8

Introduction to Option Pricing

In the previous chapters we have prepared mathematical tools that allow us to model in continuous time the dynamics of financial securities, for instance, stocks. Now, we shall study prices of derived financial securities. A *derivative security*, for instance an option, is a financial instrument whose value is dependent upon the values of an underlying more fundamental security. In this chapter we give an introduction into derivatives, in particular, European options. For simplicity, we focus our discussion on options under the BS model. Furthermore, we introduce at the end of the chapter important results on squared Bessel processes because these will be crucial for the understanding of the following chapters.

8.1 Options

Options have been introduced to provide some optionality to the buyer or seller of a security. In the simplest case the holder of an *option* has the right but not the obligation to buy or sell an underlying security for an agreed price at a preset date. We discuss now options as a particular type of derivative to highlight important general features of derivative securities.

European Call Option

Let us denote by S_t, the price of a security at time $t \in [0, \infty)$, measured in units of the domestic currency. This can be, for instance, a stock index. We call $S = \{S_t,\, t \in [0, \infty)\}$ the price process of the underlying security. A *European call option* on an underlying security S gives the owner the *right to buy* the security at a preset *strike price* K at the *expiration date* $T \in (0, \infty)$. The price at time t for this right is the *European call option price* $c_{T,K}(t, S_t)$, which is paid when the option contract is entered at time t. Note that there is an initial payment at the time when the contract is signed. An *American option* has the same payoff function as a European option. However, the holder has the

E. Platen, D. Heath, *A Benchmark Approach to Quantitative Finance*,
Springer Finance,

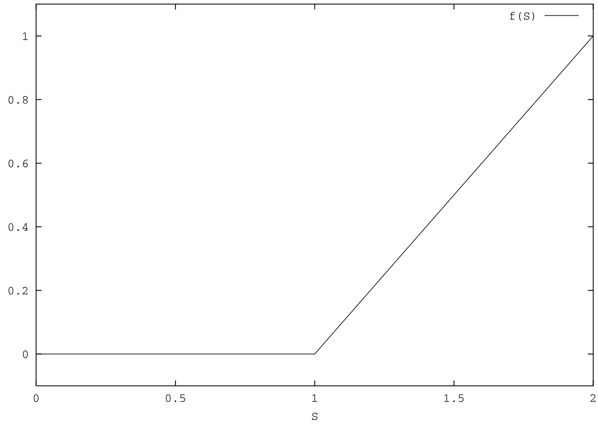

Fig. 8.1.1. Payoff function of a European call option for $K = 1$

right to exercise it at any time before the maturity date. Figure 8.1.1 shows the payoff function of a *call option*

$$H(S) = (S - K)^+ \tag{8.1.1}$$

with strike price $K = 1$, where we use the notation $a^+ = \max(0, a)$.

A European call option with expiration date $T \in (0, \infty)$ is at time $t \in [0, T]$ said to be *in-the-money*, *at-the-money* or *out-of-the-money*, if $S_t > K$, $S_t = K$ or $S_t < K$, respectively. The function

$$H(S_t) = (S_t - K)^+ \tag{8.1.2}$$

is called the *intrinsic value* of the call option at time $t \in [0, T]$.

As an example, consider a European call option at the beginning of 1995 on the S&P500 index, displayed in Fig. 3.1.1, with a strike price of $K = 400$ and expiration date at the end of 1995. Figure 3.1.1 shows that the S&P500 was at the end of 1995 approximately at \$500. This means that the value of the option was at the end of 1995 at a level of about \$100. We shall see from the theoretical pricing formulas presented in this chapter that the realized payoff of about \$100 would have considerably exceeded the original price of the option at the beginning of 1995. Of course, if the S&P500 stayed below \$400 during 1995, then the owner of the call option would have received nothing and would have lost the original option price that he or she paid when the option contract was written. This shows that there is substantial leverage involved when using options.

European Put Option

For market participants who aim to sell an underlying security at a future date, the purchase of a *European put option* might be of advantage. This

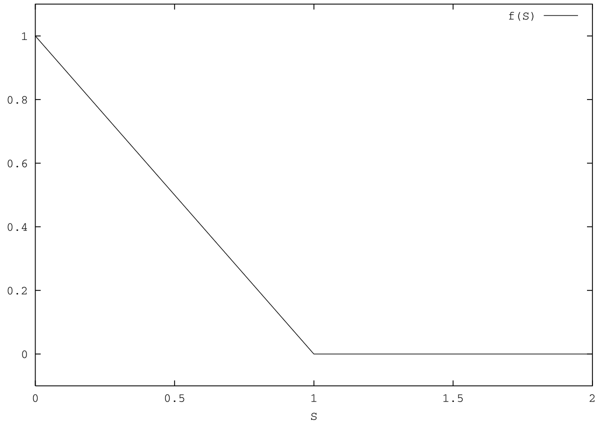

Fig. 8.1.2. Payoff function of a European put option, $K = 1$

financial contract is similar to the European call option but gives the holder the *right to sell* a security for a specified strike price K at an expiration date T. We denote the *European put option price* by $p_{T,K}(t, S_t)$. Figure 8.1.2 displays the payoff function of a European put option

$$H(S_T) = (K - S_T)^+ \tag{8.1.3}$$

with strike price $K = 1$. A European put option is at time $t \in [0, T]$ *in-the-money*, *at-the-money* or *out-of-the money* if $S_t < K$, $S_t = K$ or $S_t > K$, respectively. The quantity

$$H(S_t) = (K - S_t)^+ \tag{8.1.4}$$

is called the *intrinsic value* of a put option at time $t \in [0, T]$.

It is important to specify whether someone is the owner of an underlying security or derivative. A market participant is *long* in a security, if he or she is the owner of that security. On the other hand, one is *short* in a security if one borrows it, sells it and has the obligation of giving it back at a later date. Owning a negative unit of a security is therefore possible through the practice of *short-selling*.

Combinations of European Put and Call Options

To implement special hedging or speculative trading strategies it is common to form portfolios that consist of combinations of European call and put options. As an example, a *butterfly spread* is constructed by buying a call with strike price K_1, selling two calls with strike price $K_2 > K_1$ and buying another call with strike price $K_3 > K_2$. Figure 8.1.3 shows the resulting payoff function $H(S)$ of a butterfly spread with $K_1 = 0.6$, $K_2 = 1$, $K_3 = 1.4$. The butterfly spread has zero payoff outside the interval $[K_1, K_3]$. It allows to create at maturity a cash flow when the underlying security is near the strike price.

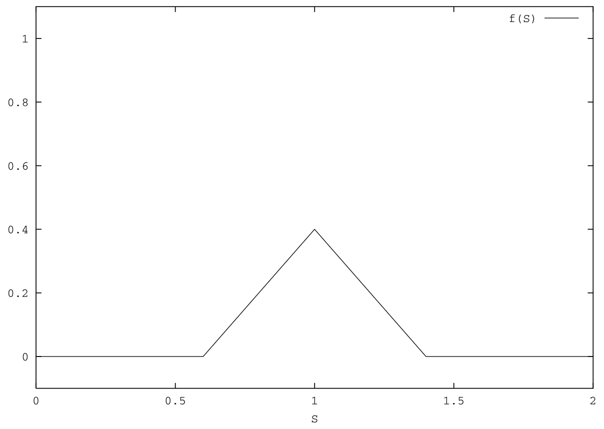

Fig. 8.1.3. Payoff of a butterfly spread

Theoretically one can approximate almost any reasonable payoff function at a given expiration date by portfolios of European calls and puts because a corresponding portfolio of butterfly spreads can concentrate a desired payoff close to each possible value of the underlying security.

Options

More generally, we call a derivative a *European option* if it gives the right to realize a given payoff according to a given function $H : [0, \infty) \to \Re$ of the underlying S_T at a specified expiration date $T \in [0, \infty)$. If the payoff can be exercised on or before the expiration date, then the contract is called an *American option*. The call and put options, introduced previously, are examples of European options. An American option is, in general, more expensive than a corresponding European option because it provides additionally the right to exercise early. One can show that the price of an American call option on an underlying security that pays no dividend is the same as its European counterpart.

In the following we denote by $V(t, S_t)$ the *value* at time $t \in [0, T]$ of a European option with payoff function H and maturity date $T \in [0, \infty)$. Here H has to fulfill some integrability condition which we do not specify at this stage. The *pricing function* $V : [0, T) \times [0, \infty) \to \Re$ for a European option can, in general, be shown to be differentiable with respect to time and twice differentiable with respect to the underlying security. This smoothness property will be exploited later for its computation. The efficient evaluation of this function is of importance both for the pricing and the hedging of these contracts. We shall show later that in certain cases explicit pricing formulas are available. However, in general, one needs to apply numerical methods.

8.2 Options under the Black-Scholes Model

We now consider options for the particular dynamics of the Black-Scholes (BS) model for the underlying security.

Black-Scholes Model

For simplicity, let us use the BS model, see Sect. 7.5, as a description for the dynamics of the underlying security. It has been established historically as the *standard market model* for option pricing, see Black & Scholes (1973). This model supposes that the underlying security price $S = \{S_t,\, t \in [0,T]\}$ follows a geometric Brownian motion, see (6.3.6), with time dependent, deterministic appreciation rate $a = \{a_t,\, t \in [0,T]\}$ and strictly positive, deterministic volatility $\sigma = \{\sigma_t,\, t \in [0,T]\}$, that is

$$dS_t = a_t\, S_t\, dt + \sigma_t\, S_t\, dW_t \tag{8.2.1}$$

for $t \in [0,T]$ with given initial value $S_0 > 0$. Here W denotes a standard Wiener process $W = \{W_t,\, t \in [0,T]\}$. Furthermore, there is a domestic savings account $B = \{B_t,\, t \in [0,T]\}$, which accrues the deterministic interest $r = \{r_t,\, t \in [0,T]\}$. We assume

$$dB_t = r_t\, B_t\, dt \tag{8.2.2}$$

for $t \in [0,T]$ with initial value

$$B_0 = 1. \tag{8.2.3}$$

The domestic savings account is also called the *locally riskless* asset since there is no noise term in its differential equation (8.2.2). Typically, in the standard BS model one sets the volatility σ, the appreciation rate a and the short rate r to be constant, which yields the basic model for option pricing. In the following analysis we typically allow these parameters to be time dependent. We shall later show in Sect. 10.6 that the savings account can be defined more precisely as a limit of a roll-over short term bond account.

Hedge Portfolio

For the following let us fix the maturity date at T. From the practical point of view it is most important to realize that the writer of a European option can *replicate* the payoff $H(S_T)$ at the expiration date T. To achieve this, a *hedge portfolio* has to be established, which consists at time t of δ_t^1 units of the underlying security S_t and δ_t^0 units of the domestic savings account B_t. At time $t \in [0,T]$ the value of this portfolio is then set to the value $V(t, S_t)$ of the option. That is, the hedge portfolio has the value

$$V(t, S_t) = \delta_t^0\, B_t + \delta_t^1\, S_t \tag{8.2.4}$$

at time $t \in [0,T]$. By the Itô formula (6.4.11) we obtain

$$dV(t,S_t) = \delta^0_t \, dB_t + \delta^1_t \, dS_t + B_t \, d\delta^0_t + S_t \, d\delta^1_t + d[\delta^1, S]_t \tag{8.2.5}$$

at time $t \in [0,T]$.

Self-Financing Portfolios

We assume that the hedge portfolio is *self-financing*. This means that all changes in the value of the portfolio are caused by gains from trade, that is, by changes in the savings account B and the underlying security S. We can express the self-financing property of the portfolio $V(t,S_t)$ in differential form by assuming the SDE

$$dV(t,S_t) = \delta^0_t \, dB_t + \delta^1_t \, dS_t \tag{8.2.6}$$

for $t \in [0,T]$. Note that by (8.2.6) and (8.2.5) for the above hedge portfolio to be self-financing we have to satisfy the condition

$$B_t \, d\delta^0_t + S_t \, d\delta^1_t + d[\delta^1, S]_t = 0 \tag{8.2.7}$$

for all time $t \in [0,T]$.

We call the process $\delta = \{\delta_t = (\delta^0_t, \delta^1_t)^\top, \; t \in [0,T]\}$ a *self-financing strategy* if $\delta^0 = \{\delta^0_t, \; t \in [0,T]\}$ and $\delta^1 = \{\delta^1_t, \; t \in [0,T]\}$ are predictable processes and both are such that the hedge portfolio, whose value is given in (8.2.4), satisfies (8.2.6). We say that the hedge portfolio *replicates* the payoff $H(S_T)$ at the expiration date T, if

$$V(T,S_T) = H(S_T). \tag{8.2.8}$$

Furthermore, we need to assume the existence of the involved gains from trade or, equivalently, the corresponding Itô integrals. For our setup it is sufficient to assume that $\delta^1(\cdot)\sigma(\cdot)S(\cdot)$, $\sqrt{\delta^1(\cdot)\, a(\cdot)\, S(\cdot)}$ and $\sqrt{|\delta^0(\cdot)\, r(\cdot)\, B(\cdot)|}$ are in $\mathcal{L}^2_T$, see (5.4.1). Note however, for other models one may require weaker integrability conditions. Without further mentioning, we consider in the following only self-financing portfolios and strategies and omit the phrase self-financing.

We allow the hedge portfolio to be rebalanced continuously. Furthermore, we assume, for simplicity, that there are no additional costs, such as transaction costs, involved in hedging. One typically characterizes this setup as *continuous hedging* in a *frictionless market*.

