6

The Itô Formula

The price of a security, for instance, a zero coupon bond which generates some future payoff at a maturity date, is often dependent on the value of an underlying process. In many applications, the effect of changes in the underlying process on this price needs to be quantified. In deterministic calculus this type of problem is handled by the chain rule. In stochastic calculus the corresponding generalization of the chain rule is given by the Itô formula. This stochastic chain rule contains terms reflecting the effect due to the stochastic processes involved having non-zero quadratic variation. In this chapter we introduce, apply and derive the Itô formula. It is widely regarded as the main tool in stochastic calculus and is therefore highly important in quantitative finance.

6.1 The Stochastic Chain Rule

The Classical Chain Rule

First consider an example, where the classical deterministic chain rule applies. Suppose we observe in the market the price of a savings account $B_t = \exp\{r\,t\}$, where r denotes a constant continuously compounding interest rate. Then

$$dB_t = r\,B_t\,dt \tag{6.1.1}$$

for $t \in [0, \infty)$ with $B_0 = 1$. Also suppose that we are interested in a financial quantity $u(B_t)$, where $u : \Re \to \Re$ is some differentiable function. For instance, such a quantity could be the square of the value of the savings account, that is, $u(B_t) = (B_t)^2$. Furthermore, suppose that we need to express the evolution of this quantity in terms of properties of u and B with respect to time. In this case, by using the well-known chain rule of deterministic calculus, we can write the equations

$$u(B_t) = u(B_0) + \int_0^t u'(B_s)\,dB_s = u(B_0) + \int_0^t u'(B_s)\,r\,B_s\,ds \tag{6.1.2}$$

E. Platen, D. Heath, *A Benchmark Approach to Quantitative Finance*,
Springer Finance,

for $t \in [0, \infty)$. Note from the first line in (6.1.2) that the value of the quantity $u(B_t)$ can be interpreted as the gains from trade with integrand $u'(B_t)$ and integrator B_t for $t \in [0, \infty)$. This means for our simple deterministic example that

$$(B_t)^2 = (B_0)^2 + 2 \int_0^t B_s \, dB_s \tag{6.1.3}$$

for $t \in [0, \infty)$.

A Stochastic Example

In Sect. 5.3 we considered the Itô integral

$$I_{W,W}(t) = \int_0^t W_s \, dW_s,$$

which is the double Wiener integral for a Wiener process $W = \{W_t, \; t \in [0, \infty)\}$. This stochastic integral was interpreted as the gains from trade, where the number of shares held in the asset whose price was W was equal to its price. By rewriting equation (5.3.8) we obtain

$$(W_t)^2 = 2 \int_0^t W_s \, dW_s + [W]_t = 2 \int_0^t W_s \, dW_s + \int_0^t ds \tag{6.1.4}$$

for $t \in [0, \infty)$. Using the Itô differentials dW_t and $d(W_t)^2$ the equation (6.1.4) can be expressed in the equivalent Itô differential form

$$d(W_t)^2 = 2 \, W_t \, dW_t + dt \tag{6.1.5}$$

for $t \in [0, \infty)$ with initial value $(W_0)^2 = 0$. As previously explained, the equation (6.1.5) is nothing more than an abbreviated form of the stochastic integral equation (6.1.4). This integral equation involves an Itô integral, which is well defined, as discussed in the previous chapter. As a rule in stochastic calculus we shall see later that one can treat $(dW_t)^2$ as $d[W]_t = dt$, see (5.4.5). Note however that $d(W_t)^2$ is different to $(dW_t)^2$. Another rule will suggest setting $(dt)^2 = d[\cdot]_t = 0$ and $dW_t dt = d[W, t]_t = 0$.

