9

Various Approaches to Asset Pricing

A fundamental result of this chapter is that prices can be generally obtained under the benchmark approach in situations where other approaches are not available. This chapter also clarifies relationships between real world pricing under the benchmark approach and the pricing by other means in the areas of finance and insurance. Furthermore, it presents the Girsanov transformation, the change of numeraire technique and the Feynman-Kac formula, which are all highly relevant to derivative pricing.

9.1 Real World Pricing

Various Pricing Approaches

In the literature, pricing concepts for risky securities have been developed in several seemingly different approaches. Often one determines the price of an asset by reference to its underlying economic value. General equilibrium based models, such as the *intertemporal capital asset pricing model* (ICAPM), see Merton (1973a), provide examples of this approach. The *actuarial pricing approach*, see Bühlmann (1970) and Gerber (1997), which is common in insurance and accounting, provides another important example in this direction. The above mentioned approaches aim to provide an economic explanation for the value of prices and why asset prices move if changes in economic variables occur.

A much less ambitious question is asked in pricing approaches which arise when one is marking to market. Given the prices of some assets that securitize uncertainty in the market, one analyzes under such approach what consequences this has for the values of other securities in this market. The securities to be priced are typically derivatives. The previously described option pricing methodology of Chap. 8 provides an example for such a pricing approach, which is based on the assumption that there is *no arbitrage*, see Ross (1976), Harrison & Kreps (1979) and Harrison & Pliska (1981). As we

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shall see in Sect. 9.4, within the *arbitrage pricing theory* (APT) the *risk neutral* pricing approach has been developed that allows convenient *numeraire changes* and corresponding changes of pricing measures.

It is a challenge to reconcile presently used different pricing approaches and to highlight their specific features in a consistent framework. The *benchmark approach* uses the *growth optimal portfolio* (GOP) as reference unit or numeraire. As we shall see, the GOP is the portfolio that maximizes expected logarithmic utility from terminal wealth, see Kelly (1956) and Long (1990). The GOP exists in all financial market models that we shall consider. In the next chapter it will be made clear for a continuous market what is the composition of the GOP. Chapter 14 will generalize this result to jump diffusion markets. For the purpose of this chapter we keep the market model as BS model and, therefore, the GOP very simple. We shall unify in a natural way some of the mentioned pricing methods under the benchmark approach by using the concept of *real world pricing*. To illustrate the different asset pricing methodologies we explore in this chapter various alternative ways to price a future payoff. We discuss the different approaches typically in the context of the BS model, which considerable simplifies our presentation. However, most of the conclusions apply also for other models, as we shall see later in Chaps. 10–14.

First, we introduce in the following the real world pricing concept that allows prices for payoffs to be obtained as conditional expectations under the real world probability measure. We then show in later sections how the benchmark approach relates to other pricing concepts. The advantage of the benchmark approach is that as soon as the GOP exists one can always perform real world pricing. Other approaches may have extra conditions to satisfy which may not allow to form derivative prices for certain models of interest.

Portfolios under the BS Model

In the previous chapter, we have identified via no-arbitrage and hedging arguments a price for a European option under the BS model. If one wants to exclude arbitrage, then there is no alternative to this price. We now translate this result into a pricing concept that is based on some conditional expectation. To achieve this we express this price as a conditional expectation of the option payoff. The expectation will be taken under the real world probability measure P. This is the probability measure that models the market as it evolves and as we can observe it by exploiting empirical evidence. Only under this measure one can estimate model parameters historically. We shall show later in Sect. 10.6 how to obtain the GOP without reference to a specific model and the estimation of particular parameters. The key question that has to be resolved is: In the denomination of which *numeraire* should one express the payoff to apply an expectation under the real world probability measure?

With this goal in mind, we ask whether there exists a strictly positive process, for instance, a market index, which when used as numeraire or *bench-*

mark, generates realistic benchmarked derivative price processes that are martingales with respect to the real world probability measure. This means that benchmarked derivative prices then represent the best forecast of their future benchmarked values. In this way a natural pricing method could be established via conditional expectation under the real world probability measure. The described use of the GOP as numeraire portfolio follows the line of arguments in Long (1990), Bajeux-Besnainou & Portait (1997), Becherer (2001) and Bühlmann & Platen (2003). Since in the case of a European option under the BS model we have already identified the corresponding no-arbitrage price, we now aim to identify the corresponding benchmark that, when used as numeraire, yields this price that allows to replicate the given payoff.

As already indicated, for simplicity, we consider here a simple Black-Scholes (BS) market. It contains an underlying security with price process $S = \{S_t,\ t \in [0,T]\}$, as given by (8.2.1), which satisfies the SDE

$$dS_t = a_t\, S_t\, dt + \sigma_t\, S_t\, dW_t \tag{9.1.1}$$

for $t \in [0,T]$ with $S_0 > 0$, where $T \in [0,\infty)$. Furthermore, our BS model has a domestic savings account with value process $B = \{B_t,\ t \in [0,T]\}$, see (8.2.2), where

$$dB_t = r_t\, B_t\, dt \tag{9.1.2}$$

for $t \in [0,T]$ and $B_0 = 1$.

A self-financing strategy $\delta = \{\delta_t = (\delta_t^0, \delta_t^1)^\top,\ t \in [0,T]\}$, see (8.2.4)–(8.2.7), with δ_t^0 units held at time t in the domestic savings account and δ_t^1 units invested in the underlying security, has the corresponding portfolio value

$$S_t^\delta = \delta_t^0\, B_t + \delta_t^1\, S_t \tag{9.1.3}$$

with

$$\begin{aligned} dS_t^\delta &= \delta_t^0\, dB_t + \delta_t^1\, dS_t \\ &= (\delta_t^0\, r_t\, B_t + \delta_t^1\, a_t\, S_t)\, dt + \delta_t^1\, \sigma_t\, S_t\, dW_t \\ &= S_t^\delta \left((\pi_\delta^0(t)\, r_t + \pi_\delta^1(t)\, a_t)\, dt + \pi_\delta^1(t)\, \sigma_t\, dW_t \right) \end{aligned} \tag{9.1.4}$$

for $t \in [0,T]$. Note that the SDE (9.1.4) is such that it guarantees the self-financing property of the portfolio, where all changes of its value are due to changes in the securities. Here we use the corresponding *fractions*

$$\pi_\delta^0(t) = \delta_t^0\, \frac{B_t}{S_t^\delta} \tag{9.1.5}$$

and

$$\pi_\delta^1(t) = \delta_t^1\, \frac{S_t}{S_t^\delta} \tag{9.1.6}$$

that are held in the respective securities. Obviously, these fractions add up to one, that is,

$$\pi_\delta^0(t) + \pi_\delta^1(t) = 1 \tag{9.1.7}$$

for $t \in [0,T]$. Note that the notion of a fraction makes only sense as long as the portfolio value is not zero.

Growth Optimal Portfolio

Let us derive for the given BS market the *growth optimal portfolio* (GOP) which will be shown in Chap. 10 to be the portfolio that maximizes the drift of its logarithm, see Long (1990), Karatzas & Shreve (1998) or Platen (2002). By the Itô formula we obtain from (9.1.4) and (9.1.7) for the logarithm $\ln(S_t^\delta)$ of a strictly positive portfolio the SDE

$$d\ln(S_t^\delta) = g_t^\delta\, dt + \pi_\delta^1(t)\, \sigma_t\, dW_t \tag{9.1.8}$$

with *growth rate*

$$g_t^\delta = r_t + \pi_\delta^1(t)\,(a_t - r_t) - \frac{1}{2}\,(\pi_\delta^1(t))^2\, \sigma_t^2 \tag{9.1.9}$$

for $t \in [0,T]$.

Definition 9.1.1. *Under the BS model the GOP is the portfolio process $S^{\delta_*} = \{S_t^{\delta_*},\ t \in [0,T]\}$ with optimal growth rate $g_t^{\delta_*}$ at time t such that*

$$g_t^\delta \le g_t^{\delta_*} \tag{9.1.10}$$

almost surely for all $t \in [0,T]$ and strictly positive portfolio processes S^δ.

Let us now choose the fraction $\pi_\delta^1(t)$ such that the growth rate g_t^δ is maximized for each $t \in [0,T]$, which will give us the GOP. Note that the choice of the reference unit is *not* relevant for the corresponding optimization problem. By application of the first order condition to maximize the growth rate g_t^δ in (9.1.9) with respect to the fraction $\pi_\delta^1(t)$ we obtain the condition

$$\frac{\partial g_t^\delta}{\partial \pi_\delta^1(t)} = a_t - r_t - \pi_{\delta_*}^1(t)\, \sigma_t^2 = 0 \tag{9.1.11}$$

for $t \in [0,T]$. Therefore, we obtain the *optimal fraction* in the underlying security

$$\pi_{\delta_*}^1(t) = \frac{a_t - r_t}{\sigma_t^2} \tag{9.1.12}$$

and, thus, by (9.1.7) the optimal fraction in the savings account

$$\pi_{\delta_*}^0(t) = 1 - \pi_{\delta_*}^1(t) \tag{9.1.13}$$

for $t \in [0,T]$. Because of (9.1.9) and (9.1.12) the *optimal growth rate* is then of the form

$$g_t^{\delta_*} = r_t + \frac{1}{2}\left(\frac{a_t - r_t}{\sigma_t}\right)^2 \tag{9.1.14}$$

for $t \in [0,T]$. Now, we obtain from (9.1.4), (9.1.12) and (9.1.13) the GOP as the wealth process $S^{\delta_*} = \{S_t^{\delta_*},\, t \in [0,T]\}$, which satisfies the SDE

$$dS_t^{\delta_*} = S_t^{\delta_*}\, \left((r_t + \theta_t^2)\, dt + \theta_t\, dW_t\right) \tag{9.1.15}$$

with initial value $S_0^{\delta_*} > 0$ and GOP volatility

$$\theta_t = \pi_{\delta_*}^1(t)\, \sigma_t = \frac{a_t - r_t}{\sigma_t} \tag{9.1.16}$$

for $t \in [0,T]$. The quantity θ_t in (9.1.16) is the, so-called, *market price of risk* at time t.

According to (9.1.14) and (9.1.16) the optimal growth rate for the given BS model equals

$$g_t^{\delta_*} = r_t + \frac{1}{2}\,\theta_t^2 \tag{9.1.17}$$

for $t \in [0,T]$. This reveals a close link between the squared volatility and the optimal growth rate of the GOP. For the *discounted* GOP

$$\bar{S}_t^{\delta_*} = \frac{S_t^{\delta_*}}{B_t} \tag{9.1.18}$$

we derive by the Itô formula with (9.1.15) and (9.1.2) the SDE

$$d\bar{S}_t^{\delta_*} = \bar{S}_t^{\delta_*}\, \theta_t\, (\theta_t\, dt + dW_t) \tag{9.1.19}$$

for $t \in [0,T]$, see (10.2.8). Note that the drift of the discounted GOP is determined as the square of its diffusion coefficient. This observation is crucial and holds also more generally for continuous financial markets, as we shall see in Chap. 10. Within this chapter we keep our BS market very simple. Therefore, the GOP is here only a composition of two securities.

Benchmarked Savings Account

Let us now introduce the notion of *benchmarking*. Any security when expressed in units of the GOP we call a *benchmarked security*. For instance, the savings account B, when denominated in units of the GOP, is called the *benchmarked savings account* $\hat{S}^0 = \{\hat{S}_t^0,\, t \in [0,T]\}$, where

$$\hat{S}_t^0 = \frac{B_t}{S_t^{\delta_*}} \tag{9.1.20}$$

for $t \in [0,T]$. By application of the Itô formula (6.2.11) to the inverse of $\bar{S}_t^{\delta_*}$ in (9.1.19) or the relation (9.1.20), the differential equation (9.1.2) and the SDE (9.1.15), it follows for the benchmarked savings account $\hat{S}_t^0$ the SDE

$$d\hat{S}^0_t = -\theta_t\, \hat{S}^0_t\, dW_t \tag{9.1.21}$$

for $t \in [0,T]$. This means that the benchmarked savings account is driftless.

Since the process $\hat{S}^0 = \{\hat{S}^0_t,\ t \in [0,T]\}$ is a geometric Brownian motion it follows by (5.4.1) and (7.3.8) that $\hat{S}^0 \in \mathcal{L}^2_T$. Thus, the Itô integral on the right hand side of the integral version of (9.1.21) is, by the martingale property (5.4.3) of Itô integrals, a martingale. This means that the benchmarked savings account process $\hat{S}^0$ is under the given BS model an $(\underline{\mathcal{A}}, P)$-martingale. We shall see later that this is a particular property of the BS model and may not hold for other models.

Benchmarked Underlying Security

Let us now benchmark in our BS market the underlying security S. That is, we consider the *benchmarked* security price

$$\hat{S}^1_t = \frac{S_t}{S^{\delta_*}_t} \tag{9.1.22}$$

for $t \in [0,T]$. Then by the Itô formula (6.2.11) together with (9.1.1), (9.1.15) and equation (9.1.16) the SDE for $\hat{S}^1_t$ becomes

$$\begin{aligned} d\hat{S}^1_t &= \hat{S}^1_t \big((a_t - r_t - \sigma_t\, \theta_t)\, dt + (\sigma_t - \theta_t)\, dW_t\big) \\ &= \hat{S}^1_t\, (\sigma_t - \theta_t)\, dW_t \end{aligned} \tag{9.1.23}$$

for $t \in [0,T]$. Consequently, according to (9.1.23), the GOP when used as benchmark, has the property that the resulting SDE for the benchmarked security $\hat{S}^1$ is driftless. By similar arguments as above one can show that the geometric Brownian motion $\hat{S}^1$ is in $\mathcal{L}^2_T$. The process $\hat{S}^1$ is, therefore, by (5.4.3) an $(\underline{\mathcal{A}}, P)$-martingale.

Benchmarked Option Price

Consider a European option on the underlying security S under the BS model with value for its hedge portfolio

$$V(t) = V(t, S_t), \tag{9.1.24}$$

as determined in Sect. 8.2 by equation (8.2.4). Using (8.2.9) and (9.1.20), we obtain for the benchmarked European option price the expression

$$\hat{V}(t) = \frac{V(t)}{S^{\delta_*}_t} = \frac{V(t, S_t)}{S^{\delta_*}_t} = \bar{V}(t, \bar{S}_t)\, \hat{S}^0_t \tag{9.1.25}$$

for $t \in [0,T]$. Here we have the benchmarked payoff

$$\hat{V}(T) = \frac{H(S_T)}{S_T^{\delta_*}} \tag{9.1.26}$$

at maturity T. By application of the Itô formula (6.2.11) we obtain from (8.2.11) and (8.2.21) for the discounted value $\bar{V}(t, \bar{S}_t) = \frac{V(t,S_t)}{B_t}$ of the option with $\bar{S}_t = \frac{S_t}{B_t}$ the SDE

$$\begin{aligned} d\bar{V}(t,\bar{S}_t) &= \left(\frac{\partial \bar{V}(t,\bar{S}_t)}{\partial t} + (a_t - r_t)\,\bar{S}_t\,\frac{\partial \bar{V}(t,\bar{S}_t)}{\partial \bar{S}} + \frac{1}{2}\,\sigma_t^2\,\bar{S}_t^2\,\frac{\partial^2 \bar{V}(t,\bar{S}_t)}{\partial \bar{S}^2} \right) dt \\ &\quad + \sigma_t\,\bar{S}_t\,\frac{\partial \bar{V}(t,\bar{S}_t)}{\partial \bar{S}}\,dW_t \\ &= \frac{\partial \bar{V}(t,\bar{S}_t)}{\partial \bar{S}}\,\bar{S}_t\,((a_t - r_t)\,dt + \sigma_t\,dW_t) \end{aligned} \tag{9.1.27}$$

for $t \in [0,T]$. On the other hand, by using the Itô formula (6.2.11) and also the relations (9.1.25), (9.1.27), (8.2.9), (9.1.16) and (9.1.21) we obtain the SDE

$$\begin{aligned} d\hat{V}(t) &= d(\bar{V}\,\hat{S}_t^0) \\ &= \hat{S}_t^0\,d\bar{V} + \bar{V}\,d\hat{S}_t^0 + d[\bar{V}, \hat{S}^0]_t \\ &= \hat{S}_t^0\,(a_t - r_t)\,\bar{S}_t\,\frac{\partial \bar{V}}{\partial \bar{S}}\,dt + \hat{S}_t^0\,\sigma_t\,\bar{S}_t\,\frac{\partial \bar{V}}{\partial \bar{S}}\,dW_t \\ &\quad - \theta_t\,\bar{V}\,\hat{S}_t^0\,dW_t - \sigma_t\,\bar{S}_t\,\frac{\partial \bar{V}}{\partial \bar{S}}\,\hat{S}_t^0\,\theta_t\,dt \end{aligned} \tag{9.1.28}$$

for $t \in [0,T)$, where, for simplicity, we have suppressed in our notation the dependence of $\bar{V}$ on $(t, \bar{S}_t)$. By using (9.1.16) the SDE (9.1.28) can be rewritten in the form

$$d\hat{V}(t) = \hat{S}_t^0 \left(\sigma_t\,\bar{S}_t\,\frac{\partial \bar{V}(t,\bar{S}_t)}{\partial \bar{S}} - \theta_t\,\bar{V}(t,\bar{S}_t) \right) dW_t \tag{9.1.29}$$

for $t \in [0,T)$. Note that the SDE for the benchmarked option price is driftless. Also here one can show that the diffusion coefficient in (9.1.29) is in $\mathcal{L}_T^2$. Therefore, by (5.4.3), the benchmarked option price process $\hat{V} = \{\hat{V}(t),\, t \in [0,T]\}$, is an $(\underline{\mathcal{A}}, P)$-martingale.

