L_2 = T_2 - U_2$, $T_2 \geqslant 0$. Its oscillations take a particularly simple form in special coordinates, which are called principal or normal.

Theorem 8.1. *A quadratic Lagrange function can be reduced by a linear change of coordinates $Q = Cq$ to a diagonal form*

$$L_2 = \frac{1}{2}(\dot{Q}_1^2 + \cdots + \dot{Q}_n^2) - \frac{1}{2}(\lambda_1 Q_1^2 + \cdots + \lambda_n Q_n^2), \tag{8.2}$$

and the equations of motion, correspondingly, to the form

$$\ddot{Q}_i = -\lambda_i Q_i, \qquad i = 1, \ldots, n. \tag{8.3}$$

The eigenvalues λ_i are the roots of the characteristic equation

$$\det(B - \lambda A) = 0,$$

where $T_2 = \frac{1}{2}(A\dot{q}, \dot{q})$ and $U_2 = \frac{1}{2}(Bq, q)$.

◁ The pair of quadratic forms T_2 and U_2, one of which (T_2) is positive definite, can be reduced to principal axes by a simultaneous linear change of variables. The new coordinates can be chosen so that the form T_2 is reduced to the sum of squares. ▷

Corollary 8.1. *A system performing linear oscillations is a direct product of n linear one-dimensional systems.*

For each one-dimensional system (8.3) there are three possible cases:

1) $\lambda_i = \omega^2$; the solution is $Q = c_1 \cos \omega t + c_2 \sin \omega t$ (oscillations);
2) $\lambda_i = 0$; the solution is $Q = c_1 + c_2 t$ (neutral equilibrium);
3) $\lambda_i = -k^2 < 0$; the solution is $Q = c_1 \cosh kt + c_2 \sinh kt$ (instability).

Corollary 8.2. *Suppose that one of the eigenvalues is positive: $\lambda = \omega^2 > 0$. Then the system can perform a periodic oscillation of the form*

$$q(t) = (c_1 \cos \omega t + c_2 \sin \omega t)\xi,$$

where ξ is an eigenvector corresponding to λ: $B\xi = \lambda A\xi$.

This periodic motion is called a *characteristic oscillation* (or a *principal oscillation*, or a *normal mode*), and the number ω a *characteristic* (or *principal*, or *normal*) *frequency*.

These results are also valid when there are multiple eigenvalues: in contrast to a general system of differential equations (and even a general Hamiltonian system), in a natural Lagrangian system no resonant terms of the form $t \sin \omega t$, etc., can appear even in the case of multiple eigenvalues (only for $\lambda = 0$ Jordan blocks of order 2 appear).

8.2.2 Rayleigh–Fisher–Courant Theorems on the Behaviour of Characteristic Frequencies when Rigidity Increases or Constraints are Imposed

Of two linear Lagrangian systems with equal kinetic energies, the more rigid (or *stiff*) is by definition the one that has higher potential energy.

Theorem 8.2. *As the rigidity of a system performing small oscillations increases, all the characteristic frequencies increase.*

A natural Lagrangian system with $n-1$ degrees of freedom is said to be obtained from a system with n degrees of freedom performing small oscillations by imposition of a linear constraint if its kinetic and potential energies are the restrictions of the kinetic and potential energies of the original system to an $(n-1)$-dimensional subspace.

Theorem 8.3. *The characteristic frequencies ω_i', $i = 1, \ldots, n-1$ of the system with constraint separate the characteristic frequencies ω_i of the original system* (*Fig.* 8.1).

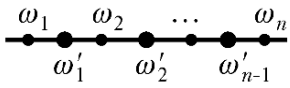

Fig. 8.1.

8.2.3 Normal Forms of Quadratic Hamiltonians

We consider a Hamiltonian system with a quadratic Hamiltonian function

$$\dot{z} = I\frac{\partial H}{\partial z}, \qquad z \in \mathbb{R}^{2n}, \qquad H = \frac{1}{2}(\Omega z, z), \qquad I = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}.$$

The roots of the characteristic equation

$$\det(I\Omega - \lambda E_{2n}) = 0$$

are called the eigenvalues of the Hamiltonian.

Theorem 8.4. *The eigenvalues of the Hamiltonian are situated on the plane of complex variable λ symmetrically with respect to the coordinate cross (Fig. 8.2): if λ is an eigenvalue, then $\bar{\lambda}$, $-\lambda$, $-\bar{\lambda}$ are also eigenvalues. The eigenvalues λ, $\bar{\lambda}$, $-\lambda$, $-\bar{\lambda}$ have equal multiplicities and the corresponding Jordan structures are the same.*

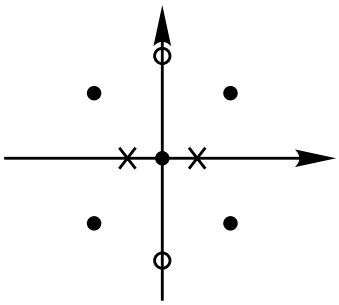

Fig. 8.2.

$\lhd$ The matrices $I\Omega$ and $(-I\Omega)^T$ are similar: $I\Omega = I^{-1}(-I\Omega)^T I$ (because $I^2 = -1$). $\rhd$

Corollary 8.3. *In a Hamiltonian system stability is always neutral: if an equilibrium is stable, then the real parts of all the eigenvalues are equal to zero.*

Corollary 8.4. *If there is a purely imaginary simple eigenvalue, then it remains on the imaginary axis under a small perturbation of the Hamiltonian. Similarly, a real simple eigenvalue remains real under a small perturbation.*

Corollary 8.5. *If $\lambda = 0$ is an eigenvalue, then it necessarily has even multiplicity.*

According to Theorem 8.4, eigenvalues can be of four types: real pairs $(a, -a)$, purely imaginary pairs $(ib, -ib)$, quadruplets $(\pm a \pm ib)$, and zero eigenvalues.

For Hamiltonian systems the following assertion replaces the theorem on reduction of the matrix of a linear differential equation to the Jordan form.

Theorem 8.5 (Williamson [50])**.** *There exists a real symplectic linear change of variables reducing the Hamiltonian to a sum of partial Hamiltonians (functions of disjoint subsets of conjugate variables), and the matrix of the system, correspondingly, to a block-diagonal form. Each partial Hamiltonian corresponds either to a real pair, or to an imaginary pair, or to a quadruplet of eigenvalues, or to a zero eigenvalue. The partial Hamiltonians are determined, up to a sign, by the Jordan blocks of the operator $I\Omega$.*

The list of partial Hamiltonians is given in [10, 240].

All the eigenvalues of a generic Hamiltonian are simple. To a simple real pair $(a, -a)$ there corresponds the partial Hamiltonian $\mathscr{H} = -ap_1q_1$; to a simple purely imaginary pair $(ib, -ib)$, the Hamiltonian $\mathscr{H} = \pm b\big(p_1^2 + q_1^2\big)/2$ (the Hamiltonians with the upper and lower sign cannot be transformed into one another); to a quadruplet $(\pm a \pm ib)$, the Hamiltonian $\mathscr{H} = -a(p_1q_1 + p_2q_2) + b(p_1q_2 - p_2q_1)$. For an imaginary pair one often uses *symplectic polar coordinates* ρ, φ: $p = \sqrt{2\rho}\cos\varphi$, $q = \sqrt{2\rho}\sin\varphi$. Then the Hamiltonian is $\mathscr{H} = \pm b\rho$, where $\rho = (p^2 + q^2)/2$.

Corollary 8.6. *Let $\lambda = i\omega$ be a simple purely imaginary eigenvalue. Then the system can perform a periodic oscillation of the form*

$$z = \operatorname{Re}\left(\xi \exp\left(i\omega(t + t_0)\right)\right),$$

where ξ is a corresponding eigenvector: $(I\Omega - i\omega E_{2n})\,\xi = 0$. This motion is called a characteristic oscillation, and ω a characteristic frequency.

Corollary 8.7. *If the eigenvalues are all distinct and purely imaginary, then the Hamiltonian can be reduced to the normal form*

$$H = \frac{1}{2}\omega_1\big(p_1^2 + q_1^2\big) + \cdots + \frac{1}{2}\omega_n\big(p_n^2 + q_n^2\big) \tag{8.4}$$

or, in symplectic polar coordinates, $H = \omega_1\rho_1 + \cdots + \omega_n\rho_n$. The motion is a sum of characteristic oscillations.

Remark 8.1. If the Hamiltonian has the form (8.4), then the equilibrium is stable regardless of whether the Hamiltonian is positive definite or not (for a natural Lagrangian linear system an equilibrium is stable only if the total energy is positive definite).

