1 Fundamentals of acoustics

“Waves” are well-known to everybody, even when they are not cognitively identified as waves. While reading this book, light waves as an example of electromagnetic waves are scattered from the white paper and absorbed by printed letters. Electromagnetic waves were predicted by James Clerk Maxell’s theory in 1864 and experimentally discovered by Heinrich Hertz in 1888. Their spectrum from gamma radiation, X-rays, ultraviolet, the visible range, infrared toward spectra for technical communication systems offers a fascinatingly wide area of natural phenomena and technical applications. Waves are to be considered local oscillations in a physical “field” with the inherent effect of energy and information transport and are found in numerous areas of physics. The common approach in these areas is that small perturbations of the equilibrium yield linear or approximately linear forces and oscillating states of permanently recycled potential and kinetic energy.[3]

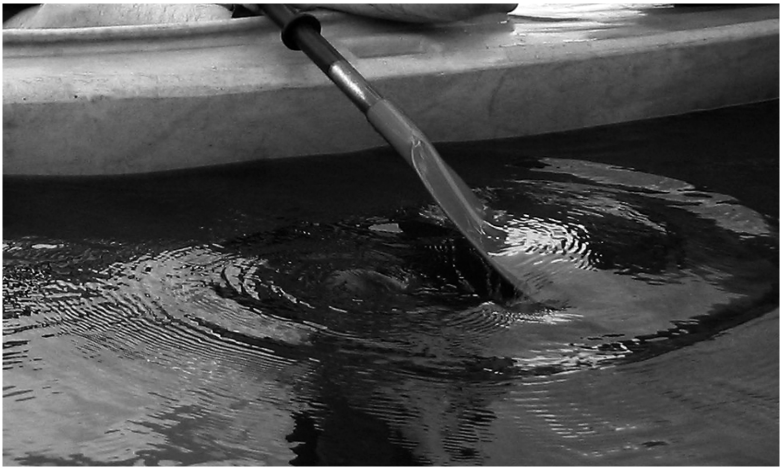


Fig. 1.1. Waves in water

[3] A row of dominos falling is an example where a kind of transport wave is observed without energetic equilibrium. Energy is not recycled. This effect is not a wave in the physical sense we discuss here!

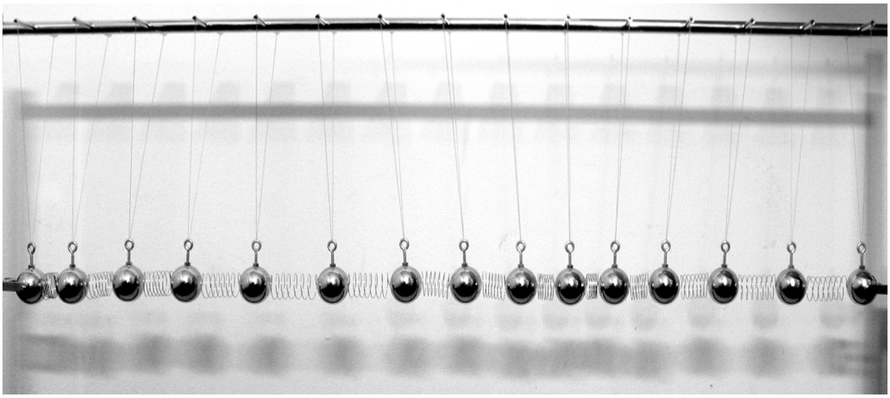


Fig. 1.2. Photograph of metal spheres connected by springs

Although water waves are possibly one of the most enjoyable examples, we now focus on a mechanical system to illustrate of the nature of waves. A chain of masses is connected by springs. In this example, a one-dimensional wave is excited. When one mass is moved (by forced excitation), it takes kinetic energy, transfers this energy to the attached spring, which is compressed (and stores potential energy), recycles and transfers its energy to the next mass, and so on. By intuition, we can imagine easily that heavy masses with large inertia and soft springs with high flexibility provide this transport effect at slower speed than lightweight masses and stiff springs. We can also think of a row of children holding hands. When the first child pushes his neighbour, the neighbour will move to the side, pushing the next neighbour and so on. When all children now stiffen their arm muscles, the wave movement in the row runs faster. By this analogy, the phenomena of energy transport in a wave and the microscopic nature of the wave speed are already understood.