We have seen that the property of the GOP when used as numeraire or benchmark, to convert benchmarked prices into martingales, seems to apply quite generally under the BS model. In the literature the GOP is therefore also known as the *numeraire portfolio*, see Long (1990).

Real World Pricing

Summarizing the above analysis, we conclude under the BS model that the GOP is the numeraire portfolio for the domestic savings account B, the underlying security price S and the option price V. When used as denominator it makes the corresponding benchmarked price processes $\hat{S}^0$, $\hat{S}^1$ and $\hat{V}$ into $(\underline{\mathcal{A}}, P)$-martingales. This implies, by the martingale property (5.1.2), that these prices, when expressed in units of the GOP, are the best forecast of their future benchmarked values. We obtain this remarkable fact as a consequence of outstanding properties of the GOP, which we shall discuss in the next chapter.

Intuitively, the martingale property of benchmarked prices relates to the common notion of what constitutes a fair price. The following definition will be applied generally throughout the book for all models and not only for the BS model.

Definition 9.1.2. *A security price process* $V = \{V_t,\, t \in [0, \infty)\}$ *is called* fair *if its benchmarked value* $\hat{V}_t = \frac{V_t}{S_t^{\delta_*}}$ *forms an* $(\underline{\mathcal{A}}, P)$*-martingale.*

This leads by application of the martingale property of $\hat{V}$ directly to the following pricing formula.

Corollary 9.1.3. *For any fair security price process* $V = \{V_t,\, t \in [0, \infty)\}$ *one has for any time* $t \in [0, \infty)$ *and* $T \in (t, \infty)$ *the* real world pricing formula

$$V_t = S_t^{\delta_*}\, E\left(\frac{V_T}{S_T^{\delta_*}} \,\bigg|\, \mathcal{A}_t \right). \tag{9.1.30}$$

It is most important to emphasize that the expectation in (9.1.30) is taken under the real world probability measure P. The numeraire is here the GOP. Note that by application of the optional sampling theorem, see (5.1.19), it follows that T can also be a bounded stopping time in the real world pricing formula (9.1.30).

Under the BS model the savings account, the underlying security and European option price form fair price processes since their benchmarked price processes are $(\underline{\mathcal{A}}, P)$-martingales. Note that a real world option price forms a fair price process and is under the given BS model consistent with the hedging arguments previously applied in Chap. 8. In this sense the real world or fair option price is a no-arbitrage price.

As we shall see later in this chapter, *real world pricing* can be generally applied and will turn out to be the natural pricing concept under the benchmark approach. It only requires the existence of a GOP, as can be seen from the real world pricing formula (9.1.30).

A Martingale Representation

The SDE (9.1.29) for the benchmarked option price process $\hat{V}$ can be rewritten by using (9.1.20), (8.2.9) and (8.2.10) in the integral form

$$\hat{V}(T) = \hat{V}(t) + \int_t^T \left(\sigma_z\, \frac{S_z}{S_z^{\delta_*}}\, \frac{\partial \bar{V}(z, \bar{S}_z)}{\partial \bar{S}} - \theta_z\, \hat{V}(z) \right) dW_z \tag{9.1.31}$$

for $t \in [0, T]$. This provides a representation of the benchmarked option payoff

$$\hat{V}(T) = \frac{H(S_T)}{S_T^{\delta_*}}.$$

Since $\hat{V}$ is a martingale under the real world probability P, we call (9.1.31) the *real world martingale representation* of $\frac{H(S_T)}{S_T^{\delta_*}}$. By taking the conditional expectation $E(\cdot \,|\, \mathcal{A}_t)$ on both sides of equation (9.1.31), it follows by the martingale property of $\hat{V}$ that

$$E\left(\hat{V}(T) \,\middle|\, \mathcal{A}_t\right) = \hat{V}(t) \tag{9.1.32}$$

for $t \in [0, T]$. Now, when we multiply both sides of equation (9.1.32) by $S_t^{\delta_*}$, then we obtain by (9.1.25) the fair option price $V(t)$ at time t in the form

$$V(t) = S_t^{\delta_*}\, \hat{V}(t) = S_t^{\delta_*}\, E\left(\hat{V}(T) \,\middle|\, \mathcal{A}_t\right) \tag{9.1.33}$$

for $t \in [0, T]$. Therefore, due to (9.1.25) and (8.2.24), we can express the European option price with payoff $H(S_T)$ at maturity T by

$$V(t) = S_t^{\delta_*}\, E\left(\frac{H(S_T)}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t\right) \tag{9.1.34}$$

for all $t \in [0, T]$. This recovers the real world pricing formula (9.1.30). It is most important to emphasize that this pricing formula uses the conditional expectation under the real world probability measure P and not under any transformed measure.

The fair price $V(t)$, when expressed in units of the domestic currency at time t, is simply obtained by multiplying the fair benchmarked price $\hat{V}(t)$ by the GOP value $S_t^{\delta_*}$, that is

$$V(t) = S_t^{\delta_*}\, \hat{V}(t) \tag{9.1.35}$$

for $t \in [0, T]$, as is described by the real world pricing formula (9.1.34), see also (9.1.25) and (9.1.33).

We shall apply the real world pricing formula later quite generally when determining the fair price of derivatives. Once the GOP is identified in a model one can determine the fair value of any integrable benchmarked payoff by the real world pricing formula. As we shall see, it is possible to derive from this pricing formula several other common derivative pricing and asset pricing rules.

Benchmarked Portfolios

For a given general portfolio S^δ we can also compute the SDE for its benchmarked value

$$\hat{S}_t^\delta = \frac{S_t^\delta}{S_t^{\delta_*}} \tag{9.1.36}$$

for $t \in [0, T]$. It follows from (9.1.4), (9.1.15), (9.1.16), (9.1.6) and by application of the Itô formula that

$$d\hat{S}_t^\delta = \hat{S}_t^\delta \left(\pi_\delta^1(t)\, \sigma_t - \theta_t\right) dW_t = \left(\delta_t^1\, \hat{S}_t^1\, \sigma_t - \hat{S}_t^\delta\, \theta_t\right) dW_t \tag{9.1.37}$$

for $t \in [0, T]$ with $\hat{S}_0^\delta = \frac{S_0^\delta}{S_0^{\delta_*}}$. Obviously, $\hat{S}^\delta$ is an $(\underline{\mathcal{A}}, P)$-local martingale, see Lemma 5.4.1. It follows by Lemma 5.2.3 that under the given BS model any nonnegative benchmarked portfolio is an $(\underline{\mathcal{A}}, P)$-supermartingale.

In the case when $\hat{S}^\delta$ is such that the conditions (ii) or (iii) of Lemma 5.2.2 are satisfied, then $\hat{S}^\delta$ is also a true $(\underline{\mathcal{A}}, P)$-martingale and not just a supermartingale. This indicates that there may exist portfolio processes that when benchmarked are not martingales.

An Unfair Portfolio

The following simple example demonstrates that even in a simple BS market there exist perfectly reasonable nonnegative portfolio processes that are *unfair*, which means that they are not fair. Since we have above observed that nonnegative benchmarked portfolios are always supermartingales an unfair portfolio is a supermartingale that is not a martingale.

To provide an example, let us introduce the inverse $Z = \{Z_t,\, t \in [0, T]\}$ of a squared Bessel process of dimension four, which we have shown in Sect. 8.7 to be a strict local martingale, see also Revuz & Yor (1999). It satisfies the SDE

$$dZ_t = -2\,(Z_t)^{\frac{3}{2}}\, dW_t \tag{9.1.38}$$

for $t \in [0, T]$, where we set $Z_0 = 1$. The process Z is an $(\underline{\mathcal{A}}, P)$-local martingale but not an $(\underline{\mathcal{A}}, P)$-martingale. By Lemma 5.2.3 it is a strict $(\underline{\mathcal{A}}, P)$-supermartingale.

We can now identify a strategy δ with initial benchmarked portfolio value

$$\hat{S}_0^\delta = Z_0 = 1 \tag{9.1.39}$$

that matches in the SDE (9.1.37) the diffusion coefficient such that

$$\hat{S}_t^\delta \left(\pi_\delta^1(t)\, \sigma_t - \theta_t\right) = -2\,(Z_t)^{\frac{3}{2}} \tag{9.1.40}$$

for all $t \in [0, T]$. Then it follows for the fraction

$$\pi_\delta^1(t) = \left(\theta_t - \frac{2\,(Z_t)^{\frac{3}{2}}}{\hat{S}_t^\delta}\right) \frac{1}{\sigma_t} \tag{9.1.41}$$

of wealth that is invested in the underlying security that the resulting self-financing portfolio S^δ has at time t the benchmarked value

$$\hat{S}^\delta_t = Z_t \tag{9.1.42}$$

for all $t \in [0,T]$. This means, $\hat{S}^\delta$ equals the strict supermartingale Z, see (8.7.21). Note by (9.1.42) and (9.1.41) that the fraction simplifies in this case to the expression

$$\pi^1_\delta(t) = \frac{1}{\sigma_t}\left(\theta_t - 2\sqrt{Z_t}\right), \tag{9.1.43}$$

for $t \in [0,T]$. We emphasize that this yields a perfectly reasonable self-financing portfolio. As we have pointed out in Sect. 8.7, the formula (8.7.17) for the first negative moment of a squared Bessel processes of dimension four yields

$$E\left(\hat{S}^\delta_t \,\middle|\, \mathcal{A}_0\right) = E\left(\hat{S}^\delta_t\right) = \hat{S}^\delta_0\left(1 - \exp\left\{\frac{-1}{2\,\hat{S}^\delta_0\,t}\right\}\right) < \hat{S}^\delta_0 \tag{9.1.44}$$

for $t \in (0,T]$.

By the strict inequality (9.1.44) we see that the $(\underline{\mathcal{A}}, P)$-supermartingale $\hat{S}^\delta$ is here not a martingale. This example demonstrates that even under a BS model not all integrable, nonnegative, benchmarked portfolios are $(\underline{\mathcal{A}}, P)$-martingales.

We shall see later that generally under the benchmark approach all nonnegative benchmarked portfolios are supermartingales. This is a fundamental property for the wide class of financial market models that we consider in this book. There exist several popular pricing concepts that we shall discuss below. For some of these we can show that they correspond to real world pricing in the sense that their benchmarked price processes are martingales under the real world probability measure.

9.2 Actuarial Pricing

In this section we show that the common actuarial pricing or net present value pricing methodology, which is widely used in insurance and accounting, can be derived from real world pricing.

Setup for the GOP

To illustrate the actuarial pricing method in a simple, familiar setting we consider again a BS market. The underlying security S_t is not of relevance for the following analysis since the payoff H that shall be priced, will be assumed to be independent of the GOP. However, it will be essential for our arguments that the financial market model has a GOP $S^{\delta_*} = \{S^{\delta_*}_t,\, t \in [0,T]\}$ which, for simplicity, we assume to be of the same form as in the SDE (9.1.15). Since the GOP satisfies then a Black-Scholes dynamics of the type (7.3.12) we obtain from (7.3.3) an explicit expression for the GOP value at time t in the form

$$S_t^{\delta_*} = S_0^{\delta_*} \exp\left\{\int_0^t \left(r_s + \frac{\theta_s^2}{2}\right) ds + \int_0^t \theta_s \, dW_s\right\} \tag{9.2.1}$$

for $t \in [0, T]$.

Fair Zero Coupon Bond

To illustrate the actuarial pricing methodology let us at first determine at time t the fair value of a *zero coupon bond*. This is the value at time t for the payment of one monetary unit at time T, obtained under the real world pricing formula (9.1.34). Obviously, this corresponds to a European payoff $H = 1$. If we denote the fair value of this payoff at time $t \in [0, T]$ by $P(t, T)$, then we obtain by (9.1.34) the *fair zero coupon bond* price in the form

$$P(t, T) = S_t^{\delta_*} \, E\left(\frac{1}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t\right). \tag{9.2.2}$$

In the given case we can explicitly compute this value. Since r and θ are deterministic, it follows from (9.2.2) and (9.2.1) that

$$\begin{aligned} P(t, T) &= E\left(\exp\left\{-\int_t^T r_s \, ds - \frac{1}{2}\int_t^T \theta_s^2 \, ds - \int_t^T \theta_s \, dW_s\right\} \,\middle|\, \mathcal{A}_t\right) \\ &= \exp\left\{-\int_t^T r_s \, ds\right\} E\left(\exp\left\{-\int_t^T \frac{\theta_s^2}{2} \, ds - \int_t^T \theta_s \, dW_s\right\} \,\middle|\, \mathcal{A}_t\right) \end{aligned} \tag{9.2.3}$$

for $t \in [0, T]$. Using the Laplace transform (1.3.76) of a Gaussian random variable it follows that the conditional expectation on the right hand side of (9.2.3) equals the real value one. Alternatively, we can use the fact that the exponential under the conditional expectation forms an $(\underline{\mathcal{A}}, P)$-martingale, see Sect. 5.1. Therefore, we obtain as fair zero coupon bond price at time t the value

$$P(t, T) = \exp\left\{-\int_t^T r_s \, ds\right\} = \frac{B_t}{B_T}. \tag{9.2.4}$$

Note that the value $\exp\{-\int_t^T r_s ds\}$, if invested at time $t = 0$ in a savings account, has the value of one monetary unit at time T. Under the benchmark approach it will be always possible to establish the fair price of a zero coupon bond. However, if the exponential under the conditional expectation in (9.2.3) is a strict supermartingale, then the conditional expectation is less than one and $P(t, T)$ is less than the right hand side of (9.2.4). This is similar to the effect that yielded inequality (9.1.44). We shall study such cases later in more detail.

Fair Price of an Independent Payoff

Now, let us consider at the fixed maturity date T a random $\mathcal{A}_T$-measurable payoff $H > 0$, which is *independent* of the GOP value $S_T^{\delta_*}$. For instance, this could be a life insurance claim or a payoff based on a weather index. Such a claim may be by its nature independent of the GOP. The payoff H at time T could also model operational failures in a company during a period that finishes at maturity T. Alternatively, it could, for instance, model the total sum of insurance claims from a particular group of cars in the year prior to T. The key assumption is here that the above random payoff H is independent of the random value $S_T^{\delta_*}$ of the GOP at the maturity date T. To be precise, we assume that H is independent of $S_T^{\delta_*}$, see (1.1.13) and (1.4.22), and that the expectation of the benchmarked payoff

$$E\left(\left|\frac{H}{S_T^{\delta_*}}\right|\right) < \infty \tag{9.2.5}$$

is finite.

Then we can compute the fair price $U_H(t)$ at time $t \in [0, T]$ for the payoff H according to the real world pricing formula (9.1.30). We obtain its fair price in the form

$$U_H(t) = S_t^{\delta_*} \, E\left(\frac{H}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t\right).$$

Recall that the expectation of a product of independent random variables is the product of their expectations, see (1.4.25). Since we have assumed that H is independent of $S_T^{\delta_*}$ we obtain by this property the expression

$$U_H(t) = S_t^{\delta_*} \, E\left(\frac{1}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t\right) E\left(H \,\middle|\, \mathcal{A}_t\right).$$

By using now the fair zero coupon bond price $P(t, T)$ in (9.2.2), it follows the widely used *actuarial pricing formula*

$$U_H(t) = P(t, T) \, E\left(H \,\middle|\, \mathcal{A}_t\right). \tag{9.2.6}$$

Under this formula one computes the conditional expectation of a future cash flow at time T and discounts it back to the present time t by using the corresponding fair zero coupon bond price. This takes into account the evolution of the time value of money. The procedure is also known as *net present value* calculation. It is widely used in practice. Thus, we recover from real world pricing in the case of independence of payoff and GOP, the well-known formula of actuarial and net present value pricing.

Note that in the actuarial pricing formula (9.2.6) we do not require the knowledge of the dynamics of the GOP. We even do not need to observe the GOP in this case. One only needs to know the expectation of the payoff under the real world probability measure and the fair price of a zero coupon bond, which is given in the market.

In our simple BS model we obtain from (9.2.6) the following version of the actuarial pricing formula

$$U_H(t) = \frac{B_t}{B_T}\, E\left(H \,\middle|\, \mathcal{A}_t\right) \tag{9.2.7}$$

for $t \in [0, T]$. We see in formula (9.2.7) the simple discounting rule for the expected future payoff, as is most common in actuarial and accounting practice. We emphasize that the conditional expectations in (9.2.6) and (9.2.7) are taken with respect to the real world probability measure P and that these formulas are derived for the case when the payoff H is independent of the GOP value $S_T^{\delta_*}$. The actuarial pricing formula (9.2.6) can be shown to hold generally for payoffs independent of the GOP for the models that we consider in this book. In this sense actuarial pricing turns out to be a particular case of real world pricing. On the other hand, when starting from a benchmarked actuarial price process $\hat{U}_H = \{\hat{U}_H(t) = \frac{U_H(t)}{S_t^{\delta_*}},\, t \in [0,\infty)\}$ with H independent of $S_T^{\delta_*}$, it follows from the actuarial pricing formula (9.2.6) that the benchmarked actuarial price

$$\hat{U}_H(t) = \frac{P(t,T)}{S_t^{\delta_*}}\, E\left(H \,\middle|\, \mathcal{A}_t\right) \tag{9.2.8}$$

is, as the product of independent martingales, an $(\underline{\mathcal{A}}, P)$-martingale.

9.3 Capital Asset Pricing Model

Risk Premium for the GOP

Later we shall derive for a general continuous financial market the influential intertemporal capital asset pricing model (ICAPM), see Merton (1973a). It is the continuous time generalization of the *capital asset pricing model* (CAPM), due to Sharpe (1964), Lintner (1965) and Mossin (1966). In practice, the ICAPM has been widely used for pricing securities in an approximate sense. We illustrate in the context of the BS model how the ICAPM can be used for pricing.