It is often necessary to consider not an individual Hamiltonian but a family depending on parameters. In such a family, for some values of the parameters there can appear singularities: multiple eigenvalues and, correspondingly,

Jordan blocks of order greater than 1 in the matrix of the system; moreover, these singularities can even be unremovable by a small change of the family of Hamiltonians. For every finite l, the unremovable singularities arising in l-parameter families of Hamiltonians are indicated in [240]. Also calculated therein are the versal deformations of these singularities, that is, normal forms to which any family of quadratic Hamiltonians smoothly depending on parameters can be reduced in a neighbourhood of singular values of the parameters by means of symplectic linear changes of variables smoothly depending on the parameters. In particular, in a one-parameter family of Hamiltonians, generally speaking, only the following three singularities occur: a real pair of multiplicity two, $(\pm a)^2$, with two Jordan blocks of order 2; an imaginary pair of multiplicity two, $(\pm ib)^2$, also with two Jordan blocks of order 2; and a zero eigenvalue of multiplicity two, $(0)^2$, with one Jordan block of order 2. The versal deformations of these singularities are:

$$\begin{aligned}
(\pm a)^2\colon &\quad \mathscr{H} = -(a+\delta_2)(p_1q_1+p_2q_2) + p_1q_2 + \delta_1 p_2 q_1,\\
(\pm ib)^2\colon &\quad \mathscr{H} = \pm\frac{p_1^2+p_2^2}{2} + (b+\delta_2)\,(p_2q_1 - p_1q_2) + \frac{\delta_1\left(q_1^2+q_2^2\right)}{2},\\
(0)^2\colon &\quad \mathscr{H} = \pm\frac{p_1^2}{2} + \frac{\delta_1 q_1^2}{2}.
\end{aligned} \tag{8.5}$$

Here δ_1, δ_2 are the parameters of the deformations.

8.3 Normal Forms of Hamiltonian Systems near an Equilibrium Position

8.3.1 Reduction to Normal Form

Let the origin of coordinates be an equilibrium position of an analytic Hamiltonian system with n degrees of freedom. Suppose that the eigenvalues of the quadratic part of the Hamiltonian in a neighbourhood of the equilibrium position are all distinct and purely imaginary. In accordance with what was said in § 8.1 and § 8.2.2, we represent the Hamiltonian in the form

$$H = \frac{1}{2}\omega_1\left(p_1^2+q_1^2\right) + \cdots + \frac{1}{2}\omega_n\left(p_n^2+q_n^2\right) + H_3 + H_4 + \cdots, \tag{8.6}$$

where H_m is a form of degree m in the phase variables p, q. (Some of the frequencies ω_i can be negative.)

Definition 8.1. The characteristic frequencies $\omega_1, \ldots, \omega_n$ satisfy a *resonance relation of order* $l > 0$ if there exist integers k_i such that $k_1\omega_1 + \cdots + k_n\omega_n = 0$ and $|k_1| + \cdots + |k_n| = l$. For example, $\omega_1 = \omega_2$ is a relation of order 2.

Definition 8.2. A *Birkhoff normal form of degree L for the Hamiltonian* is a polynomial of degree L in symplectic phase variables P, Q that is actually a polynomial of degree $[L/2]$ in the variables $\rho_i = \left(P_i^2+Q_i^2\right)/2$.

Example 8.2. For a system with two degrees of freedom,

$$H = \omega_1\rho_1 + \omega_2\rho_2 + \frac{1}{2}\left(\omega_{11}\rho_1^2 + 2\omega_{12}\rho_1\rho_2 + \omega_{22}\rho_2^2\right) \tag{8.7}$$

is a Birkhoff normal form of degree 4. The terms quadratic in ρ describe the dependence of the frequencies of the oscillations on the amplitudes. △

Theorem 8.6 (Birkhoff [14])**.** *Suppose that the characteristic frequencies ω_i do not satisfy any resonance relation of order L or less. Then in a neighbourhood of the equilibrium position* 0 *there exists a symplectic change of variables $(p, q) \mapsto (P, Q)$ fixing the equilibrium position* 0 *and such that in the new variables the Hamiltonian function is reduced to a Birkhoff normal form $\mathscr{H}_L(\rho)$ of degree L up to terms of degree higher than L:*

$$H(p, q) = \mathscr{H}_L(\rho) + R, \qquad R = O(|P| + |Q|)^{L+1}. \tag{8.8}$$

Discarding the non-normalized terms R in (8.8) we obtain an integrable system whose action–angle variables are the symplectic polar coordinates ρ_i, φ_i defined by

$$P_i = \sqrt{2\rho_i}\cos\varphi_i, \qquad Q_i = \sqrt{2\rho_i}\sin\varphi_i, \tag{8.9}$$

and whose trajectories wind round the tori $\rho = \text{const}$ with frequencies $\partial\mathscr{H}_L/\partial\rho$. Most of similar tori, which are invariant under the phase flow, in the general case exist also in the original system; this follows from the results of KAM theory (§ 6.3.6.B).

Birkhoff's normalization amounts to Lindstedt's procedure for eliminating the fast phases (§ 6.2.2) if we normalize the deviations from the equilibrium position by a small quantity ε (putting $p = \varepsilon\widehat{p}$, $q = \varepsilon\widehat{q}$, $\widehat{H} = H/\varepsilon^2$) and pass to the symplectic polar coordinates.

The normalization procedure is described below for a more general case (see the proof of Theorem 8.7). The generating function of the normalizing transformation is constructed in the form of a polynomial of degree L in the phase variables. A change in the terms of degree l in the original Hamiltonian does not change the terms of degree lower than l in the normal form (and of degree lower than $l - 1$ in the normalizing transformation).

In the absence of resonances, a Hamiltonian is in normal form if and only if the Poisson bracket of the Hamiltonian and its quadratic part is identically zero (see Proposition 5.1).

Considering the normalization as $L \to \infty$ we arrive at the notion of formal normal form, which was discussed in § 5.1.3.

The definition of a normal form must be modified for the case where the characteristic frequencies satisfy some resonance relations. The same modification is also appropriate for nearly resonant frequencies. Let K be a sublattice of the integer lattice $\mathbb{Z}^n$ defining the possible resonances (cf. § 6.1.1).

Definition 8.3. A *resonant normal form of degree L for the Hamiltonian* for resonances in K is a polynomial of degree L in symplectic variables P_i, Q_i which in the polar coordinates (8.9) depends on the phases φ_i only via their combinations (k, φ) for $k \in K$.

Theorem 8.7 ([179, 271])**.** *Suppose that the characteristic frequencies do not satisfy any resonance relations of degree L or less, except, possibly, for relations $(k, \omega) = 0$ with $k \in K$. Then in a neighbourhood of the zero equilibrium position there exists a symplectic change of variables $(p, q) \mapsto (P, Q)$ fixing the zero equilibrium position and such that in the new variables the Hamiltonian function reduces to a resonant normal form of degree L for resonances in K up to terms of degree higher than L.*

$\lhd$ In the system with Hamiltonian (8.6) we perform the change of variables with a generating function $Pq + S(P, q)$, $S = S_3 + \cdots + S_L$. The new Hamiltonian has the form

$$\mathscr{H} = \frac{1}{2}\omega_1\left(P_1^2 + Q_1^2\right) + \cdots + \frac{1}{2}\omega_n\left(P_n^2 + Q_n^2\right) + \mathscr{H}_3 + \mathscr{H}_4 + \cdots,$$

where S_l and $\mathscr{H}_l$ are forms of degree l in P, q and in P, Q, respectively. The old and new Hamiltonians are connected by the relation

$$H\left(P + \frac{\partial S}{\partial q}, q\right) = \mathscr{H}\left(P, q + \frac{\partial S}{\partial P}\right).$$

Equating here the forms of the same order in P, q we obtain

$$\sum_{j=1}^{n} \omega_j \left(P_j \frac{\partial S_l}{\partial q_j} - q_j \frac{\partial S_l}{\partial P_j}\right) = \mathscr{H}_l - F_l, \qquad l = 3, \ldots, L.$$

The form F_l is uniquely determined if we know the $S_\nu, \mathscr{H}_\nu$ for $\nu \leqslant l - 1$. In the symplectic polar coordinates ρ, φ the last equation takes the form

$$\omega \frac{\partial S_l}{\partial \varphi} = \mathscr{H}_l - F_l.$$

We choose

$$S_l = \sum i \, \frac{f_k(\rho)}{(k, \omega)} \exp\left(i(k, \varphi)\right), \qquad k \notin K,$$

where the f_k are the coefficients of the Fourier series of F_l. Then $\mathscr{H}_l$ is in the required normal form. Thus we can successively determine all the $S_l, \mathscr{H}_l$. Returning to Cartesian coordinates we obtain the result. $\rhd$

Suppose that the Hamiltonian is in a resonant normal form. If the rank of the sublattice $K \subset \mathbb{Z}^n$ defining the possible resonances is equal to r, then the system has $n - r$ independent integrals in involution which are linear combinations with integer coefficients of the quantities $\rho_i = (P_i^2 + Q_i^2)/2$

(cf. Theorem 6.15 in Ch. 6). In particular, if $r = 1$, then the system in the normal form is integrable.

Resonance normalization amounts to von Zeipel's procedure for eliminating the fast non-resonant phases (§ 6.2.2) if we normalize the deviations from the equilibrium position by a small quantity ε and pass to the symplectic polar coordinates.

In the presence of resonances, a Hamiltonian is in resonant normal form if and only if the Poisson bracket of the Hamiltonian and its quadratic part is identically zero (see Proposition 5.1).

If the matrix of the linearized system is not diagonalizable, then the quadratic part of the Hamiltonian cannot be reduced to the form (8.4). However, the nonlinear terms can be reduced to the form indicated in Theorem 8.7; see [151].