1.1 Sound field equations and the wave equation

Sound is a wave phenomenon in fluid or solid media. The main areas of acoustics are accordingly called airborne sound, underwater sound and structure-borne sound. The differential equations of vibration and waves in acoustics can be derived from dynamic physical laws of continuum mechanics. The physical foundations of linear acoustics are introduced in this chapter: the one-dimensional wave equation, its solutions and the three-dimensional generalization. We start with an image of sound, taken such as a photograph at a certain time, and we observe that the medium's molecules

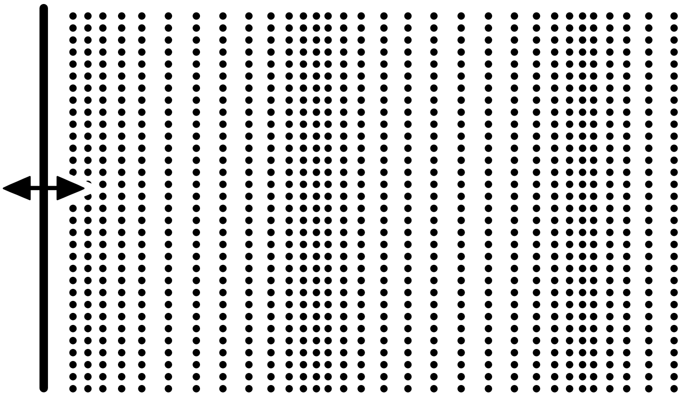


Fig. 1.3. Microscopic view of a medium with sound

or atoms are somehow displaced from their original position, of course, independent of the constant irregular thermal movement.

1.1.1 Sound field quantities

In a sound wave, the particles (gas molecules, crystal lattice atoms, etc.) follow a space- and time-dependent displacement vector, $\vec{s}$. The time derivative of this displacement, accordingly, is the particle velocity,

$$\vec{v} = \frac{\partial \vec{s}}{\partial t}, \tag{1.1}$$

with the components $v_x = \dot{\xi}$ (x, y, z, t), etc.

The displacements are neither homogeneous nor isotropic.[4] Therefore, the medium will be compressed and decompressed. ρ_{tot} is the space- and time-dependent total density, ρ_0 the density of the medium at rest. The density fluctuations due to sound are then

$$\rho = \rho_{\text{tot}} - \rho_0 , \tag{1.2}$$

and the local sound-induced pressure fluctuations, closely related to density is given by

$$p = p_{\text{tot}} - p_0 . \tag{1.3}$$

[4] Homogenous and isotropic means independent of translation or rotation, respectively.

The latter quantity, p, is particularly important. We call it "sound pressure." Note that the sound pressure is a scalar. In acoustics, the sound pressure is typically the leading quantity of interest, mainly because the human ear is sensitive to sound pressure. Hence, calculations or measurements of sound pressure yield directly the input quantity of the human hearing system.

In fluid media such as air the elasticity of the medium is described by its compressibility. In this discussion, the thermodynamic state of the medium and its capability of storing energy (heat) are of crucial importance. Particle displacement and compression affect the pressure, but so do temperature and heat transfer. For acoustic waves in air, however, we might assume that the sound-induced oscillations are so fast that diffusion of heat between local areas of the medium is not possible. This is related to the simplified model of adiabatic processes for which we can use the adiabatic Poisson equation

$$\frac{p_{\mathrm{tot}}}{p_0} = \left(\frac{\rho_{\mathrm{tot}}}{\rho_0}\right)^{\kappa}, \tag{1.4}$$

with κ denoting the adiabatic exponent, the ratio of heat capacities at constant pressure and volume, respectively. $\kappa = C_{\mathrm{p}}/C_{\mathrm{V}}$ (= 1.4 for air).