Discounted Value Function

To identify in a simple way an appropriate hedging strategy it is convenient to consider the corresponding *discounted value function* $\bar{V} : [0,T] \times [0,\infty) \to [0,\infty)$ given by

$$\bar{V}(t, \bar{S}_t) = \frac{V(t, S_t)}{B_t} \tag{8.2.9}$$

and the *discounted underlying security*

$$\bar{S}_t = \frac{S_t}{B_t} \tag{8.2.10}$$

for $t \in [0, T]$. By the Itô formula (6.2.11) we obtain from (8.2.1) and (8.2.2) the SDE

$$d\bar{S}_t = (a_t - r_t)\, \bar{S}_t\, dt + \sigma_t\, \bar{S}_t\, dW_t \tag{8.2.11}$$

for $t \in [0, T]$ with $\bar{S}_0 = S_0$. By discounting with the savings account one is taking the time value of money into account. This is extremely important for an investor who always can invest into the locally riskless asset, the savings account B. In this sense it is understandable when investors prefer to denominate a security in units of the savings account instead of denominating it in units of the currency.

Profit and Loss Process

A hedger who has an option in her or his trading book faces at time t a *profit and loss* (P&L) that is denoted by C_t for $t \in [0, T]$. The ultimate goal of the hedger is to achieve zero P&L throughout the hedge. Then the selling of options and hedging these becomes ideally a riskless business.

To take for the P&L the time value of money into account, we consider the *discounted profit and loss*

$$\bar{C}_t = \frac{C_t}{B_t} \tag{8.2.12}$$

at time t. For a given strategy δ the discounted P&L $\bar{C}_t$ at time $t \in [0, T]$ is obtained as the corresponding discounted value of the hedge portfolio minus the discounted gains from trade and minus the initial value of the discounted portfolio. It can be written in the form

$$\bar{C}_t = \bar{V}(t, \bar{S}_t) - I_{\delta^1, \bar{S}}(t) - \bar{V}(0, \bar{S}_0) \tag{8.2.13}$$

for $t \in [0, T]$. Here we use the gains from trade $I_{\delta^1, \bar{S}}$, see (5.3.11), with respect to the discounted security $\bar{S}$, which according to (8.2.11) is of the form

$$I_{\delta^1, \bar{S}}(t) = \int_0^t \delta^1_u\, d\bar{S}_u = \int_0^t \delta^1_u\, (a_u - r_u)\, \bar{S}_u\, du + \int_0^t \delta^1_u\, \sigma_u\, \bar{S}_u\, dW_u \tag{8.2.14}$$

for $t \in [0, T]$. Obviously, with respect to the constant discounted domestic savings account

$$\bar{B}_t = 1 \tag{8.2.15}$$

there is zero gains from trade $I_{\delta^0, \bar{B}}(t) = 0$ for $t \in [0, T]$.

When the option contract is established at time $t = 0$, then the hedger receives from the buyer of the option the payment $V(0, S_0)$. This is equivalent to the discounted value

$$\bar{V}(0, \bar{S}_0) = \frac{V(0, S_0)}{B_0},$$

see (8.2.9) and (8.2.3). Thus, we have according to (8.2.13) and (8.2.14) zero initial discounted P&L

$$\bar{C}_0 = 0. \tag{8.2.16}$$

The discounted P&L $\bar{C}_t$ is then the actual discounted portfolio value that a hedger holds at time t.

No-Arbitrage for P&L Process

Now, let us discuss some notion of arbitrage, which is fundamental for the modeling of financial markets. If a market participant is able to generate by her or his nonnegative total portfolio of investable securities some strictly positive wealth out of nothing, then this is interpreted as *arbitrage*. Any reasonable financial market model should avoid the modeling of arbitrage. We shall introduce a precise definition of arbitrage later in Sect. 10.2. At the present introductory level we call it an arbitrage if the market model allows to form a nonnegative portfolio that starts at zero and attains with strictly positive probability a strictly positive value at some later date. The nonnegativity of the portfolio reflects the *limited liability* of each investor for her or his total portfolio of investable wealth.

By excluding arbitrage a hedger can run a nonnegative hedge book with zero total initial value only such that its value remains always zero. This means that the P&L process of this business starts at zero and remains at zero all the time. Therefore, we aim to identify under no arbitrage a hedging strategy δ for which the discounted P&L remains zero, that is,

$$\bar{C}_t = 0 \tag{8.2.17}$$

for all $t \in [0, T]$. We call this a *perfect hedge* and the corresponding hedge portfolio $V = \{V(t, S_t),\, t \in [0, T]\}$ that returns the payoff at maturity T is then a replicating portfolio.

Discounted P&L Increments

We shall now demonstrate how an appropriate hedging strategy δ can be constructed. For this purpose we examine the increments of the discounted P&L process $\bar{C}$. With the definition of the discounted P&L given in (8.2.13) its increments can be expressed in the form

$$\bar{C}_t - \bar{C}_s = \bar{V}(t, \bar{S}_t) - \bar{V}(s, \bar{S}_s) - \int_s^t \delta_u^1 \, d\bar{S}_u \tag{8.2.18}$$

for $t \in [0,T]$ and $s \in [0,t]$. Assuming that the discounted pricing function $\bar{V}(\cdot,\cdot)$ is differentiable with respect to time and twice differentiable with respect to the discounted underlying security value, the Itô formula (6.2.11) can be applied and we obtain from (8.2.18) the relation

$$\bar{C}_t - \bar{C}_s = \int_s^t \left[\frac{\partial \bar{V}(u,\bar{S}_u)}{\partial u} + \frac{1}{2}\,\sigma_u^2\,\bar{S}_u^2\,\frac{\partial^2 \bar{V}(u,\bar{S}_u)}{\partial \bar{S}^2} + (a_u - r_u)\,\bar{S}_u \left(\frac{\partial \bar{V}(u,\bar{S}_u)}{\partial \bar{S}} - \delta_u^1 \right) \right] du + \int_s^t \sigma_u\,\bar{S}_u \left(\frac{\partial \bar{V}(u,\bar{S}_u)}{\partial \bar{S}} - \delta_u^1 \right) dW_u \tag{8.2.19}$$

for $t \in [0,T)$, $s \in [0,t]$. The formula (8.2.19) provides an explicit representation for the increments of the discounted P&L.

Discounted Black-Scholes PDE

Note that a strategy δ that minimizes the fluctuations of the discounted P&L process $\bar{C}$ is obtained if the second integral on the right hand side of (8.2.19) vanishes for the choice of the *hedge ratio* δ_t^1 given by

$$\delta_t^1 = \frac{\partial \bar{V}(t,\bar{S}_t)}{\partial \bar{S}} \tag{8.2.20}$$

for $t \in [0,T)$. It can be seen that when taking (8.2.20) into account, then the first term in (8.2.19) disappears if the discounted value function $\bar{V}$ satisfies the PDE

$$\frac{\partial \bar{V}(t,\bar{S})}{\partial t} + \frac{1}{2}\,\sigma_t^2\,\bar{S}^2\,\frac{\partial^2 \bar{V}(t,\bar{S})}{\partial \bar{S}^2} = 0 \tag{8.2.21}$$

for $t \in [0,T)$ and $\bar{S} \in (0,\infty)$. The resulting PDE (8.2.21) is not sufficient to determine fully the function $\bar{V}(\cdot,\cdot)$. However, it would keep by (8.2.19) and (8.2.20) any discounted P&L constant. Additionally, some condition at the terminal time T needs to be specified to make sure that we start from a zero initial discounted P&L. To ensure this and, thus, the replication of the payoff at the expiration date T, see (8.2.8) and (8.2.9), we have to satisfy the *terminal condition*

$$\bar{V}(T,\bar{S}) = \frac{H(\bar{S}\,B_T)}{B_T} = \frac{H(S)}{B_T} \tag{8.2.22}$$

for $\bar{S} \in (0,\infty)$. We call the PDE (8.2.21) together with its terminal condition (8.2.22) the *discounted Black-Scholes partial differential equation* (discounted BS-PDE). This PDE determines a discounted pricing function $\bar{V}(\cdot,\cdot)$ that allows a perfect hedge for the corresponding European payoff.

For instance, for European call and put options it can be shown that the discounted BS-PDE has a unique solution and, thus, determines uniquely the option price. The uniqueness of the solution of a PDE in the above form is, in general, not trivially established, as we shall see in Chap. 12.

Black-Scholes PDE

By a transformation of variables, see (8.2.9) and (8.2.10), the above discounted BS-PDE can be rewritten for the undiscounted option pricing function $V(\cdot,\cdot)$ in the form

$$\frac{\partial V(t,S)}{\partial t} + r_t\, S\, \frac{\partial V(t,S)}{\partial S} + \frac{1}{2}\,\sigma_t^2\, S^2\, \frac{\partial^2 V(t,S)}{\partial S^2} - r_t\, V(t,S) = 0 \qquad (8.2.23)$$

for $t \in [0,T)$ and $S \in (0,\infty)$ with terminal condition, see (8.2.8),

$$V(T,S) = H(S) \qquad (8.2.24)$$

for $S \in (0,\infty)$. We call (8.2.23) together with (8.2.24) the BS-PDE. Note that the BS-PDE and the discounted BS-PDE do not depend on the values of the appreciation rate a_t of the underlying security. This is a remarkable fact, which results from the choice of δ_t^1 in (8.2.20) that eliminated in (8.2.19) any potential impact of a_t.

Option Price

In the formula (8.2.20) for the hedge ratio we describe the number δ_t^1 of units to be held in the underlying security. By dividing equation (8.2.4) on both sides by the savings account and using equations (8.2.10) and (8.2.9), we can now determine the number of units that needs to be held in the domestic savings account. It is given by the relation

$$\delta_t^0 = \delta_t^0\, \bar{B}_t = \bar{V}(t,\bar{S}_t) - \delta_t^1\, \bar{S}_t \qquad (8.2.25)$$

for $t \in [0,T]$.

The option price obtained at time t is, of course, just $V(t,S_t)$. The appropriate value of the hedge portfolio at time t in units of the domestic currency can, therefore, be calculated, see (8.2.9), via the formula

$$V(t,S_t) = \bar{V}(t,\bar{S}_t)\, B_t \qquad (8.2.26)$$

for $t \in [0,T]$.

Numeraire Invariance

Let us now check whether the above construction of a hedge portfolio identifies a self-financing strategy δ. As mentioned previously, this is a strategy that changes the portfolio value only through changes in gains from trade, see (8.2.6) and (8.2.7). For our discounted securities we have from (8.2.18) because of zero discounted P&L $\bar{C}_t = 0$ for all $t \in [0,T]$ that

$$d\bar{V}(t,\bar{S}_t) = \delta_t^1\, d\bar{S}_t \qquad (8.2.27)$$

for all $t \in [0, T]$. This means that the portfolio $\bar{V}(t, \bar{S}_t)$ is self-financing when denominated in units of the savings account, because all changes in $\bar{V}(t, \bar{S}_t)$ are due to changes in $\bar{S}_t$. It is now of interest that the portfolio is also shown to be self-financing when using other numeraires, for instance, if denominated in units of the domestic currency. For this case we multiply $\bar{V}(t, S_t)$ by the savings account B_t and obtain from (8.2.26) and (8.2.27) by the integration-by-parts formula (6.3.1) the SDE

$$\begin{aligned}
dV(t, S_t) &= d(\bar{V}(t, \bar{S}_t)\, B_t) \\
&= B_t\, d\bar{V}(t, \bar{S}_t) + \bar{V}(t, \bar{S}_t)\, dB_t + d[B_\cdot, \bar{V}(\cdot, \bar{S}_\cdot)]_t \\
&= B_t\, \delta^1_t\, d\bar{S}_t + (\delta^0_t + \delta^1_t\, \bar{S}_t)\, dB_t + \delta^1_t\, d[B, \bar{S}]_t \\
&= \delta^0_t\, dB_t + \delta^1_t\, \left(B_t\, d\bar{S}_t + \bar{S}_t\, dB_t + d[B, \bar{S}]_t\right) \\
&= \delta^0_t\, dB_t + \delta^1_t\, d(B_t\, \bar{S}_t) \\
&= \delta^0_t\, dB_t + \delta^1_t\, dS_t
\end{aligned} \tag{8.2.28}$$

for $t \in [0, T]$. This proves the condition (8.2.7), which ensures that the resulting portfolio is self-financing when expressed in units of the domestic currency.

Consequently, the changes in the portfolio value are only a result of gains from trade in the underlying security S and the savings account B. The above result in (8.2.28) is important, since it shows that a portfolio that is self-financing in one denomination is also self-financing in another denomination. Note that such a result holds more generally, as will be shown in (9.6.18) and towards the end of Chap. 14. This means that a change in numeraire does not impact on the self-financing property. We could select any strictly positive portfolio as numeraire and would see, similarly as above, that a portfolio, which is self-financing in one denomination is also self-financing under this numeraire.

The discounted P&L process $\bar{C}_t$ starts at zero, see (8.2.16), and has zero increments, see (8.2.19)–(8.2.21). Therefore, it is zero for the above identified hedging strategy. The undiscounted P&L process $C = \{C_t,\, t \in [0, T]\}$ with

$$C_t = \bar{C}_t\, B_t = 0 \tag{8.2.29}$$

for $t \in [0, T]$, see (8.2.12) and (8.2.17), equals then also zero. Consequently, the resulting nonnegative P&L process does not permit arbitrage, as was required.

The above hedging approach for determining the value of an option is essentially based on the Itô formula. This fundamental tool allows us to obtain in continuous time a perfect hedging strategy together with the corresponding option price. Note that no expectation has been taken to determine the option price.