8.3.2 Phase Portraits of Systems with Two Degrees of Freedom in a Neighbourhood of an Equilibrium Position at a Resonance

Any system with two degrees of freedom whose Hamiltonian is in resonant normal form is integrable. One can reduce such a system to a system with one degree of freedom depending on the constant value of the first integral as a parameter, and then draw the phase portraits. If the coefficients of the lower terms of the normal form are generic, then for the given resonance there are only finitely many types of phase portraits, and these types are determined by the lower terms of the normal form. The phase portraits are qualitatively different only for finitely many resonances. Description of the portraits provides exhaustive information about the motion near the resonance for systems in a normal form in the generic case. Correspondingly, we obtain considerable information on the motion for systems in which the lower terms of the Hamiltonian can be reduced to this normal form. Below we give the list of phase portraits and their bifurcations. For lack of space we confine ourselves to the case where the frequencies ω_1 and ω_2 have different signs, since this case is more interesting from the viewpoint of stability theory (if $\omega_1\omega_2 > 0$, then an energy level $H = h \ll 1$ is a sphere, and the equilibrium is stable). The information requisite for constructing these portraits is contained in a series of papers of Alfriend, Henrard, van der Burgh, Duistermaat, Markeev, Roels, Sanders, Schmidt, et al. The complete information is presented in [217]. The portraits for resonances of order higher than 4 can be found in [534].

Let k_1, k_2 be coprime positive coefficients of a resonance relation. There exist coprime integers l_1, l_2 such that $k_1 l_2 - k_2 l_1 = 1$. In a neighbourhood of the equilibrium position we pass to the canonical polar coordinates ρ, φ given by (8.9) and then perform the change of variables

$$(\rho_1, \rho_2, \varphi_1, \varphi_2) \mapsto (G, I, \psi, \chi)$$

with the generating function

$$S = (k_1\varphi_1 + k_2\varphi_2)\, G + (l_1\varphi_1 + l_2\varphi_2)\, I,$$

so that

$$\psi = k_1\varphi_1 + k_2\varphi_2, \qquad G = l_2\rho_1 - l_1\rho_2,$$
$$\chi = l_1\varphi_1 + l_2\varphi_2, \qquad I = -k_2\rho_1 + k_1\rho_2.$$

Since by the assumption the Hamiltonian is in a normal form, it is independent of χ; correspondingly, I is an integral of the problem. We perform the isoenergetic reduction on an energy level $H = h$ (see [10]); as the new time we introduce the phase χ. We obtain the reduced system with one degree of freedom whose Hamiltonian depends on the parameter h. It is the phase portrait of this system that must be analysed. In the generic case the portrait depends essentially on one more parameter – the resonance detuning $\delta = k_1\omega_1 + k_2\omega_2$.

A neighbourhood of the origin on the plane h, δ is partitioned into the domains corresponding to different types of the phase portrait. These partitions for different resonances are shown in Fig. 8.3*a*–8.8*a*, and the bifurcations of the phase portrait for going around the origin clockwise are shown in Fig. 8.3*b*–8.8*b*, respectively. The numbering of the portraits corresponds to the numbering of the domains on the plane of parameters. The unnumbered portraits correspond to the curves separating the domains; they are given only in Fig. 8.3–8.5.

The normal forms for which the bifurcations are given have the form

$$H_{k_1,k_2} = \omega_1\rho_1 + \omega_2\rho_2 + F(\rho_1, \rho_2) + B\rho_1^{k_1/2}\rho_2^{k_2/2}\cos(k_1\varphi_1 + k_2\varphi_2 + \psi_0).$$

Here F is a polynomial in ρ_1, ρ_2 beginning with the quadratic form $F_2(\rho_1, \rho_2)$ (in the Hamiltonian $H_{2,1}$ the term F must be omitted), and B, ψ_0 are constants. The required genericity conditions are $B \neq 0$, $A = F_2(k_1, k_2) \neq 0$, and, for the Hamiltonian $H_{3,1}$, $|A| \neq 3\sqrt{3}|B|$. The pictures correspond to the case $\omega_1 > 0$, $A > 0$, $B > 0$ (this does not cause a loss of generality). The pictures are given for the following resonant vectors (k_1, k_2): (2,1) in Fig. 8.3; (3,1) in Fig. 8.4 if $A < 3\sqrt{3}B$, and in Fig. 8.5 if $A > 3\sqrt{3}B$; (4,1) in Fig. 8.6; (3,2) in Fig. 8.7; and (4,3) in Fig. 8.8. For the resonant vectors $(n, 1)$, $n \geqslant 5$, the bifurcations are the same as for (4,1); for $(n, 2)$, $n \geqslant 5$, the same as for (3,2) but the domain (5) is skipped in Fig. 8.7; for $(n, 3)$, $n \geqslant 5$, the same as for (4,3); and for (n, m), $n \geqslant 5$, $m \geqslant 4$, the same as for (4,3) but the domain (2) is skipped in Fig. 8.8. (Of course, the number of singular points of each type must be changed taking into account the symmetry of the Hamiltonian). The axis $\delta = 0$ is not a bifurcation line. The positions of the bifurcation lines with respect to this axis may be different from those shown in the pictures.

We make several further remarks on the presentation of the information. To ensure that the phase portraits have no singularities, we depicted them for $h > 0$ in the polar coordinates $\sqrt{\rho_2}$, ψ/k_2, and for $h < 0$ in the polar coordinates $\sqrt{\rho_1}$, ψ/k_1. For $h = 0$ the phase portrait in Fig. 8.3–8.5 is depicted

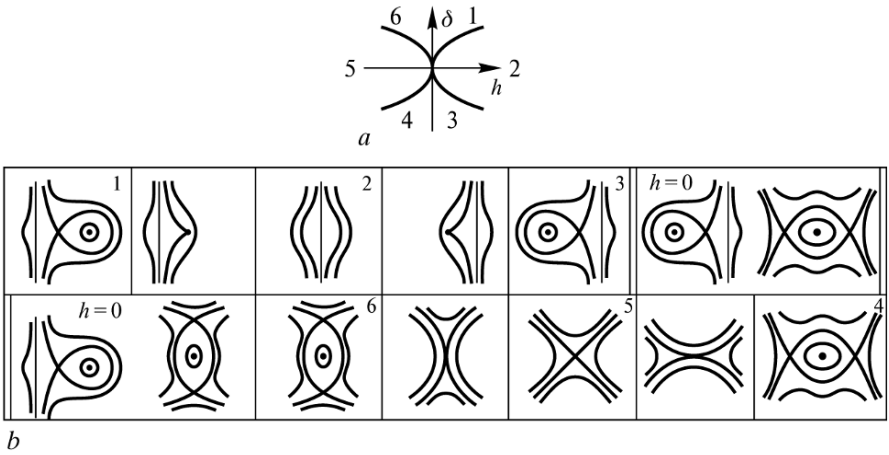

Fig. 8.3.

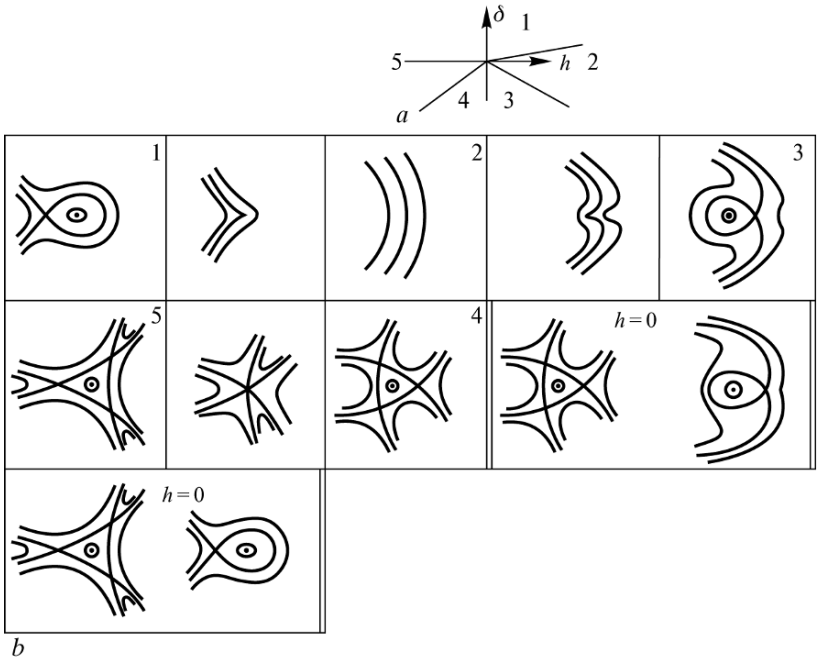

Fig. 8.4.

in both sets of coordinates. For $h > 0$ ($h < 0$) the portrait may be thought of as the section of a three-dimensional energy-level manifold by the plane $\varphi_1 = 0$ (respectively, $\varphi_2 = 0$). To the equilibrium positions on the phase

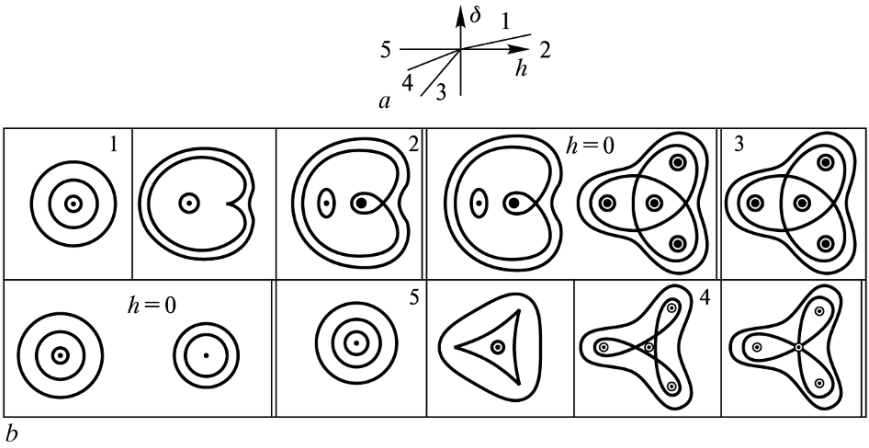

Fig. 8.5.