1.1.2 Derivation of the wave equation

We consider a small volume element of thickness Δx in a one-dimensional fluid medium bounded by a tube with cross section S. In the tube is an acoustic source pushing and pulling periodically the volume element with a strength of $qS\mathrm{d}x$ (in units of [m^3/s]). It excites a small disturbance from the pressure equilibrium. The source can be assumed anywhere in the tube, for instance, represented by a small piston mounted flush in the tube wall. The pressure and particle velocity on the left-hand side of our test element,

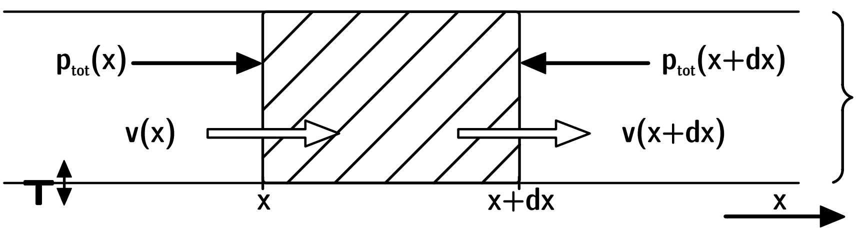


Fig. 1.4. Volume element in a one-dimensional fluid medium. Pressure variations are induced by a "pumping" source on the left side

at x, might differ from the conditions at the right hand side at $x+\Delta x$. A pressure difference will lead to a net force on the volume element.

According to the Euler equation of all forces involved, we obtain a movement of medium mass:

$$\left[p_{\mathrm{tot}}(x) - p_{\mathrm{tot}}(x + \mathrm{d}x)\right] \cdot S = \rho_{\mathrm{tot}} S \mathrm{d}x \cdot \frac{\mathrm{d}v}{\mathrm{d}t} . \tag{1.5}$$

Since v is a function of $x(t)$, we must apply the chain rule:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\partial v}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial v}{\partial x} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} , \tag{1.6}$$

and, on the one hand, we find for infinitesimal Δx by setting $\Delta p/\Delta x \to \partial p/\partial x$

$$-\frac{\partial p_{\mathrm{tot}}}{\partial x} = \rho_{\mathrm{tot}}\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) . \tag{1.7}$$

On the other hand, a movement of mass must comply with conservation of mass. Mass reduction or increase within the volume element, thus, must correspond to a change in mass density. Mass changes can be induced by medium flow due to density differences or due to injection caused by the source.

$$S\left[\left(\rho_{\mathrm{tot}} v\right)_{x+dx} - \left(\rho_{\mathrm{tot}} v\right)_x\right] = -S\,\mathrm{d}x\frac{\partial \rho_{\mathrm{tot}}}{\partial t} + \rho_{\mathrm{tot}} q S \mathrm{d}x , \tag{1.8}$$

or, again with $\Delta x \to \partial x$,

$$\frac{\partial\left(\rho_{\mathrm{tot}} v\right)}{\partial x} = -\frac{\partial \rho_{\mathrm{tot}}}{\partial t} + \rho_{\mathrm{tot}} q . \tag{1.9}$$

Equations (1.7) and (1.9) allow the derivation of an acoustic theory. They are, however, coupled in three variables, pressure, density and particle velocity, and they are nonlinear. Two linear equations can be easily found if the effects of sound are assumed small. We will see later that this is a quite sufficient approximation for almost all sound events of interest in this book. Now, if the sound pressure is small compared with the static pressure, $p \ll p_0$, and the densities follow the same prerequisite, $\rho \ll \rho_0$, we can decompose Eqs. (1.7) and (1.9) into Taylor series and neglect small terms of higher order. A pure factor ρ_{tot} can also be replaced by ρ_0.

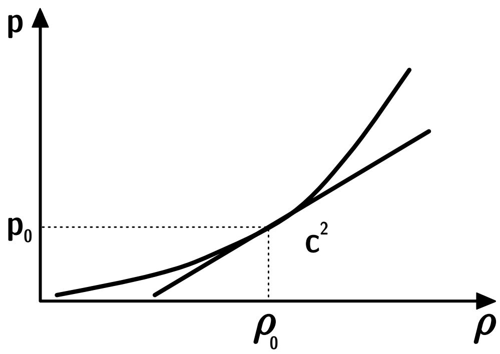


Fig. 1.5. Linearization of small amplitudes of density as a function of pressure

Furthermore, the fact that the density and the pressure are linked in the adiabatic process lets us change the variable ρ to p:

$$p = \left(\frac{\mathrm{d}p_{tot}}{\mathrm{d}\rho_{tot}} \right)_{\mathrm{ad}} \cdot \rho = c^2 \cdot \rho \,, \tag{1.10}$$

with a constant c^2 as an abbreviation of $\mathrm{d}p/\mathrm{d}\rho$.

