3

Symmetry Groups and Order Reduction

3.1 Symmetries and Linear Integrals

3.1.1 Nöther's Theorem

Let (M, L) be a Lagrangian system and v a smooth field on M. The field v gives rise to the one-parameter group g of diffeomorphisms $g^\alpha\colon\ M \to M$ defined by the differential equation

$$\frac{d}{d\alpha} g^\alpha(x) = v(g^\alpha(x)) \tag{3.1}$$

and the initial condition $g^0(x) = x$.

Definition 3.1. The Lagrangian system (M, L) *admits the group* $\{g^\alpha\}$ if the Lagrangian L is invariant under the maps $g^\alpha_*\colon\ TM \to TM$. The group g can be naturally called a *symmetry group*, and the field v a *symmetry field*.

Let $\gamma\colon\ \Delta \to M$ be a motion of the Lagrangian system (M, L). Then the composition $g^\alpha \circ \gamma\colon\ \Delta \to M$ is also a motion for every value of α.

In the non-autonomous case the Lagrangian L is a smooth function on the tangent bundle of the extended configuration space $'M = M \times \mathbb{R}$. We call a group of diffeomorphisms $'g^\alpha\colon\ 'M \to {}'M$ a symmetry group of the system $('M, L)$ if $'g^\alpha(x, t) = (y, t)$ for all $(x, t) \in M \times \mathbb{R}$ and the maps $'g^\alpha_*$ preserve L. The group $\{'g^\alpha\}$ gives rise to the smooth field on $'M$

$$'v(x, t) = \frac{d}{d\alpha}('g^\alpha(x, t))_{\alpha=0}.$$

It is obvious that $'v(x, t) = (v(x, t), 0) \in T_{(x,\,t)}(M \times \mathbb{R})$ and $v(x, t)$ can be interpreted as a field on M smoothly depending on t.

Lemma 3.1. *The system (M, L) admits the symmetry group $\{g^\alpha\}$ if and only if*

$$(p \cdot v)^{\dot{}} = [L] \cdot v. \tag{3.2}$$

◁ This follows from the identity

$$\left.\frac{d}{d\alpha}\right|_{\alpha=0} L(g_*^\alpha \dot{x}) = (L'_{\dot{x}} \cdot v)^{\cdot} - [L] \cdot v. \tag{3.3}$$

▷

Lemma 3.1 is also valid in the non-autonomous case. Equality (3.2) implies the following.

Theorem 3.1. *If the system (M, L) admits the group $\{g^\alpha\}$, then $I = p \cdot v$ is a first integral of the equations of motion.*[1]

Let $(M, \langle\ ,\ \rangle, V)$ be a natural mechanical system. The Lagrangian $L = \langle \dot{x}, \dot{x}\rangle/2 + V(x)$ is invariant under the action of the group g if and only if this property is enjoyed by the Riemannian metric $\langle\ ,\ \rangle$ and the potential V. For natural systems the integral I is clearly equal to $\langle v, \dot{x}\rangle$; it depends linearly on the velocity.

Example 3.1. If in some coordinates $x_1, \dots, x_n$ on M the Lagrangian L is independent of x_1, then the system (M, L) admits (locally) the symmetry group $g^\alpha \colon\ x_1 \mapsto x_1 + \alpha,\ \ x_k \mapsto x_k\ \ (k \geqslant 2)$. This group corresponds to the vector field $v = \partial/\partial x_1$. By Theorem 3.1 the quantity $I = p \cdot v = p_1 = L'_{\dot{x}_1}$ is conserved. In mechanics, x_1 is called a *cyclic coordinate*, and the integral I a *cyclic integral.* In particular, the energy integral is a cyclic integral of a certain extended Lagrangian system. In order to show this we introduce a new time variable τ by the formula $t = t(\tau)$ and define a function $'L\colon\ T\,'M \to \mathbb{R}$ ($'M = M \times \mathbb{R}$) by the formula

$$'L(x', t', x, t) = L(x'/t', x, t)t', \qquad (\cdot)' = \frac{d}{d\tau}(\cdot).$$

It follows from Hamilton's variational principle and the equality

$$\int\limits_{\tau_1}^{\tau_2} {'L}\, d\tau = \int\limits_{t_1}^{t_2} L\, dt$$

that if $x\colon\ [t_1, t_2] \to M$ is a motion of the system (M, L), then $(x, t)\colon\ [\tau_1, \tau_2] \to {'M}$ is a motion of the extended Lagrangian system $('M, 'L)$. In the autonomous case, time t is a cyclic coordinate and the cyclic integral

$$\frac{\partial' L}{\partial t'} = L - \frac{\partial L}{\partial \dot{x}} \cdot \dot{x} = \text{const}$$

coincides with the energy integral. △

[1] In this form this theorem was first stated by E. Nöther in 1918. The connection between the laws of conservation of momentum and angular momentum and the groups of translations and rotations was already known to Lagrange and Jacobi. Theorem 3.1 for natural systems was published by Levy in 1878.

Theorem 3.2. *If* $v(x_0) \neq 0$*, then in a small neighbourhood of the point* x_0 *there exist local coordinates* $x_1, \ldots, x_n$ *such that* $I = p \cdot v = p_1$*.*

This assertion is a consequence of the *theorem on rectification* of a vector field.

Theorem 3.3. *Suppose that* $I = p \cdot v$ *is a first integral of the equation of motion* $[L] = 0$*. Then the phase flow of equation* (3.1) *is a symmetry group of the Lagrangian system* (M, L)*.*

Theorems 3.2 and 3.3 imply the following.

Corollary 3.1. *Integrals of natural systems that are linear in the velocities locally are cyclic.*

If there are several symmetry fields $v_1, \ldots, v_k$, then the equation of motion admits as many first integrals $I_1 = p{\cdot}v_1, \ldots, \; I_k = p{\cdot}v_k$. Assuming that (M, L) is a natural Lagrangian system we use the Legendre transformations to pass to Hamilton's equations on T^*M. The functions $I_1, \ldots, I_k \colon \; T^*M \to \mathbb{R}$ are independent and in involution (in the standard symplectic structure on T^*M) if and only if the fields $v_1, \ldots, v_k$ are independent and commute on M. The existence of linear integrals imposes restrictions not only on the Riemannian metric and the potential of the force field, but also on the topology of the configuration space.

Theorem 3.4. *Let* M *be a connected compact orientable even-dimensional manifold. If a Hamiltonian natural system on* T^*M *has at least* $(\dim M)/2$ *independent linear integrals in involution, then the Euler–Poincaré characteristic of* M *is non-negative:* $\chi(M) \geqslant 0$*.*[2]

Corollary 3.2. *Suppose that* $\dim M = 2$*. If the natural system has a first integral that is linear in the velocity, then* M *is diffeomorphic to the sphere or the torus.*

In the non-orientable case one must add the projective plane and the Klein bottle.

◁ We now prove Corollary 3.2. If $\chi(M) < 0$, then the symmetry field v has singular points. Since the phase flow of the equation $\dot{x} = v(x)$ is a group of isometries of the two-dimensional Riemannian manifold $(M, \langle \, , \rangle)$, the singular points x_s are isolated and are of elliptic type. By Poincaré's formula, $\chi(M) = \sum_s \operatorname{ind}(x_s) > 0$, a contradiction. ▷

We now consider a more general situation where an arbitrary Lie group G acts (on the left) on M. Let $\mathscr{G}$ be the Lie algebra of G and let $\mathscr{G}^*$ be the dual vector space of the space of the algebra $\mathscr{G}$. We shall now define a natural map $I_G \colon \; TM \to \mathscr{G}^*$ that associates with each point $\dot{x} \in TM$ a linear function on $\mathscr{G}$.

[2] This assertion was obtained by Bolotin and Abrarov (see [56]).

To each vector $X \in \mathscr{G}$ there corresponds a one-parameter subgroup g_X, whose action on M generates a tangent field v_X. The map $X \mapsto v_X$ is a homomorphism of the algebra $\mathscr{G}$ into the Lie algebra of all vector fields on M. We set $I_G(\dot{x}) = L'_{\dot{x}} \cdot v_X$; this function is linear in X.

Definition 3.2. The map $I_G\colon\ TM \to \mathscr{G}^*$ is called the *momentum map* of the Lagrangian system (M, L) *for the action of the group* G (or simply momentum if this causes no confusion).

Along with the momentum map $I_G\colon\ TM \to \mathscr{G}^*$ we have the map $P_G\colon$ $T^*M \to \mathscr{G}^*$ defined by the formula $P_G(p) = p \cdot v_X$. The momentum map I_G is the composition of the map P_G and the Legendre transformation.

Example 3.2. Consider n free material points $(\mathbf{r}_k, m_k)$ in three-dimensional Euclidean space. Let $SO(2)$ be the group of rotations of the space around the axis given by a unit vector $\mathbf{e}$. The group $SO(2)$ acts on the configuration space $\mathbb{R}^3\{\mathbf{r}_1\} \times \cdots \times \mathbb{R}^3\{\mathbf{r}_n\}$; the corresponding vector field is

$$(\mathbf{e} \times (\mathbf{r}_1 - {}'\mathbf{r}_1), \ldots, \mathbf{e} \times (\mathbf{r}_n - {}'\mathbf{r}_n)),$$

where ${}'\mathbf{r}_k$ is the position vector of the kth point with initial point at some point of the rotation axis. Since

$$L = \frac{1}{2} \sum m_k \langle \dot{\mathbf{r}}_k, \dot{\mathbf{r}}_k \rangle + V(\mathbf{r}_1, \ldots, \mathbf{r}_n),$$

the momentum

$$I_{SO(2)} = \sum m_k \, \langle \dot{\mathbf{r}}_k, \mathbf{e} \times (\mathbf{r}_k - {}'\mathbf{r}_k) \rangle = \Big\langle \mathbf{e}, \sum m_k (\mathbf{r}_k - {}'\mathbf{r}_k) \times \dot{\mathbf{r}}_k \Big\rangle$$

coincides with the already known angular momentum of the system with respect to the axis.

Now let $G = SO(3)$ be the group of rotations around some point o. The dual space $\mathscr{G}^* = (so(3))^*$ can be canonically identified with the algebra of vectors of three-dimensional oriented Euclidean space where the commutator is defined as the ordinary cross product. Then $I_{SO(3)}$ will clearly correspond to the angular momentum of the system with respect to the point o. △

Definition 3.3. A group G is called a *symmetry group* of the Lagrangian system (M, L) if $L(g_* \dot{x}) = L(\dot{x})$ for all $\dot{x} \in TM$ and $g \in G$.

Theorem 3.5. *Suppose that the system* (M, L) *admits* G *as a symmetry group. Then the momentum map* I_G *is a first integral* (*that is,* I_G *takes constant values on the motions of the Lagrangian system* (M, L)).

This assertion is a consequence of Theorem 3.1.

Example 3.3. We already saw in Ch. 1 that the equations of the problem of n gravitating bodies admit the Galilean transformation group. However, the Lagrange function

$$L = \frac{1}{2}\sum m_k\left(\dot{x}_k^2 + \dot{y}_k^2 + \dot{z}_k^2\right) + \sum_{i<j}\frac{\gamma m_i m_j}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}}$$

is not invariant under the whole Galilean group. This function admits translations of the time axis and isometries of three-dimensional Euclidean space. Translations of the time axis correspond to the conservation of the total energy; translations of Euclidean space, to the conservation of the momentum; and the group of rotations, to the conservation of the angular momentum. We consider in addition the group of homotheties

$$(x, y, z) \mapsto (\alpha x, \alpha y, \alpha z), \qquad \alpha > 0. \tag{3.4}$$

This group is generated by the vector field

$$\mathbf{v} = \sum_k x_k \frac{\partial}{\partial x_k} + y_k \frac{\partial}{\partial y_k} + z_k \frac{\partial}{\partial z_k}.$$

For $\alpha = 1$ we have the identity transformation. The Lagrangian of the n-body problem does not admit the group of homotheties. However, we can use identity (3.3) for $\alpha = 1$. Since $T \mapsto \alpha^2 T$ and $V \mapsto \alpha^{-1} V$ under the change of variables (3.4), equality (3.3) gives the already known Lagrange's identity:

$$\left.\frac{dL}{d\alpha}\right|_{\alpha=1} = (\mathbf{p}\cdot\mathbf{v})^{\boldsymbol{\cdot}} \quad\Leftrightarrow\quad 2T - V = \sum_k m(x_k\dot{x}_k + y_k\dot{y}_k + z_k\dot{z}_k)^{\boldsymbol{\cdot}} = \frac{\ddot{I}}{2},$$

where $I = \sum m\left(x_k^2 + y_k^2 + z_k^2\right)$. △

3.1.2 Symmetries in Non-Holonomic Mechanics

Suppose that (M, S, L) is a non-holonomic system acted upon by additional non-conservative forces $F(\dot{x}, x)\colon\ T_xM \to T_x^*M$. The motions are defined by the d'Alembert–Lagrange principle: $([L] - F)\cdot\xi = 0$ for all virtual velocities ξ.