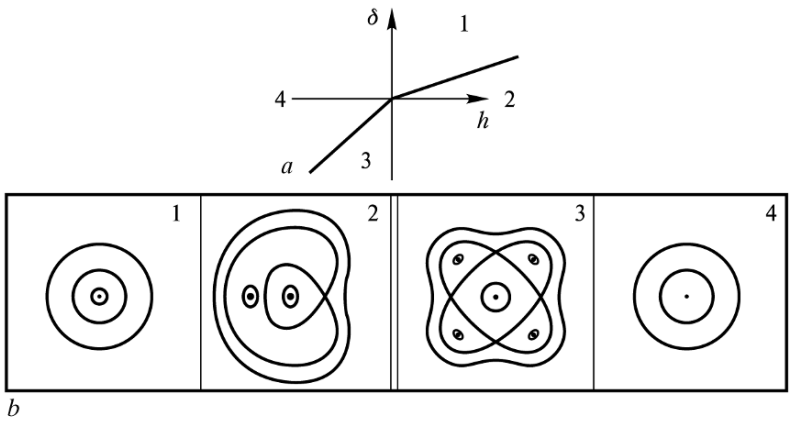

Fig. 8.6.

portrait there correspond periodic solutions[1] of the original system with two degrees of freedom, and to the closed curves there correspond two-dimensional invariant tori. Here to equilibrium positions obtained from one another by a rotation by angle $2\pi/k_2$ in the domain $h > 0$, or $2\pi/k_1$ in the domain $h < 0$, there corresponds one and the same periodic solution piercing the surface of the section k_2 times (respectively, k_1 times). Exactly the same is true for the two-dimensional tori.

To complete the analysis of resonances in systems with two degrees of freedom it remains to consider the resonances that are essential already in the

[1] For $h = 0$ to the equilibrium position at the centre of the portrait there corresponds an equilibrium of the original system.

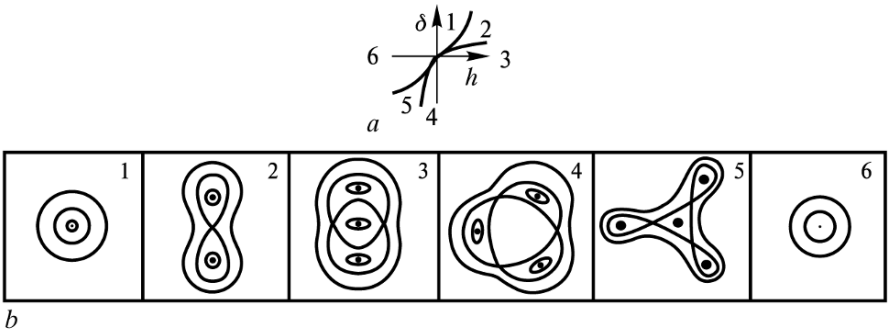

Fig. 8.7.

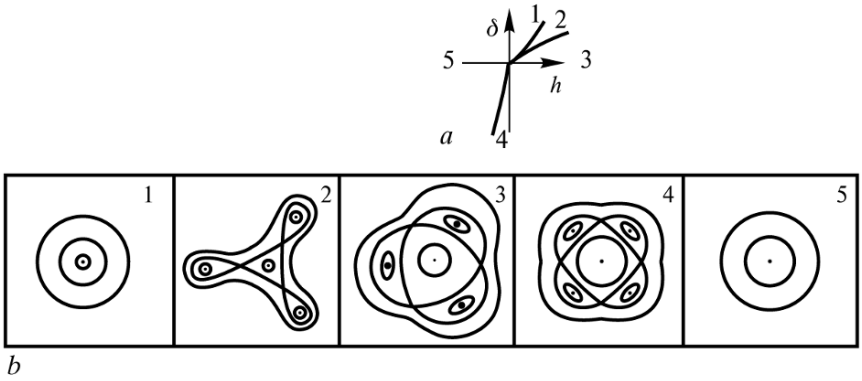

Fig. 8.8.

quadratic terms of the Hamiltonian: the case of multiple eigenvalues and the case of a zero eigenvalue.

For multiple eigenvalues, in the typical case the matrix of a linear Hamiltonian system has two Jordan blocks of order 2 (see § 8.2.3). If there are nearly multiple eigenvalues, then the quadratic part of the Hamiltonian can be reduced to the form $(\pm ib)^2$ in (8.5). According to [561], in this case the terms of the Hamiltonian of order up to and including 4 can be reduced to the following form, which is also called a normal form:

$$H = \frac{a(p_1^2+p_2^2)}{2} + \omega(p_2q_1 - p_1q_2) + \frac{\delta(q_1^2+q_2^2)}{2} + \\ + \left(q_1^2+q_2^2\right)\left[D\left(q_1^2+q_2^2\right) + B\left(p_2q_1 - p_1q_2\right) + C\left(p_1^2+p_2^2\right)\right], \qquad (8.10)$$

$$a = \pm 1.$$

The formal normal form is a series in $q_1^2 + q_2^2$, $p_1^2 + p_2^2$, and $p_2q_1 - p_1q_2$. Following [320, 563] we pass to the polar coordinates r, χ on the plane q_1, q_2 and introduce the corresponding momenta P, I defined by

$$\begin{aligned} q_1 &= r\cos\chi, \qquad & p_1 &= P\cos\chi - I\sin\frac{\chi}{r}, \\ q_2 &= r\sin\chi, \qquad & p_2 &= P\sin\chi + I\cos\frac{\chi}{r}. \end{aligned} \tag{8.11}$$

In the new variables the Hamiltonian (8.10) takes the form

$$H = \frac{1}{2}a\left(P^2 + \frac{I^2}{r^2}\right) + \omega I + r^2\left(\frac{\delta}{2} + Dr^2 + BI + C\left(P^2 + \frac{I^2}{r^2}\right)\right). \tag{8.12}$$

Since the Hamiltonian is independent of the angle χ, the momentum I is an integral, and for P, r we obtain a system with one degree of freedom depending on the two parameters I and δ. Since we consider a neighbourhood of the equilibrium position $p = q = 0$, we can neglect the term $Cr^2(P^2+I^2/r^2)$ in (8.12): this term is much smaller than the term in the first bracket in (8.12). The bifurcation diagram of the resulting system is given in Fig. 8.9 for the case $a = 1$ and $D > 0$, and in Fig. 8.10 for the case $a = 1$ and $D < 0$. It is assumed that $I \geqslant 0$, which does not cause any loss of generality.

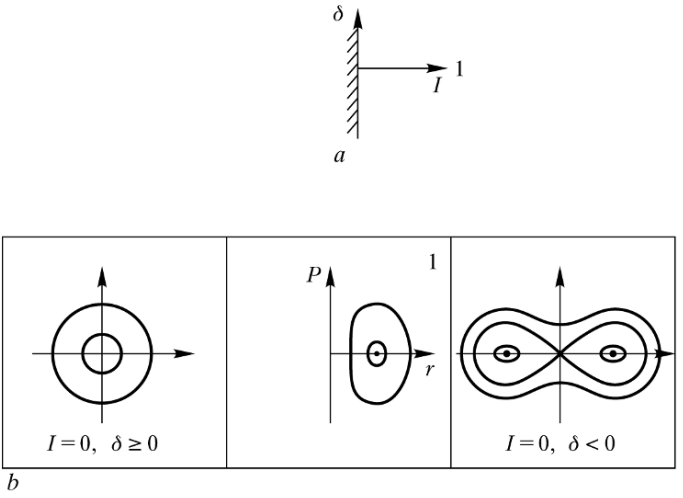

Fig. 8.9.

The left- and right-most phase portraits in Fig. 8.9, 8.10 correspond to $I = 0$. To ensure that they have no singularities we have to assume that r takes values of both signs. The curves on the portraits which are symmetric with respect to the axis $r = 0$ correspond to the same invariant surfaces in the phase space of the system with two degrees of freedom.