We shall see later in Chap. 10 that the above approach for finding a perfect hedge and a corresponding price for a derivative security can be generalized to more complex payoff structures and more general asset price models. Certain PDEs, similar to those given in (8.2.21) and (8.2.22), arise also for other payoffs and security dynamics. What differs are the volatility specification and the boundary conditions.

8.3 The Black-Scholes Formula

In this section we study the solution of the BS-PDE (8.2.23) with its terminal condition (8.2.24) in the case of a European call option.

Black-Scholes Formula

Let us describe the price of a European call option for an underlying security $S = \{S_t,\, t \in [0,T]\}$ that follows the SDE (8.2.1). The payoff is according to (8.1.1) of the form

$$H(S) = (S - K)^+ \tag{8.3.1}$$

for $S \in (0, \infty)$ with strike price $K > 0$ and matures at the terminal date T.

In their Nobel prize winning work Black, Scholes and Merton provided the explicit description of the price $c_{T,K}(t, S_t)$ at time t for the European call option with expiry date T and strike price K, see Black & Scholes (1973) and Merton (1973b). This result is widely known as the *Black-Scholes formula* (BS formula). It takes the form

$$c_{T,K}(t, S_t) = S_t\, N(d_1(t)) - K\, \frac{B_t}{B_T}\, N(d_2(t)) \tag{8.3.2}$$

with

$$d_1(t) = \frac{\ln\left(\frac{S_t}{K}\right) + \int_t^T \left(r_s + \frac{1}{2}\,\sigma_s^2\right)\, ds}{\sqrt{\int_t^T \sigma_s^2\, ds}} \tag{8.3.3}$$

and

$$d_2(t) = d_1(t) - \sqrt{\int_t^T \sigma_s^2\, ds} \tag{8.3.4}$$

for $t \in [0, T)$. Here B_t is again the domestic savings account at time t, see (8.2.2). Furthermore, $N(\cdot)$ denotes the standard Gaussian distribution function, see (1.2.7), with density

$$N'(x) = \frac{1}{\sqrt{2\pi}}\, \exp\left\{-\frac{x^2}{2}\right\} \tag{8.3.5}$$

for all $x \in \Re$, see (1.2.8). It can be shown by direct calculation that the above European call option pricing function $c_{T,K}(\cdot,\cdot)$ solves the BS-PDE given in (8.2.23) for the payoff function (8.3.1), see Exercise 8.1. One observes in the

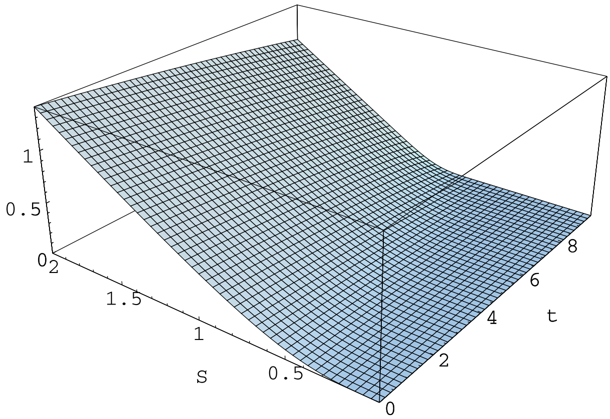

Fig. 8.3.1. Black-Scholes European call option price

BS formula that the option price does not depend on the specific choice of the appreciation rate a_t of the underlying security, which we explained earlier.

Noting the form of the BS formula (8.3.2), a heuristic guess for the number δ_t^1 of units of the risky asset to be held in the hedge portfolio would be $N(d_1(t))$. We show below that this is correct. However, this result is not as obvious as it may seem because $d_1(t)$ and $d_2(t)$ depend on S_t.

For small values of S_t, the expressions $d_1(t)$ and $d_2(t)$ and also $N(d_1(t))$ and $N(d_2(t))$ are small, see Fig. 1.2.4. Thus, for small S_t the European call option has almost no value. However, for large underlying security price S_t the quantities $d_1(t)$ and $d_2(t)$ are both large so that $N(d_1(t))$ and $N(d_2(t))$ are approximately one, as can be seen in Fig. 1.2.4. Consequently, by (8.3.2) the option value equals approximately $S_t - K\,\frac{B_t}{B_T}$ in this case.

The BS formula (8.3.2) can be interpreted as being an analytical formula. However, the Gaussian distribution function $N(\cdot)$ needs still to be approximated by other more basic functions or obtained by numerical evaluation of the integral of the Gaussian density $N'(\cdot)$ given in (8.3.5). In (1.2.7) a reasonably accurate and efficient approximation for the standard Gaussian distribution function has been provided.

European Call Option Price

To give an idea about the shape of the pricing function $c_{T,K}$ we show in Fig. 8.3.1 the European call option price as a function of time t and the underlying security price S with volatility $\sigma = 0.2$, strike price $K = 1$, expiration date $T = 10$ years and short rate $r = 0.05$. Figure 8.3.1 depicts prices for up to ten years to display some long term features of the typical Black-Scholes option price. Note that close to the expiration date $T = 10$ the option price has approximately the value of the hockey stick like payoff function (8.3.1). As previously mentioned, for small values of the underlying security the option price remains close to zero and for large security prices S the option has a price close to $S - K\,\frac{B_t}{B_T}$.

8.4 Sensitivities for European Call Option

The pricing function $c_{T,K}$, see (8.3.2), for the European call option depends on several variables. These are the underlying security price S_t, the time to maturity $T-t$, the volatility σ of the underlying security, the interest rate r and the strike price K. Changes in any of these variables influence the option price. Therefore, it is of practical importance to know how sensitive the pricing function $c_{T,K}$ is with respect to these variables.

It is informative to use the classical Taylor formula to expand the increments of the value of the derivative security over a small time interval $[t, t+h]$ in dependence on the above mentioned variables. By omitting higher order terms one obtains

$$\begin{aligned}
&V(t+h, S_{t+h}) - V(t, S_t)\\
&\quad\approx \frac{\partial V(t,S_t)}{\partial S}\,(S_{t+h}-S_t) + \frac{1}{2}\,\frac{\partial^2 V(t,S_t)}{\partial S^2}\,(S_{t+h}-S_t)^2\\
&\qquad + \frac{\partial V(t,S_t)}{\partial t}\,h + \frac{\partial V(t,S_t)}{\partial \sigma}\,(\sigma_{t+h}-\sigma_t) + \frac{\partial V(t,S_t)}{\partial r}\,(r_{t+h}-r_t)\\
&\quad= \varDelta(S_{t+h}-S_t) + \frac{1}{2}\,\varGamma\,(S_{t+h}-S_t)^2 - \varTheta\,h + \mathcal{V}\,(\sigma_{t+h}-\sigma_t) + \varrho\,(r_{t+h}-r_t),
\end{aligned} \tag{8.4.1}$$

where

$$\varDelta = \frac{\partial V(t,S_t)}{\partial S}, \quad \varGamma = \frac{\partial^2 V(t,S_t)}{\partial S^2}, \quad \varTheta = \frac{\partial V(t,S_t)}{\partial t},$$
$$\mathcal{V} = \frac{\partial V(t,S_t)}{\partial \sigma} \quad \text{and} \quad \varrho = \frac{\partial V(t,S_t)}{\partial r}$$

for $t \in [0,T]$. Here the letters $\varDelta$, $\varGamma$, $\varTheta$, $\mathcal{V}$ and ϱ denote the corresponding partial derivatives which are called *sensitivities* or *greeks*. The expansion (8.4.1) shows how the above greeks influence the increments of the Black-Scholes option price. Note that for obtaining a first order approximation one needs to include the second order derivative $\varGamma$ since the conditional expectation

$$E\left((S_{t+h}-S_t)^2 \,\middle|\, \mathcal{A}_t\right) \approx \sigma_t^2\,S_t^2\,h.$$

is of order h.

In the following we discuss some of the above greeks for the Black-Scholes European call option price. In the figures displayed below, we choose as default parameter the strike price $K = 1$ and maturity date $T = 10$. We consider the

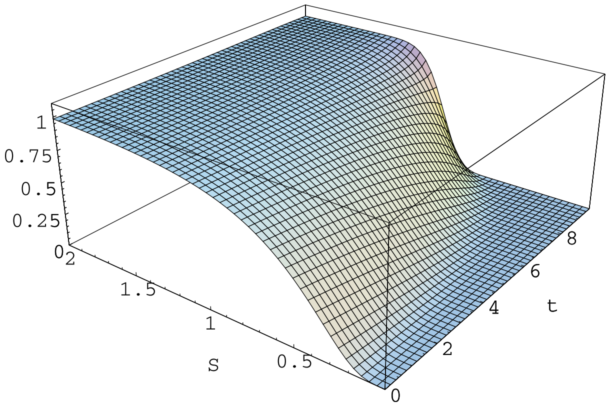

Fig. 8.4.1. Delta as a function of t and S_t

parameters a, σ and r to be constant and fix T and K, unless we study a sensitivity with respect to such a parameter. This means, we study sensitivities for the standard BS model.

Delta

The *delta* has been previously mentioned as hedge ratio, see (8.2.20). It measures the sensitivity of the option price with respect to changes in the price of the underlying security S_t. We set

$$\Delta = \frac{\partial V(t, S_t)}{\partial S} = \delta_t^1 \tag{8.4.2}$$

and obtain from (8.3.2) and (8.3.3) the expression

$$\Delta = N(d_1(t)), \tag{8.4.3}$$

which can be shown to equal the partial derivative appearing in (8.2.20), see Exercise 8.2.

Figure 8.4.1 shows for constant $\sigma = 0.2$ and $r = 0.05$ the delta for the European call option as a function of time t and asset price S_t.

Note that the delta for a European call option is always positive and bounded by one. Close to expiration and strike price $K = 1$, delta behaves almost like a step function moving from level zero to one. This makes hedging quite difficult in this situation.

Gamma

The sensitivity of the hedge ratio delta, with respect to the security price S_t is called *gamma*. This greek is important for the length of re-balancing intervals in practical hedging under transaction costs. A large gamma reflects

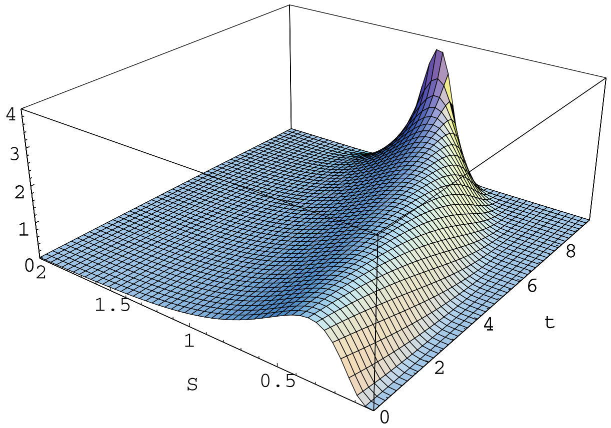

Fig. 8.4.2. Gamma as a function of t and S_t

large changes in the hedge ratio and thus typically large transaction costs. The gamma is set to

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V(t, S_t)}{\partial S^2}. \tag{8.4.4}$$

Using (8.4.3) and (8.3.3), it can be shown that

$$\Gamma = N'(d_1(t)) \, \frac{1}{S_t \, \sigma \, \sqrt{T - t}} \tag{8.4.5}$$

for $t \in [0, T)$. Note that gamma is always positive. Figure 8.4.2 displays gamma for the European call option as a function of time t and the underlying security price S_t, using the same parameter values as in Fig. 8.4.1.

Close to maturity the gamma has a profile in spatial direction similar to that of the bell shaped curve of the Gaussian density, see Fig. 1.2.3. It becomes extremely large close to expiration for security prices that are near the strike price, which is here set to K=1.

Theta

The *theta* of a hedge portfolio measures the dependence of the option price on the remaining time to expiration $(T - t)$. The parameter theta, often called the time decay of the portfolio, provides an estimate of the time sensitivity of the option price and is given by the expression

$$\Theta = -\frac{\partial V(t, S_t)}{\partial (T - t)}. \tag{8.4.6}$$

From (8.3.2) we obtain

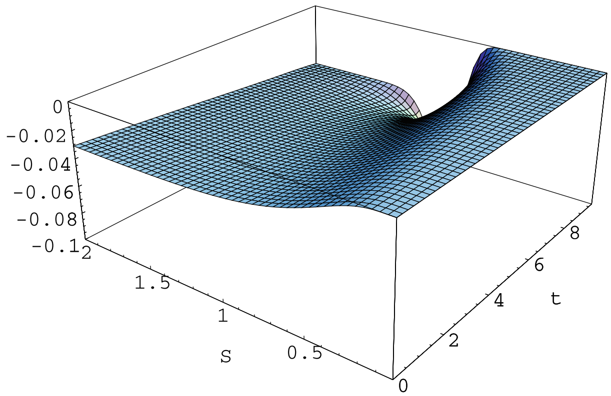

Fig. 8.4.3. Theta as a function of t and S_t

$$\Theta = -N'(d_1(t))\,\frac{S_t\,\sigma}{2\,\sqrt{T-t}} - r\,K\,\exp\{-r(T-t)\}\,N(d_2(t)) \tag{8.4.7}$$

for $t \in [0, T)$. Figure 8.4.3 displays theta for the European call option as a function of time t and security price S_t for the same parameter values as used in Fig. 8.4.1.