Definition 3.4. The Lie group G is called a *symmetry group* of the non-holonomic system (M, S, L) if

1) G preserves L,
2) the vector fields v_X, $X \in \mathscr{G}$, are fields of virtual velocities.

Definition 3.5. The *moment of the force F relative to the group G* is the map $\Phi_G\colon\ TM \to \mathscr{G}^*$ defined by the formula $\Phi_G(\dot{x}) = F\cdot v_X$.

Theorem 3.6. *If* (M, S, L) *admits* G *as a symmetry group, then* $(I_G)^{\cdot} = \Phi_G$.

Corollary 3.3. *If* $F \equiv 0$*, then under the hypotheses of Theorem* 3.6 *the momentum* I_G *is conserved.*

One can derive Theorem 3.6 from the d'Alembert–Lagrange principle using identity (3.3).

We now apply these general considerations to the dynamics of systems of material points in three-dimensional oriented Euclidean space. We assume that a force $\mathbf{F}$ acts on a point $(\mathbf{r}, m)$. We consider the group of translations along a moving straight line with directional vector $\mathbf{e}(t)$: $\mathbf{r} \mapsto \mathbf{r} + \alpha\mathbf{e}$, $\alpha \in \mathbb{R}$.

Theorem 3.7 ([353])**.** *Suppose that the following conditions hold:*

1) *the vectors* $\boldsymbol{\xi}_k = \mathbf{e}$ *(for* $1 \leqslant k \leqslant n$*) are virtual velocities,*
2) $\langle \mathbf{P}, \dot{\mathbf{e}} \rangle = 0$*, where* $\mathbf{P} = \sum m\dot{\mathbf{r}}$ *is the total momentum.*

Then $\langle \mathbf{P}, \mathbf{e} \rangle^{\cdot} = \left\langle \sum \mathbf{F}, \mathbf{e} \right\rangle$.

Corollary 3.4. *Suppose that the vectors* $\boldsymbol{\xi}_k = \dot{\boldsymbol{\eta}} = \left(\sum m\mathbf{r} / \sum m\right)^{\cdot}$ *(for* $1 \leqslant k \leqslant n$*) are virtual velocities at each instant. If the system moves freely* $(\mathbf{F} \equiv 0)$*, then the velocity of its centre of mass* $\dot{\boldsymbol{\eta}}$ *is constant.*

Example 3.4. Consider a balanced skate sliding on the horizontal plane and a homogeneous disc rolling so that its plane is always vertical. By Corollary 3.4 the velocities of their centres of mass are constant. △

We also consider the group of rotations of Euclidean space around a moving straight line l with directional unit vector $\mathbf{e}(t)$ passing through a point with position vector $\mathbf{r}_0(t)$. Let $\mathbf{K}$ be the angular momentum of a system of material points with respect to the fixed origin of reference, and let K_l and M_l be, respectively, the angular momentum and the moment of forces with respect to the moving axis l.

Theorem 3.8 ([353])**.** *Suppose that the following conditions hold:*

1) *when the system rotates as a rigid body around the axis* l*, the velocity vectors of the material points are virtual velocities at each instant,*
2) $\langle \mathbf{P}, (\mathbf{r}_0 \times \mathbf{e})^{\cdot} \rangle + \langle \mathbf{K}, \mathbf{e} \rangle^{\cdot} = 0$.

Then $\dot{K}_l = M_l$.

In particular, if the axis l does not change its direction in space ($\mathbf{e}(t) =$ const), then condition 2) becomes Chaplygin's condition (1897):

$$\langle \mathbf{e}, \dot{\mathbf{r}}_0 \times \dot{\boldsymbol{\eta}} \rangle = 0,$$

where $\dot{\boldsymbol{\eta}}$ is the velocity of the centre of mass. In the case where $\mathbf{r}_0 = \boldsymbol{\eta}$ condition 2) can be simplified to $\langle \mathbf{K} + \mathbf{r}_0 \times \mathbf{P}, \dot{\mathbf{e}} \rangle = 0$. This condition is automatically satisfied under the additional assumption that $\mathbf{e}(t) =$ const. For example, a balanced skate rotates around the vertical axis with constant angular velocity.

Example 3.5. Consider *Chaplygin's problem* of the rolling on the horizontal plane of a dynamically asymmetric ball whose centre of mass coincides with its geometric centre. Let o be the contact point of the ball with the plane and let $\mathbf{K}_0$ be the angular momentum of the ball with respect to the point o. Chaplygin's problem admits the group $SO(3)$ of rotations around the contact point. The momentum map $I_{SO(3)}$ is of course equal to $\mathbf{K}_0$, and the moment of forces is zero: $\Phi_G = 0$. Consequently, $\mathbf{K}_0 = \mathrm{const}$ by Theorem 3.6. This observation allows us to form a closed system of differential equations of rolling of the ball. Let $\mathbf{k}_0$ be the angular momentum in the moving space attached to the rigid body, $\boldsymbol{\omega}$ the angular velocity of rotation of the ball, and $\boldsymbol{\gamma}$ the unit vertical vector. The fact that the vectors $\mathbf{k}_0$ and $\boldsymbol{\gamma}$ are constant in the fixed space is equivalent to the equations

$$\dot{\mathbf{k}}_0 + \boldsymbol{\omega} \times \mathbf{k}_0 = 0, \qquad \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = 0. \tag{3.5}$$

Let A be the inertia tensor of the body with respect to the centre of mass, m the mass of the ball, and a its radius. Then $\mathbf{k}_0 = A\boldsymbol{\omega} + ma^2\boldsymbol{\gamma} \times (\boldsymbol{\omega} \times \boldsymbol{\gamma})$. This relation turns equations (3.5) into a closed system of differential equations with respect to $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$. Equations (3.5) have four independent integrals: $F_1 = \langle \mathbf{k}_0, \mathbf{k}_0 \rangle$, $F_2 = \langle \mathbf{k}_0, \boldsymbol{\gamma} \rangle$, $F_3 = \langle \boldsymbol{\gamma}, \boldsymbol{\gamma} \rangle = 1$, $F_4 = \langle \mathbf{k}_0, \boldsymbol{\omega} \rangle$. The last integral expresses the constancy of the kinetic energy of the rolling ball. Using these integrals one can integrate equations (3.5) by quadratures (Chaplygin, 1903). △

3.1.3 Symmetries in Vakonomic Mechanics

Let (M, S, L) be a vakonomic system, and G a Lie group acting on M.

Definition 3.6. The group G is called a *symmetry group of the vakonomic system* (M, S, L) if

1) the group G takes $S \subset TM$ to S,
2) G preserves the restriction of L to S.

Definition 3.7. The *momentum map* I_G *of the vakonomic system for the action of the group* G is the map $T^*M \to \mathscr{G}^*$ defined by the formula $p \mapsto p \cdot v_X$, $X \in \mathscr{G}$, where p is the vakonomic momentum.

Example 3.6. Suppose that the system (M, S, L) is natural and the kinetic energy is given by a Riemannian metric $\langle\, , \rangle$. If the constraint S is given by the equation $\langle a(x), \dot{x} \rangle = 0$, then

$$I = \langle v, \dot{x} \rangle + \langle p, a \rangle (a \cdot v)/\langle a, a \rangle. \qquad \triangle$$

Theorem 3.9. *If the vakonomic system* (M, S, L) *admits* G *as a symmetry group, then* $I_G = \mathrm{const}$.

The function I_G is not observable in the general case. However, if the symmetry fields v_X, $X \in \mathscr{G}$, are fields of virtual velocities, then I_G is equal to $L_{\dot{x}} \cdot v_X$ and therefore is observable.

Example 3.7. A skate on the horizontal plane regarded as a vakonomic system admits the group of translations, but does not admit the group of rotations around the vertical axis. Consequently, the vakonomic momentum of the skate is conserved. However, this quantity is not observable. The vakonomic momentum map for the action of the group of rotations of the skate coincides with the ordinary angular momentum, which is not a first integral of the equations of motion. △

3.1.4 Symmetries in Hamiltonian Mechanics

Let (M, ω^2) be a symplectic connected manifold and suppose that a group $g = \{g^s\}$ acts on M as a group of symplectic diffeomorphisms. The group g gives rise to the vector field

$$v = \left.\frac{d}{ds}\right|_{s=0} g^s.$$

This field is locally Hamiltonian: the 1-form $\omega^2(\cdot, v)$ is closed. Hence, locally $\omega^2(\cdot, v) = dF$. Extension of the function F to the entire manifold M leads, as a rule, to a multivalued Hamiltonian function.

Example 3.8. Let N be a smooth manifold, and $\{g^s\}$ a group of diffeomorphisms of N generated by a vector field u. Since each diffeomorphism of the manifold N takes 1-forms to 1-forms, the group $\{g^s\}$ acts also on the cotangent bundle $M = T^*N$. Recall that M has the standard symplectic structure $\omega^2 = dp \wedge dq = d(p \cdot dq)$, where p, q are "canonical" coordinates on M. Since the group $\{g^s\}$ preserves the 1-form $p \cdot dq$, it preserves the 2-form ω^2 and therefore is a group of symplectic diffeomorphisms of the manifold M. The action of $\{g^s\}$ on M is generated by the single-valued Hamiltonian function $F = p \cdot u$. △

Theorem 3.10. *A group of symplectic diffeomorphisms $\{g^s\}$ with a single-valued Hamiltonian function F preserves a function $H\colon\ M \to \mathbb{R}$ if and only if F is a first integral of the Hamiltonian system with Hamiltonian function H.*

◁ The proof is based on the formula

$$\left.\frac{d}{ds}\right|_{s=0} H(g^s(x)) = \{H, F\}(x).$$

▷

We now suppose that a Lie group G has a symplectic action on M such that to each element X of the Lie algebra $\mathscr{G}$ of G there corresponds a one-parameter subgroup with a single-valued Hamiltonian function F_X. These Hamiltonians are defined up to constant summands.

Definition 3.8. A symplectic action of G on M is called a *Poisson action* if the correspondence $X \mapsto F_X$ can be chosen so that

1) F_X depends linearly on X,
2) $\{F_X, F_Y\} = F_{[X,Y]}$ for all $X, Y \in \mathscr{G}$.

Example 3.9. Let N be a smooth manifold, and G a Lie group acting on N. We lift the action of G on N to a symplectic action of G on T^*N as described in Example 3.8. The action thus constructed is a Poisson action. This follows from the linearity of the function $p \cdot v_X$ and the formula $\{p \cdot v_X,\, p \cdot v_Y\} = p \cdot [v_X, v_Y] = p \cdot v_{[X,Y]}$. △

A Poisson action of the group G on M defines the natural map $P_G\colon\ M \to \mathscr{G}^*$ that associates with a point x the linear function $F_X(x)$ of the variable $X \in \mathscr{G}$ on the algebra $\mathscr{G}$. We call this map the *momentum map for the Poisson action* of the group G.