First, let us define what we mean by a risk premium. The *risk premium* $p_V(t)$ at time t for a security price process $V = \{V(t),\, t \in [0,T]\}$ is defined as the *expected excess return* above the short rate r_t, which is given as the almost sure limit

$$p_V(t) \overset{\text{a.s.}}{=} \lim_{h \downarrow 0} \frac{1}{h}\, E\left(\frac{V(t+h) - V(t)}{V(t)} \,\middle|\, \mathcal{A}_t\right) - r_t \tag{9.3.1}$$

for $t \in [0, T]$.

The GOP value $S_t^{\delta_*}$ at time t satisfies according to (9.1.15) the SDE

$$dS_t^{\delta_*} = S_t^{\delta_*} \left((r_t + p_{S^{\delta_*}}(t))\, dt + \sqrt{p_{S^{\delta_*}}(t)}\, dW_t \right) \tag{9.3.2}$$

for $t \in [0, T]$ with *risk premium*

$$p_{S^{\delta_*}}(t) = \theta_t^2 = \left(\frac{a_t - r_t}{\sigma_t} \right)^2, \tag{9.3.3}$$

see (9.1.16). Note that the risk premium $p_{S^{\delta_*}}(t)$ in the SDE (9.3.2) of the GOP equals the square of its volatility.

Risk Premium of the Underlying Security

By using in the SDE (9.1.1) of the underlying security S_t the formula (9.1.16) for the GOP volatility θ_t, we obtain the SDE

$$dS_t = S_t \left(r_t\, dt + \sigma_t \left(\theta_t\, dt + dW_t \right) \right) \tag{9.3.4}$$

for $t \in [0, T]$. It follows that the risk premium $p_S(t)$ of the underlying security S equals according to (9.3.1) and (9.3.4) the product

$$p_S(t) \overset{\text{a.s.}}{=} \lim_{h \downarrow 0} \frac{1}{h} E \left(\frac{S_{t+h} - S_t}{S_t} \,\middle|\, \mathcal{A}_t \right) - r_t \overset{\text{a.s.}}{=} \sigma_t\, \theta_t \tag{9.3.5}$$

almost surely for all $t \in [0, T]$. Note that the risk premium of the underlying security equals, as $h \to 0$, the normalized covariance of the returns of the underlying security and the GOP, that is,

$$\begin{aligned} p_S(t) &\overset{\text{a.s.}}{=} \lim_{h \downarrow 0} \frac{1}{h} E \left(\left(\frac{S_{t+h} - S_t}{S_t} \right) \left(\frac{S_{t+h}^{\delta_*} - S_t^{\delta_*}}{S_t^{\delta_*}} \right) \,\middle|\, \mathcal{A}_t \right) \\ &\overset{\text{a.s.}}{=} \lim_{h \downarrow 0} \frac{1}{h} E \left(\int_t^{t+h} \sigma_s\, dW_s \int_t^{t+h} \theta_s\, dW_s \,\middle|\, \mathcal{A}_t \right) \overset{\text{a.s.}}{=} \sigma_t\, \theta_t \end{aligned} \tag{9.3.6}$$

almost surely for $t \in [0, T]$.

There is also an alternative way of characterizing the risk premium (9.3.5). The risk premium can be obtained by forming the time derivative of the covariation between the logarithm $\ln(S_t)$ of the underlying security and the logarithm $\ln(S_t^{\delta_*})$ of the GOP. More precisely, by the Itô formula and the covariation property (5.4.5) of Itô integrals we can express the risk premium of S in the form

$$p_S(t) = \frac{d}{dt} \left[\ln(S), \ln(S^{\delta_*}) \right]_t = \sigma_t\, \theta_t \tag{9.3.7}$$

for $t \in [0, T]$. We shall see later that such a result holds generally in a continuous financial market.

Risk Premium of a Portfolio

Now, let us calculate risk premia for portfolios. It follows for a portfolio value S_t^δ at time t from the SDE (9.1.4) and equation (9.1.7) and (9.1.16) the SDE

$$dS_t^\delta = S_t^\delta \, \big(r_t \, dt + \pi_\delta^1(t) \, \sigma_t \, (\theta_t \, dt + dW_t)\big) \tag{9.3.8}$$

for $t \in [0, T]$. As defined above in (9.3.1), its risk premium $p_{S^\delta}(t)$ at time t equals the expected excess return

$$p_{S^\delta}(t) \overset{\text{a.s.}}{=} \lim_{h \downarrow 0} \frac{1}{h} \, E\left(\frac{S_{t+h}^\delta - S_t^\delta}{S_t^\delta} \,\middle|\, \mathcal{A}_t \right) - r_t \tag{9.3.9}$$

for $t \in [0, T)$. It follows from (9.3.8) and (9.3.9) that we obtain for the fraction $\pi_\delta^1(t)$ the risk premium

$$p_{S^\delta}(t) = \pi_\delta^1(t) \, \sigma_t \, \theta_t \tag{9.3.10}$$

at time $t \in [0, T]$. The risk premium of a portfolio equals the product of market price of risk and portfolio volatility. As in (9.3.7), it follows from the form of the portfolio SDE (9.3.8) that this risk premium equals the normalized covariance between the return of the portfolio and that of the GOP. We have then

$$p_{S^\delta}(t) \overset{\text{a.s.}}{=} \lim_{h \downarrow 0} \frac{1}{h} \, E\left(\left(\frac{S_{t+h}^\delta - S_t^\delta}{S_t^\delta} \right) \left(\frac{S_{t+h}^{\delta_*} - S_t^{\delta_*}}{S_t^{\delta_*}} \right) \,\middle|\, \mathcal{A}_t \right) \overset{\text{a.s.}}{=} \pi_\delta^1(t) \, \sigma_t \, \theta_t \tag{9.3.11}$$

for $t \in [0, T)$. Alternatively, by the Itô formula and the covariation property (5.4.5) of Itô integrals it also follows

$$p_{S^\delta}(t) = \frac{d}{dt} \, \big[\ln(S^\delta), \ln(S^{\delta_*})\big]_t = \pi_\delta^1(t) \, \sigma_t \, \theta_t \tag{9.3.12}$$

for $t \in [0, T]$. As we shall see later, this type of formula holds in a general continuous financial market and not only under the BS model.

Note that it follows from the above formula (9.3.12) that the risk premium $p_B(t)$ of the savings account is zero, as should be expected.

Portfolio Beta

The ICAPM uses the *market portfolio* (MP) as reference portfolio. One can choose, for instance, the MP as the portfolio of all tradable securities. In practice, this is convenient but difficult to specify explicitly. One can always argue about the exact composition of the MP. In any case, in reality the MP is a reasonably broadly diversified portfolio. The Morgan Stanley capital weighted world stock accumulation index (MSCI) arises as a possible proxy for the MP. We shall show later, see also Platen (2005b), that diversified portfolios can be expected in reality to be close to each other and also close to the GOP. This means, under general assumptions we shall see that diversified portfolios approximate the GOP. This fundamental fact is model independent. In the following, we use the GOP as proxy for the MP. Its movements can be interpreted to model the movements of the market as a whole, thus, modeling *general market risk* or *systematic risk*. We shall later consider *specific market risk*, which describes the movements of a portfolio that are not in line with those of the market index, see Platen & Stahl (2003).

When using the ICAPM one aims to measure for a given portfolio S_t^δ its *systematic risk parameter* $\beta_{S^\delta}(t)$, which is the, so-called, *beta*. The beta equals the ratio of covariations

$$\beta_{S^\delta}(t) = \frac{\frac{d[\ln(S^\delta),\ln(S^{\delta_*})]_t}{dt}}{d[\ln(S^{\delta_*})]_t} \tag{9.3.13}$$

for $t \in [0,T]$. Obviously, the beta equals one if the portfolio S^δ moves similarly to the market as a whole. If S^δ moves totally independent of the GOP, then its beta is zero. By using (9.3.12) it follows that

$$\beta_{S^\delta}(t) = \frac{\pi_\delta^1(t)\,\sigma_t\,\theta_t}{\theta_t^2} = \frac{p_{S^\delta}(t)}{p_{S^{\delta_*}}(t)} \tag{9.3.14}$$

for $t \in [0,T]$. This means that the portfolio beta is the normalized risk premium, where the normalizing quantity is the risk premium of the MP.

Obviously, the beta for the savings account is zero, that is

$$\beta_B(t) = 0 \tag{9.3.15}$$

for $t \in [0,T]$. This expresses the fact that there is no systematic risk in the savings account. Under the given BS model the beta of the underlying security S is by (9.3.14) and (9.3.5) obtained as

$$\beta_S(t) = \frac{\sigma_t}{\theta_t} \tag{9.3.16}$$

for $t \in [0,T]$. This beta is close to one if the underlying security fluctuates similarly to the GOP and, thus, the MP.

A portfolio has a small absolute value of beta if its fluctuations are almost independent of those of the GOP. This means that there is then little systematic or general market risk in this portfolio.

ICAPM Pricing Rule

By using relation (9.3.14) we obtain for the risk premium $p_{S^\delta}(t)$ of a portfolio S^δ the *ICAPM formula*

$$p_{S^\delta}(t) = \beta_{S^\delta}(t)\,p_{S^{\delta_*}}(t) \tag{9.3.17}$$

for $t \in [0, T]$. The portfolio beta $\beta_{S^\delta}(t)$, as defined in (9.3.14), has in the given case for a portfolio S^δ with fraction $\pi_\delta^1(t)$ the value

$$\beta_{S^\delta}(t) = \pi_\delta^1(t)\,\frac{\sigma_t}{\theta_t} \tag{9.3.18}$$

for $t \in [0, T]$.

Under the given BS model a portfolio beta is all that needs to be known about the portfolio's risk characteristics when using the ICAPM formula. The formula (9.3.17) does not contain prices explicitly. It only refers to risk premia. However, the ICAPM can be used in practice for approximate asset pricing. To explain this, we go back to the definition of a return. By the ICAPM formula (9.3.17) we have approximately for a portfolio S^δ with fraction $\pi_\delta^1(t)$ over a small period $[t, t+h]$ the expected return

$$E\left(\frac{S^\delta_{t+h} - S^\delta_t}{S^\delta_t}\,\bigg|\,\mathcal{A}_t\right) = \frac{E\left(S^\delta_{t+h}\,\big|\,\mathcal{A}_t\right) - S^\delta_t}{S^\delta_t}$$
$$\approx (r_t + p_{S^\delta}(t))\,h = (r_t + \beta_{S^\delta}(t)\,p_{S^{\delta_*}}(t))\,h.$$

Therefore, it follows for small $h > 0$ by (9.3.3) and (9.3.18) that approximately

$$\frac{E\left(S^\delta_{t+h}\,\big|\,\mathcal{A}_t\right)}{S^\delta_t} \approx 1 + (r_t + \beta_{S^\delta}(t)\,p_{S^{\delta_*}}(t))\,h = 1 + \left(r_t + \frac{\delta^1_t\,S_t\,\sigma_t\,\theta_t}{S^\delta_t}\right)h.$$

This yields the *ICAPM pricing rule*

$$S^\delta_t \approx \frac{E\left(S^\delta_{t+h}\,\big|\,\mathcal{A}_t\right)}{1 + (r_t + \beta_{S^\delta}(t)\,\theta^2_t)\,h} \tag{9.3.19}$$

or, similarly, by using the above relations and (9.1.16), the self-interpreting pricing rule

$$S^\delta_t \approx \frac{E\left(S^\delta_{t+h}\,\big|\,\mathcal{A}_t\right) - \delta^1_t\,S_t\,(a_t - r_t)\,h}{1 + r_t\,h} \tag{9.3.20}$$

for $t \in [0, T]$. We emphasize that (9.3.19) and (9.3.20) are approximate formulas. It is interesting to note that the ICAPM pricing rule (9.3.19) uses the portfolio beta and the expected future value of the portfolio as main inputs. Notice that the conditional expectation of the future portfolio value is taken under the real world probability measure, as is the case under real world pricing.

Via the benchmark approach we derive in Sect. 11.2 under general assumptions the ICAPM for continuous financial markets. This means we shall provide the basis for the ICAPM pricing rule (9.3.19). This approximate pricing formula is, of course, not fully consistent with real world pricing. However, it is a reasonable description of the fair price when h is small. The ICAPM pricing rule (9.3.19) is widely applied in practice. It provides another example where commonly accepted relationships in finance, insurance or accounting can be naturally derived under the benchmark approach by using the GOP as central building block.

9.4 Risk Neutral Pricing

By referring to the results from Chap. 8 on option pricing under the BS model, we now illustrate the widely used standard *risk neutral* pricing methodology, which one could interpret as the core of the *arbitrage pricing theory* (APT) and its generalizations, see for instance, Black & Scholes (1973), Ross (1976), Harrison & Kreps (1979), Harrison & Pliska (1981), Föllmer & Sondermann (1986), Föllmer & Schweizer (1991) and Delbaen & Schachermayer (1994, 1998, 2006).

Drifted Wiener Process

In the classical literature on derivative pricing it has been standard to use the domestic savings account $B = \{B_t,\, t \in [0,T]\}$ as reference unit or numeraire. For obtaining an option price one introduces an appropriate probability measure, the *risk neutral probability measure* P_θ. It allows to interpret the Black-Scholes pricing formula as a conditional expectation under this measure. As we shall see, this method provides an elegant and compact description of option prices in the case of the BS model. We shall show that the change to the risk neutral probability measure P_θ is equivalent to a *change of variables* with a corresponding probabilistic interpretation. Most importantly, we shall emphasize the fact that a number of assumptions have to be made to perform this change of variables, which may not be satisfied for realistic models.

Let us reformulate the SDE (9.1.1) for the underlying security S under the BS model, where we assume now, for simplicity, constant short rate r, constant volatility σ, constant appreciation rate a and, therefore, also constant market price of risk θ. We perform this change of variable in such a way that the SDE (9.1.1) shows formally the same appreciation rate r as the domestic savings account B, see (9.1.2). To achieve this it is necessary to introduce a corresponding driving process W_θ that no longer equals the Wiener process W. This is the, so-called, *drifted Wiener process* $W_\theta = \{W_\theta(t),\, t \in [0,T]\}$ with

$$W_\theta(t) = W_t + \theta\, t \tag{9.4.1}$$

for $t \in [0,T]$. Recall that the market price of risk is for our BS model of the form

$$\theta = \frac{a-r}{\sigma}, \tag{9.4.2}$$

see (9.1.16). This allows us to rewrite the SDE (9.1.1) for the underlying security in the form

$$dS_t = (a - \sigma\,\theta)\, S_t\, dt + \sigma\, S_t\, dW_\theta(t) = r\, S_t\, dt + \sigma\, S_t\, dW_\theta(t) \tag{9.4.3}$$

for $t \in [0,T]$. According to (6.3.7), (6.3.6) and (9.4.1), the geometric Brownian motion $S = \{S_t,\, t \in [0,T]\}$ has then the explicit representation

$$S_t = S_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right) t + \sigma\, W_\theta(t)\right\} \tag{9.4.4}$$

for $t \in [0,T]$. Note that for $\theta \neq 0$ the process W_θ is *not* a Wiener process under the real world probability measure P.

Radon-Nikodym Derivative

We shall show that the process W_θ is a standard Wiener process under the *risk neutral measure* P_θ. This measure is characterized by its *Radon-Nikodym derivative*

$$\Lambda_\theta(T) = \frac{dP_\theta}{dP}\bigg|_{\mathcal{A}_T} = \frac{\hat{S}^0_T}{\hat{S}^0_0}. \tag{9.4.5}$$

Recall that

$$\hat{S}^0_T = \frac{B_T}{S^{\delta_*}_T}$$

is the benchmarked domestic savings account at time T, see (9.1.22). The Radon-Nikodym derivative $\Lambda_\theta(T)$ defines the risk neutral measure P_θ, which is given in the form

$$P_\theta(A) = \int_A \Lambda_\theta(T)\, dP(\omega) = \int_A \frac{dP_\theta(\omega)}{dP(\omega)}\, dP(\omega) \tag{9.4.6}$$

for all subsets $A \in \Omega$.

Note that the measure P_θ is not automatically a probability measure. For risk neutral pricing to be useful in practice, we need the property that the risk neutral measure P_θ is a probability measure. This is equivalent to the request that a corresponding change of variables in an integration can be performed.

The following definition will be used generally throughout the book.

Definition 9.4.1. *Two measures are* equivalent *if they have the same sets of events of measure zero.*

The equivalence of the risk neutral and the real world probability measure is a fundamental requirement of the risk neutral approach. In the case of the above BS model one is able to apply the risk neutral approach since the measure P_θ is a probability measure and equivalent to P. The model generates geometric Brownian motions on $[0,T]$ under P and under P_θ with the same sets of events that have probability zero under both measures. However, there is already a problem even under the BS model if one wants to extend the time horizon T to infinity and aims to consider asymptotics for $T \to \infty$. Details on a construction allowing some risk neutral pricing in such a case can be found, for instance, in Karatzas & Shreve (1998).

As discussed in Sect. 9.1, under the BS model the benchmarked savings account $\hat{S}^0 = \{\hat{S}^0_t,\ t \in [0,T]\}$ is an $(\underline{\mathcal{A}}, P)$-martingale with initial value

$$\hat{S}_0^0 = \frac{1}{S_0^{\delta_*}}. \tag{9.4.7}$$

Thus, for the BS model due to (9.4.5) the *Radon-Nikodym derivative process* $\Lambda_\theta = \{\Lambda_\theta(t),\ t \in [0,T]\}$ with

$$\Lambda_\theta(t) = \frac{\hat{S}_t^0}{\hat{S}_0^0} \tag{9.4.8}$$

is an $(\underline{\mathcal{A}}, P)$-martingale that starts at $\Lambda_\theta(0) = 1$.