The result of linearization and replacing density is the set of two linear sound field equations:

$$-\frac{\partial p}{\partial x} = \rho_0 \frac{\partial v}{\partial t} \tag{1.11}$$

$$-\rho_0 \frac{\partial v}{\partial x} = \frac{1}{c^2} \frac{\partial p}{\partial t} + \rho_0 q \,. \tag{1.12}$$

One can easily eliminate one variable (the particle velocity v) from these two equations to achieve one differential equation containing our variable of highest interest, the sound pressure p:

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = -\rho_0 \frac{\partial q}{\partial t} \,, \tag{1.13}$$

or in short,

$$\Delta p - \frac{1}{c^2} \ddot{p} = -\rho_0 \dot{q} \,. \tag{1.14}$$

The equation is well known in mathematics and physics as a wave equation. In the same formal notation, it can be derived for particle velocity, density, or temperature. In three dimensions, it is extended with the spatial differential operator, the Laplace operator, in Cartesian coordinates (x, y, z) according to common notation used in mathematics and physics

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \tag{1.15}$$

in cylindrical coordinates (r, φ, z),

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}, \tag{1.16}$$

or in polar coordinates (r, ϑ, φ),

$$\Delta = \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right). \tag{1.17}$$

1.2 Plane waves in fluid media

The direct solution of the example of one-dimensional sound propagation in Sect. 1.1 is the plane wave. We will find terms of wave acoustics such as speed of sound, sound intensity, energy density and can define the "sound pressure level." Equation (1.13) holds for the case that the spatial displacements, particle velocities, their gradients or density or pressure have only a component in the x direction.

The wave equation is solved by any function f with a variable in the form of $x-ct$, or g as function of $x+ct$ (d'Alembert's solution); see, for instance, (Kuttruff 2007):

$$p(x,t) = f(x-ct) + g(x+ct)\,. \tag{1.18}$$

The first term, f, describes the propagation of the local state of sound pressure $p(x, t)$ in space and time in the positive x direction, the latter term, g, in the negative x direction. This can be easily understood by considering the case of propagation to the right side assuming a function $f(x)$ with a maximum at $x=0$ at time zero. After time t has passed, the maximum will be found at the location $x=ct$.

The speed of propagation is c, the speed of sound. As described in 1.1.1 the constant c is calculated in a first approach of ideal gas theory from

$$c^2 = \left(\frac{\mathrm{d}p_{\text{tot}}}{\mathrm{d}\rho_{\text{tot}}}\right)_{\text{ad}} = \frac{\kappa p_{\text{tot}}}{\rho_{\text{tot}}}\,. \tag{1.19}$$

Taking into account more thermodynamic effects, the humidity, altitude, etc., it can be estimated rather precisely (ISO9613). For most cases of sound propagation in air, the approximation

$$c = 343.2\sqrt{\frac{273.15+\theta}{293.15}}\ \text{m/s} \tag{1.20}$$

is a sufficient estimate (θ is the temperature in degrees Celsius).

Liquid media in hydrostatics are usually considered incompressible. Acoustic waves in liquids result from a small perturbation of the zero compressibility. Liquids are thus characterized by their adiabatic compressibility:

$$\beta_{\text{ad}} = \frac{1}{\rho_0}\left(\frac{\mathrm{d}\rho_{tot}}{\mathrm{d}p_{tot}}\right)_{\text{ad}}, \tag{1.21}$$

and the speed of sound is

$$c = \frac{1}{\sqrt{\rho_0 \beta_{\text{ad}}}}. \tag{1.22}$$

Now we come back to the elementary solution $p(x, t)$. Assuming $g = 0$, we now consider a wave in the positive x direction. All locations (y, z) in planes parallel to the x direction have the same conditions. By using Eq. (1.11), we can calculate the particle velocity:

$$v = -\frac{1}{\rho_0}\int f'\,\mathrm{d}t = \frac{1}{\rho_0 c} f \tag{1.23}$$

and

$$\frac{p}{v} = \rho_0 c = Z_0. \tag{1.24}$$

This ratio, Z_0, is called the wave impedance or characteristic impedance of plane waves. It is an important reference. It can be interpreted as the characteristic resistance of the medium against pressure excitation in some kind of cause-and-effect interpretation: The amount of driving pressure needed to set the medium's particles into motion.