Proposition 3.1. *Under the momentum map P the Poisson action of the connected Lie group G is projected to the coadjoint action of the group G on $\mathscr{G}^*$ in the sense that the following diagram is commutative:*

$$\begin{array}{ccc} M & \xrightarrow{g} & M \\ {\scriptstyle P}\big\downarrow & & \big\downarrow{\scriptstyle P} \\ \mathscr{G}^* & \xrightarrow{Ad^*_{g^{-1}}} & \mathscr{G}^* \end{array} .$$

Suppose that (N, L) is a Lagrangian system and a Lie group G acts on N. The Lagrangian L defines the Legendre transformation $TN \to T^*N$. The composition of the momentum map $P_G\colon\ T^*N \to \mathscr{G}^*$ for the lifted Poisson action of G on the symplectic manifold T^*N and the Legendre transformation coincides with the momentum map $I_G\colon\ TN \to \mathscr{G}^*$ of the Lagrangian system (N, L) for the action of G defined earlier.

If a function $H\colon\ M \to \mathbb{R}$ is invariant under the Poisson action of the group G, then by Theorem 3.10 the momentum map P_G is a first integral of the system with Hamiltonian function H.

In conclusion we discuss symmetries in Dirac's generalized Hamiltonian mechanics. Suppose that (M, ω^2, H, N) is a Hamiltonian system with constraints, where $H\colon\ M \to \mathbb{R}$ is the Hamiltonian function, and N a submanifold of M (see § 1.5.1).

Theorem 3.11. *Suppose that we are given a Poisson action of a Lie group G on the symplectic manifold (M, ω^2) such that G preserves the function H and the submanifold N. Then the momentum map P_G takes constant value on the motions of the Hamiltonian system with constraints.*

3.2 Reduction of Systems with Symmetries

3.2.1 Order Reduction (Lagrangian Aspect)

If a Lagrangian system (M, L) admits a symmetry group $\{g^\alpha\}$, then it turns out that it is possible to diminish the number of the degrees of freedom of the system. To the group g there corresponds the first integral I_g, which is always cyclic locally. First we consider the classical *Routh's method for eliminating cyclic coordinates*; then we discuss the global order reduction.

Suppose that the Lagrangian $L(\dot{q}, \dot{\lambda}, q)$ does not involve the coordinate λ. Using the equality $L'_{\dot{\lambda}} = c$ we represent the cyclic velocity $\dot{\lambda}$ as a function of $\dot{q}$, q, and c. Following Routh we introduce the function

$$R_c(\dot{q}, q) = L(\dot{q}, \dot{\lambda}, q) - c\dot{\lambda}\big|_{\dot{q},\, q,\, c}\,.$$

Theorem 3.12. *A vector-function $(q(t), \lambda(t))$ is a motion of the Lagrangian system (M, L) with the constant value of the cyclic integral $I_g = c$ if and only if $q(t)$ satisfies Lagrange's equation $[R_c] = 0$.*

If there are several cyclic coordinates $\lambda_1, \ldots, \lambda_k$, then for the *Routh function* one should take $R_{c_1,\ldots,c_k} = L - \sum c_s \dot{\lambda}_s$.

A small neighbourhood U of a non-singular point of the symmetry field v is "regularly" foliated into the orbits of the group g (integral curves of the field v): the quotient space $N = U/g$ is a smooth manifold with Cartesian coordinates q. It is natural to call the pair (N, R_c) the (*locally*) *reduced Lagrangian system*. For example, the elimination of the polar angle in Kepler's problem (see § 2.1.1) is an example of order reduction by Routh's method.

Cyclic coordinates are not uniquely determined: among the new variables $Q = q$, $\Lambda = \lambda + f(q)$ the coordinate Λ is also a cyclic coordinate. Let $\widehat{L}(\dot{Q}, \dot{\Lambda}, Q) = L(\dot{q}, \dot{\lambda}, q)$. Then, obviously, $\widehat{L}'_{\dot{\Lambda}} = L'_{\dot{\lambda}} = c$. The Routh function corresponding to the new cyclic coordinate Λ is $\widehat{R}_c(\dot{Q}, Q) = R_c(\dot{q}, q) + cf'_q \cdot \dot{q}$. In view of the identity $[\dot{f}] \equiv 0$ the summand $c(f'_q \cdot \dot{q})$ of course does not affect the form of the equation $[R_c] = 0$. But this means that the Routh function is not uniquely determined for $c \neq 0$. These observations prove to be useful in the analysis of the global order reduction, which we shall now consider. For definiteness we shall consider the case of natural Lagrangian systems.

Let (M, N, pr, S, G) be a fibre bundle with total space M, base space N, projection $pr\colon\ M \to N$ (the rank of the differential pr_* is equal to $\dim N$ at all points of M), fibre S, and structure group G. The group G acts on the left on the fibre S freely and transitively. This action can be extended to a left action of G on M; then all the orbits of G will be diffeomorphic to S. In the case of a principle bundle, the manifold S is diffeomorphic to the space of the group G. The base space N can be regarded as the quotient space of the manifold M by the equivalence relation defined by the action of the group G. The tangent vectors v_X, $X \in \mathscr{G}$, to the orbits of the group G are vertical: $pr_*(v_X) = 0$.

Suppose that G is a symmetry group of a natural mechanical system $(M, \langle\ ,\ \rangle, V)$. We define on the bundle (M, N, pr, S, G) the "canonical" connection by declaring as horizontal the tangent vectors to M that are orthogonal in the metric $\langle\ ,\ \rangle$ to all the vectors v_X, $X \in \mathscr{G}$. This connection is compatible with the structure group G: the distribution of horizontal vectors is mapped to itself under the action of G on M. A smooth path $\gamma\colon [t_1, t_2] \to M$ is said to be *horizontal* if the tangent vectors $\dot\gamma(t)$ are horizontal for all $t_1 \leqslant t \leqslant t_2$. It is easy to verify that for any smooth path $\widetilde\gamma\colon [t_1, t_2] \to N$ and any point $x_1 \in M$ lying over $\widetilde\gamma(t_1)$ (that is, such that $pr(x_1) = \gamma(t_1)$) there exists only one horizontal path $\gamma\colon [t_1, t_2] \to M$ covering $\widetilde\gamma$.

We equip the manifold $N = M/G$ with the "quotient metric" $\widetilde{\langle\, ,\rangle}$ by first restricting the original metric on M to the distribution of horizontal vectors and then pushing it down onto N. Since the potential $V\colon M \to \mathbb{R}$ is constant on the orbits of the group G, there exists a unique smooth function $\widetilde V\colon N \to \mathbb{R}$ such that the following diagram is commutative:

$$\begin{tikzcd} M \arrow[rr, "pr"] \arrow[dr, "V"'] & & N \arrow[dl, "\widetilde V"] \\ & \mathbb{R} & \end{tikzcd} .$$

Theorem 3.13. *The motions of the natural system $(M, \langle\, ,\rangle, V)$ with zero value of the momentum map I_G are uniquely projected to the motions of the reduced system $(N, \widetilde{\langle\, ,\rangle}, \widetilde V)$.*

$\lhd$ Let $\widetilde\gamma\colon [t_1, t_2] \to N$ be a motion of the reduced system, and $\widetilde\gamma_\alpha$ its variation with fixed ends. Let $\gamma_\alpha\colon [t_1, t_2] \to M$ be a horizontal lifting of the path $\widetilde\gamma_\alpha$ such that $\gamma_\alpha(t_1) = \gamma_0(t_1)$ for all α. The variation field u of the family of paths γ_α is such that $u(t_1) = 0$ and $u(t_2)$ is a vertical vector. If L (respectively, $\widetilde L$) is the Lagrangian of the original (reduced) system, then by the first variation formula,

$$\delta \int_{t_1}^{t_2} \widetilde L\, dt = \delta \int_{t_1}^{t_2} L\, dt = \langle \dot\gamma_0, u\rangle\Big|_{t_1}^{t_2} = 0. \qquad \rhd$$

Example 3.10. Consider the motion of a material point m in a central force field. In this problem we have the bundle $\big(\mathbb{R}^3 \setminus \{0\},\ \mathbb{R}^+,\ pr,\ S^2,\ SO(3)\big)$; the projection $pr\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^+$ is defined by the formula $(x, y, z) \mapsto \sqrt{x^2 + y^2 + z^2}$. The Lagrangian $L = m|\dot{\mathbf{r}}|^2/2 + V(|\mathbf{r}|)$ admits the group $SO(3)$ of rotations around the point $x = y = z = 0$. If the angular momentum $I_{SO(3)}$ is equal to zero, then on $\mathbb{R}^+ = \{s > 0\}$ we obtain a one-dimensional reduced system with the Lagrangian $\widetilde L = m\dot s^2/2 + V(s)$. $\triangle$

We now consider order reduction when the momentum map I_G is non-zero. We assume the group G to be commutative (Routh's method can be applied only in this case). Moreover, we assume that (M, N, pr, G) is a principal bundle; in particular, the group G acts freely on M. Apart from the quotient metric $\widetilde{\langle\, ,\rangle}$ on the base space we shall also need the curvature form of the canonical connection. We remind the reader of the construction of this form. First we introduce the connection 1-form ω on M with values in the Lie algebra $\mathscr{G}$. This form is defined as follows: if $u \in TM$, then $\omega(u)$ is equal to $X \in \mathscr{G}$

such that v_X coincides with the vertical component of the vector u. In the case of a principle bundle the kernel of the homomorphism of the Lie algebra $\mathscr{G}$ into the algebra of vector fields on M is zero; hence the connection form is well defined. For example, if $\dim G = 1$, then one can set $\omega(u) = \langle u, v\rangle / \langle v, v\rangle$, where v is the symmetry field. The *curvature form* Ω is a $\mathscr{G}$-valued 2-form such that $\Omega(u_1, u_2) = d\omega(u_1^\perp, u_2^\perp)$, where $u^\perp$ is the horizontal component of a tangent vector u. Since G is a commutative symmetry group, the form Ω can be pushed down to N. Let $I_G = c \in \mathscr{G}^*$. Since Ω takes values in $\mathscr{G}$, the value of the momentum map on the curvature form is well defined: $\Omega_c = c \cdot \Omega$. The form Ω_c is an $\mathbb{R}$-valued form on the base space N. According to *Cartan's structural equation* $\Omega = d\omega + [\omega, \omega]$, the forms Ω and Ω_c are closed.

Lemma 3.2. *Let $c \in \mathscr{G}^*$. Then for every point $x \in M$ there exists a unique vertical tangent vector $w_c \in T_x M$ such that $I_G(w_c) = c$.*

Indeed, w_c is the unique element in the set $\{w \in T_x M\colon\ I_G(w) = c\}$ that has minimum length in the $\langle\ ,\ \rangle$-metric. This assertion is valid for an arbitrary group G.

Definition 3.9. The *effective* (or *amended*, or *reduced*) *force function* of the natural system with the symmetry group G corresponding to a constant value $I_G = c$ of the momentum map is the function $V_c\colon\ M \to \mathbb{R}$ equal to $V - \langle w_c, w_c\rangle/2$.

Lemma 3.3. *The function V_c is invariant under G_c, where $G_c \subset G$ is the isotropy subgroup of the coadjoint action of G on $\mathscr{G}^*$ at the element $c \in \mathscr{G}^*$* (*see Proposition* 3.1).

Corollary 3.5. *If G is commutative, then V_c is constant on the orbits of the group G.*

This assertion allows us to define correctly the effective potential $\widetilde{U}_c = -\widetilde{V}_c$ as a function on the base space N.

Theorem 3.14. *A function $\gamma\colon\ \Delta \to M$ is a motion of the natural system $(M, \langle\ ,\ \rangle, V)$ with a constant value $I_G = c$ of the momentum map if and only if the projection $\mu = pr \circ \gamma\colon\ \Delta \to N$ satisfies the differential equation*

$$[L_c]_\mu = F_c(\dot{\mu}), \tag{3.6}$$

where $L_c = \widetilde{\langle \dot{\mu}, \dot{\mu}\rangle}/2 + \widetilde{V}_c$ and $F_c(v) = \Omega_c(\cdot, v)$.

Theorem 3.14 can be derived, for example, from Theorem 3.9.

Equation (3.6) can be regarded as the equation of motion of the natural system $(N, \widetilde{\langle\ ,\ \rangle}, \widetilde{V}_c)$ under the action of the additional non-conservative forces F_c. Since $F_c(v) \cdot v = \Omega_c(v, v) = 0$, these forces do not perform work on the real motion. They are called *gyroscopic forces*.