We shall see later that the martingale property of the Radon-Nikodym derivative process is crucial for the risk neutral approach. It makes sure that the measure P_θ is having a total mass of one, allowing it to be a probability measure.

We remark that the Radon-Nikodym derivative process is referred to in the literature also as *state price density*, *pricing kernel*, *deflator* or *stochastic discount factor*, see Hansen & Jagannathan (1991), Constatinides (1992), Rogers (1997), Cochrane (2001) and Duffie (2001).

Later it will become clear that the just mentioned Radon-Nikodym process simply expresses the benchmarked savings account when normalized to one. The corresponding risk neutral pricing method can, thus, be derived from real world pricing.

Risk Neutral Measure Transformation

To illustrate the measure transformation that is performed under the risk neutral approach, let us demonstrate under the given BS model that W_θ is a Wiener process under the risk neutral probability measure P_θ. The above Radon-Nikodym derivative process Λ_θ, see (9.4.8), has the representation

$$\Lambda_\theta(t) = \exp\left\{-\frac{1}{2}\,\theta^2\,t - \theta\,W_t\right\} \tag{9.4.9}$$

for $t \in [0,T]$. By using the Laplace transform (1.3.76) of a Gaussian random variable we have by the martingale property of Λ_θ the total risk neutral probability

$$P_\theta(\Omega) = E(\Lambda_\theta(T)) = E(\Lambda_\theta(T)\,\big|\,\mathcal{A}_0) = \Lambda_\theta(0) = 1. \tag{9.4.10}$$

This shows that P_θ is a probability measure.

For fixed $\tilde{y} \in \Re,\ t \in [0,T]$ and $s \in [0,t)$ let A be the event

$$A = \{\omega \in \Omega :\ W_\theta(t,\omega) - W_\theta(s,\omega) < \tilde{y}\}.$$

Here we indicate in the notation $W_\theta(t,\omega)$ its dependence on the outcome $\omega \in \Omega$. Using relation (9.4.1), this event can equivalently be written in the form

$$A = \{\omega \in \Omega : \, W(t,\omega) - W(s,\omega) < \tilde{y} - \theta\,(t-s)\}\,.$$

Since W is a Wiener process on $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$, then $A \in \mathcal{A}_t$ where A is independent of $\mathcal{A}_s$. Combining these facts, it follows by using the indicator function $\mathbf{1}_A$, with E_θ denoting expectation under P_θ, that

$$\begin{aligned} P_\theta(A) &= E_\theta(\mathbf{1}_A) = E(\Lambda_\theta(T)\,\mathbf{1}_A) \\ &= E\left(\Lambda_\theta(t)\,\mathbf{1}_A\,\frac{\Lambda_\theta(T)}{\Lambda_\theta(t)}\right) = E(\Lambda_\theta(t)\,\mathbf{1}_A) \\ &= E\left(\Lambda_\theta(s)\,\frac{\Lambda_\theta(t)}{\Lambda_\theta(s)}\,\mathbf{1}_A\right) = E\left(\Lambda_\theta(s)\right)\,E\left(\frac{\Lambda_\theta(t)}{\Lambda_\theta(s)}\,\mathbf{1}_A\right) \\ &= E\left(\frac{\Lambda_\theta(t)}{\Lambda_\theta(s)}\,\mathbf{1}_A\right). \end{aligned} \tag{9.4.11}$$

We know that $W_t - W_s$ is Gaussian distributed with mean zero and variance $(t-s)$. Therefore, we obtain for the event A with (9.4.9) the P_θ-probability

$$\begin{aligned} P_\theta(A) &= \int_{-\infty}^{\tilde{y}-\theta(t-s)} \exp\left\{-\frac{\theta^2}{2}\,(t-s) - \theta\,y\right\} \frac{1}{\sqrt{2\,\pi\,(t-s)}}\,\exp\left\{-\frac{y^2}{2\,(t-s)}\right\} dy \\ &= \int_{-\infty}^{\tilde{y}-\theta(t-s)} \frac{1}{\sqrt{2\,\pi\,(t-s)}}\,\exp\left\{-\frac{(y+\theta\,(t-s))^2}{2\,(t-s)}\right\} dy \\ &= \int_{-\infty}^{\tilde{y}} \frac{1}{\sqrt{2\,\pi\,(t-s)}}\,\exp\left\{-\frac{z^2}{2\,(t-s)}\right\} dz. \end{aligned} \tag{9.4.12}$$

This equation shows that $W_\theta(t) - W_\theta(s)$ is Gaussian distributed under P_θ with mean zero and variance $(t-s)$. Note that we have only changed variables for the integration in (9.4.12), which is permitted due to the properties of the Gaussian density. From the properties (3.2.6) of the Wiener process W and relation (9.4.1) we conclude under the given BS model that $W_\theta(0) = 0$. Using arguments similar to those applied in (9.4.11), it follows that W_θ has independent increments. Therefore, by using (3.2.6), we can formulate the following simple version of the following *Cameron-Martin Girsanov Theorem* when applied to the BS model.

Theorem 9.4.2. (Cameron-Martin Girsanov) *Under the BS model the process W_θ is a standard Wiener process in the filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P_\theta)$, which is defined under the risk neutral probability measure P_θ.*

In the next section we shall provide a more general version of this theorem.

Risk Neutral Pricing Formula

As previously mentioned, the probability measure P_θ, which is called the risk neutral probability measure, can be used for option pricing. This is the probability measure under which the process W_θ, see (9.4.1), becomes an $(\underline{\mathcal{A}}, P_\theta)$-Wiener process. Let us now obtain from the real world pricing formula (9.1.34) the price of a European option with payoff $H(S_T)$ under the given BS model.

Using (9.4.5), (9.4.9) and the explicit expression (9.4.4) for the geometric Brownian motion S, see (9.4.3), we can rewrite the real world pricing formula (9.1.34) for $t = 0$ in the form

$$
\begin{aligned}
V(0, S_0) &= E\left(\frac{S_0^{\delta_*}}{S_T^{\delta_*}} H(S_T) \,\Big|\, \mathcal{A}_0\right) = E\left(\frac{\frac{B_T}{S_T^{\delta_*}}}{\frac{B_0}{S_0^{\delta_*}}} \left(\frac{H(S_T)}{B_T}\right) \,\Big|\, \mathcal{A}_0\right) \\
&= E\left(\frac{\hat{S}_T^0}{\hat{S}_0^0} \left(\frac{H(S_T)}{B_T}\right) \,\Big|\, \mathcal{A}_0\right) = E\left(\frac{\Lambda_\theta(T)}{\Lambda_\theta(0)} \left(\frac{H(S_T)}{B_T}\right) \,\Big|\, \mathcal{A}_0\right) \\
&= E\left(\exp\left\{-\frac{\theta^2}{2} T - \theta\, W_T\right\} \left(\exp\{-r\,T\}\, H(S_T)\right) \,\Big|\, \mathcal{A}_0\right) \\
&= \int_{-\infty}^{\infty} \left[\exp\{-r\,T\}\, H\left(S_0 \exp\left\{\left(a - \frac{1}{2}\sigma^2\right) T + \sigma\, y\right\}\right)\right] \\
&\qquad \times \exp\left\{-\frac{\theta^2}{2} T - \theta\, y\right\} \frac{1}{\sqrt{T}}\, N'\left(\frac{y}{\sqrt{T}}\right) dy.
\end{aligned}
$$

With the change of variables $\tilde{y} = y + \theta\, T = y + (\frac{a-r}{\sigma}) T$ we then obtain

$$
\begin{aligned}
V(0, S_0) = \exp\{-r\,T\} \int_{-\infty}^{\infty} &\left[H\left(S_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right) T + \sigma\, \tilde{y}\right\}\right)\right] \\
&\times \frac{1}{\sqrt{T}}\, N'\left(\frac{\tilde{y}}{\sqrt{T}}\right) d\tilde{y}.
\end{aligned}
$$

When written in the following form, the above result provides the *risk neutral pricing formula*

$$
V(0, S_0) = \exp\{-r\,T\}\, E_\theta\left(H(S_T) \,\big|\, \mathcal{A}_0\right) = E_\theta\left(\frac{H(S_T)}{B_T} \,\Big|\, \mathcal{A}_0\right). \tag{9.4.13}
$$

Here E_θ denotes the expectation with respect to the risk neutral probability measure P_θ and $N'(\cdot)$ is the standard Gaussian density, see (1.2.8).

The above derivation shows that the fair price at time $t = 0$ of an option under the BS model can be rewritten as a conditional expectation E_θ under the risk neutral probability measure P_θ of a savings account discounted payoff $\frac{H(S_T)}{B_T}$. The above risk neutral pricing formula has been widely used in derivative pricing. In the current literature the risk neutral pricing formula appears to be the standard pricing tool. However, note that certain assumptions need to be satisfied to apply this pricing formula.

We exploited a number of mathematical properties that are automatically guaranteed under the BS model. As we shall see later, for certain more realistic asset price models, for instance the MMM, the martingale property of the Radon-Nikodym derivative Λ_θ, does not hold and an equivalent risk neutral probability measure does not exist. Since the real world pricing concept does not require the existence of an equivalent risk neutral probability measure one can always apply the real world pricing formula as long as the GOP exists and the expectation of the benchmarked payoff is finite.

Risk Neutral SDEs

Note that under the risk neutral probability measure P_θ the discounted underlying security price $\bar{S}$, see (8.2.10), satisfies under the BS model according to (9.4.3) and by application of the Itô formula the SDE

$$d\bar{S}_t = \sigma\,\bar{S}_t\,dW_\theta(t) \tag{9.4.14}$$

for $t \in [0, T]$. Thus, $\bar{S}$ is driftless under P_θ and can be shown for the BS model to be an $(\underline{\mathcal{A}}, P_\theta)$-martingale, see Exercise 9.1. Furthermore, it follows by the Itô formula, (8.2.21) and (9.4.1) that the SDE for the discounted option price $\bar{V}$, see (8.2.9), is given by

$$d\bar{V}(t, \bar{S}_t) = \frac{\partial \bar{V}(t, \bar{S}_t)}{\partial \bar{S}}\,\sigma\,\bar{S}_t\,dW_\theta(t) \tag{9.4.15}$$

for $t \in [0, T]$. This means that also the SDE for $\bar{V}$ is driftless under P_θ. One can show for the given BS model that $\bar{V}$ is an $(\underline{\mathcal{A}}, P_\theta)$-martingale, see Exercise 9.2. Obviously, the discounted savings account $\bar{B}$, see (8.2.15), is a constant and, thus, trivially an $(\underline{\mathcal{A}}, P_\theta)$-martingale. For both $(\underline{\mathcal{A}}, P_\theta)$-martingales $\bar{V}$ and $\bar{B}$ it is easy to see from Sect. 9.1 that their benchmarked values $\hat{V}(t) = \frac{\bar{V}(t,\bar{S}_t)}{\bar{S}^{\delta_*}_t}$ and $\hat{S}^0_t = \frac{\bar{B}_t}{\bar{S}^{\delta_*}_t}$ form $(\underline{\mathcal{A}}, P)$-martingales.

Risk Neutral SDE for Portfolios

Generally, for the above BS model all discounted portfolio prices can be shown to form $(\underline{\mathcal{A}}, P_\theta)$-local martingales. This property follows from the SDE (9.1.37) and Lemma 5.4.1 since by application of Itô's formula

$$\begin{aligned} d\bar{S}^\delta_t &= d(\bar{S}^{\delta_*}_t\,\hat{S}^\delta_t) \\ &= \bar{S}^\delta_t\,\pi^1_\delta(t)\,\sigma\,(\theta\,dt + dW_t) \\ &= \bar{S}^\delta_t\,\pi^1_\delta(t)\,\sigma\,dW_\theta(t) \end{aligned} \tag{9.4.16}$$

for $t \in [0, T]$. By Lemma 5.2.3 any nonnegative discounted portfolio is, therefore, an $(\underline{\mathcal{A}}, P)$-supermartingale. If $\bar{S}^\delta$ is an $(\underline{\mathcal{A}}, P_\theta)$-martingale, then the risk

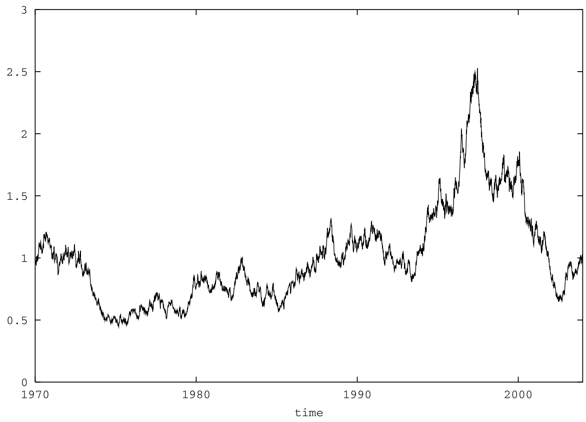

Fig. 9.4.1. A Radon-Nikodym derivative process for a BS model

neutral pricing formula (9.4.13) holds for $\bar{S}^\delta$. Furthermore, since the Radon-Nikodym derivative process Λ_θ is here an $(\underline{\mathcal{A}}, P)$-martingale, we shall see later that in this case the benchmarked portfolio value $\hat{S}^\delta_t = \frac{\bar{S}^\delta}{\bar{S}^{\delta_*}_t}$ forms an $(\underline{\mathcal{A}}, P)$-martingale.

We have seen that the real world pricing formula (9.1.34) does not hold for an unfair portfolio as constructed in (9.1.38)–(9.1.43). Similarly, for such a portfolio also the risk neutral pricing formula fails. Thus, one should *not* expect all discounted portfolios to be automatically $(\underline{\mathcal{A}}, P_\theta)$-martingales under the risk neutral probability measure P_θ, even under a simple BS model. Unfortunately, some literature gives the impression that this is the case.

Observe in the derivation of (9.4.13) that we have performed a change of variables from W_t to $W_\theta(t)$ with the interpretation that W and W_θ are Wiener processes under P and P_θ, respectively. The only variable that is random in the risk neutral pricing formula (9.4.13) is S_T, as compared to the real world pricing formula (9.1.34), where also the random GOP value $S^{\delta_*}_T$ is involved. Thus, the computation of option prices by using the risk neutral approach is simplified for the case of the BS model. This simplification relies on the existence of the equivalent risk neutral probability measure P_θ under the BS model.

We shall see in the next chapter that the benchmark approach, with its real world pricing concept, handles more general models than those permitted under the risk neutral approach. An equivalent risk neutral probability measure need not exist under the benchmark approach. This freedom in modeling will become important when we are going to model realistically the typical market dynamics.

In Fig. 9.4.1 we show a path of an exponential martingale from a geometric Brownian motion with volatility $\theta = 0.2$. We know that the path in Fig. 9.4.1 is that of a martingale. Here the actual value is the best forecast of future values. Similar to equation (9.4.8) one can show that the candidate Radon-

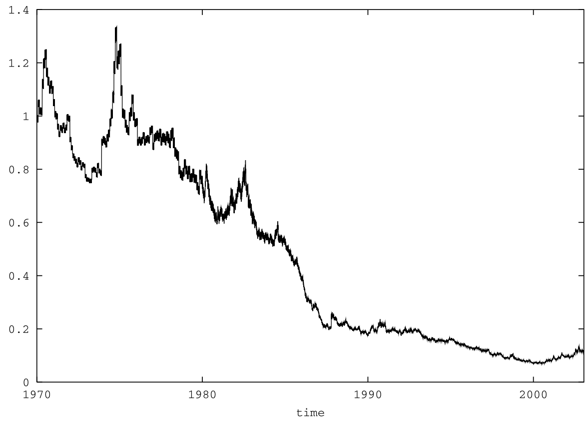

Fig. 9.4.2. Candidate Radon-Nikodym derivative of hypothetical risk neutral measure

Nikodym derivative process $\Lambda = \{\Lambda_t,\ t \in [0,T]\}$ for a hypothetical equivalent risk neutral probability measure for a range of continuous financial markets is given by the benchmarked savings account, see (9.4.8) and Karatzas & Shreve (1998), normalized at the initial time to one. An indication for the potential nonexistence of an equivalent risk neutral probability measure for the real market is given by the following important observation:

If the GOP is proxied by a diversified world stock index, as we shall suggest in the next chapter, then one can observe the benchmarked savings account for the world market and, thus, the candidate Radon-Nikodym derivative of its hypothetical risk neutral measure. We show in Fig. 9.4.2 the candidate Radon-Nikodym derivative of the hypothetical risk neutral measure of the world stock market with respect to the US dollar as domestic currency when using the Morgan Stanley capital weighted world stock accumulation index (MSCI) as proxy for the GOP. The path of this process seems to trend systematically downward, which is not typical for a martingale. However, for economic reasons the graph in Fig. 9.4.2 is rather typical, as we shall discuss below. In the long run the benchmarked savings account must be expected to decline systematically in reality. Otherwise, investors have no reason to invest in the stock market. This has been empirically confirmed by Dimson, Marsh & Staunton (2002), who showed that the market capitalization weighted world stock index, when discounted by the US dollar savings account, showed an annually discretely compounded net growth rate of about 0.049 over the last century. From economic reasoning it does not appear to be natural that the trajectory of the benchmarked savings account should form in reality a martingale. However, this martingale property is needed for the application of the Cameron-Martin Girsanov Theorem.

The downward trending trajectory in Fig. 9.4.2 resembles more the path of a strict supermartingale. Of course, a single path cannot prove that the candidate Radon-Nikodym derivative of the hypothetical risk neutral measure is a strict supermartingale. However, based on the economic argument that stock market investments grow in the long term faster than a savings account, one should be prepared to acknowledge such possibility, when developing long term market models. Of course, even if we agree that the benchmarked savings account is not a martingale under the real world probability measure, this is insufficient to infer that no equivalent risk neutral probability measure exists. But it is certainly enough evidence for us to consider this possibility seriously, which we acknowledge by working under the benchmark approach with its real world pricing concept.