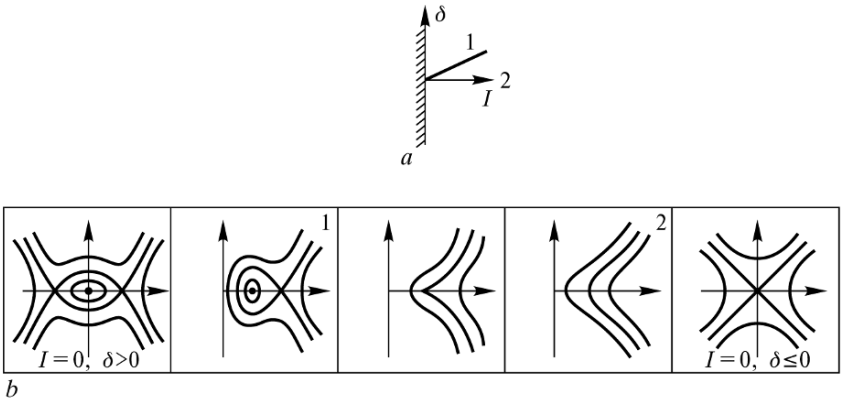

Fig. 8.10.

Finally, we consider the case of a zero eigenvalue (a degenerate equilibrium). This case appears already in systems with one degree of freedom; it is such a system that we shall consider.[2] We assume that in the linearized system to the zero eigenvalue there corresponds a Jordan block of order 2 (see § 8.2.3). If the equilibrium is nearly degenerate, then it cannot be shifted to the origin by a change of variables that is smooth in the parameters of the problem. Hence the linear part remains in the Hamiltonian. The terms of the Hamiltonian of order up to and including 3 can be reduced to the form

$$H = \delta q + \frac{ap^2}{2} + bq^3, \qquad a = \pm 1. \tag{8.13}$$

Suppose that $a = 1$ and $b > 0$. The bifurcation of the phase portrait in the transition from negative δ to positive is shown in Fig. 8.11. The two equilibrium positions merge and disappear.

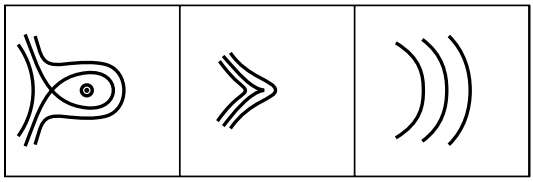

Fig. 8.11.

The diagrams given here exhaust all the resonance-related bifurcations that occur in one-parameter families of generic Hamiltonians with two degrees of freedom and can be calculated from the normal form.

[2] For two degrees of freedom, the order can be reduced to one by using the integral corresponding to the non-zero characteristic frequency.

These diagrams are also useful for a higher number of degrees of freedom. Indeed, suppose that in a system with n degrees of freedom there is a single resonance relation approximately satisfied by two frequencies. Then its normal form has $n-2$ integrals $\rho_i = \text{const}$ and is reducible to a system with two degrees of freedom. As a result we obtain one of the normal forms considered above, whose coefficients depend on the parameters $\rho_i \ll 1$.

The study of multiple resonances in systems with many degrees of freedom is presently in its early stage. In [53] the case with frequency ratio $1:2:1$ was studied, its periodic solutions and additional integrals appearing for special values of the parameters were found. In [54] it was shown that for the resonance $1:2:2$ the normal form of order 3 has an additional symmetry, and the corresponding system is completely integrable. In [218] it was shown that for the resonance $1:1:2$ the normal form of order 3 generates a non-integrable system.[3]

8.3.3 Stability of Equilibria of Hamiltonian Systems with Two Degrees of Freedom at Resonances

Studying the normal form provides considerable information about the motion of the original system for which the lower terms of the Hamiltonian can be reduced to this form. For example, if the normal form has a non-degenerate periodic solution, then the original system has a periodic solution close to that one. This follows from the implicit function theorem. Most of the invariant tori that exist for the normal form also exist, in the general case, for the original system. This follows from the results of KAM theory (one must use Theorem 6.17 in § 6.3). As always in systems with two degrees of freedom, the existence of invariant tori allows us to draw conclusions on stability.

If the characteristic frequencies of a system with two degrees of freedom do not satisfy resonance relations of order up to and including 4, then the equilibrium is stable (under the additional condition of isoenergetic non-degeneracy); this result was already discussed in § 6.3.6.B. For the remaining finitely many resonant cases the following result holds.

Theorem 8.8 ([191, 320, 408, 561, 562, 563])**.** *If the characteristic frequencies satisfy a resonance relation of order* $\leqslant 4$*, and the conditions of generality of position of* § 8.3.2 *hold, then the equilibrium of the original system is stable or unstable simultaneously with the equilibrium of the normal form.*

The stability can be proved by using KAM theory, and the instability, by comparing the rate of moving away from the equilibrium position for the original system and the normal form, or by constructing a Chetaev function.

[3] In these papers it is assumed that to the multiple characteristic frequency there correspond, in the matrix of the linearized system, four Jordan blocks of order 1, rather than two blocks of order 2, that is, there is additional degeneracy: to obtain this case in a generic system four parameters are required.

In the notation of § 8.3.2 we have the following results.

Corollary 8.8 ([408]). *For the resonance* $(2, 1)$ *the equilibrium is unstable if* $B \neq 0$ (*Fig.* 8.3).

Corollary 8.9 ([408]). *For the resonance* $(3, 1)$ *the equilibrium is stable if* $|A| > 3\sqrt{3}\,|B| > 0$ (*Fig.* 8.5), *and unstable if* $0 < |A| < 3\sqrt{3}\,|B|$ (*Fig.* 8.4).

Corollary 8.10 ([320, 561, 563]). *If the linearized system has a multiple non-zero frequency with a pair of Jordan blocks of order* 2, *then the equilibrium of the full system is stable if* $aD > 0$ (*Fig.* 8.9), *and unstable if* $aD < 0$ (*Fig.* 8.10).

Corollary 8.11 ([191, 562]). *If the linearized system has a zero characteristic frequency with a Jordan block of order* 2, *then the equilibrium of the full system is unstable if* $b \neq 0$ (*Fig.* 8.11).

When some of the conditions of generality of position stated above are violated, the problem of stability was analysed in [191, 408, 561, 562].

The separatrices on the phase portraits of the normal form, generally speaking, split on passing to the exact system, as described in § 6.3.3.B.

8.4 Normal Forms of Hamiltonian Systems near Closed Trajectories

8.4.1 Reduction to Equilibrium of a System with Periodic Coefficients

Suppose that a Hamiltonian system with $n + 1$ degrees of freedom has a closed trajectory which is not an equilibrium position. Such trajectories are not isolated but, as a rule, form families. We now reduce the problem of the oscillations in a neighbourhood of this family to a convenient form.

Proposition 8.1 (see, for example, [154]). *In a neighbourhood of a closed trajectory there exist new symplectic coordinates* $\varphi \bmod 2\pi$, J, *and* $z \in \mathbb{R}^{2n}$ *such that* $J = 0$ *and* $z = 0$ *on the trajectory under consideration, and going around this trajectory changes* φ *by* 2π; *on the trajectory itself,* $\dot{\varphi} = \text{const}$. *In the new coordinates the Hamiltonian function takes the form* $H = f(J) + \mathscr{H}(z, \varphi, J)$, *where* $f'_J \neq 0$ *and the expansion of* $\mathscr{H}$ *in* z, J *begins with terms of the second order of smallness.*

We now perform the isoenergetic reduction (see [10]) choosing, on an energy level $H = h$, the phase φ for the new time (which we now denote by t). The Hamiltonian of the problem takes the form $F = F(z, t, h)$. For $h = 0$ the origin is an equilibrium position of the system. Suppose that this equilibrium is non-degenerate (all the multipliers are distinct from 1; the degenerate case is considered in § 8.4.3). Then for small h the system also has a non-degenerate equilibrium. By a change of variables smooth in the parameter one can shift this equilibrium to the origin. The Hamiltonian takes the form

$$F = \frac{1}{2}(\Xi(t, h)z, z) + G(z, t, h), \tag{8.14}$$

where the expansion of G in z begins with terms of the third order of smallness; the Hamiltonian has period 2π in t.

We now consider the linearized system.

Theorem 8.9 (see, for example, [614]**).** *A linear Hamiltonian system that is 2π-periodic in time can be reduced to an autonomous form by a linear symplectic change of variables. If the system has no negative real multipliers, then the reducing change of variables can be chosen to be 2π-periodic in time, and if the system has negative real multipliers, then 4π-periodic. If the system depends smoothly on a parameter, then the change of variables can also be chosen to be smooth in this parameter.*

Suppose that all the multipliers of the linearized system lie on the unit circle and are all distinct. Then, by the theorem stated above and by § 8.2.2, the Hamiltonian (8.14) can be reduced by a linear 2π-periodic change of variables to the form

$$\Phi = \frac{1}{2}\omega_1\left(p_1^2 + q_1^2\right) + \cdots + \frac{1}{2}\omega_n\left(p_n^2 + q_n^2\right) + \Psi(p, q, t, h), \tag{8.15}$$

where the expansion of Ψ in the phase variables begins with terms of the third order of smallness, and Ψ has period 2π in time t.

8.4.2 Reduction of a System with Periodic Coefficients to Normal Form

Definition 8.4. The characteristic frequencies $\omega_1, \ldots, \omega_n$ satisfy a *resonance relation of order $l > 0$ for 2π-periodic systems* if there exist integers $k_0, k_1, \ldots, k_n$ such that $k_1\omega_1 + \cdots + k_n\omega_n + k_0 = 0$ and $|k_1| + \cdots + |k_n| = l$.