Heuristic Derivation of the Itô Formula

One of the key features of the Itô integral with respect to the Wiener process is its martingale property, described in (5.4.3), which makes it an essential tool for pricing in finance. However, as previously indicated, this fundamental property does not come freely, namely the chain rule of classical calculus does not apply when using Itô integrals. Instead, the stochastic chain rule, the *Itô formula*, has to be applied. We now provide a heuristic derivation of the Itô formula. In Sect. 6.6 a proof of this formula will be presented.

Let $X = \{X_t,\, t \in [0, \infty)\}$ be a stochastic process that is characterized by the Itô differential

$$dX_t = e_t\, dt + f_t\, dW_t \tag{6.1.6}$$

for $t \in [0, \infty)$ with initial value $X_0 = x_0$, see (5.3.15). Here $e = \{e_t,\, t \in [0, \infty)\}$ and $f = \{f_t,\, t \in [0, \infty)\}$ are two stochastic processes with appropriate measurability and integrability properties. Consider a finite difference approximation of the Itô differential (6.1.6) of the form

$$\Delta X_{t_k} = X_{t_{k+1}} - X_{t_k} \approx e_{t_k}\, h + f_{t_k}\, (W_{t_{k+1}} - W_{t_k}) \tag{6.1.7}$$

for t_k from an equidistant time discretization $\{t_\ell = \ell\, h,\, \ell \in \{0, 1, \ldots\}\}$ with step size $h > 0$, as introduced in (5.2.1).

We focus our attention on changes in the value $u(t, X_t)$ for a function $u : [0, \infty) \times \Re \to \Re$, resulting from changes in the time t and the value of the underlying X_t. Assume that u is differentiable with respect to time t and twice continuously differentiable with respect to the spatial component x, that is, the functions $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ exist and are continuous.

To quantify the changes in $u(t, X_t)$ caused by changes in X_t we consider over small time intervals $[t_k, t_{k+1})$ the differences

$$\Delta u(t_k, X_{t_k}) = u(t_{k+1}, X_{t_{k+1}}) - u(t_k, X_{t_k}) \tag{6.1.8}$$

for $k \in \{0, 1, \ldots\}$. Since u is assumed to be sufficiently differentiable we can apply a Taylor expansion to obtain the expansion

$$\begin{aligned} \Delta u(t_k, X_{t_k}) = {} & \frac{\partial u(t_k, X_{t_k})}{\partial t}\, h + \frac{\partial u(t_k, X_{t_k})}{\partial x}\, \Delta X_{t_k} \\ & + \frac{1}{2}\, \frac{\partial^2 u(t_k, X_{t_k})}{\partial x^2}\, (\Delta X_{t_k})^2 + R_{t_k}, \end{aligned} \tag{6.1.9}$$

where R_{t_k} is the corresponding remainder term.

If the quadratic variation of X were zero, then $h \to 0$ would imply $(\Delta X_{t_k})^2 \to 0$ asymptotically and hence the corresponding term in (6.1.9) would not influence the movements of $u(t, X_t)$. However, in the given stochastic setting this is not the case and, therefore, we need to consider the approximation

$$(\Delta X_{t_k})^2 \approx [X]_{h,t_{k+1}} - [X]_{h,t_k} \approx (f_{t_k})^2\, h, \tag{6.1.10}$$

where $[X]_{h,t}$ denotes the approximate quadratic variation, see (5.2.3). Substituting this expression, together with (6.1.7) into (6.1.9) yields the relation

$$\begin{aligned} \Delta u(t_k, X_{t_k}) = {} & \left(\frac{\partial u(t_k, X_{t_k})}{\partial t} + e_{t_k} \frac{\partial u(t_k, X_{t_k})}{\partial x} + \frac{1}{2} (f_{t_k})^2 \frac{\partial^2 u(t_k, X_{t_k})}{\partial x^2} \right) h \\ & + f_{t_k}\, \frac{\partial u(t_k, X_{t_k})}{\partial x}\, (W_{t_{k+1}} - W_{t_k}) + \bar{R}_{t_k}, \end{aligned} \tag{6.1.11}$$

where $\bar{R}_{t_k}$ is the corresponding remainder term.