Table 1.1. Characteristic acoustic data for air and water

At normal conditions, 20°C	Sound speed in m/s	Characteristic impedance in kg/m^2s
Air	344	414
Water	1484	$1{,}48 \cdot 10^6$

1.3 Plane harmonic waves

We get a harmonic wave of sound pressure in the positive x direction by choosing a harmonic function representing $f(x-ct)$ (without loss in generality g is set to zero). In complex form,

$$\underline{p}(x,t) = \hat{p}\mathrm{e}^{-jk(x-ct)} = \hat{p}\mathrm{e}^{j(\omega t-kx)}, \tag{1.25}$$

$\hat{p}$ is called pressure amplitude, k wave number and ω angular frequency ($\omega = kc$). Note the symmetry between the terms kx and ωt in the harmonic function and that the space and time domains are coupled by a factor of c. Now, it is well known that the period of harmonic functions is 2π. Hence, the obvious role of the wave number and angular frequency is to rescale the periods of the wave in space and time, respectively, to 2π. The period in space is called wavelength, λ. We obtain it from

$$k = \frac{2\pi}{\lambda}. \tag{1.26}$$

With the important relation,

$$c = f \cdot \lambda = \frac{\lambda}{T}, \tag{1.27}$$

we can introduce the temporal period, T, of the wave:

$$\omega = \frac{2\pi}{T} = 2\pi f. \tag{1.28}$$

f denotes the frequency in the unit Hertz.

1.4 Wideband waves and signals

From the harmonic wave, we can directly construct any other wave function or, referring to one measurement point in space, a time function of sound pressure, called an acoustic “signal.” The procedure of superposition of harmonic signals into a more complex waveform is given by the Fourier transformation.

For continuous periodic pressure-time functions with period T_0, the transformation into the frequency domain yields the set of Fourier coefficients, $\underline{S}_m$, which are the complex amplitudes of the respective harmonic signal components. The set of complex amplitudes is called a “spectrum.”

$$\underline{S}_m = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} p(t) e^{-jm2\pi f_0 t} \mathrm{d}t \,. \tag{1.29}$$

Vice versa, the pressure-time signal, $p(t)$, can be reconstructed by Fourier synthesis, i. e., by adding all harmonic components:

$$p(t) = \sum_{m=-\infty}^{\infty} \underline{S}_m \mathrm{e}^{jm2\pi f_0 t} \,. \tag{1.30}$$

Periodic signals have a specific fundamental frequency, f_0, and the spectrum is composed of discrete components at multiples of the fundamental frequency. It is called a line spectrum and has accordingly a frequency resolution of $\Delta f = f_0 = 1/T_0$. The same concept can be extended toward aperiodic signals ($T_0 \to \infty$), which have then a continuous spectrum (line spacing $\to 0$). More details will be discussed in Sect. 7.2.

When the spectrum contains several frequencies, we talk about wide-band or broadband sound. Typical spectra of acoustic signals are shown in Fig. 1.6.

1.5 Energy and level

Usually the sound pressure is not presented in linear form in its units of pascals (1 Pa = 1 N/m^2). In daily life, the strength of sound is indicated by “decibels.” One reason is the enormous range of sound pressures in music, speech, and the urban and working environment; another is the somewhat better match with human hearing sensation (at least in a first approximation; see also Chap. 6). The range of sound pressures should be also discussed in relation to our initial assumption that sound is a small displacement or pressure or density fluctuation compared with static conditions. If we just assume a static pressure of 100 kPa in the atmosphere at sea level, we have to deal with sound pressures of orders of magnitude lower, between 0.00001 Pa and 1000 Pa. The lower limit is near the “hearing threshold,” the limit of sensation of human hearing. The upper limit is called the threshold of pain which needs no further explanation. Thus typical sound pressures are lower than the static pressure p_0 by orders of magnitude.