What we have just observed creates serious concerns about the practical applicability of the risk neutral pricing methodology that has been the prevailing approach in finance for several decades. Within this book we aim to provide with the benchmark approach a framework that allows to handle not only models that have an equivalent risk neutral probability measure but also models for which this is *not* the case. The real world pricing concept makes financial modeling, derivative pricing and calibration less complicated since a measure transformation is not required.

Under the benchmark approach some potential model risk is removed which could be caused by the fact that an equivalent risk neutral probability measure may not exist for the existing financial market.

9.5 Girsanov Transformation and Bayes Rule (*)

In the previous section, an equivalent probability measure transformation was applied, which is also known as *Girsanov transformation*. The following section describes such transformation more generally. It will also introduce *Bayes's Theorem*, which is needed to interpret conditional expectations under a given measure by using those defined under another measure. Both results are important for equivalent probability measure changes.

Change of Probability Measure (*)

We denote by $\boldsymbol{W} = \{\boldsymbol{W}_t = (W_t^1, \ldots, W_t^m)^\top, \, t \in [0,T]\}$ an m-dimensional standard Wiener process on a filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$, as given in Sect. 5.1, with $\mathcal{A}_0$ being the trivial σ-algebra, augmented by the sets of zero probability. For an $\underline{\mathcal{A}}$-predictable m-dimensional stochastic process $\boldsymbol{\theta} = \{\boldsymbol{\theta}_t = (\theta_t^1, \ldots, \theta_t^m)^\top, \, t \in [0,T]\}$ with

$$\int_0^T \sum_{i=1}^m (\theta_t^i)^2 \, dt < \infty \tag{9.5.1}$$

almost surely, we assume that the strictly positive *Radon-Nikodym derivative process* $\Lambda_{\boldsymbol{\theta}} = \{\Lambda_{\boldsymbol{\theta}}(t), \, t \in [0,T]\}$, where

$$\Lambda_{\boldsymbol{\theta}}(t) = \exp\left\{-\int_0^t \boldsymbol{\theta}_s^\top \, d\boldsymbol{W}_s - \frac{1}{2}\int_0^t \boldsymbol{\theta}_s^\top \, \boldsymbol{\theta}_s \, ds\right\} < \infty \tag{9.5.2}$$

almost surely for $t \in [0, T]$ is an $(\underline{\mathcal{A}}, P)$-*martingale*. By the Itô formula (6.2.11) it follows from (9.5.2) that

$$\Lambda_{\boldsymbol{\theta}}(t) = 1 - \sum_{i=1}^{m} \int_0^t \Lambda_{\boldsymbol{\theta}}(s) \, \theta_s^i \, dW_s^i \tag{9.5.3}$$

for $t \in [0, T]$. Since $\Lambda_{\boldsymbol{\theta}}$ is by the above assumption an $(\underline{\mathcal{A}}, P)$-martingale we have

$$E\left(\Lambda_{\boldsymbol{\theta}}(t) \,\middle|\, \mathcal{A}_s\right) = \Lambda_{\boldsymbol{\theta}}(s) \tag{9.5.4}$$

for $t \in [0, T]$ and $s \in [0, t]$ and, in particular,

$$E\left(\Lambda_{\boldsymbol{\theta}}(t) \,\middle|\, \mathcal{A}_0\right) = \Lambda_{\boldsymbol{\theta}}(0) = 1. \tag{9.5.5}$$

Now, we define a measure $P_{\boldsymbol{\theta}}$ via the Radon-Nikodym derivative

$$\frac{dP_{\boldsymbol{\theta}}}{dP} = \Lambda_{\boldsymbol{\theta}}(T), \tag{9.5.6}$$

by setting

$$P_{\boldsymbol{\theta}}(A) = E(\Lambda_{\boldsymbol{\theta}}(T) \, \mathbf{1}_A) = E_{\boldsymbol{\theta}}(\mathbf{1}_A) \tag{9.5.7}$$

for $A \in \mathcal{A}_T$. Recall that $\mathbf{1}_A$ is the indicator function for A and $E_{\boldsymbol{\theta}}$ means expectation with respect to $P_{\boldsymbol{\theta}}$.

Note that $P_{\boldsymbol{\theta}}$ is not just a measure but also a probability measure because

$$P_{\boldsymbol{\theta}}(\Omega) = E(\Lambda_{\boldsymbol{\theta}}(T)) = E\left(\Lambda_{\boldsymbol{\theta}}(T) \,\middle|\, \mathcal{A}_0\right) = \Lambda_{\boldsymbol{\theta}}(0) = 1 \tag{9.5.8}$$

due to the martingale property of $\Lambda_{\boldsymbol{\theta}}$. This indicates why the martingale property of the Radon-Nikodym derivative is so important. It guarantees that the resulting risk neutral measure is a probability measure.

If the Radon-Nikodym derivative for the candidate risk neutral measure is a strict supermartingale, then the equality (9.5.8) does not hold and $P_{\boldsymbol{\theta}}(\Omega)$ is strictly less than one. As we shall see, this case arises, for instance, under the MMM, see Fig. 13.3.2.

Bayes's Theorem (*)

As seen in the risk neutral pricing formula (9.4.13), it is useful to be able to change the probability measure for conditional expectations. For a simple case this is indicated by formula (9.5.7). There exists a general tool, which is the following *Bayes rule*, that allows one to establish a relationship between conditional expectations with respect to different equivalent probability measures.

Theorem 9.5.1. (Bayes) *Assume that a given strictly positive Radon-Nikodym derivative process $\Lambda_{\boldsymbol{\theta}}$ is an $(\underline{\mathcal{A}}, P)$-martingale determining a corresponding equivalent probability measure $P_{\boldsymbol{\theta}}$. Then for any given stopping time $\tau \in [0, T]$ and any $\mathcal{A}_\tau$-measurable random variable Y, satisfying the integrability condition*

$$E_{\boldsymbol{\theta}}(|Y|) < \infty, \tag{9.5.9}$$

one can apply the Bayes rule

$$E_{\boldsymbol{\theta}}\left(Y \,\middle|\, \mathcal{A}_s\right) = \frac{E\left(\Lambda_{\boldsymbol{\theta}}(\tau)\, Y \,\middle|\, \mathcal{A}_s\right)}{E\left(\Lambda_{\boldsymbol{\theta}}(\tau) \,\middle|\, \mathcal{A}_s\right)} \tag{9.5.10}$$

for $s \in [0, \tau]$.

Proof of Bayes's Theorem (*)

We indicate here a proof of Bayes's Theorem. For a stopping time $\tau \in [s, T]$ and given fixed time $s \in [0, T]$ one can prove Bayes's theorem by using formula (9.5.7) for the probability $P_{\boldsymbol{\theta}}(A)$ together with the properties (1.3.63)–(1.3.66) of conditional expectations and the martingale property of $\Lambda_{\boldsymbol{\theta}}$. Then for each $\mathcal{A}_\tau$-measurable random variable Y and a set $A \in \mathcal{A}_s$ with some fixed time $s \in [0, T]$ we can show that both sides of (9.5.10) are identical for any such set A, that is,

$$\begin{aligned}
\mathbf{1}_A\, E_{\boldsymbol{\theta}}\left(Y \,\middle|\, \mathcal{A}_s\right) &= E_{\boldsymbol{\theta}}\left(\mathbf{1}_A\, Y \,\middle|\, \mathcal{A}_s\right) = E\left(\mathbf{1}_A\, Y\, \Lambda_{\boldsymbol{\theta}}(T) \,\middle|\, \mathcal{A}_s\right) \\
&= E\left(\mathbf{1}_A\, Y\, \Lambda_{\boldsymbol{\theta}}(\tau) \,\middle|\, \mathcal{A}_s\right) = E\left(\mathbf{1}_A\, E\left(Y\, \Lambda_{\boldsymbol{\theta}}(\tau) \,\middle|\, \mathcal{A}_s\right) \,\middle|\, \mathcal{A}_s\right) \\
&= E\left(\Lambda_{\boldsymbol{\theta}}(s) \left(\frac{\mathbf{1}_A}{\Lambda_{\boldsymbol{\theta}}(s)}\, E\left(Y\, \Lambda_{\boldsymbol{\theta}}(\tau) \,\middle|\, \mathcal{A}_s\right)\right) \,\middle|\, \mathcal{A}_s\right) \\
&= E_{\boldsymbol{\theta}}\left(\frac{\mathbf{1}_A}{\Lambda_{\boldsymbol{\theta}}(s)}\, E\left(Y\, \Lambda_{\boldsymbol{\theta}}(\tau) \,\middle|\, \mathcal{A}_s\right) \,\middle|\, \mathcal{A}_s\right) \\
&= E_{\boldsymbol{\theta}}\left(\mathbf{1}_A\, \frac{E\left(\Lambda_{\boldsymbol{\theta}}(\tau)\, Y \,\middle|\, \mathcal{A}_s\right)}{E\left(\Lambda_{\boldsymbol{\theta}}(\tau) \,\middle|\, \mathcal{A}_s\right)} \,\middle|\, \mathcal{A}_s\right) = \mathbf{1}_A\, \frac{E\left(\Lambda_{\boldsymbol{\theta}}(\tau)\, Y \,\middle|\, \mathcal{A}_s\right)}{E\left(\Lambda_{\boldsymbol{\theta}}(\tau) \,\middle|\, \mathcal{A}_s\right)}.
\end{aligned}$$

This proves Theorem 9.5.1. □

Girsanov Theorem (*)

The following important result is known as *Girsanov Theorem* for which we shall indicate a proof at the end of the section. A simple version of the Girsanov Theorem has been already given with the Cameron-Martin Girsanov Theorem, see Theorem 9.4.2. The Girsanov Theorem allows us to perform a measure transformation, which transforms an $(\underline{\mathcal{A}}, P)$-drifted Wiener process, as given in (9.4.1), into a Wiener process under a new probability measure $P_{\boldsymbol{\theta}}$. Such a transformation is called Girsanov transformation.

Theorem 9.5.2. (Girsanov) *If for $T \in (0, \infty)$ a given strictly positive Radon-Nikodym derivative process $\Lambda_{\boldsymbol{\theta}}$ is an $(\underline{\mathcal{A}}, P)$-martingale, then the m-dimensional process $\boldsymbol{W}_{\boldsymbol{\theta}} = \{\boldsymbol{W}_{\boldsymbol{\theta}}(t),\, t \in [0,T]\}$, given by*

$$\boldsymbol{W}_{\boldsymbol{\theta}}(t) = \boldsymbol{W}_t + \int_0^t \theta_s \, ds \tag{9.5.11}$$

for all $t \in [0,T]$, is an m-dimensional standard Wiener process on the filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P_{\boldsymbol{\theta}})$.

Note that certain assumption needs to be satisfied before one can apply the above Girsanov Theorem. The sole key assumption is that $\Lambda_{\boldsymbol{\theta}}$ must be a strictly positive $(\underline{\mathcal{A}}, P)$-martingale. For instance, if the Radon-Nikodym derivative process is almost surely only a strictly positive local martingale, then this does not guarantee that $P_{\boldsymbol{\theta}}$ is a probability measure.

Novikov Condition (*)

As just mentioned, a key assumption of the risk neutral approach is that $\Lambda_{\boldsymbol{\theta}}$ has to be a strictly positive $(\underline{\mathcal{A}}, P)$-martingale. A sufficient condition for the Radon-Nikodym derivative process $\Lambda_{\boldsymbol{\theta}}$ to be an $(\underline{\mathcal{A}}, P)$-martingale is the *Novikov condition*, see Novikov (1972), which requires that

$$E\left(\exp\left\{\frac{1}{2}\int_0^T \boldsymbol{\theta}_s^\top \boldsymbol{\theta}_s \, ds\right\}\right) < \infty. \tag{9.5.12}$$

This condition is fulfilled for the BS model, as was given in (9.1.1), since the market price of risk θ, given in (9.1.16), is a constant. For the case, when $\Lambda_{\boldsymbol{\theta}}$ is already known to be a strictly positive $(\underline{\mathcal{A}}, P)$-local martingale, then some other sufficient conditions can potentially be applied. Some conditions of this kind are given in Lemma 5.2.2. Further conditions can be found in Revuz & Yor (1999).

Proof of the Girsanov Theorem (*)

For simplicity, we only indicate the proof of Theorem 9.5.2 for the one-dimensional case, that is $m = 1$. Furthermore, we assume for simplicity that $\Lambda_\theta \theta, \Lambda_\theta, \Lambda_\theta W_\theta \theta,\, \Lambda_\theta (W_\theta)^2 \theta \in \mathcal{L}_T^2$ and that P is equivalent to P_θ. The general case is obtained by similar arguments, see Karatzas & Shreve (1991).

1. First, let us show that P_θ is a probability measure. It follows by application of the Itô formula (6.2.11) to the expression (9.5.2) that

$$d\Lambda_\theta(t) = -\Lambda_\theta(t)\, \theta_t \, dW_t \tag{9.5.13}$$

with $\Lambda_\theta(0) = 1$. For the strictly positive process Λ_θ we have a.s. the inequality $\Lambda_\theta(t) > 0$ and from equation (9.5.5) the property

$$E(\Lambda_\theta(t)) = 1 \tag{9.5.14}$$

for all $t \in [0,T]$. From equation (9.5.7) we conclude for any event $A \in \mathcal{A}$ that

$$P_\theta(A) = \int_\Omega \mathbf{1}_A(\omega)\, \Lambda_\theta(T)\, dP(\omega) \geq 0, \tag{9.5.15}$$

where $\mathbf{1}_A(\omega)$ is the indicator function for ω being in A. This combined with the property (9.5.14) shows that

$$P_\theta(\Omega) = \int_\Omega \Lambda_\theta(T)\, dP(\omega) = E(\Lambda_\theta(T)) = 1. \tag{9.5.16}$$

Therefore, $P_\theta(\cdot)$ is a well-defined probability measure on $(\Omega, \mathcal{A})$.

2. We now consider the product $\Lambda_\theta(t)\, W_\theta(t)$ and show that it forms a martingale. By the Itô formula (6.2.11) and equations (9.5.11) and (9.5.13) the SDE for $\Lambda_\theta W_\theta$ can be written in the form

$$\begin{aligned} d(\Lambda_\theta(t)\, W_\theta(t)) &= \Lambda_\theta(t)\, dW_\theta(t) + W_\theta(t)\, d\Lambda_\theta(t) + d[\Lambda_\theta, W_\theta]_t \\ &= \Lambda_\theta(t)\, dW_t + \Lambda_\theta(t)\, \theta_t\, dt - W_\theta(t)\, \Lambda_\theta(t)\, \theta_t\, dW_t - \Lambda_\theta(t)\, \theta_t\, dt \\ &= \Lambda_\theta(t)\, (1 - W_\theta(t)\, \theta_t)\, dW_t \end{aligned} \tag{9.5.17}$$

for $t \in [0,T]$. Thus, since $\Lambda_\theta(1 - W_\theta \theta) \in \mathcal{L}^2_T$ it follows by the martingale property (5.4.3) of Itô integrals that the process $\Lambda_\theta W_\theta$ is an $(\underline{\mathcal{A}}, P)$-martingale.

3. For $t \in [0,T]$ and $s \in [0,t]$, using the equivalence of P_θ, we obtain with Theorem 9.5.1 from the martingale property of $\Lambda_\theta W_\theta$ the conditional expectation

$$\begin{aligned} E_\theta\left(W_\theta(t)\,\middle|\,\mathcal{A}_s\right) &= E\left(\Lambda_\theta(t)\, W_\theta(t)\,\middle|\,\mathcal{A}_s\right) \\ &= E\left(\Lambda_\theta(s)\, W_\theta(s)\,\middle|\,\mathcal{A}_s\right) \\ &= E_\theta\left(W_\theta(s)\,\middle|\,\mathcal{A}_s\right) = W_\theta(s). \end{aligned} \tag{9.5.18}$$

Note that W_θ is an $(\underline{\mathcal{A}}, P_\theta)$-martingale. Note that it is not only a martingale with respect to the filtration that it generates.

4. Let us now show that W_θ is under P_θ a continuous square integrable martingale. Note that we obtain from (9.5.17) and (9.5.11) by the Itô formula

$$d\left(\Lambda_\theta(t)\, (W_\theta(t))^2\right) = \Lambda_\theta(t)\, dt + \Lambda_\theta(t)\, (W_\theta(t))^2\, \theta_t\, dW_t \tag{9.5.19}$$

for $t \in [0,T]$. Now, the square integrability of W_θ under P_θ follows, so $(W_\theta)^2\, \Lambda_\theta\, \theta \in \mathcal{L}^2_T$. From (9.5.19) we can conclude that W_θ is a continuous, square integrable $(\underline{\mathcal{A}}, P_\theta)$-martingale, see (5.1.2) with $W_\theta(0) = 0$, see (9.5.11).

5. The quadratic variation process $[W_\theta] = \{[W_\theta]_t,\ t \in [0,T]\}$, see (5.2.2) and (5.2.8), of the continuous $(\underline{\mathcal{A}}, P_\theta)$-martingale W_θ is, according to (9.5.11), of the form

$$[W_\theta]_t = t \tag{9.5.20}$$

for $t \in [0,T]$. It then follows by Lévy's Theorem, see Theorem 6.5.1, that W_θ is a standard Wiener process on the probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P_\theta)$. □

9.6 Change of Numeraire (*)

It became clear in our previous discussion on real world pricing and risk neutral pricing that there exist equivalent ways of obtaining derivative prices as conditional expectations under certain probability measures by using corresponding numeraires. This has been formalized in Geman, El Karoui & Rochet (1995). Each of these alternative choices of numeraires result in corresponding SDEs for the prices. Often, different numeraires can be used to characterize the same derivative price. Some numeraire choices can provide significant analytic or computational advantages. The expectations involved are simply different ways of representing the same integral value. What actually happens in a numeraire change is a change of variables in an integration. We emphasize that certain conditions have to be satisfied to perform a numeraire change. This is analogous to the well-known fact that not all changes of variables are feasible for certain integrations.