Theorem 8.10 (Birkhoff [14]**).** *Suppose that the characteristic frequencies ω_i of the 2π-periodic system* (8.15) *do not satisfy any resonance relation of order L or less. Then there is a symplectic change of variables that is 2π-periodic in time and reduces the Hamiltonian function to the same Birkhoff normal form of degree L as if the system were autonomous, with the only difference that the remainder terms of degree $L + 1$ and higher depend 2π-periodically on time.*

The normalization procedure is similar to the one described in § 8.3.1. If the system depends smoothly on a parameter, then the normalizing transformation can also be chosen to be smooth in the parameter.

For resonant cases one uses resonant normal forms. Let K be a sublattice of the integer lattice $\mathbb{Z}^{n+1}$ defining the possible resonances (cf. § 6.1.1).

Definition 8.5. A *non-autonomous resonant normal form of degree L* for a Hamiltonian for resonances in K is a polynomial of degree L in symplectic variables P_i, Q_i which in the polar coordinates (8.9) depends on the phases φ_i and time t only via their combinations $k_1\varphi_1 + \cdots + k_n\varphi_n + k_0 t$ with $(k_1, \ldots, k_n, k_0) \in K$.

Theorem 8.11. *Suppose that the characteristic frequencies do not satisfy any resonance relations of order L or less, except, possibly, for relations $k_1\omega_1 + \cdots + k_n\omega_n + k_0 = 0$ with $(k_1, \ldots, k_n, k_0) \in K$. Then there exists a symplectic 2π-periodic change of variables reducing the Hamiltonian to a non-autonomous resonant normal form of degree L for resonances in K up to terms of degree higher than L.*

If the rank of the sublattice K is equal to r, then a system in a normal form for resonances in K has $n - r$ independent integrals in involution which are linear combinations of the quantities $\rho_i = \left(P_i^2 + Q_i^2\right)/2$ with integer coefficients. In particular, if there is only one resonance relation, then the system is integrable.

8.4.3 Phase Portraits of Systems with Two Degrees of Freedom near a Closed Trajectory at a Resonance

In a system with two degrees of freedom the oscillations about a closed trajectory are described by a time-periodic system with one degree of freedom depending on a parameter (§ 8.4.1). A system having a resonant normal form for such a problem reduces to a system with one degree of freedom; its phase portraits can be drawn. If the coefficients of the lower terms of the normal form are generic, then there exists only finitely many types of phase portraits for this resonance, and these types are determined by the lower terms of the normal form. The phase portraits differ qualitatively only for finitely many resonances. The list of them and the description of the bifurcations that the portraits undergo when the parameters of the system pass through an exact resonance are contained in [150, 152] and are reproduced below.

The normal forms H_{k,k_0} for resonances (k, k_0) in the variables ρ, $\psi = \varphi + k_0 t/k + \psi_0$ have the form

$$H_{3,k_0} = \delta\rho + B\rho^{3/2}\cos 3\psi,$$
$$H_{k,k_0} = \delta\rho + \rho^2 A(\rho) + B\rho^{k/2}\cos k\psi, \qquad k \geqslant 4.$$

Here ρ and ψ are conjugate phase variables, $\delta = \omega + k_0/k$ is the resonance detuning, A is a polynomial in ρ, and B, ψ_0 are constants. The required genericity conditions are $B \neq 0$, $A(0) \neq 0$ for $k \geqslant 4$, $|A(0)| \neq |B|$ for $k = 4$. All the coefficients depend also on a parameter h. We assume that $d\delta/dh \neq 0$, so that we can use δ instead of h. Under these conditions a small change in B and in the coefficients of A does not cause bifurcations; hence we can ignore the dependence of A and B on the parameter. We assume that $B > 0$ and $A(0) > 0$; this does not cause any loss of generality.

The metamorphosis of the phase portrait as δ increases passing through zero is shown for $k = 3$ in Fig. 8.12*a*; for $k = 4$ in Fig. 8.12*b* if $A(0) < B$, and in Fig. 8.12*c* if $A(0) > B$; and for $k = 5$ in Fig. 8.12*d*.

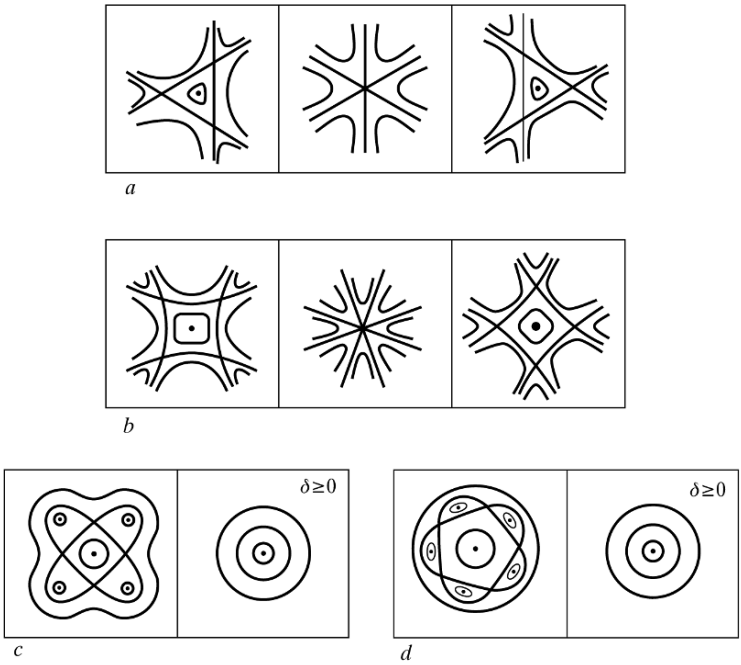

Fig. 8.12.

For $k \geqslant 6$ the metamorphosis is the same as for $k = 5$, only there are $2k$ singular points around the origin, rather than 10. For $k \geqslant 5$ these singular points are at a distance of order $\sqrt{\delta}$ from the origin. The "oscillation islands" surrounding stable points have width of order $\delta^{(k-2)/4}$. Consequently, for $k \geqslant 5$ these islands occupy only a small proportion of the neighbourhood of the origin under consideration, and the other phase curves are close to circles.

There are two more resonant cases which are already related to the quadratic terms of the Hamiltonian. These are the cases where the multipliers of a closed trajectory are equal to -1 or 1.

If the multipliers are close to -1 (resonance $(2, k_0)$), then in the typical case the lower terms of the Hamiltonian can be reduced by a 4π-periodic change of variables to the normal form

$$H = \delta q^2 + \frac{ap^2}{2} + Dq^4, \qquad a = \pm 1.$$

The metamorphosis is shown in Fig. 8.13*a* for $a = 1$ and $D > 0$, and in Fig. 8.13*b* for $a = 1$ and $D < 0$.

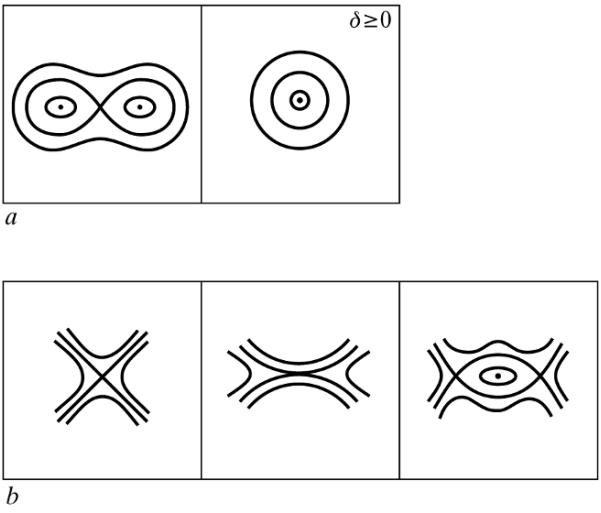

Fig. 8.13.

If the multipliers are close to 1 (resonance $(1, k_0)$), then the lower terms of the Hamiltonian can be reduced to the normal form

$$H = \delta q^2 + \frac{ap^2}{2} + bq^3, \qquad a = \pm 1.$$

The metamorphosis is shown in Fig. 8.14 (under the assumption that $a = 1$ and $b > 0$).

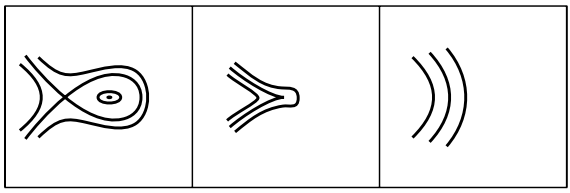

Fig. 8.14.

The phase portraits constructed here allow one to determine many properties of the original system when its lower terms can be reduced to the corresponding normal form. For example, to non-degenerate equilibrium positions on the portraits there correspond periodic trajectories of the full system going over the original periodic trajectory k times. For a resonance of order 3 there is only one such trajectory, it is unstable and merges with the original one at the instant of the exact resonance ($\delta = 0$). For a resonance of order $k \geqslant 5$ there are two such trajectories, one is stable, the other is unstable; they branch off from the original trajectory at passing through the resonance along the δ-axis in one definite direction. For a resonance of order 4, depending on the values of the parameters, the picture is either the same as for order 3, or as for order $k \geqslant 5$. At passing through a resonance of order 2 (the multipliers are equal to -1) the original trajectory loses or acquires stability, and a periodic trajectory branches off which goes twice over it. Finally, for a resonance of order 1 (the multipliers are equal to 1) the original trajectory vanishes merging with another trajectory with the same period (or, if we move in the opposite direction along the parameter, two periodic trajectories are born).