Vega

In the standard BS model a constant volatility σ is assumed. However, in practice volatility is difficult to estimate and changes over time. It is important to see how differences in volatilities influence derivative prices. The sensitivity of the option price with respect to volatility is called *vega*, which is given by

$$\mathcal{V} = \frac{\partial V(t, S_t)}{\partial \sigma}. \tag{8.4.8}$$

Using (8.3.2), it can be shown that

$$\mathcal{V} = N'(d_1(t))\,S_t\,\sqrt{T-t} \tag{8.4.9}$$

for $t \in [0, T)$. Figure 8.4.4 shows vega as a function of the volatility σ and the time t, where we have set $r = 0.05$, $K = 1$ and $S_t = 1$. Vega is positive and decreases substantially close to expiration. Its maximum value can be found for a volatility value that is close to $\sqrt{2\,r}$.

Rho

In the standard BS model interest rates are assumed to be constant. However, in practice interest rates vary. The sensitivity of the option price with respect to the interest rate r can be analyzed through *rho*, which is defined as

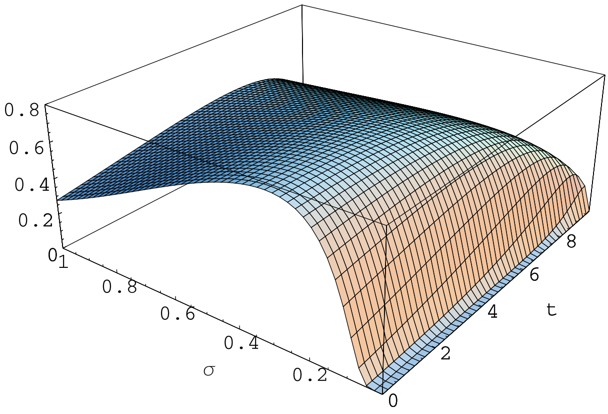

Fig. 8.4.4. Vega as a function of t and σ

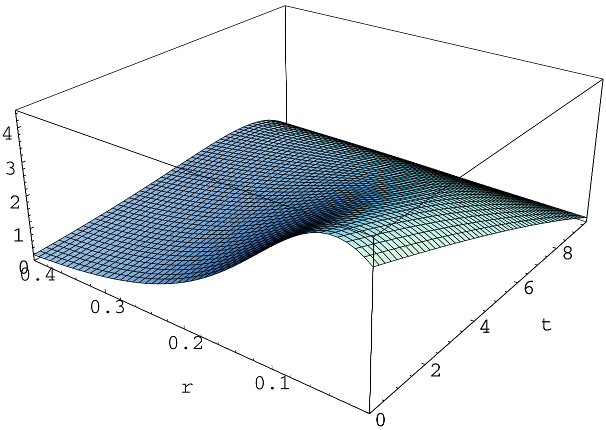

Fig. 8.4.5. Rho as a function of t and r

$$\varrho = \frac{\partial V(t, S_t)}{\partial r}.$$

From (8.3.2), the rho can be derived as the expression

$$\varrho = N(d_2(t))\,(T - t)\,K\,\exp\{-r\,(T - t)\} \tag{8.4.10}$$

for $t \in [0, T)$. Note that rho is always positive for a European call option. Figure 8.4.5 shows rho as a function of time t and interest rate r for $\sigma = 0.2$, $K = 1$ and $S_t = 1$. Rho appears to be larger for large time to maturity and largest for an interest rate close to $\frac{1}{2}\sigma^2$.

8.5 European Put Option

In this section we present a key relationship between European put and call options under the BS model. Additionally, the greeks of put options and their properties will be discussed.

Put-Call Parity

Put-call parity provides a simple way to determine the price of a European put option if the corresponding call option price for the same strike and maturity has been already computed. For the BS model the put-call parity relation can be expressed in the form

$$c_{T,K}(t, S_t) = p_{T,K}(t, S_t) + S_t - K \frac{B_t}{B_T} \tag{8.5.1}$$

for $t \in [0, T]$. This relation can be derived from the fact that the payoff function for the terminal value of the quantity on the left hand side of equation (8.5.1) equals the payoff function of that on the right hand side, which is

$$(S_T - K)^+ = (K - S_T)^+ + S_T - K. \tag{8.5.2}$$

For a wide range of models a similar put-call parity holds. This property of put and call prices is not restricted to the BS model because it reflects the general relationship (8.5.2) between their payoffs.

European Put Option Price

Using put-call parity, the pricing function $p_{T,K}$ for a *European put option* for the BS model with constant volatility σ and constant interest rate r is given by the formula

$$\begin{aligned} p_{T,K}(t, S_t) &= S_t \left(N(d_1(t)) - 1\right) - K \frac{B_t}{B_T} \left(N(d_2(t)) - 1\right) \\ &= -S_t\, N(-d_1(t)) + K \frac{B_t}{B_T}\, N(-d_2(t)), \end{aligned} \tag{8.5.3}$$

where $d_1(t)$ and $d_2(t)$ are given in (8.3.3) and (8.3.4). Figure 8.5.1 shows the European put option price as a function of time t and security price S_t for volatility $\sigma = 0.2$, strike price $K = 1$, expiration date $T = 10$ and interest rate $r = 0.05$.

It is interesting to compare the European call option price in Fig. 8.3.1 with the corresponding put option price displayed in Fig. 8.5.1. Inspection of both figures shows that, at the expiration date, the corresponding ramp like payoff functions are matched by the pricing functions.

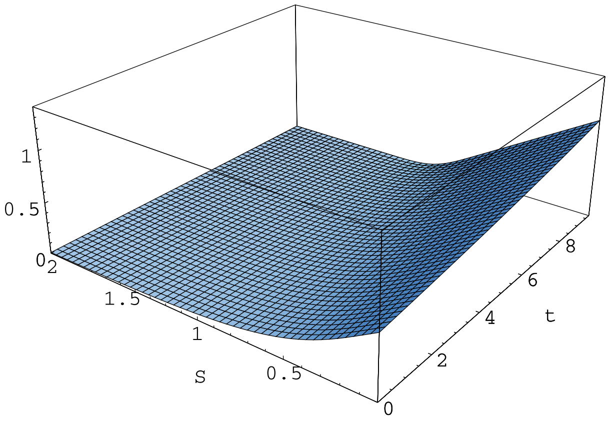

Fig. 8.5.1. European put option price

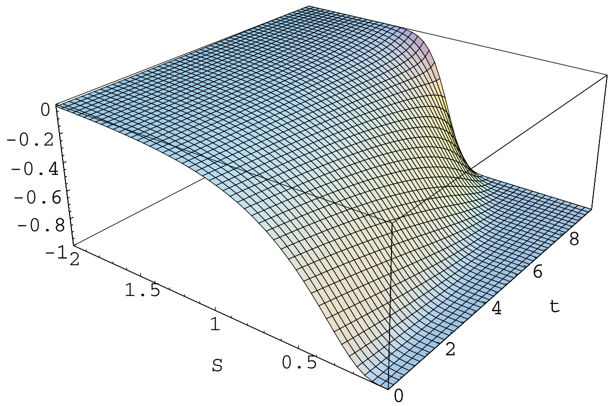

Fig. 8.5.2. Delta for the European put as a function of t and S_t

Greeks for the European Put Option

As with European calls, the sensitivities of the Black-Scholes European put option price (8.5.3) can be examined with respect to changes in various variables using the same notation. By using the put-call parity in (8.5.1) or the European put price (8.5.3) one obtains easily the corresponding sensitivities. The delta for the European put option is, according to (8.4.2) and (8.5.1), given by

$$\varDelta = \frac{\partial p_{T,K}}{\partial S} = N(d_1(t)) - 1. \tag{8.5.4}$$

Figure 8.5.2 shows the delta for the European put option as a function of time t and security price S_t. Note for the European put option that delta is always negative and bounded between -1 and 0. Comparing formulas (8.4.3) with (8.5.4) reveals that the delta of the put equals that of the call minus one.

The sensitivity of the delta with respect to the underlying security price S_t is again called gamma, which for European puts, see (8.4.4) and (8.5.4), is given by the expression

$$\varGamma = \frac{\partial^2 p_{T,K}}{\partial S^2} = N'(d_1(t))\, \frac{1}{S_t\, \sigma\, \sqrt{T-t}}. \tag{8.5.5}$$

This is the same formula as that for the European call given in (8.4.5). Thus, Fig. 8.4.2 also shows the shape of the gamma for the European put.

The other greeks for European puts, similar to those mentioned earlier, are given by the relations:

$$\varTheta = \frac{\partial p_{T,K}}{\partial (T-t)} = N'(d_1(t))\, \frac{S_t\, \sigma}{2\sqrt{T-t}} + r\, K\, \exp\{-r\,(T-t)\}\,(N(d_2(t)) - 1), \tag{8.5.6}$$

$$\mathcal{V} = \frac{\partial p_{T,K}}{\partial \sigma} = N'(d_1(t))\, S_t\, \sqrt{T-t} \tag{8.5.7}$$

and

$$\varrho = \frac{\partial p_{T,K}}{\partial r} = (T-t)\, K\, \exp\{r\,(T-t)\}\,(N(d_2(t)) - 1). \tag{8.5.8}$$

Bounds for European Calls and Puts

There exist some simple bounds for European call and put option prices on a stock that pays no dividends. From the BS formula (8.3.2) it follows for the European call

$$c_{T,K}(t, S_t) \leq S_t \tag{8.5.9}$$

for $t \in [0, T]$. By forming at time t a portfolio that consists of a European call together with $K\, \frac{B_t}{B_T}$ units of the savings account, the payoff at maturity T will be

$$\max(S_T, K) \geq S_T.$$

Therefore, it follows that

$$c_{T,K}(t, S_t) \geq \left(S_t - K\, \frac{B_t}{B_T} \right)^+ \tag{8.5.10}$$

for $t \in [0, T]$. This holds for all European call option prices. Any derivative that gives the holder more rights is more expensive. Therefore, an American call price $C_{T,K}(t, S_t)$ at time t with maturity T and strike K is larger than the corresponding European call and by (8.5.10) we obtain

$$C_{T,K}(t, S_t) \geq c_{T,K}(t, S_t) \geq \left(S_t - K\, \frac{B_t}{B_T} \right)^+ \geq (S_t - K)^+ \tag{8.5.11}$$

for $t \in [0, T]$. This means, the American call price $C_{T,K}(t, S_t)$, see Sect. 8.1, is always larger than the intrinsic value $(S_t - K)^+$ and will therefore never be early exercised. Thus, we have

$$C_{T,K}(t, S_t) = c_{T,K}(t, S_t) \tag{8.5.12}$$

for $t \in [0, T]$. This interesting feature is not model dependent.

On the other hand, from (8.5.3) the upper bound

$$p_{T,K}(t, S_t) \leq K \frac{B_t}{B_T} \tag{8.5.13}$$

for $t \in [0, T]$ can be obtained. By put-call parity and the positivity of call prices it also follows that

$$p_{T,K}(t, S_t) \geq K \frac{B_t}{B_T} - S_t. \tag{8.5.14}$$

for $t \in [0, T]$.

Note that the above bounds do, in principle, not depend on the choice of the model for the underlying security dynamics if one substitutes $\frac{B_t}{B_T}$ by the corresponding zero coupon bond of the respective model. They hold generally because they are a consequence of the shape of the put and call payoff functions.

8.6 Hedge Simulation

In Sect. 8.1 we identified by hedging arguments the discounted BS-PDE for discounted option prices. This led in Sect. 8.3 to the BS formula, which provides the solution for the BS-PDE. By using a *hedge simulation* we show now how a hedge portfolio works in detail. This type of continuous trading is called *delta hedging*. In the following, we construct a hedge portfolio for a European call option under the standard BS model with constant appreciation rate a, volatility σ and short rate r. We examine the evolution of the hedge portfolio for two different scenarios along an equidistant time discretization with $t_k = k\,h$, $k \in \{0, 1, \ldots\}$, for some small time step size $h > 0$. In this sense we shall perform an approximate hedge, which can be interpreted as a continuous hedge in a frictionless market. For each of the two scenarios we shall check whether the payoff is replicated by the hedge portfolio and the P&L remains approximately zero as predicted by the theoretical results presented in Sect. 8.1.

Hedging Strategy

The hedge ratio, that is the delta δ_t^1, has according to (8.2.20), (8.4.3) and (8.3.3) the value

$$\begin{aligned} \delta_t^1 &= N(d_1(t)) \\ &= N\left(\frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\,\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}} \right) \end{aligned} \tag{8.6.1}$$

for $t \in [0, T)$. For the number of units held in the domestic savings account we obtain from (8.2.25), (8.2.26), (8.2.10), (8.3.2) and (8.6.1) the relation

$$\begin{aligned} \delta_t^0 &= -\frac{K}{B_T} N(d_2(t)) \\ &= -\frac{K}{B_T} N\left(\frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \end{aligned} \tag{8.6.2}$$

for $t \in [0, T]$. Recall that the price of the call option at time $t \in [0, T)$, see (8.2.4), is given by

$$c_{T,K}(t, S_t) = \delta_t^1 S_t + \delta_t^0 B_t. \tag{8.6.3}$$

Furthermore, from (8.2.13) the discounted P&L takes the form

$$\bar{C}_t = \bar{V}(t, \bar{S}_t) - \int_0^t \delta_s^1 \, d\bar{S}_s - \bar{V}(0, \bar{S}_0) \tag{8.6.4}$$

for $t \in [0, T)$. Let us now recall that the discounted P&L remains zero. This follows, for instance, by a straightforward application of the Itô formula (6.2.11) for $\bar{V}$, where we obtain

$$\bar{V}(t, \bar{S}_t) = \bar{V}(0, \bar{S}_0) + \int_0^t \delta_s^1 \, d\bar{S}_s \tag{8.6.5}$$

and, thus, with (8.6.4) it must hold

$$\bar{C}_t = 0 \tag{8.6.6}$$

for $t \in [0, T]$. As previously explained, this is a consequence of the fact that the terms on the right hand side of (8.2.19) vanish by the choice of the hedging strategy.