Benchmarked PDE (*)

To illustrate the change of numeraire technique, let us recall from the real world pricing formula (9.1.34) that the benchmarked option price can be expressed as conditional expectation of the benchmarked payoff. We shall now show for the BS model, as introduced in Sect. 9.1, that the benchmarked pricing function $\hat{V} : [0,T] \times (0,\infty) \times (0,\infty) \to [0,\infty)$, obtained as $\hat{V}(t, S_t, S_t^{\delta_*}) = \hat{V}(t)$, can be expressed as a PDE solution.

With a view on (9.1.32)–(9.1.34) let us determine whether there exists a sufficiently often differentiable benchmarked pricing function $\hat{V}(\cdot,\cdot,\cdot)$ such that

$$\hat{V}(t) = \hat{V}(t, S_t, S_t^{\delta_*}) = E\left(\frac{H(S_T)}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t\right), \tag{9.6.1}$$

for $t \in [0,T]$ with S_t and $S_t^{\delta_*}$ satisfying the SDEs (9.1.1) and (9.1.15), respectively.

Application of the Itô formula to the function $\hat{V}(t, S, S^{\delta_*})$ yields, as in (9.1.31), the equation

$$\begin{aligned}\frac{H(S_T)}{S_T^{\delta_*}} &= \hat{V}(T, S_T, S_T^{\delta_*}) \\ &= \hat{V}(t, S_t, S_t^{\delta_*}) + \int_t^T \tilde{L}^0\, \hat{V}(s, S_s, S_s^{\delta_*})\, ds \\ &\quad + \int_t^T \left(\frac{\partial \hat{V}(s, S_s, S_s^{\delta_*})}{\partial S}\, \sigma_s\, S_s + \frac{\partial \hat{V}(s, S_s, S_s^{\delta_*})}{\partial S^{\delta_*}}\, \theta_s\, S_s^{\delta_*}\right) dW_s\end{aligned} \tag{9.6.2}$$

with operator

$$\tilde{L}^0\,\hat{V}(t,S,S^{\delta_*}) = \frac{\partial \hat{V}(t,S,S^{\delta_*})}{\partial t} + a_t\,S\,\frac{\partial \hat{V}(t,S,S^{\delta_*})}{\partial S} + \frac{1}{2}\,\sigma_t^2\,S^2\,\frac{\partial^2 \hat{V}(t,S,S^{\delta_*})}{\partial S^2}$$
$$+ (r_t+\theta_t^2)\,S^{\delta_*}\,\frac{\partial \hat{V}(t,S,S^{\delta_*})}{\partial S^{\delta_*}} + \frac{1}{2}\,\theta_t^2\,(S^{\delta_*})^2\,\frac{\partial^2 \hat{V}(t,S,S^{\delta_*})}{\partial (S^{\delta_*})^2}$$
$$+ \sigma_t\,\theta_t\,S\,S^{\delta_*}\,\frac{\partial^2 \hat{V}(t,S,S^{\delta_*})}{\partial S\,\partial S^{\delta_*}} \tag{9.6.3}$$

for $t \in [0,T)$ and $S, S^{\delta_*} \in (0,\infty)$.

Since the process $\hat{V} = \{\hat{V}(t,S_t,S_t^{\delta_*}),\ t\in[0,T]\}$ is an $(\underline{\mathcal{A}},P)$-martingale, see (9.6.1), it follows from (9.6.2) that we obtain the *benchmarked PDE*

$$\tilde{L}^0\,\hat{V}(t,S,S^{\delta_*}) = 0 \tag{9.6.4}$$

for $(t,S,S^{\delta_*}) \in (0,T)\times(0,\infty)\times(0,\infty)$ with *benchmarked terminal condition*

$$\hat{V}(T,S,S^{\delta_*}) = \frac{H(S)}{S^{\delta_*}} \tag{9.6.5}$$

for $(S,S^{\delta_*}) \in (0,\infty)\times(0,\infty)$. Note that we have linked the conditional expectation (9.6.1) to the PDE (9.6.4)–(9.6.5). Such a relationship is generally known as a Feynman-Kac formula, which we shall describe in the next section. In the above case the numeraire at time t is the GOP $S_t^{\delta_*}$ and the pricing measure is the real world probability measure P.

Recovering the BS-PDE (*)

Now, we use a transformation of variables to confirm that the benchmarked PDE (9.6.4)–(9.6.5) is for the given BS model simply a transformation of the BS-PDE (8.2.23)–(8.2.24). Using the formula (9.1.25), we obtain

$$\hat{V}(t,S,S^{\delta_*}) = \frac{V(t,S)}{S^{\delta_*}} \tag{9.6.6}$$

for $(t,S,S^{\delta_*}) \in (0,T)\times(0,\infty)\times(0,\infty)$. Then the PDE (9.6.4)–(9.6.5) becomes

$$\frac{1}{S^{\delta_*}}\left(\frac{\partial V(t,S)}{\partial t} + a_t\,S\,\frac{\partial V(t,S)}{\partial S} + \frac{1}{2}\,\sigma_t^2\,S^2\,\frac{\partial^2 V(t,S)}{\partial S^2}\right.$$
$$\left. - (r_t+\theta_t^2)\,V(t,S) + \theta_t^2\,V(t,S) - \sigma_t\,\theta_t\,S\,\frac{\partial V(t,S)}{\partial S}\right) = 0 \tag{9.6.7}$$

for $(t,S,S^{\delta_*}) \in (0,T)\times(0,\infty)\times(0,\infty)$ with terminal condition

$$V(T,S) = H(S) \tag{9.6.8}$$

for $S \in (0,\infty)$. Consequently, by (9.1.16) and (9.6.7), the function $V(t,S)$ must satisfy the PDE

$$\frac{\partial V(t,S)}{\partial t} + r_t\, S\, \frac{\partial V(t,S)}{\partial S} + \frac{1}{2}\,\sigma_t^2\, S^2\, \frac{\partial^2 V(t,S)}{\partial S^2} - r_t\, V(t,S) = 0 \qquad (9.6.9)$$

for $(t, S) \in (0, T) \times (0, \infty)$ with terminal condition (9.6.8). This recovers the BS-PDE (8.2.23) with terminal condition (8.2.24). It confirms that the benchmark approach provides an alternative way of obtaining the BS-PDE for the pricing function of a European option.

Risk Neutral PDE (*)

By using the savings account B as numeraire in the BS model, let us now recall what we obtained under the risk neutral probability measure P_θ. We have established through the risk neutral pricing formula a link between the conditional expectation (9.4.13) under P_θ and the BS-PDE (8.2.21)–(8.2.22).

By similar arguments that provided (9.4.13), it holds for the discounted option price $\bar{V}(t, \bar{S}_t)$ that

$$\bar{V}(t,\bar{S}_t) = E_\theta\left(\frac{H(\bar{S}_T\, B_T)}{B_T}\,\middle|\,\mathcal{A}_t\right) \qquad (9.6.10)$$

for $t \in [0, T]$. On the other hand, we obtain for the discounted pricing function $\bar{V}(t, \bar{S})$ by (8.2.21)–(8.2.22) the, so-called, *risk neutral* PDE

$$\frac{\partial \bar{V}(t,\bar{S})}{\partial t} + \frac{1}{2}\,\sigma_t^2\, \bar{S}^2\, \frac{\partial^2 \bar{V}(t,\bar{S})}{\partial \bar{S}^2} = 0 \qquad (9.6.11)$$

for $(t, \bar{S}) \in [0, T) \times (0, \infty)$ with terminal condition

$$\bar{V}(T,\bar{S}) = \frac{H(\bar{S}\, B_T)}{B_T} \qquad (9.6.12)$$

for $\bar{S} \in (0, \infty)$. As we shall see in Sect. 9.7, also the conditional expectation (9.6.10) refers to a Feynman-Kac formula, here under the risk neutral probability measure P_θ. In the above case the numeraire is the savings account B and the pricing measure is the risk neutral probability measure P_θ.

Change of Numeraire Technique (*)

The above discussed possibility to use various strictly positive portfolios as numeraire to compute option prices, provides theoretical and computational freedom for finding convenient ways of derivative pricing. This has been observed by practitioners and researchers who realized that the risk neutral probability measure is not necessarily the most convenient probability measure for pricing certain payoffs. Geman et al. (1995) developed this into a general technique which is called the *change of numeraire technique*.

In general, a *numeraire* $S^{\bar{\delta}} = \{S_t^{\bar{\delta}},\, t \in [0, T]\}$ is in this book a strictly positive portfolio process with a corresponding strategy $\bar{\delta} = \{\bar{\delta}_t,\, t \in [0, T]\}$. Intuitively, a numeraire is used as a reference to normalize all other portfolios with respect to it. By choosing a numeraire $S^{\bar{\delta}}$ one considers the relative price of a portfolio $\frac{S_t^\delta}{S_t^{\bar{\delta}}}$.

Self-Financing under Numeraire Change (*)

Now, we shall show that *self-financing* portfolios remain self-financing after a numeraire change. This is a desirable but not obvious feature of continuous time financial market models. We have seen an example of this kind in (8.2.28). To illustrate this property more generally, consider under the given BS model a numeraire $S^{\bar{\delta}}$ and a portfolio S^{δ}. Then we have by (9.1.3)

$$S_t^{\delta} = \delta_t^0 \, B_t + \delta_t^1 \, S_t \tag{9.6.13}$$

and by (9.1.4)

$$dS_t^{\delta} = \delta_t^0 \, dB_t + \delta_t^1 \, dS_t \tag{9.6.14}$$

and

$$dS_t^{\bar{\delta}} = \bar{\delta}_t^0 \, dB_t + \bar{\delta}_t^1 \, dS_t \tag{9.6.15}$$

for $t \in [0, T]$. By the Itô formula it follows for the ratio $\frac{S_t^{\delta}}{S_t^{\bar{\delta}}}$ that

$$d\left(\frac{S_t^{\delta}}{S_t^{\bar{\delta}}}\right) = \frac{1}{S_t^{\bar{\delta}}} \, dS_t^{\delta} + S_t^{\delta} \, d\left(\frac{1}{S_t^{\bar{\delta}}}\right) + d\left[\frac{1}{S^{\bar{\delta}}}, S^{\delta}\right]_t. \tag{9.6.16}$$

By (9.6.14) and (9.6.13) we obtain

$$\begin{aligned} d\left(\frac{S_t^{\delta}}{S_t^{\bar{\delta}}}\right) &= \delta_t^0 \left(\frac{1}{S_t^{\bar{\delta}}} \, dB_t + B_t \, d\left(\frac{1}{S_t^{\bar{\delta}}}\right)\right) \\ &\quad + \delta_t^1 \left(\frac{1}{S_t^{\bar{\delta}}} \, dS_t + S_t \, d\left(\frac{1}{S_t^{\bar{\delta}}}\right) + d\left[\frac{1}{S^{\bar{\delta}}}, S\right]_t\right). \end{aligned} \tag{9.6.17}$$

Application of the Itô formula to the ratios $\frac{B_t}{S_t^{\bar{\delta}}}$ and $\frac{S_t}{S_t^{\bar{\delta}}}$ allows us to conclude that

$$d\left(\frac{S_t^{\delta}}{S_t^{\bar{\delta}}}\right) = \delta_t^0 \, d\left(\frac{B_t}{S_t^{\bar{\delta}}}\right) + \delta_t^1 \, d\left(\frac{S_t}{S_t^{\bar{\delta}}}\right) \tag{9.6.18}$$

for $t \in [0, T]$. This confirms that the portfolio S^{δ}, when denominated in units of the numeraire $S^{\bar{\delta}}$, is changing its value only due to the gains from trade in $\frac{B}{S^{\bar{\delta}}}$ and $\frac{S}{S^{\bar{\delta}}}$. Thus, the portfolio is also in the denomination of another numeraire $S^{\bar{\delta}}$ a self-financing portfolio. By using the Itô formula this property can be shown to hold generally for any model that we consider.

Numeraire Pairs (*)

When presenting the above pricing rules we always have considered *numeraire pairs* $(S^{\bar{\delta}}, P_{\theta_{\bar{\delta}}})$. This means, when we selected a numeraire $S^{\bar{\delta}}$, then there was also a corresponding candidate for a related pricing measure $P_{\theta_{\bar{\delta}}}$. In the real world pricing formula (9.1.34) this pair consists of the GOP S^{δ_*} as numeraire and the real world probability measure P as pricing measure, thus, resulting in the numeraire pair (S^{δ_*}, P). This is the only case where we are always sure that the pricing measure is an equivalent probability measure because there is no measure change involved.

In the derivation of the risk neutral measure P_θ in Sect. 9.4 we used the savings account B as numeraire, which yields the numeraire pair (B, P_θ). This is just another possible choice for a numeraire. Note that we have to make sure that P_θ is an equivalent probability measure when using this numeraire pair.

There can be also other numeraires that are convenient for the pricing of certain classes of derivatives, for instance, for the computation of interest rate term structure derivatives.

The following result provides a useful tool for the construction of numeraire pairs. From the real world pricing formula (9.1.34) it follows that

$$\frac{V(t)}{S_t^{\delta_*}} = E\left(\frac{H(S_T)}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t\right) \tag{9.6.19}$$

for all $t \in [0, T]$. We now introduce a strictly positive portfolio $S^{\bar{\delta}}$, which we use as numeraire. The numeraire, when benchmarked and normalized to the initial value one, has the form

$$\Lambda_{\theta_{\bar{\delta}}}(t) = \frac{\hat{S}_t^{\bar{\delta}}}{\hat{S}_0^{\bar{\delta}}} = \frac{S_t^{\bar{\delta}}}{S_t^{\delta_*}} \frac{S_0^{\delta_*}}{S_0^{\bar{\delta}}} \tag{9.6.20}$$

for $t \in [0, T]$. Then we can write by using (9.6.19) and (9.6.20)

$$\frac{V(t)}{S_t^{\bar{\delta}}} = E\left(\frac{S_t^{\delta_*}}{S_t^{\bar{\delta}}} \frac{S_T^{\bar{\delta}}}{S_T^{\delta_*}} \frac{H(S_T)}{S_T^{\bar{\delta}}} \,\middle|\, \mathcal{A}_t\right) = E\left(\frac{\Lambda_{\theta_{\bar{\delta}}}(T)}{\Lambda_{\theta_{\bar{\delta}}}(t)} \frac{H(S_T)}{S_T^{\bar{\delta}}} \,\middle|\, \mathcal{A}_t\right). \tag{9.6.21}$$

The benchmarked numeraire $\Lambda_{\theta_{\bar{\delta}}}(t)$ satisfies by (9.1.37) the SDE

$$d\Lambda_{\theta_{\bar{\delta}}}(t) = d\left(\frac{\hat{S}_t^{\bar{\delta}}}{\hat{S}_0^{\bar{\delta}}}\right) = \Lambda_{\theta_{\bar{\delta}}}(t)\left(\pi_{\bar{\delta}}^1(t)\,\sigma_t - \theta_t\right) dW_t \tag{9.6.22}$$

for $t \in [0, T]$. Note by Lemma 5.4.1 that $\Lambda_{\theta_{\bar{\delta}}}$ is an $(\underline{\mathcal{A}}, P)$-local martingale because the SDE (9.6.22) is driftless. Assume now that we have chosen a numeraire $S^{\bar{\delta}}$ such that $\Lambda_{\theta_{\bar{\delta}}}$ is an $(\underline{\mathcal{A}}, P)$-martingale. This allows us to show that $P_{\theta_{\bar{\delta}}}$ is a probability measure when defined via the Radon-Nikodym derivative

$$\frac{dP_{\theta_{\bar{\delta}}}}{dP} = \Lambda_{\theta_{\bar{\delta}}}(T). \tag{9.6.23}$$

We then can introduce the drifted Wiener process $W_{\theta_{\bar{\delta}}} = \{W_{\theta_{\bar{\delta}}}(t),\, t \in [0, T]\}$ with

$$dW_{\theta_{\bar{\delta}}}(t) = dW_t + \theta_{\bar{\delta}}(t)\,dt, \tag{9.6.24}$$

where

$$\theta_{\bar{\delta}}(t) = \theta_t - \pi_{\bar{\delta}}^1(t)\,\sigma_t \tag{9.6.25}$$

for $t \in [0,T]$. Now, we are in a position to apply Theorem 9.5.2 to conclude that by the Girsanov transformation (9.6.24) $W_{\theta_{\bar{\delta}}}$ is a standard Wiener process under the probability measure $P_{\theta_{\bar{\delta}}}$. This provides us, rather generally, with the numeraire pair $(S^{\bar{\delta}}, P_{\theta_{\bar{\delta}}})$.

Obviously, there is no measure transformation involved if we choose the GOP S^{δ_*} as numeraire since in this case we have from (9.6.25)

$$\theta_{\delta_*}(t) = 0$$

for all $t \in [0,T]$.

If we use the savings account B as numeraire, then $\pi_{\bar{\delta}}^1(t) = 0$ and we obtain from (9.6.25)

$$\theta_{\bar{\delta}}(t) = \theta_t.$$

This is the risk neutral measure change, where the probability measure $P_{\theta_{\bar{\delta}}} = P_\theta$ equals the risk neutral probability measure.