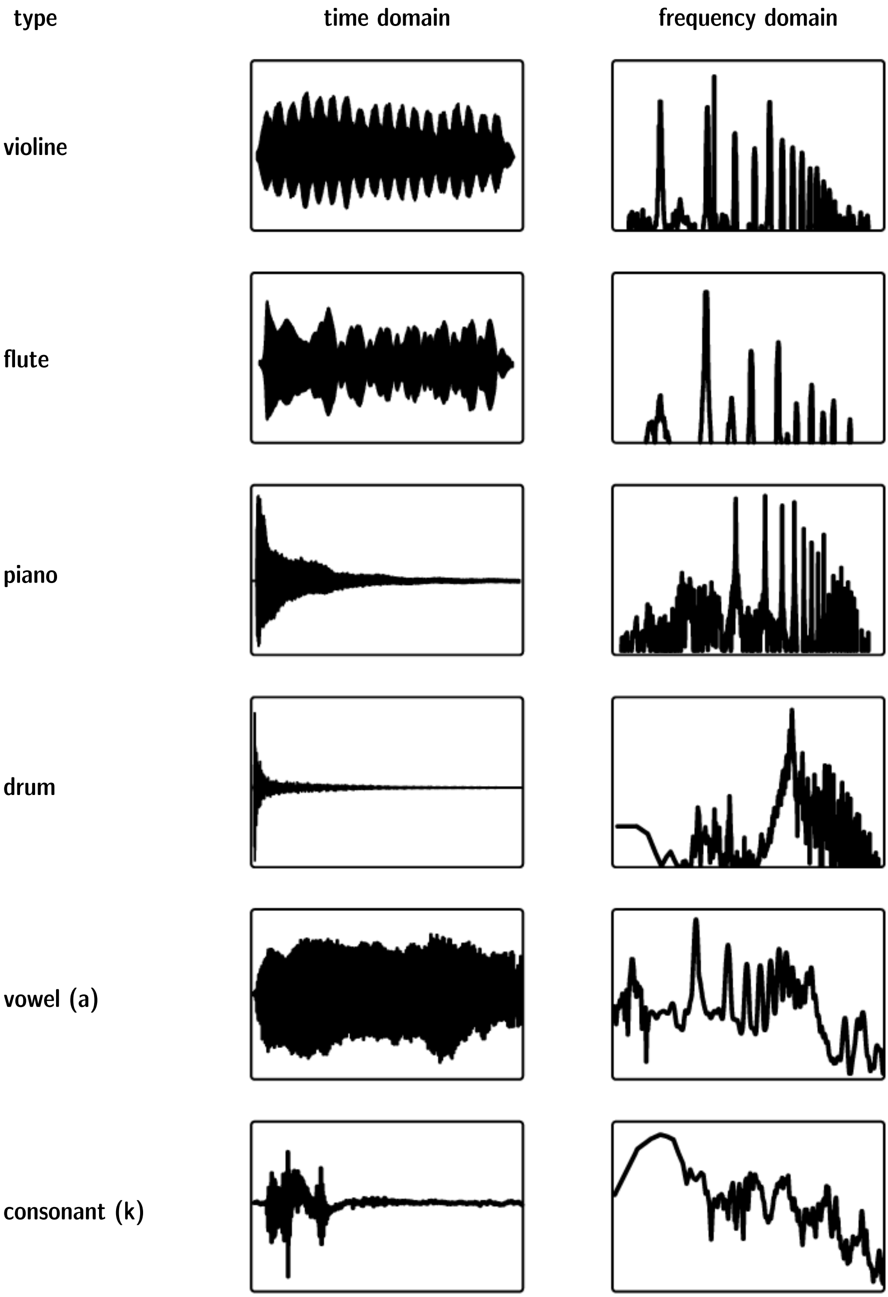


Fig. 1.6. Typical acoustic signals and their spectra

In practical acoustics, this enormous range is mapped to a logarithmic scale between about 0 dB and 130 dB, more or less similar to a scale counted in percent with a resolution practically in steps of 1. This resolution is appropriate for discussing the differences in audible sounds and it corresponds with the just noticeable difference, jnd, of about 1 decibel.