In-the-Money Scenario

For illustration, let us generate linearly interpolated values of the underlying security price S_t, say a stock index, from a sample path of a geometric Brownian motion starting at $S_0 = 1$ with appreciation rate $a = 0.05$ and volatility $\sigma = 0.2$, using the time points $t_i = i\,h \in [0, 10]$ for $i \in \{0, 1, \ldots, 500\}$ with time step size $h = 0.02$. This path is shown in Fig. 8.6.1 together with the corresponding hedge ratio δ_t^1, see (8.6.1), for the European call option with expiration date $T = 10$, strike price $K = 1$ and interest rate $r = 0.05$. Note that for the given sample path the security price ends up in-the-money, that is we have $S_T > K$. For this scenario we observe that the hedge ratio converges to the value $\delta_T^1 = 1$ as t tends to T. This is the correct value for the hedge ratio since the security price ends up in-the-money and the option will be exercised.

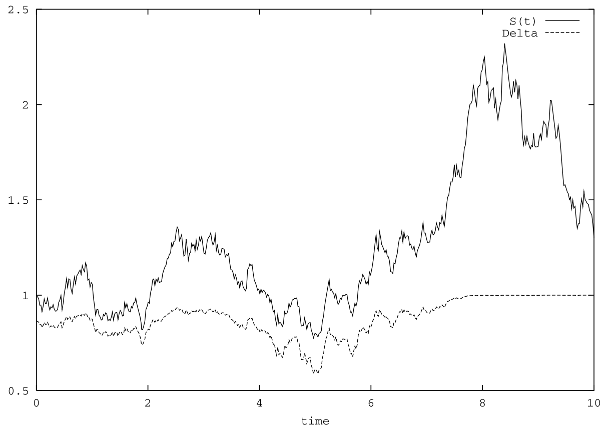

Fig. 8.6.1. Underlying security and hedge ratio for in-the-money call

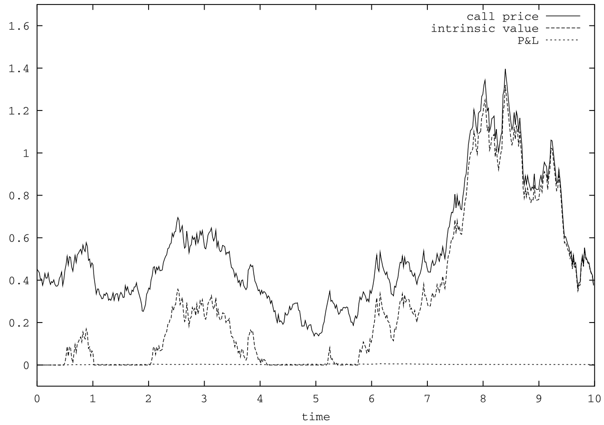

Fig. 8.6.2. Price, intrinsic value and P&L for in-the-money call

The evolution of the value of the corresponding hedge portfolio, which equals the call option price $c_{T,K}(t, S_t)$, is shown in Fig. 8.6.2 in dependence on time t. For comparison, Fig. 8.6.2 also displays the intrinsic value of the call option, that is the value

$$H(S_t) = (S_t - K)^+$$

for $t \in [0, T]$, see (8.1.2). Figure 8.6.2 shows for this sample path that the hedge portfolio replicates the payoff of the option at the expiration date $T = 10$ since the option price converges to its intrinsic value for t approaching T.

For illustration, Fig. 8.6.2 also displays for the obtained self-financing strategy δ the P&L C_t of the hedge portfolio. According to (8.2.13), the discounted P&L equals the value of the discounted portfolio minus the gains from trade in the discounted security $\bar{S}$ using the strategy δ minus the initial price. Note in Fig. 8.6.2 that the P&L remains almost perfectly zero over time as is expected from equation (8.6.6).

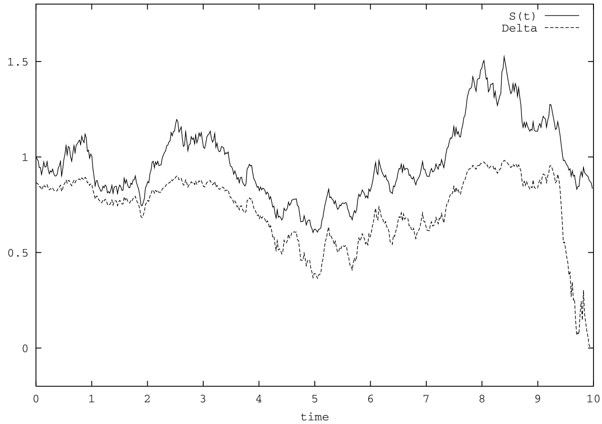

Fig. 8.6.3. Underlying security and hedge ratio for out-of-the-money call

Out-of-the-Money Scenario

The replication of the payoff through delta hedging does not depend on the sample path of the underlying security, as can be seen from (8.2.19). To illustrate this we change the sample path by assuming a zero appreciation rate $a = 0$ in the above example. This brings the previous sample path of the underlying security down, as is evident from Fig. 8.6.3 and Fig. 8.6.1. It shows that the call option expires now out-of-the-money, that is $S_T < K = 1$. Consequently, the delta, that is the hedge ratio δ_t^1, converges to zero for t tending towards T. Figure 8.6.4 shows the corresponding sample path of the call option price $c_{T,K}(t, S_t)$ and its intrinsic value

$$H(S_t) = (S_t - K)^+$$

together with the P&L for the hedge portfolio. It is apparent that also in this case the payoff is replicated at the expiration date $T = 10$ and the P&L remains approximately zero over time, see (8.6.6).

We have used the same sample path of the driving Wiener process to generate both the in- and out-of-the-money scenarios for $a = 0.05$ and $a = 0$, respectively. The hedge simulations can be compared with each other via the corresponding graphs in Figs. 8.6.1–8.6.4. Note that the initial option prices $c_{T,K}(0, S_0)$ at time $t = 0$ for both scenarios are the same. Changing the appreciation rate a in the BS model has not altered any part of our formulas and final hedging results. This striking phenomenon is a key feature of hedging. Independently of the realized scenario and the underlying appreciation rate the previously identified perfect hedge replicates the given payoff.

Hedging a European Call Option on the S&P500

Let us now apply the above Black-Scholes delta hedging to an S&P500 index option. Figure 3.1.1 shows this index for the years from 1993 up until 1998

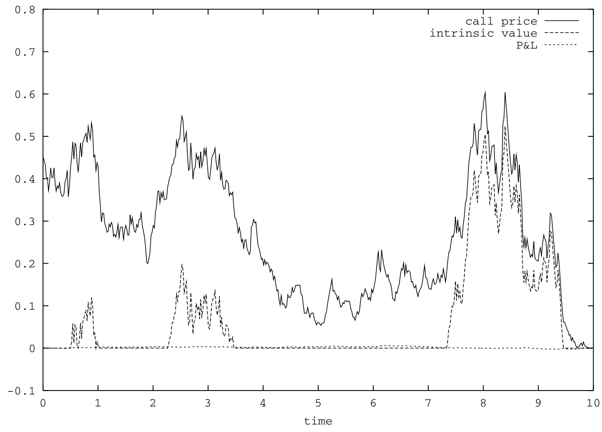

Fig. 8.6.4. Price, intrinsic value and P&L for out-of-the-money call

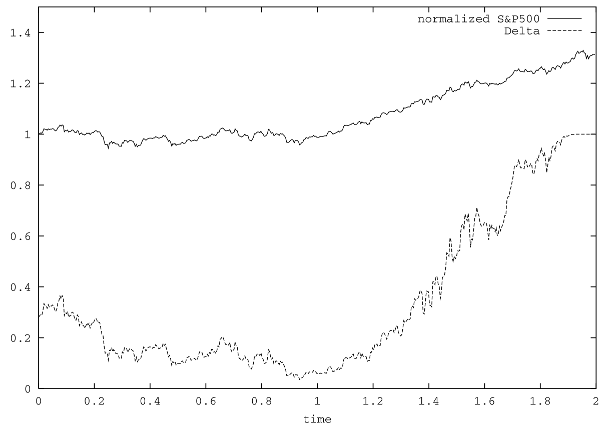

Fig. 8.6.5. Normalized S&P500 and hedge ratio for $K = 1.2$

and Fig. 5.2.6 its logarithm and quadratic variation. To make the following study similar to the above hedge simulation we divide the S&P500 data by its value at January, 3, 1994 and use the 520 observations of the normalized index for the years 1994 and 1995 as scenario of the underlying security. Figure 8.6.5 depicts the normalized S&P500 values for these two years and Fig. 8.6.6 the approximate quadratic variation, see (5.2.3), of the logarithm of the normalized S&P500. The quadratic variation seems to be reasonably linear for this period. According to formula (5.2.14), which provides some definition of volatility, we can read off from the plotted graph of the quadratic variation in Fig. 8.6.6 an average volatility of approximately $\sigma \approx \sqrt{\frac{0.016}{2}} \approx 0.09$. Furthermore, we set the USD short rate to the constant value $r \approx 0.05$, which is reasonable for the period under consideration.

Now, let us consider a European call option on the normalized S&P500 sample path as underlying security, which expires at the end of the period,

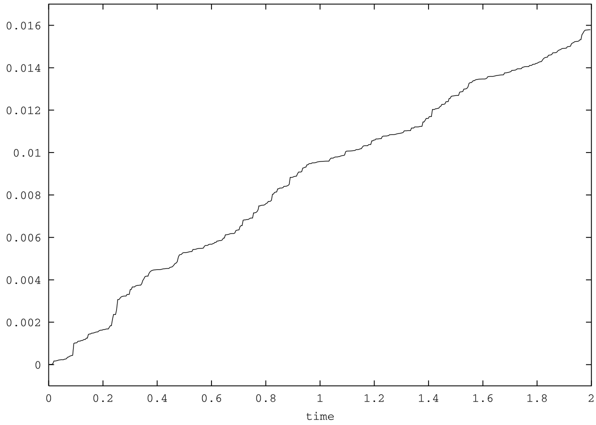

Fig. 8.6.6. Quadratic variation of log-S&P500

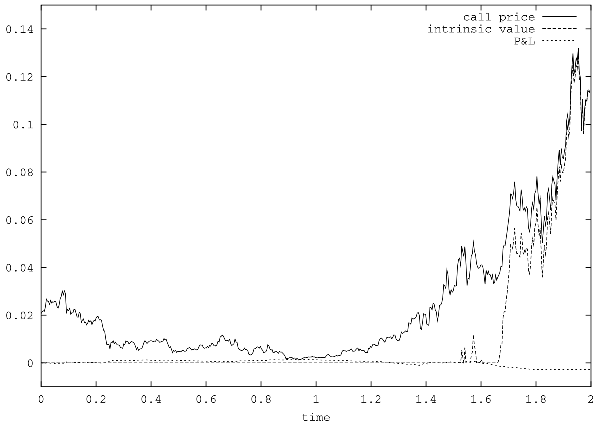

Fig. 8.6.7. Call price on S&P500, intrinsic value and P&L

that is in December 1995. We perform delta hedging according to what would be obtained for the BS model with the above parameters and use the same procedure that was employed for the previous hedge simulation. To study an in-the-money call option scenario we consider first a strike price of $K = 1.2$. Figure 8.6.5 shows the hedge ratio δ_t^1 for this European call option. Figure 8.6.7 displays the corresponding evolution of the call option price, intrinsic value and P&L similar to Fig. 8.6.2 and Fig. 8.6.4. Note that the payoff is reasonably well replicated at the expiration date. However, the P&L is not as close to zero as was the case for the simulated BS model, previously examined. Note from Fig. 8.6.6 that the volatility of the underlying security was not fluctuating greatly during the chosen period. For longer dated options over longer time periods changes in the P&L can be shown to be more dramatic and the BS model is then not sufficient to provide an acceptable hedge. The nonzero P&L is clearly a consequence of the fact that the S&P500 does not exactly follow the BS model. The result can only be improved by using alternative asset price models which allow a volatility that is stochastic and reflects better reality. A paper by Bakshi, Cao & Chen (1997) shows that for the hedging of short dated options the BS model performs reasonably well. However, for the prices of these options the authors pointed out that the BS model seems to be not sufficiently accurate.

We shall later study various models that generate volatility which is stochastic. Some of these models involve squared Bessel processes, which we introduce in the following section.

8.7 Squared Bessel Processes (*)

As we shall see, many quantities that involve Bessel processes can be expressed in terms of Bessel functions. This gives this class of processes its name. We summarize in this section important results on squared Bessel processes because some of these will be crucial for the understanding of the following chapters presenting the benchmark approach.

To facilitate the explicit computation of derivative prices and other quantities under various models, including the CIR model, the CEV model and the MMM, we list in this section properties of square root and squared Bessel processes. Most of these properties are scattered in the literature. Some of them can be found, for instance, in Karatzas & Shreve (1991), Revuz & Yor (1999) or Jeanblanc, Yor & Chesney (2009).

The following results on time transformed squared Bessel processes will also be important for the understanding of the typical dynamics of financial markets. We shall give an example for a local martingale that is not a martingale. This example will turn out to be potentially closely linked to the real market dynamics.