Since the form Ω_c is closed, we have locally $\Omega_c = d\omega_c$. Consequently, (3.6) is Lagrange's equation $[R_c] = 0$, where $R_c = L_c - \omega_c$. Routh's function R_c is defined globally on TN only if the form Ω_c is exact.

Example 3.11. Consider the rotation of a rigid body with a fixed point in an axially symmetric force field. The kinetic energy and the potential admit the group $SO(2)$ of rotations around the symmetry axis of the field. In this problem, M is diffeomorphic to the underlying space of the group $SO(3)$. The reduction $SO(3)/SO(2)$ was first carried out by Poisson as follows. Let $\mathbf{e}$ be a unit vector of the symmetry axis of the force field regarded as a vector of the moving space. The action of the subgroup $SO(2)$ on $SO(3)$ by right translations leaves $\mathbf{e}$ invariant. The set of all positions of the vector $\mathbf{e}$ in the moving space forms a two-dimensional sphere S^2, called the "Poisson sphere". The points of S^2 "number" the orbits of the rotation group $SO(2)$. Thus, we have the fibre bundle $SO(3)$ with structure group $SO(2)$ and base space S^2. The symmetry group $SO(2)$ generates a first integral: the projection of the angular momentum of the rigid body onto the axis with directional vector $\mathbf{e}$ is conserved. By fixing a constant value of this projection we can simplify the problem to the study of the reduced system with configuration space S^2. Here Routh's function is not defined globally, since the curvature form $\varOmega$ is not exact:

$$\int_{S^2} \varOmega = 4\pi \neq 0$$

for all values of the principal moments of inertia. We shall give explicit order-reduction formulae below. △

The theory of order reduction for Lagrangian systems can be carried over, with obvious modifications, to non-holonomic mechanics. To carry out the reduction of a non-holonomic system to the quotient system by a symmetry group we need the additional assumption that the constraints be invariant under the action of this group. An example is provided by Chaplygin's problem of a ball rolling on a horizontal plane (see Example 3.5). This problem admits the group $SO(2)$ of rotations of the ball around the vertical line passing through its centre. The group $SO(2)$ preserves the constraints, and the field generating this group is a virtual velocity field. In fact we have eliminated the rotation group in Example 3.5 using Poisson's method.

In conclusion we also mention the "problem of *hidden motions*" or the "problem of *action at a distance*", which agitated physicists at the end of 19th century. Suppose that a natural mechanical system with $n+1$ degrees of freedom moves freely and that its Lagrangian, representing only the kinetic energy, admits a symmetry group with field v. Reducing the order of the system we see that Routh's function, which is the Lagrangian of the reduced system with n degrees of freedom, contains the summand (the effective potential) $\widetilde{U}_c = \langle w_c, w_c\rangle/2 = c^2/2\langle v, v\rangle$, which is independent of the velocities. This summand can be interpreted as the potential of certain forces acting on the reduced system. Helmholtz, Thomson, Hertz insisted that every mechanical quantity that manifests itself as a "potential energy" is caused by hidden "cyclic" motions. A typical example is the rotation of a symmetric top: since the rotation of the top around the symmetry axis cannot be detected, one can regard the top as non-rotating and explain its strange behaviour by the action of additional conservative forces.

Since $U_c = \langle w_c, w_c\rangle/2 > 0$, Routh's method can produce only positive potentials. However, since a potential is defined up to an additive constant, this limitation is inessential if the configuration space is compact.

Theorem 3.15. *Let* $(M, \langle\ ,\ \rangle, V, \Omega)$ *be a mechanical system with a closed form of gyroscopic forces* Ω. *If* M *is compact, then there exists a principal bundle with base space* M *and structure symmetry group* $\mathbb{T}^k$, $k \leqslant \operatorname{rank} H^2(M,\mathbb{R})$, *such that after the reduction according to Routh, for some constant value* $J_{\mathbb{T}^k} = c$ *of the momentum map we have the equalities* $V_c = V + \mathrm{const}$, $\Omega_c = \Omega$.

This assertion was proved by Bolotin (see [124]).

If $\Omega = 0$, then for the fibre bundle in Theorem 3.15 we can take the direct product $M \times S^1\{\varphi \bmod 2\pi\}$ with the metric $\langle \dot{x}, \dot{x}\rangle + \dot{\varphi}^2/U(x)$, where $\langle\ ,\ \rangle$ is the Riemannian metric on M. The coordinate φ is cyclic; the corresponding cyclic integral is $\dot{\varphi}/U = c$. Routh's function is $R_c = \langle \dot{x}, \dot{x}\rangle/2 - c^2U/2$. For $c = \sqrt{2}$ we have a natural system on $M \times S^1/S^1 \simeq M$ with potential U.

3.2.2 Order Reduction (Hamiltonian Aspect)

Let $F\colon\ M \to \mathbb{R}$ be a first integral of a Hamiltonian system with Hamiltonian H.

Proposition 3.2. *If* $dF(z) \neq 0$, *then in some neighbourhood of the point* $z \in M$ *there exist symplectic coordinates* $x_1, \dots, x_n, y_1, \dots, y_n$ *such that* $F(x,y) = y_1$ *and* $\omega = \sum dy_k \wedge dx_k$.

This assertion is the Hamiltonian version of the theorem on rectification of trajectories.

In the coordinates x, y the function H is independent of x_1. Consequently, if we fix a value $F = y_1 = c$, then the system of equations

$$\dot{x}_k = H'_{y_k}, \qquad \dot{y}_k = -H'_{x_k} \qquad (k \geqslant 2)$$

is a Hamiltonian system with $n-1$ degrees of freedom. Thus, one integral allows us to reduce the dimension of the *phase space* by two units: one unit vanishes when the value $F = c$ is fixed, and another vanishes due to the elimination of the cyclic variable x_1 along the orbit of the action of the symmetry group $\{g_F^\alpha\}$. This remark can be generalized: if a Hamiltonian system has s independent integrals in involution, then it can be reduced to a system with $n-s$ degrees of freedom. We remark that an effective use of the first integral F for order reduction is held up by the problem of finding the orbits of the group $\{g_F^\alpha\}$, which is related to integration of the Hamiltonian system with Hamiltonian F.

If the algebra of integrals is non-commutative, then the dimension of the Hamiltonian system can be reduced by at least double the maximum dimension of a commutative subalgebra. The number of commuting integrals can sometimes be increased by considering nonlinear functions of the first integrals.

Example 3.12. In the problem of the motion of a point in a central field in $\mathbb{R}^3$ the algebra of first integrals has a subalgebra isomorphic to the Lie algebra $so(3)$. All of its commutative subalgebras are one-dimensional. Let M_i be the projection of the angular momentum of the point onto the ith axis of a Cartesian orthogonal coordinate system. It is easy to verify that the functions M_1 and $M^2 = \sum M_i^2$ are independent and commute. Thus, this problem reduces to the study of a Hamiltonian system with one degree of freedom. $\triangle$

This method of order reduction for Hamiltonian systems is due to Poincaré, who applied it in various problems of celestial mechanics. This is essentially the Hamiltonian version of the order reduction according to Routh. If the algebra of integrals is non-commutative, then Poincaré's method does not make full use of the known integrals. This shortcoming of Poincaré's method was overcome by Cartan, who studied the general case of an infinite-dimensional Lie algebra of the first integrals (see [18]). More precisely, Cartan considered a Hamiltonian system (M, ω^2, H) with first integrals $F_1, \dots, F_k$ such that $\{F_i, F_j\} = a_{ij}(F_1, \dots, F_k)$. The set of integrals $F_1, \dots, F_k$ defines the natural map $F\colon\ M \to \mathbb{R}^k$. In the general case the functions $a_{ij}\colon\ \mathbb{R}^k \to \mathbb{R}$ are nonlinear.

Theorem 3.16 (Lie–Cartan). *Suppose that a point $c \in \mathbb{R}^k$ is not a critical value of the map F and has a neighbourhood where the rank of the matrix $\big(a_{ij}\big)$ is constant. Then in a small neighbourhood $U \subset \mathbb{R}^k$ of the point c there exist k independent functions $\varphi_s\colon\ U \to \mathbb{R}$ such that the functions $\Phi_s = \varphi_s \circ F\colon\ N \to \mathbb{R}$, where $N = F^{-1}(U)$, satisfy the following relations:*

$$\{\Phi_1, \Phi_2\} = \cdots = \{\Phi_{2q-1}, \Phi_{2q}\} = 1 \tag{3.7}$$

and all the other brackets are $\{\Phi_i, \Phi_j\} = 0$. The number $2q$ is equal to the rank of the matrix $\big(a_{ij}\big)$.

A proof can be found in [18]. Using this theorem we can now easily reduce the order. Suppose that a point $c = (c_1, \dots, c_k)$ satisfies the hypotheses of Theorem 3.16. Then, in particular, the level set $M_c = \{x \in M : \Phi_s(x) = c_s, 1 \leqslant s \leqslant k\}$ is a smooth submanifold of M of dimension $2n - k$, where $2n = \dim M$. Since the functions $\Phi_{2q+1}, \dots, \Phi_k$ commute with all the functions Φ_s, $1 \leqslant s \leqslant k$, their Hamiltonian fields are tangent to the manifold M_c. If these Hamiltonian fields are not hampered[3] on M_c, then defined

[3] A vector field is said to be *not hampered* if the motion with this field as the velocity field is defined on the time interval $(-\infty, \infty)$.

on M_c there is the action of the commutative group $\mathbb{R}^l$, $l = k - 2q$, generated by the phase flows of Hamilton's equations with Hamiltonians Φ_s, $s > 2q$. Since the functions Φ_s are functionally independent, the group $\mathbb{R}^l$ acts on M_c without fixed points. If its orbits are compact (then they are l-dimensional tori), then the quotient space $M_c/\mathbb{R}^l = \widetilde{M}_c$ is a smooth manifold called the *reduced phase space*. Since $\dim \widetilde{M}_c = (2n-k) - l = 2(n-k+q)$, the manifold $\widetilde{M}_c$ is always even-dimensional.

On the reduced phase space there exists a natural symplectic structure $\widetilde{\omega^2}$, which can be defined, for example, by a non-degenerate Poisson bracket $\widetilde{\{\ ,\ \}}$. Let $A, B\colon\ \widetilde{M}_c \to \mathbb{R}$ be smooth functions. They can be lifted to smooth functions $'A$, $'B$ defined on the level manifold $M_c \subset M$. Let $\widetilde{A}, \widetilde{B}$ be arbitrary smooth functions on M whose restrictions to M_c coincide with $'A, 'B$. We finally set $\widetilde{\{A, B\}} = \{\widetilde{A}, \widetilde{B}\}$.

Lemma 3.4. *The bracket $\widetilde{\{\ ,\ \}}$ is well defined (it is independent of the extensions of the smooth functions from the submanifold $\widetilde{M}_c$ to the whole of M) and is a Poisson bracket on $\widetilde{M}_c$.*

Let $'H$ be the restriction of the Hamiltonian function H to the integral level M_c. Since the function $'H$ is constant on the orbits of the group $\mathbb{R}^l$, there exists a smooth function $\widetilde{H}\colon\ M_c/\mathbb{R}^l \to \mathbb{R}$ such that the diagram

$$\begin{array}{ccc} M_c & \xrightarrow{pr} & \widetilde{M}_c \\ {}_{'H}\searrow & & \swarrow_{\widetilde{H}} \\ & \mathbb{R} & \end{array}$$

is commutative.