Itô Formula

Letting the time discretization become finer and finer in (6.1.11), that is $h \to 0$, results in the one-dimensional version of the *Itô formula*

$$du(t, X_t) = \left(\frac{\partial u(t, X_t)}{\partial t} + e_t \frac{\partial u(t, X_t)}{\partial x} + \frac{1}{2} (f_t)^2 \frac{\partial^2 u(t, X_t)}{\partial x^2} \right) dt + f_t \frac{\partial u(t, X_t)}{\partial x} dW_t \tag{6.1.12}$$

for $t \in [0, \infty)$. This formula will be derived rigorously towards the end of this chapter.

Note again that the Itô differential in (6.1.12) is only a shorthand notation for the *integral representation* of the Itô formula given as

$$u(t, X_t) = u(s, X_s) + \int_s^t \left(\frac{\partial u(z, X_z)}{\partial t} + e_z \frac{\partial u(z, X_z)}{\partial x} + \frac{1}{2} (f_z)^2 \frac{\partial^2 u(z, X_z)}{\partial x^2} \right) dz + \int_s^t f_z \frac{\partial u(z, X_z)}{\partial x} dW_z \tag{6.1.13}$$

for $t \in [0, \infty)$ and $s \in [0, t]$. We remark that by using the notion of quadratic variation, introduced in the previous chapter, we can write the Itô formula (6.1.12) in the compact form

$$du(t, X_t) = \frac{\partial u(t, X_t)}{\partial t} dt + \frac{\partial u(t, X_t)}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 u(t, X_t)}{\partial x^2} d[X]_t \tag{6.1.14}$$

for $t \in [0, \infty)$. One can read off the following rule

$$(dX_t)^2 = d[X]_t$$

if X is a continuous process. This generalizes the rule that we mentioned after equation (6.1.5). As shown in Föllmer (1981), the Itô formula holds very generally in a pathwise sense requiring almost no technical assumptions.

Example for a Stochastic Exponential

Let us consider the one dimensional Itô differential

$$dX_t = e_t \, dt + f_t \, dW_t \tag{6.1.15}$$

for $t \in [0, \infty)$ with initial value $X_0 = x_0$. Note by (5.4.5) that

$$d[X]_t = (f_t)^2 \, d[W]_t = (f_t)^2 \, dt.$$

The exponential

$$Y_t = u(X_t) = \exp\{X_t\} \tag{6.1.16}$$

has then by application of the Itô formula (6.1.12) the Itô differential

$$\begin{aligned} dY_t &= d(\exp\{X_t\}) \\ &= \exp\{X_t\} \left(e_t + \frac{1}{2} (f_t)^2 \right) dt + \exp\{X_t\} f_t \, dW_t \\ &= Y_t \left(e_t + \frac{1}{2} (f_t)^2 \right) dt + Y_t \, f_t \, dW_t \end{aligned} \quad (6.1.17)$$

for $t \in [0, \infty)$ with initial value $Y_0 = \exp\{x_0\}$. In the case when $e_t = e$ and $f_t = f$ are constants, the process $X = \{X_t, \, t \in [0, \infty)\}$ is a transformed Wiener process, see (3.2.7), and $Y = \{Y_t, \, t \in [0, \infty)\}$ is a geometric Brownian motion, as is employed under the BS model. One can interpret Y as a solution of a *stochastic differential equation* (SDE) since here some feedback in the drift and diffusion coefficient is built in. In the next chapter we shall study SDEs of more general form.