The decibel scale surely has its merits, although we will later emphasize that sound evaluation purely based on decibels or related quantities will not be sufficient without a more thorough investigation or with auralization.

The decibel scale is based on sound energy. Similarly, as in other wave and vibration phenomena in radio waves, in voltage and current measurements, the level is defined by the energy of the wave. Due to the local harmonic medium particle movement, the total energy contained in a small volume element can be interpreted similar to the energy in little pendulums in terms of kinetic and potential energy. The volume shall be so small that all particles in it move in the same way. At the time of maximum particle velocity the total energy is purely kinetic, whereas at zero velocity the total energy is purely potential. This approach leads to two possible equations for the total energy density:

$$w = \frac{\rho_0 \hat{v}^2}{2} = \frac{\hat{p}^2}{2\rho_0 c^2} \tag{1.31}$$

in which we used Eq. (1.24) to change from velocity to pressure in a plane wave.

With introduction of the "root mean square" sound pressure (rms) the sound pressure level can finally be written as:

$$L = 10\log\frac{\tilde{p}^2}{p_0^2} = 20\log\frac{\tilde{p}}{p_0} \ \text{dB (decibel)} \tag{1.32}$$

with

$$\tilde{p} = \sqrt{\frac{1}{T}\int_0^T p^2(t)\mathrm{d}t} = p_{rms} \tag{1.33}$$

and setting the reference sound pressure $p_0 = 20\,\mu\text{Pa}$ which is approximately the human hearing threshold in midfrequencies.

Table 1.2. Sound pressure levels of typical sound events

Event	Level in dB
Hearing threshold in midfrequencies	0
Anechoic chamber	0–15
Bedroom	25–30
Living room	40–55
Conversation	60
Office	70
Typical noise limit for factories	85
Pneumatic hammer	100
Rock concert, disco or walkman maximum	110
Jet engine, 25 m away	120
Rocket at start	> 190

1.6 Sound intensity

The microscopic energy in a sound wave is not just a static phenomenon. Energy is also transported. Wave propagation should not be mixed up with particle flow, however. At zero mean flow (such as random wind), net particle displacement is zero, while energy is transported to the neighbouring volume element and so forth. This effect allows a very deep and detailed investigation into sound fields, particularly for sound fields more complex than simple plane waves. The basic quantity for describing the mean energy flow is the energy transported per second through a reference area of 1 m^2. It is called sound intensity:

$$\vec{I} = \overline{p \cdot \vec{v}} = \frac{1}{T} \int_0^T p\vec{v} \mathrm{d}t \, . \tag{1.34}$$

In a plane wave, sound pressure and particle velocity are in phase (Eq. (1.24)) and the sound intensity formula reduces to

$$\left|\vec{I}\right| = \frac{\overline{p^2}}{\rho_0 c} = \frac{\tilde{p}^2}{\rho_0 c} \, , \tag{1.35}$$

with the direction of sound intensity in direction of propagation. Sound intensity can also be denoted with a level, the intensity level:

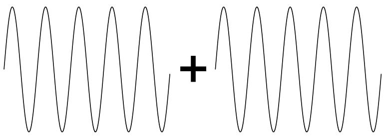
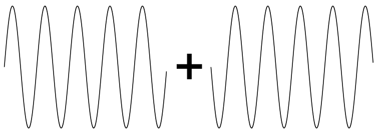
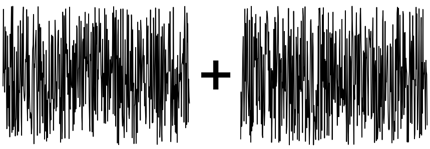
$$L_I = 10 \log \frac{\left|\vec{I}\right|}{I_0} . \tag{1.36}$$

$I_0 = 10^{-12}\,\mathrm{W/m^2}$. This choice of reference intensity is made to adjust for the same levels of sound pressure and sound intensity in a plane wave.