Definition 3.10. The Hamiltonian system $(\widetilde{M}_c, \widetilde{\omega}^2, \widetilde{H})$ is called the *reduced Hamiltonian system.*

Theorem 3.17. *A smooth map $\gamma\colon\ \Delta \to M$ with $F(\gamma(t)) = c$ is a motion of the Hamiltonian system (M_c, ω^2, H) if and only if the composition $pr \circ \gamma\colon$ $\Delta \to \widetilde{M}_c$ is a motion of the reduced Hamiltonian system $(\widetilde{M}_c, \widetilde{\omega}^2, \widetilde{H})$.*

$\lhd$ This theorem can be established by the following considerations. Formulae (3.7) show that the functions $\Phi_1, \dots, \Phi_k$ form a part of symplectic coordinates in a neighbourhood of the submanifold M_c. More precisely, in a small neighbourhood of every point of M_c one can introduce symplectic coordinates $x_1, \dots, x_n, y_1, \dots, y_n$ so that $x_i = \Phi_{2i-1}$, $y_i = \Phi_{2i}$ if $i \leqslant q$, and $y_i = \Phi_i$ if $i > 2q$. This assertion is a consequence of the well-known "completion lemma" of Carathéodory (see [10]). Since the functions Φ_s are first integrals, in variables x, y the Hamiltonian has the form $H(y, x) = H(y_{q+1}, \dots, y_n, x_{k-q+1}, \dots, x_n)$. It remains to fix the values of the cyclic integrals $y_{q+1}, \dots, y_{k-q}$ and observe that the variables x_s, y_s $(s > k-q)$ are local coordinates on $\widetilde{M}_c$ in which the form $\widetilde{\omega}^2$ becomes "canonical": $\sum_{s>k-q} dx_s \wedge dy_s$. $\rhd$

Remark 3.1. Since the $k-q$ first integrals $\Phi_2, \dots, \Phi_{2q}, \Phi_{2q+1}, \dots, \Phi_k$ commute, one can use them for reducing the order of the Hamiltonian system according to Poincaré. The dimension of the local phase space of the reduced system will be equal to $2n-2(k-q)$, that is, to the dimension of the manifold $\widetilde{M}_c$. Moreover, by Theorem 3.16 the order reductions according to Poincaré and according to Cartan give locally the same result, but the reduction by Poincaré's method can be carried out globally only under more restrictive conditions.

In degenerate cases the rank of the matrix of Poisson brackets $\big(\{F_i, F_j\}\big)$ can of course drop. One can carry out the order reduction by Cartan's scheme also in this situation if in addition the integrals $F_1, \dots, F_k$ are assumed to generate a finite-dimensional algebra (the functions $a_{ij}\colon \mathbb{R}^k \to \mathbb{R}$ are linear). Indeed, suppose that we have a Poisson action of the group G on the symplectic manifold (M, ω^2). Consider the set $M_c = P^{-1}(c)$, the inverse image of some point $c \in \mathscr{G}^*$ under the momentum map $P\colon\ M \to \mathscr{G}^*$. If c is not a critical value of the momentum map P, then M_c is a smooth submanifold of M. Since the action of the group G is Poisson, by Proposition 3.1 the elements of G take the integral levels M_c one to another. Let G_c be the isotropy subgroup at a point $c \in \mathscr{G}^*$ consisting of the $g \in G$ such that $Ad_{g^*} c = c$. The group G_c is a Lie group acting on M_c. If the orbits of G_c on M_c are compact, then the reduced phase space $\widetilde{M}_c = M_c/G_c$ is a smooth manifold. Then we can define the reduced Hamiltonian system $(\widetilde{M}_c, \widetilde{\omega}^2, \widetilde{H})$ by repeating word for word the construction of order reduction according to Cartan. The connection between the original and reduced Hamiltonian systems is again described by Theorem 3.17. The proofs can be found in the works of Souriau [565] and Marsden and Weinstein [411].

Example 3.13. The motion of a material point of unit mass in a central field can be described by the Hamiltonian system in $\mathbb{R}^6 = \mathbb{R}^3\{\mathbf{x}\} \times \mathbb{R}^3\{\mathbf{y}\}$ with the standard symplectic structure and Hamiltonian function $H(\mathbf{y}, \mathbf{x}) = |\mathbf{y}|^2/2 + U(|\mathbf{x}|)$. We fix the constant angular momentum vector $\mathbf{x} \times \mathbf{y} = \boldsymbol{\mu}$ ($\boldsymbol{\mu} \neq 0$). We may assume that $\boldsymbol{\mu} = c\,\mathbf{e}_3$, where $\mathbf{e}_3 = (0,0,1)$ and $c > 0$. The level set M_c is given by the equations $x_3 = y_3 = 0,\ \ x_1 y_2 - x_2 y_1 = c$. Clearly the vector $\boldsymbol{\mu}$ is invariant under the group $SO(2)$ of rotations around the axis with unit vector $\mathbf{e}_3$. To carry out the reduction with respect to this group we introduce in the plane $\mathbb{R}^2$ the polar coordinates r, φ and the corresponding canonical conjugate variables p_r, p_φ:

$$x_1 = r\cos\varphi, \qquad y_1 = p_r \cos\varphi - \frac{p_\varphi}{r}\sin\varphi,$$

$$x_2 = r\sin\varphi, \qquad y_2 = p_r \sin\varphi + \frac{p_\varphi}{r}\cos\varphi.$$

Obviously, in the new variables the set M_c is given by the equations $x_3 = y_3 = 0,\ \ p_\varphi = c$. The reduction with respect to the group $SO(2)$ amounts to the elimination of the angle variable φ. Thus, the reduced phase space $\widetilde{M}_c = M_c/SO(2)$ is diffeomorphic to $\mathbb{R}^+\{r\} \times \mathbb{R}\{p_r\}$; it is equipped with the reduced symplectic structure $\widetilde{\omega}^2 = dp_r \wedge dr$. The reduced Hamiltonian has the form $\widetilde{H} = (p_r^2 + c^2 r^{-2})/2 + U(r)$. $\triangle$

If an element $c \in \mathscr{G}^*$ is generic (the matrix $\bigl(a_{ij}\bigr)$ has maximum rank[4]), then the group G_c is commutative; the order reduction conducted by this scheme gives the same result as the reduction according to Cartan. If $c = 0$, then the rank of the matrix $\bigl(a_{ij}\bigr)$ drops to zero and the integral manifold M_0 has the most "symmetric" structure: the isotropy subgroup G_0 coincides with the entire group G. In this case we have the maximal possible reduction of the order of the Hamiltonian system by $2k = 2\dim G$ units (cf. Theorem 3.13).

Let $(N, \langle\ ,\ \rangle, V)$ be a natural mechanical system, and G a compact commutative symmetry group (isomorphic to $\mathbb{T}^k$) acting freely on the configuration space N. We can regard this system as a Hamiltonian system with symmetries on $M = T^*N$ and apply our scheme of order reduction. There is a Poisson action of the group G on T^*N; since this action is free, every value $c \in \mathscr{G}^*$ of the momentum map is regular. Consequently, the smooth integral level manifold M_c is defined (of codimension $k = \dim G$ in M), and the reduced phase space $\widetilde{M}_c$ (whose dimension is smaller by $2k$ than the dimension of M). On the other hand, we can define the smooth reduced configuration space $\widetilde{N}$ as the quotient of N by the orbits of the action of G. Moreover, for the same value $c \in \mathscr{G}^*$ we have the "seminatural" reduced Lagrangian system $(\widetilde{N}, \widetilde{\langle\ ,\ \rangle}, \widetilde{V}_c, \Omega_c)$ (see § 3.1.2, Theorem 3.13). It is appropriate to define the *reduced Lagrangian* $\widetilde{L}\colon T\widetilde{N} \to \mathbb{R}$ as the function given by the equality $\widetilde{L}(\dot{x}) = \widetilde{\langle \dot{x}, \dot{x} \rangle}/2 + \widetilde{V}_c(x)$.

Theorem 3.18. *For every $c \in \mathscr{G}^*$ there exists a diffeomorphism $f\colon \widetilde{M}_c \to T^*\widetilde{N}$ such that*

1) *$f^*\widetilde{\omega}^2 = \Omega + \Omega_c$, where Ω is the standard symplectic structure on T^*N,*
2) *the function $f \circ \widetilde{H}\colon T^*\widetilde{N} \to \mathbb{R}$ is the Legendre transform of the reduced Lagrangian defined by the metric $\widetilde{\langle\ ,\ \rangle}$.*

Corollary 3.6. *The manifold $\widetilde{M}_0$ is symplectically diffeomorphic to T^*N.*

If the group G is non-commutative, then the reduced phase space $\widetilde{M}_c$ in general does not coincide with the cotangent bundle of any smooth manifold.

Suppose that we have a free Poisson action of a commutative group G on a symplectic manifold (M, ω^2). In this case the passage to the reduced manifold $(\widetilde{M}_c, \widetilde{\omega}^2)$ can also be realized as follows. Consider the quotient manifold $N = M/G$ and the bracket $'\{\ ,\ \}$ on it which is the original Poisson bracket $\{\ ,\ \}$ pushed down to N. It is easy to see that the bracket $'\{\ ,\ \}$ is degenerate.

[4] In the case of a Poisson algebra of integrals one should, perhaps, better speak about the rank of the bilinear form $\{F_X, F_Y\}$, $X, Y \in \mathscr{G}$.

If $P\colon M \to \mathscr{G}^*$ is the momentum map, then there exists a smooth map $\widetilde{P}\colon N \to \mathscr{G}^*$ such that the diagram

$$\begin{tikzcd} M \arrow[rr, "pr"] \arrow[dr, "P"'] & & N \arrow[dl, "\widetilde{P}"] \\ & \mathscr{G}^* & \end{tikzcd}$$

is commutative. Since G acts freely, a point $c \in \mathscr{G}^*$ is a critical value of the map P if and only if c is a critical value of $\widetilde{P}$. Assuming that $c \in \mathscr{G}^*$ is a regular point we consider the smooth manifold $N_c = \widetilde{P}^{-1}(c)$ and the restriction of the bracket $'\{\ ,\ \}$ to N_c.

Proposition 3.3. *The restriction of the bracket $'\{\ ,\ \}$ to N_c defines a symplectic structure $'\omega^2$, and the manifolds $(\widetilde{M}_c, \widetilde{\omega}^2)$ and $(N_c,\ '\omega^2)$ are symplectically diffeomorphic.*

This remark can be generalized to the case of a non-commutative group G, but taking the quotient of M with respect to the whole group G must be replaced by the reduction with respect to the centre of G.

Example 3.14. In the problem of rotation of a rigid body with a fixed point we have $M = TSO(3) = SO(3) \times \mathbb{R}^3$. If the body rotates in an axially symmetric force field, then there is the one-parameter symmetry group $G = SO(2)$. The quotient manifold $M/SO(2)$ is diffeomorphic to $S^2 \times \mathbb{R}^3$. The equations of motion on this five-dimensional manifold can be written as the Euler–Poisson equations

$$\dot{\mathbf{k}} + \boldsymbol{\omega} \times \mathbf{k} = V' \times \mathbf{e}, \qquad \dot{\mathbf{e}} + \boldsymbol{\omega} \times \mathbf{e} = 0 \qquad (|\mathbf{e}| = 1),$$

where $\mathbf{k} = A\boldsymbol{\omega}$ is the angular momentum and $V\colon\ S^2 \to \mathbb{R}$ is the force function (see § 1.2). The bracket $'\{\ ,\ \}$ in $S^2 \times \mathbb{R}^3$ is defined by the following formulae:

$$\begin{gathered} '\{\omega_1, \omega_2\} = -\frac{A_3\omega_3}{A_1A_2}, \quad \ldots, \quad '\{\omega_1, e_1\} = 0, \\ '\{\omega_1, e_2\} = -\frac{e_3}{A_1}, \qquad '\{\omega_1, e_3\} = \frac{e_2}{A_1}, \quad \ldots, \quad '\{e_i, e_j\} = 0. \end{gathered} \tag{3.8}$$