To most of the closed curves on the phase portraits there correspond two-dimensional invariant tori of the full system carrying conditionally periodic motions (according to KAM theory).

Under the genericity conditions stated above, the stability or instability of the original closed trajectory can be determined by using the normal form (cf. Theorem 8.8). For $k = 3$ we have instability if $B \neq 0$; for $k = 4$, stability if $|A(0)| > |B| > 0$, and instability if $0 < |A(0)| < |B|$; for $k \geqslant 5$, stability if $A(0)B \neq 0$. For the multipliers equal to -1 we have stability if $aD > 0$, and instability if $aD < 0$. For the multipliers equal to 1 we have instability if $ab \neq 0$.

When we pass from the normal form to the exact system, the separatrices that are present on the phase portraits, generally speaking, split similarly to what was described in § 6.3.3.B.

8.5 Stability of Equilibria in Conservative Fields

8.5.1 Lagrange–Dirichlet Theorem

Theorem 8.12 (Lagrange–Dirichlet). *If the potential has a strict local minimum at an equilibrium position, then the corresponding equilibrium state is stable.*

◁ For a Lyapunov function we can take the total mechanical energy. ▷

The hypothesis of the Lagrange–Dirichlet theorem is not a necessary condition for stability.

Example 8.3 (Painlevé–Wintner)**.** Consider the infinitely differentiable potential $U(q) = (\cos q^{-1}) \exp(-q^{-2})$, where $q \neq 0$; $U(0) = 0$. The equilibrium position $q = 0$ is stable, although the point $q = 0$ is of course not a local minimum of the function U (Fig. 8.15).

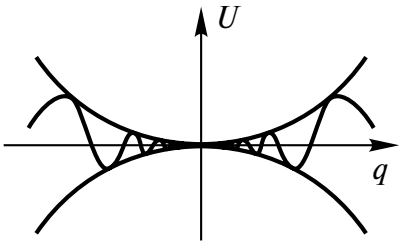

Fig. 8.15.

△

In 1892 Lyapunov posed the problem of proving the converse of Lagrange's theorem for the case in which the coefficients of the quadratic form $T = \sum a_{ij}(q)\dot{q}_i\dot{q}_j$ and the potential U are analytic functions in a neighbourhood of the equilibrium position. A detailed survey of papers on Lyapunov's problem up to 1983 is contained in [24].

Theorem 8.13. *Suppose that the equilibrium position $q = 0$ is not a local minimum of an analytic potential U. Then the equilibrium state $(\dot{q}, q) = (0, 0)$ is unstable.*

This result was established by Palamodov in [491]. Earlier he proved the converse of the Lagrange–Dirichlet theorem for systems with two degrees of freedom.

The proof of Theorem 8.13 is based on the following assertion going back to Chetaev [180]. We assume that the matrix of kinetic energy $\big(a_{ij}(q)\big)$ at the point $q = 0$ is the identity matrix. This can be achieved by a suitable linear change of coordinates.

Lemma 8.1 ([180, 24])**.** *Suppose that in some neighbourhood Q of the point $q = 0$ there exists a vector field v such that*

1) $v \in C^1(Q)$ *and* $v(0) = 0$,
2) $\langle v'\xi, \xi\rangle \geqslant \langle \xi, \xi\rangle$ *for all* $\xi \in \mathbb{R}^n$ *and* $q \in Q$,
3) $\langle v, U'\rangle = PU$, *where P is positive and continuous in* $Q \cap \{U(q) < 0\}$.

Then any motion $q(\cdot)$ of the mechanical system with negative energy leaves the domain Q in finite time.

Remark 8.2. Let $q(\cdot)$ be a motion with zero total energy. If the equilibrium position $q = 0$ is isolated, then (under the assumptions of Lemma 8.1) the point $q(t)$ either leaves some domain $|q| \leqslant \varepsilon_0$ in finite time, or tends to zero as $t \to \infty$.

The main difficulty in the proof of Palamodov's theorem is precisely in the construction of the required field v. This construction is based on the technique of resolution of singularities, which is often used in algebraic geometry.

Example 8.4. Suppose that U is a quasi-homogeneous function with quasi-homogeneity exponents $\alpha_1, \dots, \alpha_n \in \mathbb{N}$:

$$U(\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_n} x_n) = \lambda^{\alpha} U(x_1, \dots, x_n), \qquad \alpha \in \mathbb{N}.$$

Then for the field v one can take the field Aq, where $A = \operatorname{diag}(\alpha_1, \dots, \alpha_n)$. Indeed, $\langle v, U' \rangle = \alpha U$ by the Euler formula. $\triangle$

Remark 8.3. Of special interest is the case where the potential energy has a *non-strict* minimum. Laloy and Pfeiffer [365] proved that such critical points of an analytic potential of a system with two degrees of freedom are unstable equilibrium positions. This problem is so far unsolved in the multidimensional case.

The problem of converses to the Lagrange–Dirichlet theorem is interesting not only in the analytic but also in the smooth case, where the absence of a minimum of the potential energy is determined by its Maclaurin series. Let

$$U = U_2 + U_k + U_{k+1} + \cdots \tag{8.16}$$

be the formal Maclaurin series of the potential U, where U_s is a homogeneous form of degree s in $q_1, \dots, q_n$. In a typical situation, of course, $k = 3$.

If the first form U_2 does not have a minimum at the equilibrium position $q = 0$, then this equilibrium is unstable. In this case one of the eigenvalues is positive, and therefore the instability follows from the well-known theorem of Lyapunov.

Therefore we consider the case where $U_2 \geqslant 0$. We introduce the plane

$$\Pi = \{q\colon\ U_2(q) = 0\}.$$

If $\dim \Pi = 0$, then the form U_2 is positive definite and therefore the equilibrium $q = 0$ is stable by Theorem 8.12. We assume that $\dim \Pi \geqslant 1$. Let W_k be the restriction of the form U_k in the expansion (8.16) to the plane Π. We have the following.

Theorem 8.14 ([333]). *If the form W_k does not have a minimum at the point $q = 0$, then this equilibrium is unstable.*

The proof of Theorem 8.14 is based on the following idea: if the equations of motion have a solution $q(t)$ that asymptotically tends to the point $q = 0$ as $t \to +\infty$, then the equilibrium state $(q, \dot{q}) = (0, 0)$ is unstable. Indeed, in view of the reversibility property, the equations of motion have also the solution $t \mapsto q(-t)$, which asymptotically goes out of the equilibrium position.

Under the hypotheses of Theorem 8.14 an asymptotic solution can be represented as a series in negative powers of time:

$$\sum_{s=1}^{\infty} \frac{x_s(\ln t)}{t^{s\mu}}, \qquad \mu = \frac{2}{k-2}, \tag{8.17}$$

where $x_s \in \mathbb{R}^n$, and each component of the vector-function $x_s(\cdot)$ is a polynomial with constant coefficients.

Suppose that $U_2 = 0$ and the Maclaurin series (8.16) converges. Then (as established in [328, 357]) the series (8.17) converges for $t \geqslant t_0$. Furthermore, in the case of odd k the coefficients x_s are altogether independent of time.

If $U_2 \not\equiv 0$, then the series (8.17) are, as a rule, divergent even in the analytic case.

Example 8.5. Consider the system of equations

$$\ddot{x} = \frac{\partial U}{\partial x}, \qquad \ddot{y} = \dot{x}^2 - \frac{\partial U}{\partial y}; \qquad U = -4x^3 + \frac{y^2}{2}. \tag{8.18}$$

The presence of the summand $\dot{x}^2$ models the case where the kinetic energy is non-Euclidean. Equations (8.18) have the formal solution

$$x = \frac{1}{2t^2}, \qquad y = \frac{1}{t^6} \sum_{n=0}^{\infty} \frac{a_{2n}}{t^{2n}}, \qquad a_{2n} = \frac{(-1)^n (2n+5)!}{120}. \tag{8.19}$$

The radius of convergence of the power series for y is zero.

However, equations (8.18) have the following exact asymptotic solutions corresponding to the formal series (8.19):

$$x = \frac{1}{2t^2}, \qquad y(t) = -\sin t \int_t^{\infty} \frac{\cos s}{s^6}\, ds + \cos t \int_t^{\infty} \frac{\sin s}{s^6}\, ds$$

By performing successive integration by parts, from the last formula we obtain the divergent series (8.19). This series is an asymptotic expansion of the function $y(t)$ as $t \to +\infty$. △

According to Kuznetsov's theorem [364], with each series (8.17) formally satisfying the equations of motion one can associate a genuine solution for which this series is an asymptotic expansion as $t \to +\infty$:

$$q(t) - \sum_{s=1}^{N} \frac{x_s}{t^{s\mu}} = o\left(\frac{1}{t^{N\mu}}\right)$$

Note that the paper [364] appeared precisely in connection with the discussion of the range of questions related to Theorem 8.14. Problems of constructing asymptotic solutions of strongly nonlinear systems of differential equations are considered in detail in the book [350].