We could also use, for instance, the underlying security S as numeraire, where $\pi_{\bar{\delta}}^1(t) = 1$ and we obtain by (9.6.25)

$$\theta_{\bar{\delta}}(t) = \theta_t - \sigma_t.$$

This also would provide under the above BS model an appropriate measure transformation.

Note however, the situation is different, if we choose the unfair portfolio $S_t^{\bar{\delta}} = S_t^{\delta_*} Z_t$ given in (9.1.42)–(9.1.43). Obviously, by (9.1.44) this numeraire, when benchmarked is not an $(\underline{\mathcal{A}}, P)$-martingale. By (8.7.23) it is a strict supermartingale. The pricing measure $P_{\theta_{\bar{\delta}}}$ is in this case *not* a probability measure. In particular, we have

$$P_{\theta_{\bar{\delta}}}(\Omega) = E\left(\Lambda_{\theta_{\bar{\delta}}}(T)\,\middle|\,\mathcal{A}_0\right) < \Lambda_{\theta_{\bar{\delta}}}(0) = 1.$$

Consequently, the Girsanov Theorem cannot be applied.

Change of Numeraire Pricing Formula (*)

Using a strictly positive numeraire $S^{(\underline{\delta})}$ and noting that $\Lambda_{\theta_{\bar{\delta}}}(0) = 1$ we can always rewrite the real world pricing formula (9.1.34) in the form

$$V(0,S_0) = E\left(\frac{S_0^{\delta_*}}{S_T^{\delta_*}}\,H(S_T)\,\mathcal{A}_0\right) = E\left(\Lambda_{\theta_{\bar{\delta}}}(T)\,\frac{H(S_T)}{S_T^{\bar{\delta}}}\,\middle|\,\mathcal{A}_0\right). \tag{9.6.26}$$

Note that the quantity

$$\frac{S_0^{\delta_*}}{S_T^{\delta_*}} = \frac{\Lambda_{\theta_{\bar{\delta}}}(T)}{S_T^{\bar{\delta}}} \tag{9.6.27}$$

remains *numeraire invariant* under all above discussed numeraire changes. Let us compute the expectation on the right hand side of (9.6.26) by application of Bayes's Theorem and formula (9.5.10). The required corresponding conditional expectation is of the form

$$E\left(\Lambda_{\theta_{\bar{\delta}}}(T)\,\frac{H(S_T)}{S_T^{\bar{\delta}}}\,\middle|\,\mathcal{A}_0\right) = E_{\theta_{\bar{\delta}}}\left(\frac{H(S_T)}{S_T^{\bar{\delta}}}\,\middle|\,\mathcal{A}_0\right),$$

where $E_{\theta_{\bar{\delta}}}$ denotes expectation under $P_{\theta_{\bar{\delta}}}$. For this formula to be valid it is necessary that the assumptions of the Girsanov Theorem and the Bayes Theorem can be verified. This requires $\Lambda_{\theta_{\bar{\delta}}}$ to form an $(\underline{\mathcal{A}}, P)$-martingale to guarantee that $P_{\theta_{\bar{\delta}}}$ is an equivalent probability measure. If this is the case, then we obtain the *change of numeraire pricing formula*

$$V(0, S_0) = E_{\theta_{\bar{\delta}}}\left(\frac{H(S_T)}{S_T^{\bar{\delta}}}\,\middle|\,\mathcal{A}_0\right). \tag{9.6.28}$$

We learned from our previous discussion and example (9.1.38)–(9.1.43) in Sect. 9.1 that not all benchmarked numeraires form $(\underline{\mathcal{A}}, P)$-martingales. This indicates that the change of numeraire pricing formula (9.6.28) may fail to hold in certain cases. One needs to check carefully the assumptions that are needed for choosing a numeraire pair. Otherwise, an inappropriate numeraire choice, like the unfair portfolio in (9.1.42), may lead to wrong prices.

In the risk neutral case the Radon-Nikodym derivative process Λ_θ for the candidate risk neutral measure P_θ needs to be an $(\underline{\mathcal{A}}, P)$-martingale to provide the risk neutral pricing formula (9.4.13). Consequently, by (9.6.20) it is necessary that the benchmarked savings account $\frac{B_t}{S_t^{\delta_*}}$ forms an $(\underline{\mathcal{A}}, P)$-martingale to allow the use of the standard risk neutral approach.

9.7 Feynman-Kac Formula (*)

As previously shown, several of the existing pricing approaches can be expressed via pricing formulas that have the form of conditional expectations. These conditional expectations lead to pricing functions that satisfy certain PDEs, which are usually *Kolmogorov backward equations*, as was shown for real world pricing and for risk neutral pricing. The link between the conditional expectations and respective PDEs can be interpreted as an application of the, so-called, *Feynman-Kac formula*. In this section we formulate the Feynman-Kac formula under rather general assumptions, allowing also first exit times and jump diffusions. For a wide range of models this formula provides the Kolmogorov backward PDEs that characterize pricing functions of derivatives.

SDE for Factor Process (*)

At first we consider a fixed time horizon $T \in (0, \infty)$ and a d-dimensional Markov process $\boldsymbol{X}^{t,\boldsymbol{x}} = \{\boldsymbol{X}^{t,\boldsymbol{x}}_s,\ s \in [t, T]\}$ describing some factors, which satisfies the vector SDE

$$d\boldsymbol{X}^{t,\boldsymbol{x}}_s = \boldsymbol{a}(s, \boldsymbol{X}^{t,\boldsymbol{x}}_s)\, ds + \sum_{k=1}^{m} \boldsymbol{b}^{\boldsymbol{k}}(s, \boldsymbol{X}^{t,\boldsymbol{x}}_s)\, dW^k_s \tag{9.7.1}$$

for $s \in [t, T]$ with initial value $\boldsymbol{X}^{t,x}_t = \boldsymbol{x} \in \Re^d$ at time $t \in [0, T]$, see (7.8.1)–(7.8.4). The process $\boldsymbol{W} = \{\boldsymbol{W}_t = (W^1_t, \ldots, W^m_t)^\top,\ t \in [0, T]\}$ is assumed to represent an m-dimensional standard Wiener process on the filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$. One can show, similarly as in the proof of Theorem 7.8.2, that under appropriate assumptions, which will be described below, the process $\boldsymbol{X}^{t,\boldsymbol{x}}$ is a diffusion process with drift coefficient $\boldsymbol{a}(\cdot, \cdot)$ and diffusion coefficients $\boldsymbol{b}^{\boldsymbol{k}}(\cdot, \cdot)$, $k \in \{1, 2, \ldots, m\}$. In general, $\boldsymbol{a} = (a^1, \ldots, a^d)^\top$ and $\boldsymbol{b}^{\boldsymbol{k}} = (b^{1,k}, \ldots, b^{d,k})^\top$, $k \in \{1, 2, \ldots, m\}$ represent vector valued functions on $[0, T] \times \Re^d$ into $\Re^d$, such that a pathwise unique solution of the SDE (9.7.1) exists. Usually, the components of the SDE (9.7.1) are the factors in a financial market model.

Terminal Payoff Function (*)

Let us describe the case for a European option, where we have a terminal payoff $H(\boldsymbol{X}^{t,\boldsymbol{x}}_T)$ at the maturity date T with some given payoff function $H : \Re^d \to [0, \infty)$ such that

$$E(|H(\boldsymbol{X}^{t,\boldsymbol{x}}_T)|) < \infty. \tag{9.7.2}$$

We can then introduce the pricing function $u : [0, T] \times \Re^d \to [0, \infty)$

$$u(t, \boldsymbol{x}) = E\left(H(\boldsymbol{X}^{t,\boldsymbol{x}}_T) \,\middle|\, \mathcal{A}_t\right) \tag{9.7.3}$$

for $(t, \boldsymbol{x}) \in [0, T] \times \Re^d$. The *Feynman-Kac formula* for this payoff structure refers to the fact that under sufficient regularity of $\boldsymbol{a}, \boldsymbol{b}^{\boldsymbol{1}}, \ldots, \boldsymbol{b}^{\boldsymbol{m}}$ and H the function $u : (0, T) \times \Re^d \to [0, \infty)$ satisfies the PDE

$$\begin{aligned} L^0\, u(t, \boldsymbol{x}) &= \frac{\partial u(t, \boldsymbol{x})}{\partial t} + \sum_{i=1}^{d} a^i(t, \boldsymbol{x}) \frac{\partial u(t, \boldsymbol{x})}{\partial x^i} \\ &\quad + \frac{1}{2} \sum_{i,k=1}^{d} \sum_{j=1}^{m} b^{i,j}(t, \boldsymbol{x})\, b^{k,j}(t, \boldsymbol{x}) \frac{\partial^2 u(t, \boldsymbol{x})}{\partial x^i \partial x^k} \\ &= 0 \end{aligned} \tag{9.7.4}$$

for $(t, \boldsymbol{x}) \in (0, T) \times \Re^d$ with terminal condition

$$u(T, \boldsymbol{x}) = H(\boldsymbol{x}) \tag{9.7.5}$$

for $\boldsymbol{x} \in \Re^d$. This type of European payoff will be covered by a general version of the Feynman-Kac formula that we present later in this section. For instance, it can be applied to determine the discounted pricing function for risk neutral pricing with zero interest rate when the expectation is taken for the discounted payoff with respect to the equivalent risk neutral probability measure. Under the real world pricing of the benchmark approach the above version of the Feynman-Kac formula would allow the calculation of the benchmarked pricing function under the real world probability measure.

Discounted Payoff Function (*)

Let us now generalize the above payoff function by discounting it with a given *discount rate process* r, which is obtained as a function of the given vector diffusion process $\boldsymbol{X}^{t,\boldsymbol{x}}$, that is $r : [0, T] \times \Re^d \to \Re$. For instance, in a risk neutral setting the discount rate is given by the short term interest rate.

Over the period $[t, T]$ we obtain for the *discounted payoff*

$$\exp\left\{-\int_t^T r(s, \boldsymbol{X}_s^{t,\boldsymbol{x}})\, ds\right\} H(\boldsymbol{X}_T^{t,\boldsymbol{x}})$$

the pricing function

$$u(t, \boldsymbol{x}) = E\left(\exp\left\{-\int_t^T r(s, \boldsymbol{X}_s^{t,\boldsymbol{x}})\, ds\right\} H(\boldsymbol{X}_T^{t,\boldsymbol{x}}) \,\Big|\, \mathcal{A}_t\right) \tag{9.7.6}$$

for $(t, \boldsymbol{x}) \in [0, T] \times \Re^d$. Under conditions that we shall specify below, it follows that the pricing function u satisfies the PDE

$$L^0\, u(t, \boldsymbol{x}) = r(t, \boldsymbol{x})\, u(t, \boldsymbol{x}) \tag{9.7.7}$$

for $(t, \boldsymbol{x}) \in (0, T) \times \Re^d$ with terminal condition

$$u(T, \boldsymbol{x}) = H(\boldsymbol{x}) \tag{9.7.8}$$

for $\boldsymbol{x} \in \Re^d$, where the PDE operator L^0 is given in (9.7.4). Also this version of the Feynman-Kac formula is covered by a more general result that follows later.

Terminal Payoff and Payoff Rate (*)

Now, we add to the above discounted payoff structure some payoff stream, which continuously pays with a *payoff rate* $g : [0, T] \times \Re^d \to [0, \infty)$ some amount per unit of time. This can model, for instance, an income stream in a company, continuous dividend payments for a share or continuous interest payments. The corresponding *discounted payoff with payoff rate* is then at time $t \in [0,T]$ of the form

$$\exp\left\{-\int_t^T r(s, \boldsymbol{X}_s^{t,\boldsymbol{x}})\,ds\right\} H(\boldsymbol{X}_T^{t,\boldsymbol{x}}) + \int_t^T \exp\left\{-\int_t^s r(z, \boldsymbol{X}_z^{t,\boldsymbol{x}})\,dz\right\} g(s, \boldsymbol{X}_s^{t,\boldsymbol{x}})\,ds.$$

This leads to the pricing function

$$u(t,\boldsymbol{x}) = E\left(\exp\left\{-\int_t^T r(s, \boldsymbol{X}_s^{t,\boldsymbol{x}})\,ds\right\} H(\boldsymbol{X}_T^{t,\boldsymbol{x}}) + \int_t^T \exp\left\{-\int_t^s r(z, \boldsymbol{X}_z^{t,\boldsymbol{x}})\,dz\right\} g(s, \boldsymbol{X}_s^{t,\boldsymbol{x}})\,ds \,\Bigg|\, \mathcal{A}_t\right) \tag{9.7.9}$$

for $(t,\boldsymbol{x}) \in [0,T] \times \Re^d$. As we show below, this pricing function satisfies the PDE

$$L^0\, u(t,\boldsymbol{x}) + g(t,\boldsymbol{x}) = r(t,\boldsymbol{x})\, u(t,\boldsymbol{x}) \tag{9.7.10}$$

for $(t,\boldsymbol{x}) \in (0,T) \times \Re^d$ with terminal condition

$$u(T,\boldsymbol{x}) = H(\boldsymbol{x}) \tag{9.7.11}$$

for $\boldsymbol{x} \in \Re^d$.

SDE with Jumps (*)

We consider now jump diffusions. Let Γ denote an open connected subset of $\Re^d$ and $T \in (0,\infty)$ a fixed time horizon. We consider for a d-dimensional process $\boldsymbol{X}^{t,\boldsymbol{x}} = \{\boldsymbol{X}_s^{t,\boldsymbol{x}},\ s \in [t,T]\}$, see (6.4.19), the vector SDE

$$d\boldsymbol{X}_s^{t,\boldsymbol{x}} = \boldsymbol{a}(s, \boldsymbol{X}_s^{t,\boldsymbol{x}})\,ds + \sum_{k=1}^m \boldsymbol{b}^{\boldsymbol{k}}(s, \boldsymbol{X}_s^{t,\boldsymbol{x}})\,dW_s^k + \sum_{j=1}^{\ell} \int_{\mathcal{E}} \boldsymbol{c}^{\boldsymbol{j}}(v, s-, \boldsymbol{X}_{s-}^{t,\boldsymbol{x}})\, p_{\varphi_j}^j(dv, ds) \tag{9.7.12}$$

for $t \in [0,T]$, $s \in [t,T]$ and $\boldsymbol{x} \in \Gamma$ with value

$$\boldsymbol{X}_t^{t,\boldsymbol{x}} = \boldsymbol{x} \tag{9.7.13}$$

at time t, see (7.6.23). Here $\boldsymbol{W} = \{\boldsymbol{W}_t = (W_t^1, \ldots, W_t^m)^\top,\ t \in [0,T]\}$ is again an m-dimensional standard Wiener process on a filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$ as introduced in Sect. 5.1. Furthermore, $p_{\varphi_j}^j(\cdot,\cdot)$ denotes a Poisson measure, $j \in \{1,2,\ldots,\ell\}$, as introduced in Sect. 3.5, satisfying condition (3.5.14). Here $\boldsymbol{a} = (a^1, \ldots, a^d)^\top$ and $\boldsymbol{b}^{\boldsymbol{k}} = (b^{1,k}, \ldots, b^{d,k})^\top$, $k \in \{1,2,\ldots,m\}$, are vector valued functions from $[0,T] \times \Gamma$ into $\Re^d$ and $\boldsymbol{c}^{\boldsymbol{j}} = (c^{1,j}, \ldots, c^{d,j})^\top$, $j \in \{1,2,\ldots,\ell\}$, is a vector valued function on $\mathcal{E} \times [0,T] \times \Gamma$, $\mathcal{E} = \Re \backslash \{0\}$.

Feynman-Kac Formula with Jumps (*)

For the above payoff structure with discounted terminal payoff and a given payoff rate, we can form the pricing function

$$u(t,\boldsymbol{x}) = E\Bigg(\exp\left\{-\int_t^T r(s,\boldsymbol{X}_s^{t,\boldsymbol{x}})\,ds\right\} H(\boldsymbol{X}_T^{t,\boldsymbol{x}}) \\ + \int_t^T \exp\left\{-\int_t^s r(z,\boldsymbol{X}_z^{t,\boldsymbol{x}})\,dz\right\} g(s,\boldsymbol{X}_s^{t,\boldsymbol{x}})\,ds \,\Bigg|\, \mathcal{A}_t\Bigg) \quad (9.7.14)$$

for $t \in [0,T] \times \Re^d$. It turns out under appropriate conditions, as will be described below, that u satisfies the *partial integro differential equation* (PIDE)

$$L^0\, u(t,\boldsymbol{x}) + g(t,\boldsymbol{x}) = r(t,\boldsymbol{x})\, u(t,\boldsymbol{x}) \quad (9.7.15)$$

for $(t,\boldsymbol{x}) \in (0,T)$ with terminal condition

$$u(T,\boldsymbol{x}) = H(\boldsymbol{x}) \quad (9.7.16)$$

for $\boldsymbol{x} \in \Re^d$. Here the operator L^0 is given in the form

$$L^0 u(t,\boldsymbol{x}) = \sum_{i=1}^d a^i(t,\boldsymbol{x})\,\frac{\partial u(t,\boldsymbol{x})}{\partial x^i} + \frac{1}{2}\sum_{i,k=1}^d \sum_{j=1}^m b^{i,j}(t,\boldsymbol{x})\, b^{k,j}(t,\boldsymbol{x})\,\frac{\partial^2 u(t,\boldsymbol{x})}{\partial x^i\,\partial x^k} \\ + \frac{\partial u(t,\boldsymbol{x})}{\partial t} + \sum_{j=1}^{\ell}\int_{\mathcal{E}} \Big[u(s, x^1 + c^{1,j}(v,s,\boldsymbol{x}),\ldots,x^d + c^{d,j}(v,s,\boldsymbol{x})) \\ - u(s,x^1,\ldots,x^d)\Big]\ \varphi_j(dv), \quad (9.7.17)$$

where we abuse slightly the notation by writing $u(s,(x^1,\ldots,x^d)^\top) = u(s,x^1,\ldots,x^d)$. Note that an extra integral term is generated by the jumps as a consequence of the Itô formula with jumps, see (6.4.11) and (6.4.20).