The Euler–Poisson equations have the integral $\langle \mathbf{k}, \mathbf{e} \rangle = c$ generated by the symmetry group $SO(2)$. We fix a constant value of this integral and consider the four-dimensional integral level $N_c = \{\boldsymbol{\omega}, \mathbf{e}\colon \langle A\boldsymbol{\omega}, \mathbf{e}\rangle = c,\ \langle \mathbf{e}, \mathbf{e}\rangle = 1\}$, which is diffeomorphic to the tangent bundle of the Poisson sphere $S^2 = \{\mathbf{e} \in \mathbb{R}^3\colon\ \langle \mathbf{e}, \mathbf{e}\rangle = 1\}$. We set $\boldsymbol{\omega} = {}'\boldsymbol{\omega} + c\mathbf{e}/\langle A\mathbf{e}, \mathbf{e}\rangle$; the vector $'\boldsymbol{\omega}$ is a horizontal tangent vector in the canonical connection of the principal bundle $(SO(3),\ S^2,\ SO(2))$ generated by the invariant Riemannian metric $\langle A\boldsymbol{\omega}, \boldsymbol{\omega}\rangle/2$. The projection $SO(3) \to S^2$ allows us to identify the horizontal vectors $'\boldsymbol{\omega}$ with the tangent vectors to the Poisson sphere. Let $\widetilde{\langle\ ,\ \rangle}$ be the quotient metric on S^2 given by $\langle \widetilde{'\mathbf{a}, '\mathbf{b}} \rangle = \langle '\mathbf{a}, A'\mathbf{b} \rangle$. The Lagrange function of the reduced system is obviously equal to

$$\frac{1}{2}\langle A\boldsymbol{\omega}, \boldsymbol{\omega} \rangle + V(\mathbf{e}) = \frac{1}{2}\left\langle \widetilde{'\boldsymbol{\omega}, '\boldsymbol{\omega}} \right\rangle + \widetilde{V}_c(\mathbf{e}),$$

where $\widetilde{V}_c = V - c^2/2\langle A\mathbf{e}, \mathbf{e} \rangle$ is the effective force function. In the variables $'\boldsymbol{\omega}, \mathbf{e}$ the standard symplectic structure on T^*S^2 is given by (3.8). For $c \neq 0$ the reduced structure on T^*S^2 can also be defined by (3.8), only summands proportional to the constant c must be added to the right-hand sides. △

3.2.3 Examples: Free Rotation of a Rigid Body and the Three-Body Problem

First we consider the Euler problem of the free rotation of a rigid body around a fixed point (see § 1.2.4). Here $M = TSO(3) = SO(3) \times \mathbb{R}^3$, the symmetry group G is the rotation group $SO(3)$; the corresponding Poisson algebra of first integrals is isomorphic to the Lie algebra $so(3)$. We fix a value of the angular momentum $c \in \mathscr{G}^* \simeq \mathbb{R}^3$ and consider the integral level $M_c = P^{-1}_{SO(3)}(c)$. It is easy to show that for any value of c the set M_c is a three-dimensional manifold diffeomorphic to the space of the group $SO(3)$. The isotropy group G_c is the one-dimensional group $SO(2)$ of rotations of the rigid body in the stationary space around the constant vector of angular momentum. The reduced phase space $\widetilde{M}_c = SO(3)/SO(2)$ is diffeomorphic to the two-dimensional sphere.

This reduction can be realized, for example, as follows. Since the Hamiltonian vector field on M admits the group G, this field can be pushed down to the quotient space $M/G \simeq \mathbb{R}^3$. The differential equation emerging on $\mathbb{R}^3$ is the Euler equation

$$\dot{\mathbf{k}} + \boldsymbol{\omega} \times \mathbf{k} = 0, \qquad \boldsymbol{\omega} = A^{-1}\mathbf{k}.$$

This equation can be represented in the Hamiltonian form $\dot{F} = \{F, H\}$, where $H = \langle \mathbf{k}, \boldsymbol{\omega} \rangle/2$ is the kinetic energy of the rigid body, and the bracket $\{\ ,\ \}$ is defined by the equalities $\{k_1, k_2\} = -k_3$, $\{k_2, k_3\} = -k_1$, $\{k_3, k_1\} = -k_2$. However, this bracket is degenerate: the function $F = \langle \mathbf{k}, \mathbf{k} \rangle$ commutes with all the functions defined on $\mathbb{R}^3 = \{\mathbf{k}\}$. We obtain a non-degenerate Poisson bracket by restricting the bracket $\{\ ,\ \}$ to the level surface $F = |c|^2$, which is diffeomorphic to the two-dimensional sphere S^2. The required Hamiltonian system arises on the symplectic manifold S^2; its Hamiltonian function is the total energy $\langle \mathbf{k}, \boldsymbol{\omega} \rangle/2$ restricted to S^2.

We now describe the classical method of reducing the Euler problem to a Hamiltonian system with one degree of freedom based on the special canonical variables. Let $oXYZ$ be a stationary trihedron with origin at the fixed point, and let $oxyz$ be the moving coordinate system (the principal inertia axes of the body). A position of the rigid body in the fixed space is determined by the three *Euler angles*: ϑ (nutation angle) is the angle between the axes oZ and oz,

φ (proper rotation angle) is the angle between the axis ox and the intersection line of the planes oxy and oXY (called the line of nodes), ψ (precession angle) is the angle between the axis oX and the line of nodes. The angles ϑ, φ, ψ form a coordinate system on $SO(3)$ similar to the geographical coordinates on a sphere, which is singular at the poles (where $\vartheta = 0, \pi$) and multivalued on one meridian. Let $p_\vartheta, p_\varphi, p_\psi$ be the canonical momenta conjugate to the coordinates ϑ, φ, ψ. If the rigid body rotates in an axially symmetric force field with symmetry axis oZ, then the Hamiltonian function is independent of the angle ψ. The order reduction in this case can be interpreted as the "elimination of the node", that is, the elimination of the cyclic variable ψ which defines the position of the line of nodes in the fixed space.

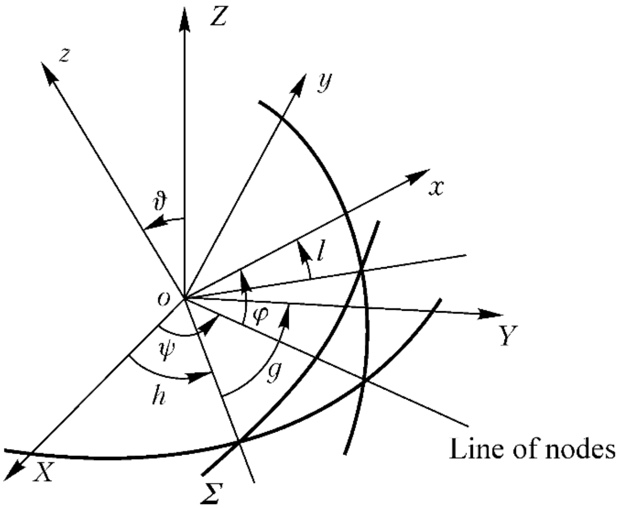

Fig. 3.1. Special canonical variables

We now introduce the "special canonical variables" L, G, H, l, g, h. Let Σ be the plane passing through the point o and perpendicular to the angular momentum vector of the body. Then L is the projection of the angular momentum onto the axis oz, G is the magnitude of the angular momentum, H is the projection of the angular momentum onto the axis oZ, l is the angle between the axis ox and the intersection line of Σ and the plane oxy, g is the angle between the intersection lines of Σ and the planes oxy and oXY, h is the angle between the axis oX and the intersection line of Σ and the plane oXY.

Proposition 3.4. *The transformation* $(\vartheta, \varphi, \psi, p_\vartheta, p_\varphi, p_\psi) \mapsto (l, g, h, L, G, H)$ *is "homogeneous" canonical:*

$$p_\vartheta\, d\vartheta + p_\varphi\, d\varphi + p_\psi\, d\psi = L\, dl + G\, dg + H\, dh.$$

This assertion is due to Andoyer; non-canonical variables similar to the elements L, G, H, l, g, h were used by Poisson in the analysis of the rotational motion of celestial bodies [65].

It is easy to obtain from the definition of the special canonical variables that $A_1\omega_1 = \sqrt{G^2 - L^2}\sin l$, $A_2\omega_2 = \sqrt{G^2 - L^2}\cos l$, and $A_3\omega_3 = L$. Consequently, in the Euler problem the Hamiltonian function reduces to the form

$$\frac{1}{2}\left(A_1\omega_1^2 + A_2\omega_2^2 + A_3\omega_3^2\right) = \frac{1}{2}\left(\frac{\sin^2 l}{A_1} + \frac{\cos^2 l}{A_2}\right)(G^2 - L^2) + \frac{L^2}{2A_3}.$$

For a fixed value of the magnitude of the angular momentum G_0, the variables L, l vary within the annulus $|L| \leqslant G_0$, $l \bmod 2\pi$. The level lines of the Hamiltonian function are shown in Fig. 3.2. The curves $L = \pm G_0$ correspond to the singular points of the Euler equations – the permanent rotations of the body around the inertia axis oz. It is natural to regard the variables L, l as geographical symplectic coordinates on the reduced phase space S^2.

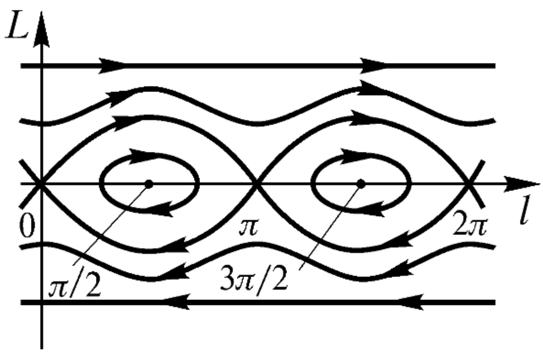

Fig. 3.2.

We now consider from the viewpoint of order reduction the three-body problem, which has 9 degrees of freedom (in the spatial case). We shall show that using the six integrals of momentum and angular momentum one can reduce the equations of motion of the three gravitating bodies to a Hamiltonian system with 4 degrees of freedom. Using also the energy integral we conclude that the three-body problem reduces to studying a dynamical system on a certain seven-dimensional manifold. In the case where the three bodies are permanently situated in a fixed plane, the dimension of this manifold is equal to five. These results go back to Lagrange and Jacobi.

We pass to a barycentric coordinate system and first use the three-dimensional commutative group of translations. Using this group we reduce the dimension of Hamilton's equations of motion from 18 to 12. The resulting reduced system, as the original one, has the symmetry group $G = SO(3)$. Fixing a value of the angular momentum we arrive at the equations of motion on a nine-dimensional integral manifold. Taking its quotient by the isotropy subgroup of rotations around the constant angular momentum vector we obtain the required Hamiltonian system with eight-dimensional phase space. Now the question is how this reduction can be carried out explicitly.

First we eliminate the motion of the centre of mass. Let $\mathbf{r}_s$ be the position vectors of the point masses m_s in a barycentric frame of reference, so that $\sum m_s \mathbf{r}_s = 0$. In order to use this relation for order reduction of the differential equations of motion

$$m_{\mathbf{s}}\ddot{\mathbf{r}}_s = V'_{\mathbf{r}_s} \qquad (1 \leqslant s \leqslant 3), \qquad V = \sum_{i<j} \frac{m_i m_j}{r_{ij}}, \tag{3.9}$$

we introduce the relative position vectors $\boldsymbol{\xi} = \mathbf{r}_2 - \mathbf{r}_1$, $\boldsymbol{\eta} = \mathbf{r}_3 - \boldsymbol{\zeta}$, where $\boldsymbol{\zeta} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1 + m_2)$ is the centre of mass of the points m_1 and m_2. We set $\mu = m_1 m_2/(m_1 + m_2)$ and $\nu = (m_1 + m_2)m_3/\sum m_s$.

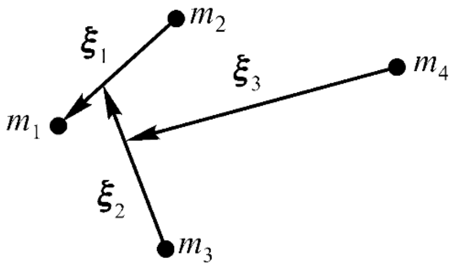

Fig. 3.3. Elimination of the centre of mass in the n-body problem

Proposition 3.5. *If $\mathbf{r}_s(t)$ is a motion of the gravitating points, then the functions $\boldsymbol{\xi}(t)$ and $\boldsymbol{\eta}(t)$ satisfy the equations*

$$\mu\ddot{\boldsymbol{\xi}} = W'_{\boldsymbol{\xi}}, \qquad \nu\ddot{\boldsymbol{\eta}} = W'_{\boldsymbol{\eta}}, \qquad W(\boldsymbol{\xi}, \boldsymbol{\eta}) = V\big|_{\boldsymbol{\xi},\, \boldsymbol{\eta}}. \tag{3.10}$$

These equations have the first integral

$$\mu(\boldsymbol{\xi} \times \dot{\boldsymbol{\xi}}) + \nu(\boldsymbol{\eta} \times \dot{\boldsymbol{\eta}}) = \sum m_s(\mathbf{r}_s \times \dot{\mathbf{r}}_s) = \mathbf{c}.$$

Equations (3.10) describe the motion of the "fictitious" material points with masses μ, ν. Proposition 3.5 can be easily generalized to the case of any $n > 3$. Equations (3.10) with 6 degrees of freedom are of course Hamiltonian.