Squared Bessel Process (*)

Let us introduce the *squared Bessel process* (BESQ^δ_x) $X = \{X_\varphi,\, \varphi \in [0,\infty)\}$ of *dimension* $\delta \geq 0$ given by the SDE

$$dX_\varphi = \delta\, d\varphi + 2\sqrt{|X_\varphi|}\, dW_\varphi \tag{8.7.1}$$

for $\varphi \in [0,\infty)$ with $X_0 = x \geq 0$, where $W = \{W_\varphi,\, \varphi \in [0,\infty)\}$ is a standard Wiener process on $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$ starting at the initial φ-time, $\varphi = 0$, at zero. This means for $\varphi \in [0,\infty)$ that

$$[W]_\varphi = \varphi$$

for all $\varphi \in [0,\infty)$. Here we assume that X is reflected at zero if it reaches the level zero. It turns out that the absolute sign under the square root in (8.7.1) can be removed. X_φ remains nonnegative in this case and (8.7.1) has a unique strong solution, see Revuz & Yor (1999).

We have the following *scaling property*:
If $X = \{X_\varphi, \varphi \in [0,\infty)\}$ is a BESQ_x^δ, then $Z = \{Z_\varphi, \varphi \in [0,\infty)\}$ with $Z_\varphi = \frac{1}{a} X_{a\varphi}$ is a $\mathrm{BESQ}_{\frac{x}{a}}^\delta$ for all $a > 0$.

For $\delta \in \mathcal{N}$ and $x \geq 0$ the dynamics of a BESQ_x^δ X can be expressed as the sum of the squares of δ independent standard Wiener processes W^1, W^2, ..., W^δ, where

$$x = \sum_{k=1}^{\delta} (w^k)^2. \tag{8.7.2}$$

Here one sets

$$X_\varphi = \sum_{k=1}^{\delta} (w^k + W_\varphi^k)^2 \tag{8.7.3}$$

for $\varphi \in [0,\infty)$. Note that this construction is invariant with respect to the particular choice of w^k, $k \in \{1, 2, \ldots, \delta\}$, when (8.7.2) is satisfied. Clearly, the function (8.7.3) is a function of components of the solution of a simple linear system of SDEs, where each component represents a Wiener process. By an application of the Itô formula we obtain

$$dX_\varphi = \delta\, d\varphi + 2 \sum_{k=1}^{\delta} \left(w^k + W_\varphi^k\right) dW_\varphi^k \tag{8.7.4}$$

for $\varphi \in [0,\infty)$ with

$$X_0 = \sum_{k=1}^{\delta} (w^k)^2 = x. \tag{8.7.5}$$

By setting

$$dW_\varphi = |X_\varphi|^{-\frac{1}{2}} \sum_{k=1}^{\delta} \left(w^k + W_\varphi^k\right) dW_\varphi^k \tag{8.7.6}$$

we obtain the SDE (8.7.1). Note that we have for W_φ the quadratic variation

$$[W]_\varphi = \int_0^\varphi \frac{1}{X_s} \sum_{k=1}^{\delta} \left(w^k + W_s^k\right)^2 ds = \varphi.$$

Thus, W_φ in (8.7.6) forms by Lévy's theorem, see Theorem 6.5.1, a Wiener process in the φ time scale.

In Fig. 8.7.1 we show the path of a squared Bessel process of dimension $\delta = 4$ in the φ time scale, which starts at $X_0 = 100$, where we set $w^k = 5$ for $k \in \{1, 2, 3, 4\}$. Note the tendency of the process to increase over time, which is typical.

Squared Bessel processes have the following important *additivity property*, see Shiga & Watanabe (1973):
Let $X = \{X_\varphi,\ \varphi \in [0,\infty)\}$ be a BESQ_x^δ and $Y = \{Y_\varphi,\ \varphi \in [0,\infty)\}$ an

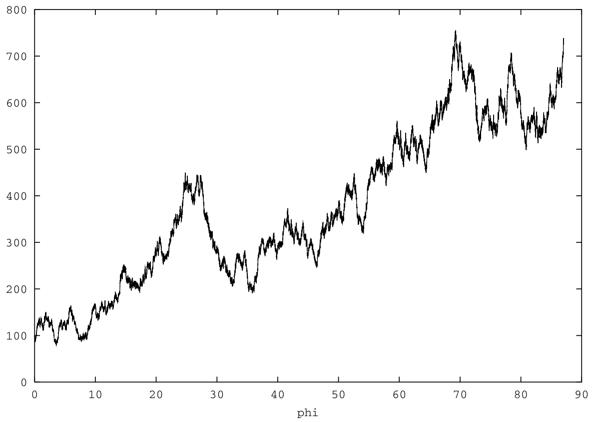

Fig. 8.7.1. Squared Bessel process of dimension $\delta = 4$ in φ-time

independent $\text{BESQ}_y^{\delta'}$ with $x, y, \delta, \delta' \geq 0$. Then the process $Z = \{Z_\varphi, \varphi \in [0, \infty)\}$ where $Z_\varphi = X_\varphi + Y_\varphi$ is a $\text{BESQ}_{x+y}^{\delta+\delta'}$.

It can be shown that a squared Bessel process BESQ_x^δ of dimension $\delta > 2$ with $X_0 = x > 0$ stays always strictly positive, that is

$$P\left(\inf_{0 \leq \varphi < \infty} X_\varphi > 0\right) = 1, \tag{8.7.7}$$

see Karatzas & Shreve (1998). In this case X_φ tends to infinity as φ goes to infinity. For the case $\delta = 2$ one has

$$P\left(\inf_{0 \leq \varphi < \infty} X_\varphi > 0\right) = 0.$$

Furthermore, for a BESQ_x^δ X process with $\delta \in [0, 2)$ and $X_0 = x > 0$, there is a strictly positive probability that X will hit zero before any fixed φ-time $\varphi' \in (0, \infty)$, that is

$$P\left(\inf_{0 \leq \varphi \leq \varphi'} X_\varphi = 0\right) > 0. \tag{8.7.8}$$

This means X_φ reaches zero in finite time with strictly positive probability.

For $\delta > 0$ and $x > 0$ the transition density for a BESQ_x^δ process X starting at the φ-time $\varrho \in [0, \infty)$ in x being at time $\varphi \in (\varrho, \infty)$ in y is given as

$$p_\delta(\varrho, x; \varphi, y) = \frac{1}{2(\varphi - \varrho)} \left(\frac{y}{x}\right)^{\frac{\nu}{2}} \exp\left\{-\frac{x + y}{2(\varphi - \varrho)}\right\} I_\nu\left(\frac{\sqrt{x\,y}}{\varphi - \varrho}\right), \tag{8.7.9}$$

see Revuz & Yor (1999), where I_ν is the modified Bessel function of the first kind, see (1.2.15), with index ν. Here the index is defined as

$$\nu = \frac{\delta}{2} - 1. \tag{8.7.10}$$

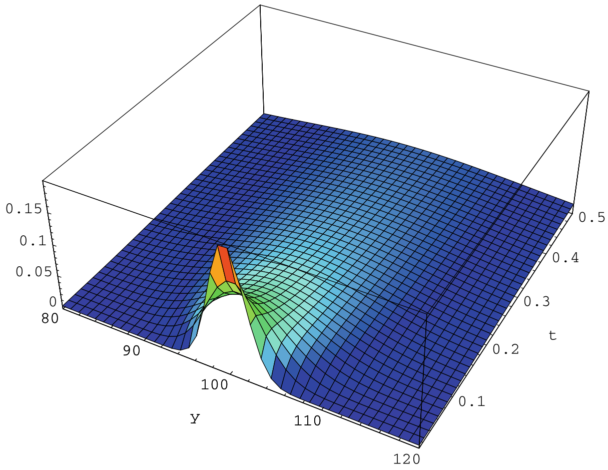

Fig. 8.7.2. Transition density of squared Bessel process for $\delta = 4$

In Fig. 8.7.2 we show the transition density of a squared Bessel process of dimension four, $\delta = 4$, which means index $\nu = 1$, starting at $x = 100$.

For small values of z one has

$$I_\nu(z) \approx \frac{1}{\nu\, \Gamma(\nu)} \left(\frac{z}{2}\right)^\nu \tag{8.7.11}$$

for $\nu > 0$. Therefore, the transition density of a BESQ_0^δ process X, which starts at time zero at $x = 0$, is

$$p_\delta(0,0;\varphi,y) = (2\,\varphi)^{-\frac{\delta}{2}} \frac{y^{\frac{\delta}{2}-1}}{\Gamma(\frac{\delta}{2})} \exp\left\{-\frac{y}{2\,\varphi}\right\}. \tag{8.7.12}$$

From (8.7.9) and (1.2.14) one notices that for fixed $\delta > 2$, $x, y \geq 0$ and $\varphi > 0$ the transition density $p_\delta(0,x;\varphi,y)$ is the density of a non-central chi-square distributed random variable $Y = \frac{X_\varphi}{\varphi}$, see (1.2.13) with dimension δ, and non-centrality parameter $\ell = \frac{x}{\varphi}$. Consequently, by (1.2.13) we obtain

$$P\left(\frac{X_\varphi}{\varphi} < u\right) = \sum_{k=0}^{\infty} \frac{\exp\left\{-\frac{\ell}{2}\right\}\left(\frac{\ell}{2}\right)^k}{k\,!} \left(1 - \frac{\Gamma\left(\frac{u}{2};\frac{\delta+2k}{2}\right)}{\Gamma\left(\frac{\delta+2k}{2}\right)}\right), \tag{8.7.13}$$

where $\Gamma(\cdot;\cdot)$ is the incomplete gamma function, see (1.2.12).

Furthermore, for $\alpha > -\frac{\delta}{2}$, $\varphi \in (0,\infty)$ and $\delta > 2$ one can show that

$$E\left(X_\varphi^\alpha \,\middle|\, \mathcal{A}_0\right) = \begin{cases} (2\,\varphi)^\alpha \exp\left\{\frac{-X_0}{2\,\varphi}\right\} \sum_{k=0}^\infty \left(\frac{X_0}{2\,\varphi}\right)^k \frac{\Gamma\left(\alpha+k+\frac{\delta}{2}\right)}{k\,!\,\Gamma\left(k+\frac{\delta}{2}\right)} & \text{for} \quad \alpha > -\frac{\delta}{2} \\ \infty & \text{for} \quad \alpha \leq -\frac{\delta}{2}, \end{cases} \tag{8.7.14}$$

see Exercise 8.8. By (8.7.1) it follows that

$$E\left(X_\varphi \,\middle|\, \mathcal{A}_0\right) = X_0 + \delta\,\varphi \tag{8.7.15}$$

for $\varphi \in [0,\infty)$. Thus, for $\alpha \in (-\frac{\delta}{2}, 0]$, $\varphi \in (0,\infty)$ and $\delta > 2$ it follows by the monotonicity of the gamma function

$$E\left(X_\varphi^\alpha \,\middle|\, \mathcal{A}_0\right) \le (2\varphi)^\alpha \exp\left\{\frac{-X_0}{2\varphi}\right\}\left(\frac{\Gamma\left(\alpha + \frac{\delta}{2}\right)}{\Gamma\left(\frac{\delta}{2}\right)} + \exp\left\{\frac{X_0}{2\varphi}\right\}\right)$$
$$< \infty, \tag{8.7.16}$$

see Exercise 8.8. Let us remark, by using the property $\frac{\Gamma(k+1)}{\Gamma(k+2)} = \frac{1}{k+1}$ of the gamma function and an expansion of the exponential function, that one obtains from (8.7.14) for $\delta = 4$ the explicit expression

$$E\left(X_\varphi^{-1} \,\middle|\, \mathcal{A}_0\right) = X_0^{-1}\left(1 - \exp\left\{\frac{-X_0}{2\,\varphi}\right\}\right) \tag{8.7.17}$$

for $\varphi \in (0,\infty)$.

If one absorbs a squared Bessel process with dimension $\delta \in [0,2)$ at zero, the transition density (8.7.9) changes, see Borodin & Salminen (2002), such that $I_{|\nu|}$ appears in the formula instead of I_ν. That is, one has for $x > 0$ and $\varphi \in [0,\infty)$

$$p_\delta(0,x;\varphi,y) = \frac{1}{2\,\varphi}\left(\frac{y}{x}\right)^{\frac{\nu}{2}} \exp\left\{-\frac{x+y}{2\,\varphi}\right\} I_{|\nu|}\left(\frac{\sqrt{x\,y}}{\varphi}\right). \tag{8.7.18}$$

Let P_x^δ denote the law of a BESQ_x^δ process $X = \{X_\varphi,\ \varphi \in [0,\infty)\}$ of dimension δ with initial value $X_0 = x > 0$ at time $\varphi = 0$. In Revuz & Yor (1999) one can find the following important result. If we introduce the stopping time $\tau = \inf\{\varphi \in [0,\infty) :\ X_\varphi = 0\}$, then for $\delta > 2$ the relation holds:

$$P_x^{4-\delta}\Big|_{\mathcal{A}_\varphi \cap \{\varphi < \tau\}} = \left(\frac{x}{X_\varphi}\right)^{\frac{\delta}{2}-1} P_x^\delta\Big|_{\mathcal{A}_\varphi} \tag{8.7.19}$$

for all $\varphi \in (0,\infty)$. In principle, on the left hand side of the above relationship we consider squared Bessel processes with absorption at zero and on the right hand side squared Bessel processes that never reach zero. The same relationship (8.7.19) yields for $\delta < 2$ the equation

$$P_x^\delta\Big|_{\mathcal{A}_\varphi \cap \{\varphi < \tau\}} = \left(\frac{x}{X_\varphi}\right)^{1-\frac{\delta}{2}} P_x^{4-\delta}\Big|_{\mathcal{A}_\varphi}, \tag{8.7.20}$$

see also Exercises 8.9 and 8.10.