1.7 Level arithmetic

If several sound pressure signals are present at a time, the total pressure is the sum of the individual pressures. In case of coherent signals, i. e., with identical frequency and specific phase relation, the pressure–time functions must be added, and the rms value and the level are calculated in the end. In the case of incoherent waves of different frequencies or frequency compositions, the superposition reduces to adding the energies. The reason for the simplification can be interpreted as the effect of the binominal formula $(p_1 + p_2)^2 = p_1^2 + p_2^2 + 2 p_1 p_2$, where the latter term of pressure signal multiplication cancels with incoherent p_1 and p_2.

Table 1.3. Example: Level addition of two signals of 50 dB

Phase relation		Total level in dB
Coherent, in phase		56
Coherent, antiphase		–∞
Incoherent		53

Thus for incoherent signals, the quadratic pressures or the energy densities can be added directly:

$$w_{\text{total}} = \sum_{i=1}^{N} w_i = \frac{1}{\rho_0 c^2} \sum_{i=1}^{N} \tilde{p}_i^2 \,, \tag{1.37}$$

or in level representation,

$$L_{\text{total}} = 10 \log \sum_{i=1}^{N} 10^{L_i/10} = 10 \log \frac{\sum_{i=1}^{N} \tilde{p}_i^2}{p_0^2} \,. \tag{1.38}$$

1.8 Frequency bands

In acoustics, standardized frequency bands are often used, typically one-third octave bands or octave bands. The midband frequencies of one-third octave bands are defined on a logarithmic frequency scale as follows (here in the example of the base-2 logarithm);

$$\begin{aligned} f_{\text{u}} &= 2^{1/3} \cdot f_{\text{l}} \\ \Delta f &= f_{\text{u}} - f_{\text{l}} = f_{\text{l}}\left(2^{1/3} - 1\right) \\ f_m &= \sqrt{f_{\text{l}} \cdot f_{\text{u}}} \\ f_{m+1} &= 2^{1/3} f_m \end{aligned} \tag{1.39}$$

with f_{l} and f_{u} as lower and upper edge frequency and f_{m}, $f_{\text{m+1}}$ as midband frequencies of the bands m and $m+1$. Similarly, for octave bands,

$$\begin{aligned} f_{\text{u}} &= 2 f_{\text{l}} \\ \Delta f &= f_{\text{u}} - f_{\text{l}} = f_{\text{l}} \\ f_m &= \sqrt{f_{\text{l}} \cdot f_{\text{u}}} = \sqrt{2} f_{\text{l}} \\ f_{m+1} &= f_m \cdot 2 \end{aligned} \tag{1.40}$$

Fractional bands with bandwidth of 1/6 or 1/12 octave are also in use. The separation of broadband spectra into several frequency bands allows discussion of transmission characteristics or noise frequency content. The band-filtered results, however, are still sound levels. The specific spectrum in interpretation of Fourier transformation theory is discussed in more detail later (see Sect. 7.2).

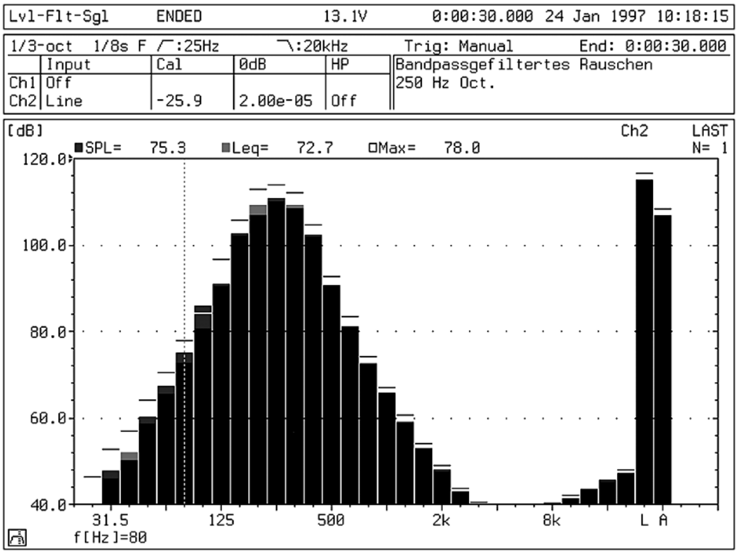


Fig. 1.7. Band-filtered spectrum of a broadband signal (courtesy of Norsonic A/S)