We point out two important consequences of Theorem 8.14.

a) As we already noted, the question of stability of non-degenerate equilibria (at which $\det\left(\partial^2 U/\partial q^2\right) \neq 0$) is decided by the Lagrange–Dirichlet theorem. By Theorem 8.14, degenerate equilibria are unstable in a typical situation. Indeed, in the general case the expansion (8.16) involves terms of degree 3, and therefore $W_3 \not\equiv 0$. It remains to observe that a non-zero form of degree 3 cannot have a minimum.

b) Equilibria of a mechanical system in a conservative force field with a harmonic potential (satisfying the Laplace equation $\Delta U = 0$) are unstable. A special case is "Earnshaw's theorem": an equilibrium of a system of electric charges in a stationary electric field is always unstable. Before the papers [328, 357] Earnshaw's theorem had been proved only for the case where the eigenvalues of the first approximation are non-zero.

Indeed, harmonic functions are analytic. We expand the potential in the convergent Maclaurin series:

$$U = U_k + U_{k+1} + \cdots, \qquad k \geqslant 2.$$

Suppose that $U_k \not\equiv 0$ (otherwise $U = 0$ and then all points q will obviously be unstable equilibrium positions). Clearly, U_k is also a harmonic function. By the mean value theorem, U_k does not have a minimum at zero, whence the instability follows (Theorem 8.14).

8.5.2 Influence of Dissipative Forces

Suppose that a mechanical system is in addition acted upon by non-conservative forces $F(q, \dot{q})$; the motion is described by Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = F, \qquad L = T - U. \tag{8.20}$$

Definition 8.6. We call the force F a *force of viscous friction with total dissipation* if $F(q, 0) = 0$ and $(T + U)^{\cdot} = F\dot{q} < 0$ for $\dot{q} \neq 0$.

Even after addition of forces of viscous friction, the equilibrium positions will again coincide with the critical points of the potential U. The equilibrium states that were stable by Lagrange's theorem remain stable with dissipation of energy taken into account. Moreover, if the potential is an analytic function, then these equilibrium states become asymptotically stable. This is the Kelvin–Chetaev theorem [181].

Theorem 8.15 (see [327]). *Suppose that the point $q = 0$ is not a local minimum of the function U, and $U(0) = 0$. The equilibrium state $(q, \dot{q}) = (0, 0)$ of system* (8.20) *is unstable if one of the following conditions holds:*

a) *the function U is analytic in a neighbourhood of the point $q = 0$;*
b) *the function U is smooth and has no critical points in the domain $\Sigma_\varepsilon = \{q\colon\ U(q) < 0,\ \ |q| < \varepsilon\}$ for some $\varepsilon > 0$.*

In the analytic case condition b) holds automatically.

$\lhd$ Consider a motion $q(\cdot)$ with negative total energy and therefore with $q(0) \in \Sigma_\varepsilon$. We claim that the point $q(t)$ leaves Σ_ϵ in a finite time. Indeed, on such a motion we have $\dot{q}(t) \not\equiv 0$. Consequently, the total energy $E = T + U$ monotonically decreases. If $q(t) \in \Sigma_\varepsilon$ for all $t > 0$, and $E(t)$ tends to a finite limit as $t \to +\infty$, then $\dot{q}(t) \to 0$. But for small values of the speed the friction forces are small compared to the conservative forces, which impart a sufficiently high velocity to the system. $\rhd$

8.5.3 Influence of Gyroscopic Forces

Suppose that, apart from dissipative forces, the mechanical system is also acted upon by additional gyroscopic forces

$$F = \Omega(\dot{q}, \cdot),$$

where Ω is a closed 2-form (the form of gyroscopic forces; see § 3.2). Since gyroscopic forces do not perform any work, the equilibrium states that were stable by the Lagrange–Dirichlet theorem remain stable after addition of gyroscopic forces. Moreover, if the dissipation is total and the potential U satisfies the hypotheses of Theorem 8.15, then equilibrium states cannot be stabilized by adding gyroscopic forces.

Suppose that $q = 0$ is a non-degenerate equilibrium. Poincaré called the Morse index of the potential U at this point the *degree of instability* of the equilibrium $q = 0$.

Theorem 8.16 (Kelvin–Chetaev [181]). *If the degree of instability is odd, then this equilibrium cannot be stabilized by adding dissipative and gyroscopic forces.*

The proof is based on verifying the fact that if the degree of instability is odd, then among the eigenvalues there necessarily exists a positive one.

However, if the degree of instability is even, then such an equilibrium in the absence of dissipative forces can be stabilized by suitable gyroscopic forces.

Example 8.6. It is well known that the motion of a charge in an electric **E** and magnetic **H** fields is described by the equation

$$m\dot{\mathbf{v}} = e\Big(\mathbf{E} + \frac{1}{c}[\mathbf{v}, \mathbf{H}]\Big), \tag{8.21}$$

where $\mathbf{v} = \dot{\mathbf{x}}$ is the velocity of the charge ($\mathbf{x} \in \mathbb{R}^3$) and c is the speed of light. We consider a stationary electromagnetic field (when **E** and **H** do not explicitly depend on time). The field **E** is conservative: $\mathbf{E} = -\operatorname{grad}\varphi$. The magnetic component of the Lorentz force is a gyroscopic force: its presence does not affect the conservation of the total energy

$$W = \frac{mv^2}{2} + \varphi.$$

If $\mathbf{H} = 0$, then all the equilibria (stationary points of the potential φ) are unstable by Earnshaw's theorem.

We now give a simple example showing that it is possible to stabilize unstable equilibria by a stationary magnetic field [347]. Suppose that the electric field **E** is created by two equal charges Q situated on the x_3-axis at a distance R from the origin O. Then the point O is an unstable equilibrium position. The potential of the electric field is equal to $\varphi_+ + \varphi_-$, where

$$\varphi_\pm = eQ\left[x_1^2 + x_2^2 + (R \pm x_3)^2\right]^{-1/2}$$

The expansion of the total energy W in the Maclaurin series has the form

$$W = \frac{m\left(v_1^2 + v_2^2 + v_3^2\right)}{2} - \frac{eQ\left(x_1^2 + x_2^2 - 2x_3^2\right)}{R^3} + \cdots .$$

If $eQ > 0$ (which we assume in what follows), then the degree of instability (the Morse index of the function W at the critical point $\mathbf{x} = \mathbf{v} = 0$) is equal to two. However, if the charges e and Q have opposite signs, then the degree of instability is odd (equal to one) and a gyroscopic stabilization is impossible by the Kelvin–Chetaev theorem.

We introduce the magnetic field $\mathbf{H} = (0, 0, \varkappa)$, $\varkappa = \text{const}$, which of course satisfies Maxwell's equations. Since the kinetic energy and the electromagnetic field are invariant under rotations around the x_3-axis, equations (8.21) admit the Nöther integral

$$\Phi = m(v_1x_2 - v_2x_1) + \frac{e\varkappa}{2c}\left(x_1^2 + x_2^2\right).$$

We seek a Lyapunov function in the form of a combination of integrals $W + \lambda\Phi$, where $\lambda = \text{const}$. Choosing λ so that this integral takes minimum value we obtain the following sufficient condition for the Lyapunov stability:

$$H^2 > \frac{8Qmc^2}{eR^3}.$$

△

Theorem 8.16 can be extended to systems of the most general form. Let v be a smooth vector field on $\mathbb{R}^n = \{x\}$. This field generates the dynamical system

$$\dot{x} = v(x), \qquad x \in \mathbb{R}^n. \tag{8.22}$$

Suppose that $x = 0$ is an equilibrium position: $v(0) = 0$. Then in a neighbourhood of this point system (8.22) has the form

$$\dot{x} = Ax + o(|x|),$$

where A is the Jacobi matrix of the field v at the point $x = 0$. We define the *degree of instability* $\deg(x = 0)$ of the equilibrium $x = 0$ to be the number of eigenvalues of the matrix A with positive real part (counting multiplicities). This definition generalizes Poincaré's definition of degree of instability for classical mechanical systems. In particular, if the degree of instability is odd, then the characteristic equation $\det(A - \lambda E) = 0$ has a positive root. We say that the equilibrium $x = 0$ is *non-degenerate* if $\det A \neq 0$.

Suppose that there exists a smooth function $F\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\dot{F} = \frac{\partial F}{\partial x} v \leqslant 0.$$

We say that such a system is *dissipative*. The function F kind of plays the role of the total energy. It is easy to verify that the non-degenerate critical points of the function F correspond to the equilibria of system (8.22).

Theorem 8.17 ([341])**.** *Suppose that $x = 0$ is a non-degenerate equilibrium which is a non-degenerate critical point of the function F. Then*

$$\deg(x = 0) = \operatorname{ind}_0 F \mod 2.$$

In this equality on the right is the Morse index of the function F at the critical point $x = 0$.

Corollary 8.12. *Suppose that F is a Morse function. Then its critical points of odd index are unstable equilibria.*

This assertion includes the Kelvin–Chetaev theorem (Theorem 8.16). Indeed, let $W = T + U$ be the energy integral of a reversible system. Its index at an equilibrium position is obviously odd. This index does not change after addition of gyroscopic forces. Since $\dot{W} \leqslant 0$ after addition of dissipative forces, the instability of the equilibrium follows from Corollary 8.12 of Theorem 8.17.