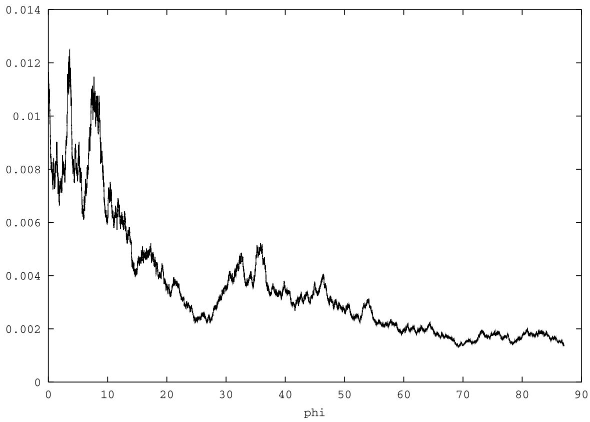

Fig. 8.7.3. Inverse of a squared Bessel process of dimension $\delta = 4$ in φ-time

Examples of Strict Local Martingales (*)

We now present an example of a local martingale that is not a martingale, see Definition 5.1.1 and Definition 5.2.1. We consider the inverse $Z = \{Z_\varphi = X_\varphi^{-1}, \varphi \in [0,\infty)\}$ of a squared Bessel process $X = \{X_\varphi, \varphi \in [0,\infty)\}$ of dimension four, as given in (8.7.1), with $X_0 > 0$. Then it follows by the Itô formula that

$$dZ_\varphi = -2\, Z_\varphi^{\frac{3}{2}}\, dW_\varphi \tag{8.7.21}$$

for $\varphi \in [0,\infty)$, where

$$Z_0 = X_0^{-1}. \tag{8.7.22}$$

By the *driftless* SDE (8.7.21) the process Z turns out to be a local martingale in φ time, see Sect. 5.2 and Sect. 5.5 or Protter (2004). From (8.7.17) it follows that

$$\begin{aligned} E\left(Z_\varphi \,\middle|\, \mathcal{A}_0\right) &= E\left(X_\varphi^{-1} \,\middle|\, \mathcal{A}_0\right) \\ &= Z_0 \left(1 - \exp\left\{\frac{-1}{2\, Z_0\, \varphi}\right\}\right) < Z_0 \end{aligned} \tag{8.7.23}$$

for $\varphi \in (0,\infty)$. This relation is *not* consistent with Z being an $(\underline{\mathcal{A}}, P)$-martingale. It actually proves that Z cannot be a martingale according to equation (5.1.2). Thus, the inverse Z of a squared Bessel process of dimension four is a continuous local martingale that is not a martingale. We say that such a local martingale is a *strict local martingale*. This observation will be very important for realistic financial modeling and derivative pricing, as we shall see later. Similarly, from relations (5.1.7) and (8.7.23) we can conclude that Z is a strict supermartingale.

In Fig. 8.7.3 we exhibit the inverse of the path of a squared Bessel process of dimension four in φ-time, which refers to the example with the path in

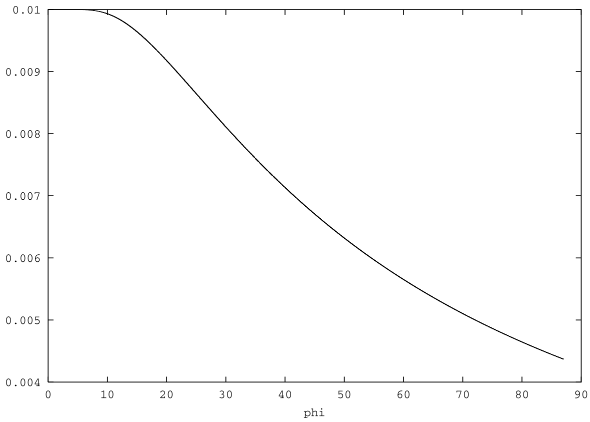

Fig. 8.7.4. Expectation of the inverse of the squared Bessel process for $\delta = 4$ in φ-time

Fig. 8.7.1. Note that this path is typical of that of a strict supermartingale. Here its current observation is larger than the best forecast of its future values.

In Fig. 8.7.4 we plot for the above example, by using formula (8.7.23), the expectation at time 0 of the inverse of the four dimensional squared Bessel process for varying φ-time. We clearly see the decline in this expectation over φ-time as was already indicated by the sample path in Fig. 8.7.3.

More generally, see Göing-Jaeschke & Yor (2003), for real valued dimension $\delta > 2$ one can show that the process

$$Z = \{Z_\varphi = X_\varphi^{1-\frac{\delta}{2}},\ \varphi \in [0, \infty)\} \tag{8.7.24}$$

is a *strict local martingale* if $X = \{X_\varphi,\ \varphi \in [0, \infty)\}$ is a $BESQ_x^\delta$ process of dimension $\delta > 2$ with $X_0 = x > 0$. One can see this from the relationship (8.7.19) since the expectation of Z_φ is strictly less than one, because of the possible absorption of a squared Bessel process of dimension $4 - \delta < 2$, see (8.7.8). Alternatively, by application of the transition density (8.7.9) it follows that

$$\begin{aligned} E\left(Z_\varphi \,\middle|\, \mathcal{A}_0\right) = E\left(X_\varphi^{1-\frac{\delta}{2}} \,\middle|\, \mathcal{A}_0\right) &= \int_0^\infty y^{1-\frac{\delta}{2}}\, p_\delta(0, x; \varphi, y)\, dy \\ &= x^{1-\frac{\delta}{2}} \int_0^\infty p_{4-\delta}(0, y; \varphi, x)\, dy \\ &= x^{1-\frac{\delta}{2}} \left(1 - \frac{\Gamma(\frac{\delta}{2} - 1; \frac{x}{2\,\varphi})}{\Gamma(\frac{\delta}{2} - 1)}\right) < x^{1-\frac{\delta}{2}} \end{aligned} \tag{8.7.25}$$

for $\varphi \in (0, \infty)$. Here $\Gamma(\cdot)$ is again the gamma function, see (1.2.10), and $\Gamma(\cdot;\cdot)$ is the incomplete gamma function, see (1.2.12). Note that for the special case $\delta = 4$ we obtain from (8.7.25) and (1.2.12) the relation (8.7.23). Furthermore, the inequality in (8.7.25) is strict for $\varphi > 0$, which shows that Z is a strict supermartingale.

Time Transformation (*)

Using a squared Bessel process we can derive by transformations more general processes. These include, for instance, the *square root* (SR) *process* that was mentioned previously in (4.4.6).

Let $b : [0, \infty) \to \Re$ and $c : [0, \infty) \to (0, \infty)$ be given deterministic functions of time. We introduce the exponential

$$s_t = s_0 \exp\left\{\int_0^t b_u \, du\right\} \tag{8.7.26}$$

and the φ-time

$$\varphi(t) = \varphi(0) + \frac{1}{4}\int_0^t \frac{c_u^2}{s_u} \, du \tag{8.7.27}$$

for $t \in [0, \infty)$ and $s_0 > 0$ in dependence on time. Note that by (8.7.27) and (8.7.26) we have for constant $b < 0$ and $c \neq 0$ that

$$\varphi(t) = \varphi(0) + \frac{c^2}{4\, b\, s_0}\,(1 - \exp\{-b\, t\}) \tag{8.7.28}$$

for $t \in [0, \infty)$ and the time

$$t(\varphi) = -\frac{1}{b}\, \ln\left(1 - \frac{4\, b\, s_0}{c^2}\,(\varphi - \varphi(0))\right) \tag{8.7.29}$$

for $\varphi \in [\varphi(0), \infty)$. For illustration we plot in Fig. 8.7.5 the time in units of φ-time, when we set $\varphi(0) = 0$, $c = 1$, $b = -0.05$ and $s_0 = 0.2$.

In Fig. 8.7.6 we show the path $X_{\varphi(t)}$ of the squared Bessel process X in dependence on time t. It will be suggested in Sect. 13.2 under the MMM that such a time transformed squared Bessel process of dimension $\delta = 4$ is closely matching the dynamics of the discounted market portfolio. For comparison we plot for the previous example of a squared Bessel process in Fig. 8.7.7 the expected value $E(X_{\varphi(t)}|\mathcal{A}_0)$ of $X_{\varphi(t)}$, see (8.7.15), in dependence on time t when using the above default parameters.

Let us visualize for the above example in Fig. 8.7.8 also the expected value of the squared Bessel process of dimension $\delta = 4$ with respect to time t. If one compares the Figs. 8.7.4 and 8.7.8, then one notes that after about five years, that is $t = 5$, the expected value of the inverse of the squared Bessel process starts to decline noticeably in our example.

We then show in the case of our example for comparison the inverse of the squared Bessel process of dimension $\delta = 4$, which we plotted in Fig. 8.7.3 in φ-time, in Fig. 8.7.9 in dependence on time t. One observes the typical systematic decline of a strict supermartingale.

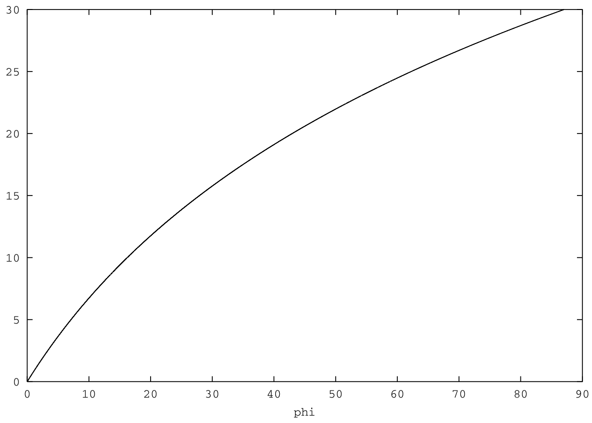

Fig. 8.7.5. Time $t(\varphi)$ against φ time

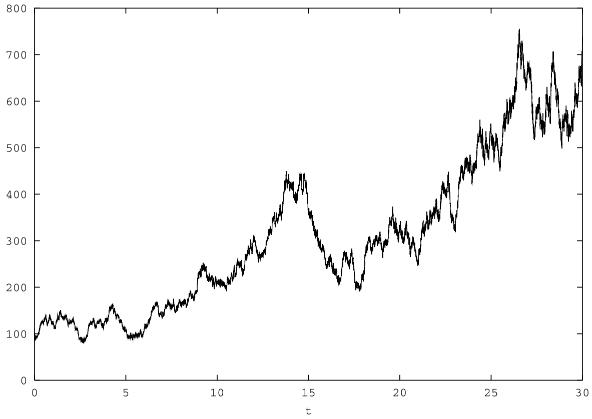

Fig. 8.7.6. Squared Bessel process in dependence on time t

Square Root Process (*)

We shall now demonstrate the close relationship of a *square root* (SR) *process* with a squared Bessel process X. Given a squared Bessel process X of dimension $\delta > 0$ and using our previous notation we introduce then the SR process

$$Y = \left\{Y_t = s_t \, X_{\varphi(t)}, \, t \in [0, \infty)\right\}$$

of dimension $\delta > 0$ in dependence on time t, by the transformation

$$Y_t = s_t \, X_{\varphi(t)} \tag{8.7.30}$$

for $t \in [0, \infty)$, see also Delbaen & Shirakawa (1997). Using (8.7.1), (8.7.26), (8.7.27), (8.7.30) and applying the Itô formula (6.2.11), the SDE for the SR process Y follows as