Elimination of the angular momentum ("elimination of the node") can be carried out for equations (3.10). However, it is easier to state the final result independently in a symmetric form with respect to the masses m_1, m_2, m_3. Let Σ be the "Laplacian invariant plane": it contains the barycentre and is perpendicular to the constant angular momentum $\mathbf{c}$. Let Π be the plane passing through the points m_1, m_2, m_3. We denote by ϑ_i the angle of the triangle $m_1m_2m_3$ at the vertex m_i, and by $\varDelta$ the area of this triangle. We have the formulae

$$\sin\vartheta_i = \frac{2\varDelta}{\rho_j\rho_k}, \quad \cos\vartheta_i = \frac{\rho_i^2-\rho_j^2-\rho_k^2}{2\rho_j\rho_k}, \quad \varDelta = \frac{\varPi\sqrt{\rho_j+\rho_k-\rho_i}}{4\sqrt{\sum\rho_s}}, \tag{3.11}$$

where i, j, k are allowed to be only the three cyclic permutations of the indices 1, 2, 3, and ρ_i is the length of the side of the triangle opposite the vertex m_i. Let γ be the angle between the planes $\varPi$ and $\varSigma$; in the planar motion, $\gamma\equiv 0$.

Proposition 3.6. *For a fixed value* $\mathbf{c}=\sum m_s(\mathbf{r}_s\times\dot{\mathbf{r}}_s)$ *of the angular momentum, in barycentric coordinates the equations of the three-body problem reduce to the following Hamilton's equations with four degrees of freedom:*

$$\dot\varGamma = -H'_\gamma, \quad \dot\gamma = H'_\varGamma; \quad \dot P_s = -H'_{\rho_s}, \quad \dot\rho_s = H'_{P_s} \quad (1\leqslant s\leqslant 3); \tag{3.12}$$

$$\begin{aligned} H(\varGamma,\, P_1,\, P_2,\, P_3,\, \gamma,\, \rho_1,\, \rho_2,\, \rho_3) &= \\ = \frac{|\mathbf{c}|^2\sin\gamma}{4\varDelta}&\sum\frac{\rho_s^2}{m_s}\sin^2\left(\frac{\varGamma}{|\mathbf{c}|\sin\gamma}+\frac{\vartheta_j-\vartheta_k}{3}\right) \\ &+\sum\frac{P_j^2+P_k^2-2P_jP_k\cos\vartheta_i}{2m_i}+|\mathbf{c}|\cos\gamma\sum\left(\frac{P_j}{\rho_k}-\frac{P_k}{\rho_j}\right)\frac{\sin\vartheta_i}{3m_i} \\ &+|\mathbf{c}|^2\cos^2\gamma\sum\frac{\rho_j^2+\rho_k^2-\rho_i^2/2}{36m_i\rho_j^2\rho_k^2}-\sum\frac{m_jm_k}{\rho_i}, \end{aligned}$$

where the quantities $\varDelta$, ϑ_1, ϑ_2, *and* ϑ_3 *are expressed by formulae* (3.11) *as functions of* ρ_1,ρ_2,ρ_3, *and* $\sum f_{ijk}$ *denotes the sum* $f_{123}+f_{231}+f_{312}$.

This proposition is due to van Kampen and Wintner [301]. The proof is based on elementary but cumbersome calculations. The expressions of the momenta $\varGamma$, P_s in terms of the coordinates and velocities of the gravitating points are very cumbersome and usually are not used.

When the motion is planar, then the first two equations (3.12) reduce to the equalities $\varGamma=\gamma=0$ and we obtain a Hamiltonian system with three degrees of freedom.

If $\mathbf{c}=\mathbf{0}$, then equations (3.12) form a natural Hamiltonian system with three degrees of freedom (cf. Theorem 3.13).

3.3 Relative Equilibria and Bifurcation of Integral Manifolds

3.3.1 Relative Equilibria and Effective Potential

We again return to the study of a Hamiltonian system (M,ω^2,H) admitting a symmetry group G with a Poisson action on the phase space M. Let $(\widetilde{M},\widetilde{\omega}^2,\widetilde{H})$ be the reduced Hamiltonian system in the sense of § 3.2.2.

Definition 3.11. The phase curves of the Hamiltonian system on M with a constant value $P_G = c$ of the momentum map that are taken by the projection $M \to \widetilde{M}_c$ to equilibrium positions of the reduced Hamiltonian system are called *relative equilibria* or *stationary motions.*

Example 3.15. Consider the rotation of a rigid body in an axially symmetric force field. Let c be a fixed value of the angular momentum of the body with respect to the symmetry axis of the force field. The equations of motion of the reduced system can be represented in the form

$$A\dot{\boldsymbol{\omega}} = A\boldsymbol{\omega} \times \boldsymbol{\omega} - \mathbf{e} \times V', \quad \dot{\mathbf{e}} = \mathbf{e} \times \boldsymbol{\omega}; \quad \langle A\boldsymbol{\omega}, \mathbf{e}\rangle = c, \quad \langle \mathbf{e}, \mathbf{e}\rangle = 1, \quad (3.13)$$

where $V(e)$ is the force function. At an equilibrium position of the reduced system we obviously have $\mathbf{e} = \text{const}$ and therefore $\boldsymbol{\omega} = \lambda\mathbf{e}$. The factor λ can be uniquely determined from the equation $\langle A\boldsymbol{\omega}, \mathbf{e}\rangle = c$, which gives $\lambda = c/\langle A\mathbf{e}, \mathbf{e}\rangle$. Since $\mathbf{e} = \text{const}$, the angular velocity $\boldsymbol{\omega}$ is also constant. From the first equation (3.13) we obtain the following equation for finding the relative equilibria with the angular momentum c:

$$c^2(A\mathbf{e} \times \mathbf{e}) + (V' \times \mathbf{e})\langle A\mathbf{e}, \mathbf{e}\rangle^2 = 0, \qquad \langle \mathbf{e}, \mathbf{e}\rangle = 1.$$

This result was first noted by Staude in 1894. In a stationary motion (a relative equilibrium) the rigid body rotates uniformly around the symmetry axis of the force field with the angular velocity $|\boldsymbol{\omega}| = |c|/\langle A\mathbf{e}, \mathbf{e}\rangle$. △

Proposition 3.7. *A phase curve $x(t)$ of the Hamiltonian system (M, ω^2, H) with the symmetry group G is a relative equilibrium if and only if $x(t) = g^t(x(0))$, where $\{g^t\}$ is a one-parameter subgroup of G.*

◁ If $x(t) = g^t(x_0)$ and $\{g^t\}$ is a subgroup of G, then the projection $M \to \widetilde{M}_c$ takes the solution $x(t)$ to an equilibrium position of the reduced system. Conversely, suppose that $x(t) = h^t(x_0)$ is a relative equilibrium of the Hamiltonian system with Hamiltonian H satisfying the initial condition $x(0) = x_0$. We claim that $\{h^t\}$ is a subgroup of G. Let $\{g^t\}$ be a one-parameter subgroup of G such that

$$\left.\frac{d}{dt}\right|_{t=0} g^t(x_0) = \dot{x}(0) \qquad \left(= \left.\frac{d}{dt}\right|_{t=0} h^t(x_0)\right). \qquad (3.14)$$

Since G is a symmetry group, the groups $\{h^s\}$ and $\{g^t\}$ commute and therefore $x(t) = g^t(x_0)$ by (3.14). ▷

In Example 3.15 above, the trajectories of stationary motions are the orbits of the group $SO(2)$ of rotations of the body around the symmetry axis of the field.

For natural mechanical systems with symmetries one can state a more effective criterion for a motion to be stationary. Let $(M, \langle\ ,\ \rangle, V)$ be a mechanical system with a symmetry group (in the sense of § 3.2.1): the manifold M is the space of a principal bundle with base space N and structure group G.

Proposition 3.8. *If the symmetry group G is commutative, then $y \in N$ is a relative equilibrium position (that is, the projection of a relative equilibrium onto the base N) with momentum constant $c \in \mathcal{G}^*$ if and only if y is a critical point of the effective potential $\widetilde{U}_c\colon\ N \to \mathbb{R}$.*

This assertion follows from Theorem 3.14 and the definition of a relative equilibrium. For example, since any smooth function on the sphere has at least two critical points, Proposition 3.8 implies the following.

Corollary 3.7. *The problem of rotation of a rigid body with a fixed point in any axially symmetric force field has at least two distinct stationary rotations for every value of the angular momentum.*

One can estimate the number of distinct stationary motions in the general case, for example, using Morse's inequalities. However, it is usually possible to obtain more precise information in concrete problems (see §§ 3.3.3–3.3.4).

3.3.2 Integral Manifolds, Regions of Possible Motion, and Bifurcation Sets

Let (M, ω^2, H, G) be a Hamiltonian system with a Poisson symmetry group G. Since the Hamiltonian H is a first integral, it is natural to combine this function with the momentum integrals $P\colon\ M \to \mathcal{G}^*$ and consider the smooth *energy–momentum map* $H \times P\colon\ M \to R \times \mathcal{G}^*$.

Definition 3.12. We define the *bifurcation set* Σ of the Hamiltonian system (M, ω^2, H, G) as the set of points in $R \times \mathcal{G}^*$ over whose neighbourhoods the map $H \times P$ is not a locally trivial bundle.

In particular, the set Σ' of critical values of the energy–momentum map is contained in Σ. However, in the general case the set Σ is not exhausted by Σ'. An example is provided by the bifurcation set of Kepler's problem considered in § 2.1.

Proposition 3.9. *The critical points of the map $H \times P\colon\ M \to \mathbb{R} \times \mathcal{G}^*$ on a regular level of the momentum map coincide with the relative equilibria.*

This simple assertion proves to be useful in the study of the structure of bifurcation sets.

Definition 3.13. For fixed values of the energy $h \in \mathbb{R}$ and the momentum map $c \in \mathcal{G}^*$ the set $I_{h,c} = (H \times P)^{-1}(h, c)$ is called the *integral manifold* of the Hamiltonian system (M, ω^2, H, G).

It is obvious that the integral levels $I_{h,c}$ may not be smooth manifolds only for $(h, c) \in \Sigma$. Since the action of the group G preserves the function H, the isotropy group G_c acts on the level $I_{h,c}$ and therefore the quotient manifold

$\widetilde{I}_{h,\,c} = I_{h,\,c}/G_c$ is defined. If c is a regular value of the momentum map, then $\widetilde{I}_{h,\,c}$ coincides with an energy level of the reduced Hamiltonian system $(\widetilde{M}_c, \widetilde{\omega}^2, \widetilde{H})$. It is therefore natural to call the set $\widetilde{I}_{h,\,c}$ the *reduced integral manifold*. For example, in the spatial three-body problem typical manifolds $\widetilde{I}_{h,c}$ are seven-dimensional, and in the planar problem their dimension is five. Since the map $H \times P$ is a bundle over each connected component of $\mathbb{R} \times \mathscr{G}^* \setminus \Sigma$, the topological type of the integral manifolds $\widetilde{I}_{h,\,c}$ can change only as the point (h, c) passes through the bifurcation set Σ.

Thus, the study of the original Hamiltonian system with symmetries reduces to the study of the map $H \times P$ and the structure of the phase flows on the reduced integral manifolds $\widetilde{I}_{h,\,c}$.