Functional with First Exit Time (*)

Assume that there is a, so-called, *continuation region* Φ, which is an open connected subset of $[0,T] \times \Gamma$. We continue to receive payments as long as the process $\boldsymbol{X}^{t,\boldsymbol{x}}$ stays in the continuation region in Φ. For instance, in the case of a, so-called, *knock-out-barrier option* this would mean that $\boldsymbol{X}_s^{t,\boldsymbol{x}}$ has to stay below a given critical barrier to receive the terminal payment. Then we define the *first exit time* τ_Φ^t from Φ after t as

$$\tau_\Phi^t = \inf\{s \in [t,T] :\ (s,\boldsymbol{X}_s^{t,\boldsymbol{x}}) \notin \Phi\}, \quad (9.7.18)$$

which is a stopping time, see (5.1.13).

To characterize a general payoff structure we use a *terminal payoff function* $H: (0,T]\times\Gamma \to [0,\infty)$ for payments at time τ^t_Φ, a *payoff rate* $g: [0,T]\times\Gamma \to [0,\infty)$ for incremental payments during the time period $[t,\tau^t_\Phi)$ and a *discount rate* $r: [0,T]\times\Gamma \to \Re$. These quantities are all assumed to be measurable functions. Assume that the process $\boldsymbol{X}^{t,\boldsymbol{x}}$ does not explode or leave Γ before time T. We then define the *pricing function* $u: \Phi \to [0,\infty)$ by

$$u(t,\boldsymbol{x}) = E\Bigg(H(\tau^t_\Phi, \boldsymbol{X}^{t,\boldsymbol{x}}_{\tau^t_\Phi}) \exp\left\{ -\int_t^{\tau^t_\Phi} r(s,\boldsymbol{X}^{t,\boldsymbol{x}}_s)\, ds \right\} + \int_t^{\tau^t_\Phi} g(s,\boldsymbol{X}^{t,\boldsymbol{x}}_s) \exp\left\{ -\int_t^s r(z,\boldsymbol{X}^{t,\boldsymbol{x}}_u)\, dz \right\} ds \,\Bigg|\, \mathcal{A}_t \Bigg) \quad (9.7.19)$$

for $(t,\boldsymbol{x}) \in \Phi$.

General Feynman-Kac Formula (*)

For the formulation of the PIDE for the function u we use the operator L^0 given in (9.7.17). Under sufficient regularity of Φ, $\boldsymbol{a}$, $\boldsymbol{b}^{\boldsymbol{1}}$, ..., $\boldsymbol{b}^{\boldsymbol{m}}$, $\boldsymbol{c}^{\boldsymbol{1}},\ldots,\boldsymbol{c}^{\boldsymbol{\ell}}$, H, g, φ_1, ..., φ_ℓ and r one can show by application of the Itô formula (6.4.11) that the pricing function u satisfies the PIDE

$$L^0 u(t,\boldsymbol{x}) + g(t,\boldsymbol{x}) = r(t,\boldsymbol{x})\, u(t,\boldsymbol{x}) \quad (9.7.20)$$

for $(t,\boldsymbol{x}) \in \Phi$ with boundary condition

$$u(t,\boldsymbol{x}) = H(t,\boldsymbol{x}) \quad (9.7.21)$$

for $(t,\boldsymbol{x}) \in ((0,T]\times\Gamma)\backslash\Phi$. This result links the functional (9.7.19) to the PIDE (9.7.20)–(9.7.21) and can again be called a Feynman-Kac formula.

The above Feynman-Kac formula also holds for a partly negative terminal payoff function H and payoff rate g. One can split these payoffs into their negative and positive parts, where each can be separately handled by the above result. The Feynman-Kac formula can be conveniently derived by application of the Itô formula (6.4.20). Due to the complexity of boundary conditions that one has to deal with, such a derivation is useful, in principle, only for particular classes of asset price models and functionals. Therefore, we do not state here an extremely general and, consequently, very technical theorem that formulates a fully general Feynman-Kac formula for SDEs with jump component. However, it is clear that under similar conditions, as we formulate for the already rather general case below, that one obtains the Feynman-Kac formula also in the case with jumps by using the smoothness of the PIDE solution, the Itô formula and the martingale property of the resulting functional.

Conditions for the Feynman-Kac Formula (*)

For the case $\Phi = (0,T) \times \Gamma$ and assuming no jumps, that is $\boldsymbol{c}^{\boldsymbol{1}} = \ldots = \boldsymbol{c}^{\boldsymbol{\ell}} = 0$ and $\tau^t_\Phi = T$, let us now formulate some technical conditions that ensure that the Feynman-Kac formula holds.

(A) The drift coefficient $\boldsymbol{a}$ and diffusion coefficients $\boldsymbol{b}^{\boldsymbol{k}}$, $k \in \{1, 2, \ldots, m\}$, are assumed to be on $[0,T] \times \Gamma$ locally Lipschitz-continuous in $\boldsymbol{x}$, uniformly in t. That is, for each compact subset Γ^1 of Γ there exists a constant $K_{\Gamma^1} < \infty$ such that

$$|\boldsymbol{a}(t,\boldsymbol{x}) - \boldsymbol{a}(t,\boldsymbol{y})| + \sum_{k=1}^{m} |\boldsymbol{b}^{\boldsymbol{k}}(t,\boldsymbol{x}) - \boldsymbol{b}^{\boldsymbol{k}}(t,\boldsymbol{y})| \leq K_{\Gamma^1} \, |\boldsymbol{x} - \boldsymbol{y}| \tag{9.7.22}$$

for all $t \in [0,T]$ and $\boldsymbol{x}, \boldsymbol{y} \in \Gamma^1$.

(B) For all $(t,\boldsymbol{x}) \in [0,T) \times \Gamma$ the solution $\boldsymbol{X}^{t,\boldsymbol{x}}$ of (9.7.12) neither explodes nor leaves Γ before T, that is

$$P\left(\sup_{t \leq s \leq T} |\boldsymbol{X}^{t,\boldsymbol{x}}_s| < \infty\right) = 1 \tag{9.7.23}$$

and

$$P(\boldsymbol{X}^{t,\boldsymbol{x}}_s \in \Gamma \text{ for all } s \in [t,T]) = 1. \tag{9.7.24}$$

(C) There exists an increasing sequence $(\Gamma_n)_{n \in \mathcal{N}}$ of bounded, open and connected domains of Γ such that $\cup_{n=1}^{\infty} \Gamma_n = \Gamma$, and for each $n \in \mathcal{N}$ the PDE

$$L^0 u_n(t,\boldsymbol{x}) + g(t,\boldsymbol{x}) = r(t,\boldsymbol{x}) \, u_n(t,\boldsymbol{x}) \tag{9.7.25}$$

has a unique solution u_n, see Friedman (1975), on $(0,T) \times \Gamma_n$ with boundary condition

$$u_n(t,\boldsymbol{x}) = u(t,\boldsymbol{x}) \tag{9.7.26}$$

on $((0,T) \times \partial\Gamma_n) \cup (\{T\} \times \Gamma_n)$, where $\partial\Gamma_n$ denotes the boundary of Γ_n.

(D) The process $b^{i,k}(\cdot, \boldsymbol{X}_\cdot) \, \frac{\partial u(\cdot, \boldsymbol{X}_\cdot)}{\partial x^i}$ is from $\mathcal{L}^2_T$ for all $i \in \{1, 2, \ldots, d\}$ and $k \in \{1, 2, \ldots, m\}$.

For the following theorem, which is similar to a result in Heath & Schweizer (2000), we shall give a proof at the end of the section.

Theorem 9.7.1. *In the case without jumps under the conditions (A), (B), (C) and (D), the function u given by* (9.7.19) *is the unique solution of the PDE* (9.7.20) *with boundary condition* (9.7.21)*, where u is differentiable with respect to t and twice differentiable with respect to the components of $\boldsymbol{x}$.*

Condition (A) is satisfied if, for instance, $\boldsymbol{a}$ and $\boldsymbol{b} = (\boldsymbol{b}^{\boldsymbol{1}}, \ldots, \boldsymbol{b}^{\boldsymbol{m}})$ are differentiable in $\boldsymbol{x}$ on the open set $(0,T) \times \Gamma$ with derivatives that are continuous on $[0,T] \times \Gamma$.

To establish condition (B) one needs to exploit specific properties of the process $\boldsymbol{X}^{t,\boldsymbol{x}}$ given by the SDE (9.7.12).

Condition (C) can be shown to be implied by the following assumptions:

(C1) There exists an increasing sequence $(\Gamma_n)_{n\in\mathcal{N}}$ of bounded, open and connected subdomains of Γ with $\Gamma_n \cup \partial\Gamma_n \subset \Gamma$ such that $\cup_{n=1}^{\infty}\Gamma_n = \Gamma$, and each Γ_n has a twice differentiable boundary $\partial\Gamma_n$.

(C2) For each $n \in \mathcal{N}$ the functions $\boldsymbol{a}$ and $\boldsymbol{b}\boldsymbol{b}^\top$ are uniformly Lipschitz-continuous on $[0,T] \times (\Gamma_n \cup \partial\Gamma_n)$.

(C3) For each $n \in \mathcal{N}$ the function $\boldsymbol{b}(t,\boldsymbol{x})\boldsymbol{b}(t,\boldsymbol{x})^\top$ is uniformly elliptic on $\Re^d$ for $(t,\boldsymbol{x}) \in [0,T] \times \Gamma_n$, that is there exists a $\delta_n > 0$ such that

$$\boldsymbol{y}^\top \boldsymbol{b}(t,\boldsymbol{x})\, \boldsymbol{b}(t,\boldsymbol{x})^\top \boldsymbol{y} \geq \delta_n\, |\boldsymbol{y}|^2 \tag{9.7.27}$$

for all $\boldsymbol{y} \in \Re^d$.

(C4) For each $n \in \mathcal{N}$ the functions r and g are uniformly *Hölder-continuous* on $[0,T]\times(\Gamma_n\cup\partial\Gamma_n)$, that is there exists a constant $\bar{K}_n$ and an exponent $q_n > 0$ such that

$$|r(t,\boldsymbol{x}) - r(t,\boldsymbol{y})| + |g(t,\boldsymbol{x}) - g(t,\boldsymbol{y})| \leq \bar{K}_n\, |\boldsymbol{x} - \boldsymbol{y}|^{q_n} \tag{9.7.28}$$

for $t \in [0,T]$ and $\boldsymbol{x},\boldsymbol{y} \in (\Gamma_n \cup \partial\Gamma_n)$.

(C5) For each $n \in \mathcal{N}$ the function u is finite and continuous on $([0,T] \times \partial\Gamma_n) \cup (\{T\} \times (\Gamma_n \cup \partial\Gamma_n))$.

Condition (D) is satisfied when

$$\int_0^T E\left(\left(b^{i,k}(t,\boldsymbol{X}_t)\,\frac{\partial u(t,\boldsymbol{X}_t)}{\partial x^i}\right)^2\right) dt < \infty$$

for all $i \in \{1,2,\ldots,d\}$ and $k \in \{1,2,\ldots,m\}$. This condition ensures that the process $u(\cdot,\boldsymbol{X}_\cdot)$ is a martingale and the PDE (9.7.20)–(9.7.21) has a unique solution.

On the Proof of Theorem 9.7.1 (*)

Let us now indicate the proof of Theorem 9.7.1. It follows from condition (A) that (9.7.12) has a unique solution up to an explosion time, see Theorem II.5.2 in Kunita (1984). Due to (B) this explosion time has to be greater than T almost surely so that the stochastic process $\boldsymbol{X}^{t,\boldsymbol{x}}$ is well defined on $[t,T]$. The expectation in (9.7.19) is then also well-defined with values in $[0,\infty)$ because H and g are nonnegative. Condition (C) implicitly contains the assumption that for all $n \in \mathcal{N}$ and $(t,\boldsymbol{x}) \in ((0,\infty) \times \partial\Gamma_n) \cup (\{T\} \times \Gamma_n)$ the function $u(t,\boldsymbol{x})$ is finite, that is $u(t,\boldsymbol{x}) < \infty$. For fixed $(t,\boldsymbol{x}) \in (0,T) \times \Gamma$ the condition (C) allows us then to find an $n \in \mathcal{N}$ such that $\boldsymbol{x} \in \Gamma_n$.

Let us denote by

$$\tau^t_{\Gamma_n} = \inf\{s \in [t,T] :\, \boldsymbol{X}^{t,\boldsymbol{x}}_s \notin \Gamma_n\} \tag{9.7.29}$$

the first exit time of $(s,\boldsymbol{X}^{t,\boldsymbol{x}}_s)$ from $[t,T) \times \Gamma_n$, see (9.7.18). Due to the continuity of $\boldsymbol{X}^{t,\boldsymbol{x}}$ it is

$$\left(\tau^t_{\Gamma_n}, \boldsymbol{X}^{t,\boldsymbol{x}}_{\tau^t_{\Gamma_n}}\right) \in ((0,T) \times \partial\Gamma_n) \cup (\{T\} \times \Gamma_n)$$

such that

$$u\left(\tau^t_{\Gamma_n}, \boldsymbol{X}^{t,\boldsymbol{x}}_{\tau^t_{\Gamma_n}}\right) < \infty.$$

We then have by application of the Itô formula (6.2.11) to u_n, conditions (9.7.25) and (9.7.26) that

$$u_n(t,\boldsymbol{x}) = E\left(u\left(\tau^t_{\Gamma_n}, \boldsymbol{X}^{t,\boldsymbol{x}}_{\tau^t_{\Gamma_n}}\right) \,\Big|\, \mathcal{A}_t\right), \tag{9.7.30}$$

where the appearing Itô integral is, due to the boundedness of Γ_n, an $(\underline{\mathcal{A}}, P)$-martingale.

Because of (A) and (B) it follows that $\boldsymbol{X}^{t,\boldsymbol{x}}$ is a strong Markov process, see Theorem IV.2.3 and the remark after Theorem IV.6.1 in Ikeda & Watanabe (1989). This means that the Markov property still holds when the present time is chosen to be a stopping time. These results are stated for $\boldsymbol{a}$ and $\boldsymbol{b}$ not depending on t and $\boldsymbol{x}$ from $\Re^d$, but the condition (B) allows us to replace $\Re^d$ by Γ. Then the results can be shown to hold for time dependent $\boldsymbol{a}$ and $\boldsymbol{b}$, as in Chap. 6 of Stroock & Varadhan (1982). Therefore, by the strong Markov property we obtain

$$E\Bigg(H(T,\boldsymbol{X}^{t,\boldsymbol{x}}_T)\,\exp\left\{-\int_t^T r(s,\boldsymbol{X}^{t,\boldsymbol{x}}_s)\,ds\right\}$$
$$-\int_t^T g(s,\boldsymbol{X}^{t,\boldsymbol{x}}_s)\,\exp\left\{-\int_t^s r(u,\boldsymbol{X}^{t,\boldsymbol{x}}_u)\,du\right\}ds\,\Bigg|\,\mathcal{A}_{\tau^t_{\Gamma_n}}\Bigg) = u\left(\tau^t_{\Gamma_n}, \boldsymbol{X}^{t,\boldsymbol{x}}_{\tau^t_{\Gamma_n}}\right)$$

and, thus, by (9.7.19) and (9.7.30)

$$u(t,\boldsymbol{x}) = E\left(u\left(\tau^t_{\Gamma_n}, \boldsymbol{X}^{t,\boldsymbol{x}}_{\tau^t_{\Gamma_n}}\right) \,\Big|\, \mathcal{A}_t\right) = u_n(t,\boldsymbol{x}).$$

Hence for all $n \in \mathcal{N}$ the functions u and u_n coincide on $(0,T)\times\Gamma_n$. This implies by (C) that u satisfies (9.7.20) on $(0,T)\times\Gamma$. From (9.7.12) and (9.7.19) we obtain then the boundary condition (9.7.21) and also the uniqueness of u, if we exploit the fact that $u(\cdot,\boldsymbol{X}_\cdot)$ is a martingale due to (D). □

9.8 Exercises for Chapter 9

9.1. Prove for the BS model with constant volatility $\sigma_t > 0$ appreciation rate a and short rate r that the domestic savings account discounted stock price process $\bar{S}$ is an $(\underline{\mathcal{A}}, P_\theta)$-martingale under the risk neutral probability measure P_θ.

9.2. Show that the discounted European call option price process for the BS model with constant parameters is a martingale under the risk neutral probability measure.

9.3. Formulate the SDE for the European put option price for the BS model with constant parameters under the risk neutral probability measure P_θ and under the original probability measure P.

9.4. Starting from the risk neutral SDE for the stock price verify that the benchmarked stock price for the BS model is an $(\underline{\mathcal{A}}, P)$-martingale.

9.5. Compute the European call option price as an expectation under the risk neutral probability measure for the BS model.

9.6. (*) Write down for the BS model the Itô SDE for the Radon-Nikodym derivative process of the risk neutral measure.

9.7. (*) Use under the BS model with constant interest rate r the zero coupon bond price $P(t,T)$ with maturity T as numeraire, $t \in [0,T]$. Describe the corresponding numeraire pair. What is the relationship of the resulting pricing measure with the risk neutral probability measure?

9.8. (*) Apply for the BS model the Feynman-Kac formula to compute the PDE for the price $V(t, S_t)$ at time t of the payoff $H(S_T) = S_T^2$ of the square of the underlying security at maturity T. Can you explicitly solve the corresponding PDE?