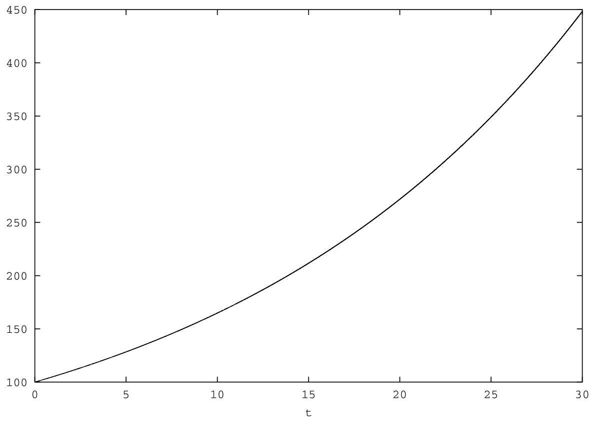

Fig. 8.7.7. Expectation of a squared Bessel process in dependence on time t

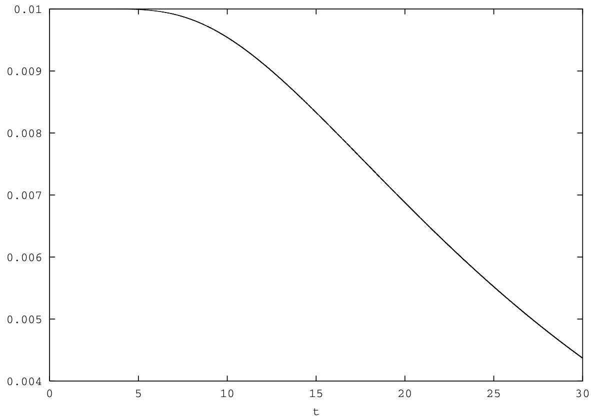

Fig. 8.7.8. Expectation of the inverse of a squared Bessel process in dependence on time t

$$\begin{aligned} dY_t &= s_t\, dX_{\varphi(t)} + X_{\varphi(t)}\, ds_t \\ &= s_t\, \delta\, d\varphi(t) + s_t\, 2\sqrt{X_{\varphi(t)}}\, dW_{\varphi(t)} + X_{\varphi(t)}\, s_t\, b_t\, dt \\ &= \left(\frac{\delta}{4}\, c_t^2 + b_t\, Y_t\right) dt + c_t\, \sqrt{Y_t}\, \sqrt{\frac{4\, s_t}{c_t^2}}\, dW_{\varphi(t)} \end{aligned} \tag{8.7.31}$$

for $t \in [0, \infty)$ and $Y_0 = s_0\, X_{\varphi(0)} > 0$. Note that $W = \{W_\varphi,\, \varphi \in [\varphi(0), \infty)\}$ is a Wiener process in the transformed φ-time $\varphi(t) \in [\varphi(0), \infty)$, which is linked to the time t by (8.7.27). The martingale $U = \{U_t,\, t \in [0, \infty)\}$ with the stochastic differential

$$dU_t = \sqrt{\frac{4\, s_t}{c_t^2}}\, dW_{\varphi(t)} \tag{8.7.32}$$

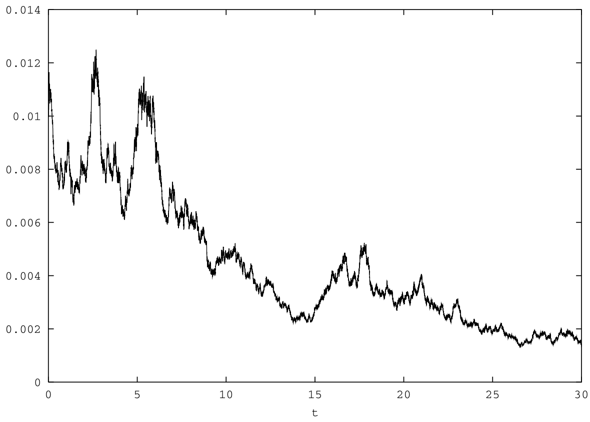

Fig. 8.7.9. Inverse of squared Bessel process in dependence on time t

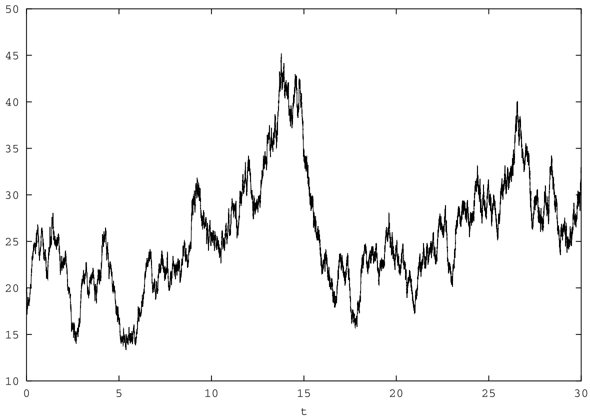

Fig. 8.7.10. Sample path of a square root process in dependence on time t

has the quadratic variation

$$[U]_t = \int_0^t \frac{4\, s_z}{c_z^2}\, d\varphi(z) = t. \tag{8.7.33}$$

By Lévy's theorem, see Theorem 6.5.1, the process $U = \{U_t,\, t \in [0, \infty)\}$ is then a Wiener process with respect to $t \in [0, \infty)$ on $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$. Thus, we have from (8.7.31) and (8.7.32) the SDE

$$dY_t = \left(\frac{\delta}{4}\, c_t^2 + b_t\, Y_t \right) dt + c_t\, \sqrt{Y_t}\, dU_t \tag{8.7.34}$$

for $t \in [0, \infty)$ for the SR process Y with $Y_0 = s_0\, X_{\varphi(0)}$. For an appropriate choice of b, c and δ the process Y expresses the SR process mentioned in (4.4.6) and (7.5.15). Figure 8.7.10 displays for our example the path of the corresponding SR process of dimension $\delta = 4$ in dependence on time t. For the visualization of the transition density of an SR process we refer to Fig. 4.4.1.

For $\delta > 2$ the transformation (8.7.30) allows us to reduce the characterization of the probability density for Y_t, see (8.7.43), to that of determining $p_\delta(\varphi(0), \frac{Y_0}{s_0}; \varphi(t), \frac{Y_t}{s_t})$, which is given in (8.7.9). It follows from (8.7.27), (8.7.14) and (8.7.16) for $\alpha > -\frac{\delta}{2}$, $t \in (0,\infty)$ and $\delta > 2$ the αth moment

$$E\left(Y_t^\alpha \,\middle|\, \mathcal{A}_0\right) = (2\,\bar{\varphi}_t\,\bar{s}_t)^\alpha \exp\left\{-\frac{Y_0}{2\,\bar{\varphi}_t}\right\} \sum_{k=0}^{\infty} \left(\frac{Y_0}{2\,\bar{\varphi}_t}\right)^k \frac{\Gamma(\alpha + k + \frac{\delta}{2})}{k!\,\Gamma(k + \frac{\delta}{2})} \tag{8.7.35}$$

and if additionally $\alpha \in (-\frac{\delta}{2}, 0)$ the estimate

$$E\left(Y_t^\alpha \,\middle|\, \mathcal{A}_0\right) \leq (2\,\bar{\varphi}_t\,\bar{s}_t)^\alpha \exp\left\{-\frac{Y_0}{2\,\bar{\varphi}_t}\right\} \left(\frac{\Gamma(\alpha + \frac{\delta}{2})}{\Gamma(\frac{\delta}{2})} + \exp\left\{\frac{Y_0}{2\,\bar{\varphi}_t}\right\}\right) < \infty, \tag{8.7.36}$$

where

$$\bar{s}_t = \frac{s_t}{s_0} = \exp\left\{\int_0^t b_u\,du\right\} \tag{8.7.37}$$

and

$$\bar{\varphi}_t = s_0\,(\varphi(t) - \varphi(0)) = \frac{1}{4}\int_0^t \frac{c_u^2}{\bar{s}_u}\,du \tag{8.7.38}$$

for $t \in [0,\infty)$. Note that $\bar{\varphi}_t$ and the above moments do not depend on the choice of the parameter s_0, which cancels due to the structure of the functions $\varphi(t)$ and s_t.

By using the SDE (8.7.34) the first moment of the SR process value Y_t can be shown to have the form

$$E\left(Y_t \,\middle|\, \mathcal{A}_0\right) = E\left(Y_0 \,\middle|\, \mathcal{A}_0\right) \exp\left\{\int_0^t b_s\,ds\right\} + \int_0^t \frac{\delta}{4}\,c_s^2 \exp\left\{\int_s^t b_z\,dz\right\} ds \tag{8.7.39}$$

for $t \in [0,\infty)$.

For the special case $\delta = 4$ and $\alpha = -1$ we obtain from (8.7.17) and (8.7.27)–(8.7.30) the first order negative moment of Y_t in the form

$$E\left(Y_t^{-1} \,\middle|\, \mathcal{A}_0\right) = \frac{1 - \exp\left\{-\frac{Y_0}{2\,\bar{\varphi}_t}\right\}}{Y_0\,\bar{s}_t}. \tag{8.7.40}$$

For $\delta > 2$, $c_t^2 = c^2 > 0$ and $b_t = b < 0$ the resulting SR process $Y = \{Y_t,\, t \in [0,\infty)\}$ with SDE (8.7.34) is ergodic, see Sect. 4.5. For the case $\delta = 4$ it has linear mean reversion with speed of adjustment parameter $-b$ and reference level $-\frac{c^2}{b}$. Thus, we obtain by (8.7.39) for an ergodic SR process Y the long term mean

$$\lim_{t\to\infty} E\left(Y_t \,\middle|\, \mathcal{A}_0\right) = -\frac{c^2}{b} \tag{8.7.41}$$

and the first order negative moment

$$\lim_{t\to\infty} E\left(Y_t^{-1} \,\middle|\, \mathcal{A}_0\right) = -2\,\frac{b}{c^2}. \tag{8.7.42}$$

We have for the SR process $Y = \{Y_t,\, t \in [0,\infty)\}$ an analytical transition density $p(s, Y_s; t, Y_t)$ that follows from (8.7.9) and (8.7.30) in the form

$$p(s, Y_s; t, Y_t) = \frac{p_\delta\left(\varphi(s), \frac{Y_s}{s_s}; \varphi(t), \frac{Y_t}{s_t}\right)}{s_t} \tag{8.7.43}$$

for $0 \le s < t < \infty$. In the case when $\delta > 2$, $b_t = b < 0$ and $c_t = c \neq 0$ the ergodic SR process Y has the transition density

$$p(0, x; t, y) = \frac{1}{2\bar{s}_t\, \bar{\varphi}_t} \left(\frac{y}{x\,\bar{s}_t}\right)^{\frac{\nu}{2}} \exp\left\{-\frac{x + \frac{y}{\bar{s}_t}}{2\,\bar{\varphi}_t}\right\} I_\nu\left(\frac{\sqrt{x\,\frac{y}{\bar{s}_t}}}{\bar{\varphi}_t}\right) \tag{8.7.44}$$

for $0 < t < \infty$ and $x, y \in (0,\infty)$, where $\nu = \frac{\delta}{2} - 1$, $\bar{s}_t = \exp\{b\,t\}$ and $\bar{\varphi}_t = \frac{c^2}{4\,b}(1 - \frac{1}{\bar{s}_t})$. It has then as stationary density a gamma density, which can be obtained via (4.5.20) in the form

$$p_{Y_\infty}(y) = \frac{\left(\frac{-2\,b}{c^2}\right)^{\frac{\delta}{2}}\, y^{\frac{\delta}{2}-1}}{\Gamma(\frac{\delta}{2})}\, \exp\left\{\frac{2\,b}{c^2}\, y\right\}. \tag{8.7.45}$$

The variance equals in this case

$$E\left((Y_t - E(Y_t))^2 \,\middle|\, \mathcal{A}_0\right) = Y_0\, \frac{c^2}{b}\,(\exp\{2\,b\,t\} - \exp\{b\,t\}) + \frac{\delta\, c^4}{8\, b^2}\,(1 - \exp\{b\,t\})^2 \tag{8.7.46}$$

for $t \in [0,\infty)$.

Affine Process (*)

Let us now further transform the above SR process given by (8.7.30) to cover the class of *affine processes*, see Duffie & Kan (1994) and Sect. 4.5. These processes have affine, that is linear, drift and linear squared diffusion coefficient functions. Here, we simply shift the SR process by a nonnegative, differentiable, deterministic function of time $a : [0,\infty) \to [0,\infty)$ defined through its derivative

$$a'_t = \frac{da_t}{dt} \tag{8.7.47}$$

for $t \in [0,\infty)$ with $a_0 \in [0,\infty)$. More precisely, we define the process $R = \{R_t,\, t \in [0,\infty)\}$ with

$$R_t = Y_t + a_t \tag{8.7.48}$$

for $t \in [0, \infty)$. Since Y is nonnegative also R remains nonnegative. By the Itô formula we obtain from (8.7.48) and (8.7.47) the SDE

$$dR_t = \left(\frac{\delta}{4}\, c_t^2 + a_t' - b_t\, a_t + b_t\, R_t \right) dt + c_t \sqrt{R_t - a_t}\, dU_t \qquad (8.7.49)$$

for $t \in [0, \infty)$ with $R_0 = Y_0 + a_0$. This means that the transform

$$R_t = s_t\, X_{\varphi(t)} + a_t \qquad (8.7.50)$$

of a squared Bessel process X of dimension δ yields an *affine diffusion process*, see (4.5.14) and (4.5.15), which satisfies the SDE (8.7.49).

8.8 Exercises for Chapter 8

8.1. Show for the BS model that the discounted European call option price of the discounted Black-Scholes formula satisfies the discounted Black-Scholes partial differential equation with corresponding terminal condition.

8.2. Derive the expression for the hedge ratio of the European call option in the BS model using the discounted BS-PDE.

8.3. Derive the gamma of a European put option for the BS model.

8.4. Compute, for a European put option under the BS model, the number of units δ_t^0 to be held at a given time t in the domestic savings account.

8.5. Transform the discounted BS-PDE for a discounted European option price into a corresponding BS-PDE for the corresponding undiscounted option price as a function of time and undiscounted underlying security.

8.6. Show for the European put option under the BS model that the corresponding P&L process is zero.

8.7. Derive the first moment of a square root process with constant parameters $c > 0$, $b < 0$ and dimension $\delta > 2$ satisfying the SDE

$$dY_t = \left(\frac{\delta}{4}\, c^2 + b\, Y_t \right) dt + c \sqrt{Y_t}\, dW_t$$

for $t \in [0, \infty)$ and $Y_0 > 0$, where W is a Wiener process.

8.8. (*) Derive the moments for the squared Bessel process with dimension $\delta > 2$ including moments of negative order, as long as they exist, and show estimates of the type (8.7.16).

8.9. (*) Show by using the transition density p_δ of a squared Bessel process of dimension $\delta > 2$ that

$$\int_0^\infty y^{1-\frac{\delta}{2}}\, p_\delta(0,x;\varphi,y)\,dy = x^{1-\frac{\delta}{2}} \int_0^\infty p_\delta(0,y;\varphi,x)\,dy.$$

8.10. (*) Show with the transition density p_δ of a squared Bessel process of dimension $\delta > 2$ that

$$\int_0^\infty p_\delta(0,y;\varphi,x)\,dy = \left(1 - \frac{\Gamma\left(\frac{\delta}{2}-1, \frac{x}{2\varphi}\right)}{\Gamma(\frac{\delta}{2}-1)}\right).$$