We consider in more detail the structure of the energy–momentum map for a natural mechanical system $(M, \langle\ ,\ \rangle, V)$ with a symmetry group G; we are not assuming the action of G on M to be free. Let Λ be the set of points $x \in M$ such that the isotropy subgroup G_x (consisting of $g \in G$ such that $g(x) = x$) has positive dimension. The set Λ is closed in M. For example, in the spatial three-body problem Λ consists of collinear triples of points. In the planar problem Λ reduces to the single point $r_1 = r_2 = r_3 = 0$ (as usual we assume that the barycentre is at the origin of reference).

Let $J\colon\ \dot{x} \to \langle \dot{x}, v_X\rangle$ be the momentum map. By Lemma 3.2, for every point $x \in M \setminus \Lambda$ and every $c \in \mathscr{G}^*$ there exists a unique vector $w_c(x)$ such that $J(w_c) = c$ and $\langle w_c, v_X\rangle = 0$ for all $X \in \mathscr{G}$. In § 3.2.1 we defined the effective potential $U_c\colon\ M \to \mathbb{R}$ to be the function $-V + \langle w_c, w_c\rangle/2$.

Proposition 3.10. *The effective potential has the following properties:*

1) $U_c(x) = \min\limits_{v \in J_x^{-1}(c)} H(v)$, *where* $H(v) = \langle v, v\rangle/2 - V(x)$ *is the total energy of the system*;
2) *on* $M \setminus \Lambda$ *the set of critical points of the map* $H\colon\ J^{-1}(c) \to \mathbb{R}$ *coincides with* $w_c(\Gamma)$, *where* Γ *is the set of critical points of the effective potential* $U_c\colon\ M \setminus \Lambda \to \mathbb{R}$;
3) $\Sigma' = \{(h, c)\colon\ h \in U_c(\Gamma)\}$;
4) $\pi(I_{h,\,c}) = U_c^{-1}(-\infty, h]$, *where* $\pi\colon\ TM \to M$ *is the projection.*

This proposition was stated by Smale; in concrete situations it had been used even earlier by various authors. Part 2) refines Proposition 3.9.

Definition 3.14. The set $\pi(I_{h,\,c}) \subset M$ is called the *region of possible motion* for the fixed values of the energy h and the momentum map c.

If the group G is commutative, then part 4) of Proposition 3.10 can be replaced by

4′) $\pi'(\widetilde{I}_{h,\,c}) \subset \widetilde{U}_c^{-1}(-\infty, h]$, *where* $\pi'\colon\ TN \to N$ *is the projection,* $N = M/G$ *is the reduced configuration space, and* $\widetilde{U}_c\colon\ N \to \mathbb{R}$ *is the effective potential.*

If M is compact, then $\Sigma = \Sigma'$ and inclusion in $4'$) can be replaced by equality. In the non-compact case this is no longer true: a counterexample is provided by the spatial n-body problem. It is interesting to note that in the planar n-body problem the region of possible motion is described by the inequality $U_c \leqslant h$ (Proposition 1.8 in § 1.1.5).

3.3.3 The Bifurcation Set in the Planar Three-Body Problem

Proposition 3.11. *For any given set of masses in the planar three-body problem,*

(1) *in the coordinates h, c, the set of critical values Σ' of the map $H \times J : TM \to \mathbb{R}^2$ consists of the four cubic curves given by equations of the form $hc^2 = \alpha_i < 0$ $(1 \leqslant i \leqslant 4)$,*
(2) *the bifurcation set Σ consists of Σ' and the coordinate axes $h = 0$ and $c = 0$.*

$\lhd$ If U is the potential energy in the three-body problem, then the effective potential U_c is clearly equal to $U + c^2/2I$, where I is the moment of inertia of the points with respect to their barycentre (cf. § 1.1). In a relative equilibrium, dU is proportional to dI and therefore the three points form a central configuration (see § 2.3.1). For a fixed value $c \neq 0$ there are exactly five such configurations: three collinear and two triangular. In the latter case the triangle is necessarily equilateral and these two triangular configurations differ only in the order of the gravitating points. Let $\boldsymbol{\omega}$ be the constant angular velocity of rotation of a central configuration. Then, obviously, $|c| = I|\boldsymbol{\omega}|$, $T = I\omega^2/2$, and

$$h = T + U = \frac{c^2}{2I} + U.$$

Since all the configurations of this type are similar, we can assume that $I = \alpha^2 I_0$ and $U = \alpha^{-1} U_0$. The similarity ratio α can be found from the equality $2T = U$, which is a consequence of Lagrange's identity $\ddot{I} = 2T - U$. The coefficient α is equal to c^2/I_0U_0 and therefore $hc^2 = \alpha_s < 0$ in a relative equilibrium. By part 2) of Proposition 3.10 the bifurcation set Σ includes the curves defined by the equations $hc^2 = \alpha_s$ $(1 \leqslant s \leqslant 5)$. Among the five numbers $\alpha_1, \dots, \alpha_5$ at least two are equal (they correspond to the triangular solutions of Lagrange). The bifurcation set obviously includes also the straight lines $h = 0$, $c = 0$ (as in Kepler's problem). As shown by Smale, the set Σ does not contain any other points (see [47]). $\rhd$

Smale's paper [47] contains information about the topological structure of the integral manifolds in various connected components of the set $\mathbb{R}^2 \setminus \Sigma$.

3.3.4 Bifurcation Sets and Integral Manifolds in the Problem of Rotation of a Heavy Rigid Body with a Fixed Point

Let $A_1 \geqslant A_2 \geqslant A_3$ be the principal moments of inertia of a rigid body, and let x_1, x_2, x_3 be the coordinates of the centre of mass relative to the principal axes. If $\boldsymbol{\omega}$ is the angular velocity of the body, and $\mathbf{e}$ the unit vertical vector (both given in the moving space), then $H = \langle A\boldsymbol{\omega}, \boldsymbol{\omega}\rangle/2 + \varepsilon\langle \mathbf{x}, \mathbf{e}\rangle$ and $J = \langle A\boldsymbol{\omega}, \mathbf{e}\rangle$, where $A = \operatorname{diag}(A_1, A_2, A_3)$. Our task is to describe the bifurcation diagram Σ in the plane $\mathbb{R}^2$ with coordinates h, c and the topological structure of the reduced integral manifolds $\widetilde{I}_{h,\,c}$. It is useful to consider first the special degenerate case where $\varepsilon = 0$ (the Euler problem). The relative equilibria are the critical points of the effective potential $\widetilde{U}_c = c^2/2\langle A\mathbf{e}, \mathbf{e}\rangle$ on the unit sphere $\langle \mathbf{e}, \mathbf{e}\rangle = 1$. If the body is asymmetric ($A_1 > A_2 > A_3$), then there are exactly six such points: $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$. These points correspond to the uniform rotations of the rigid body around the principal axes. Since $\boldsymbol{\omega} = c\mathbf{e}/\langle A\mathbf{e}, \mathbf{e}\rangle$ in a relative equilibrium of the body (see Example 3.15), the energy h and the angular momentum c are connected by one of the relations $h = c^2/2A_s$ $(1 \leqslant s \leqslant 3)$. Since the configuration space of the rigid body – the group $SO(3)$ – is compact, the bifurcation set Σ is the union of the three parabolas (Fig. 3.4).

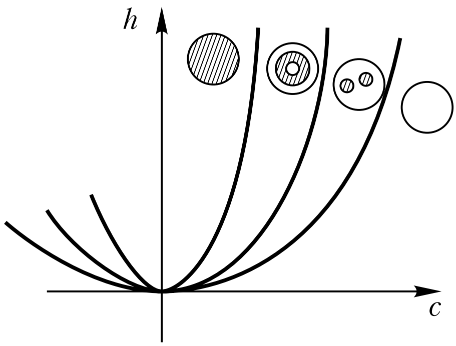

Fig. 3.4. Bifurcation diagram of the Euler problem

In the case of dynamical symmetry the number of parabolas diminishes; if $A_1 = A_2 = A_3 = A$, then Σ consists of the single parabola $h = c^2/2A$. Let $B_{h,\,c} = \{\widetilde{U}_c \leqslant h\}$ be the region of possible motion on the Poisson sphere. The classification of the regions $B_{h,\,c}$ and the reduced integral manifolds $\widetilde{I}_{h,\,c}$ in the Euler problem are given by the following.

Proposition 3.12. *Suppose that $A_1 > A_2 > A_3$. Then*

1) *if $h < c^2/2A_1$, then $B_{h,\,c} = \varnothing$ and $\widetilde{I}_{h,\,c} = \varnothing$;*
2) *if $c^2/2A_1 < h < c^2/2A_2$, then $B_{h,\,c} = D^2 \cup D^2$ and $\widetilde{I}_{h,\,c} = 2S^3$;*

3) *if* $c^2/2A_2 < h < c^2/2A_3$, *then* $B_{h,\,c} = D^1 \times S^1$ *and* $\widetilde{I}_{h,\,c} = S^2 \times S^1$;

4) *if* $c^2/2A_3 < h$, *then* $B_{h,\,c} = S^2$ *and* $\widetilde{I}_{h,\,c} = SO(3)$.

The description of the topological structure of the reduced integral manifolds is based on the following observation: $\widetilde{I}_{h,\,c}$ is diffeomorphic to the fibre bundle with base space $B_{h,\,c}$ and fibre S_1 such that the fibre over each point of the boundary $\partial B_{h,\,c}$ is identified with the point.

In the general case, where the centre of mass does not coincide with the point of suspension, the problem of a complete description of the bifurcation sets and integral manifolds is considerably more difficult. This problem was studied in detail in the papers of Katok [307], Tatarinov [579], and Kuz'mina [363].

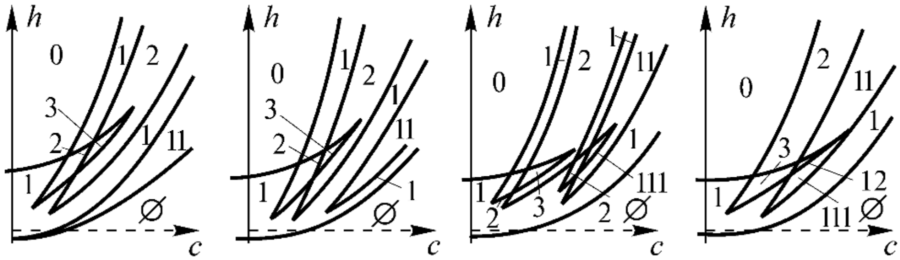

Fig. 3.5.

As an example we give a series of pictures in [579] which shows the mechanism of the transformation of the bifurcation diagram when the centre of mass passes from a generic position in the plane $x_3 = 0$ to the axis $x_1 = x_2 = 0$. The numbers in these pictures indicate the "multivalued genus" of the regions of possible motion on the Poisson sphere. We say that a connected region $B_{h,\,c}$ has genus l if $B_{h,\,c}$ is diffeomorphic to the sphere S^2 from which l non-intersecting open discs are removed. If a region of possible motion is disconnected, then we assign to it the multivalued genus $l_1,\ l_2, \dots$, where the l_s are the genera of its connected components. (Since in the situation under consideration the numbers l_s are at most three, no confusion arises.) The topology of the integral manifolds is uniquely determined by the structure of the regions of possible motion (their genera). The topological structure of maps defined by integrals (for example, momentum maps or energy–momentum maps) is described by the complex whose points are the connected components of the level manifolds of the integrals. For example, for a Hamiltonian system with one degree of freedom whose phase space is simply connected (a disc or a sphere S^2) this complex turns out to be a tree (the level lines of a function with two maxima and one saddle, like the mountain El'brus, form a complex homeomorphic to the letter Y). For a phase space that is a surface of genus g

the resulting graph has g independent cycles (the simplest function on a torus gives rise to a complex homeomorphic to the letter A).

If the number of independent integrals r is greater than 1, then the complex of connected components is no longer a graph but an r-dimensional "surface" with singularities.

The topological invariants of the components are "functions" on this complex. The study of the topological structure of integrable problems should be accompanied by the description of these complexes and "functions" on them. But this has not been done even for the simplest classical integrable systems, notwithstanding hundreds of publications (often erroneous) describing their topological structure.