Example for Powers of Processes

Let us give another example, where we start again from the Itô differential (6.1.15) for the process $X = \{X_t, \, t \in [0, \infty)\}$. Now we consider for some exponent $k \neq 0$ the power

$$Y_t = u(X_t) = (X_t)^k \quad (6.1.18)$$

for $t \in [0, \infty)$. By application of the Itô formula (6.1.12) we obtain the Itô differential

$$\begin{aligned} dY_t &= k \, (X_t)^{k-1} \, (e_t \, dt + f_t \, dW_t) + \frac{1}{2} \, k \, (k-1) \, (X_t)^{k-2} \, (f_t)^2 \, dt \\ &= k \, (Y_t)^{\frac{k-1}{k}} \, (e_t \, dt + f_t \, dW_t) + \frac{1}{2} \, k \, (k-1) \, (Y_t)^{\frac{k-2}{k}} \, (f_t)^2 \, dt \end{aligned} \quad (6.1.19)$$

for $t \in [0, \infty)$ with $Y_0 = (x_0)^k$.

6.2 Multivariate Itô Formula

In the context of financial modeling, the discussion of functionals of two or more underlying stochastic processes, such as a stock price and a stochastic interest rate, is often required. To enable us to treat such problems properly we consider multi-dimensional stochastic processes or, equivalently, vector valued Itô integrals. For this reason we introduce multi-component Itô differentials with respect to multi-dimensional standard Wiener processes. These then will appear in a multivariate version of the Itô formula to be formulated below.

Multi-Dimensional Wiener Process

Definition 6.2.1. *We call the vector process* $\boldsymbol{W} = \{\boldsymbol{W}_t = (W_t^1, W_t^2, \ldots, W_t^m)^\top,\ t \in [0,\infty)\}$ *an* m-dimensional standard Wiener process *if each of its components* $W^j = \{W_t^j,\ t \in [0,\infty)\},\ j \in \{1,2,\ldots,m\}$ *is a scalar* $\underline{\mathcal{A}}$*-adapted standard Wiener process and the Wiener processes* W^k *and* W^j *are independent for* $k \neq j,\ k,j \in \{1,2,\ldots,m\}$.

This means that according to Definition 3.2.2 of a Wiener process, each random variable W_t^j is Gaussian and $\mathcal{A}_t$-measurable with

$$E\left(W_t^j \,\middle|\, \mathcal{A}_0\right) = 0 \tag{6.2.1}$$

and we have independent increments $W_t^j - W_s^j$ such that

$$E\left(W_t^j - W_s^j \,\middle|\, \mathcal{A}_s\right) = 0 \tag{6.2.2}$$

for $t \in [0,\infty),\ s \in [0,t]$ and $j \in \{1,2,\ldots,m\}$. Moreover, we have here additionally the property that

$$E\left(\left(W_t^i - W_s^i\right)\left(W_t^j - W_s^j\right) \,\middle|\, \mathcal{A}_s\right) = \begin{cases} (t-s) & \text{for } i=j \\ 0 & \text{otherwise} \end{cases} \tag{6.2.3}$$

for $t \in [0,\infty),\ s \in [0,t]$ and $i,j \in \{1,2,\ldots,m\}$.

Note that the covariation between different components of the above standard Wiener process is zero, see (5.4.5), that is

$$[W^i, W^j]_t = \begin{cases} t & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \tag{6.2.4}$$

for $t \in [0,\infty)$ and $i,j \in \{1,2,\ldots,m\}$.

To illustrate the above notion in preparation of future examples, Fig. 6.2.1 shows the sample paths of the two components of a two-dimensional standard Wiener process. Each of these two components forms a standard one-dimensional Wiener process and both Wiener processes are independent.

Figure 6.2.2 presents a different visualization of the same pair of trajectories for the two-dimensional Wiener process. Here W_t^1 and W_t^2 represent the x and y coordinates at time t, respectively, that generate a trace similar to the motion of a pollen particle under the microscope. Recall that such a motion was originally observed by Robert Brown, giving rise to it the name Brownian motion. As indicated above, it can be modeled by two independent Wiener processes.

Vector Itô Differentials

Consider a d-dimensional vector function $\boldsymbol{e} : [0,\infty) \times \Omega \to \Re^d$ with predictable components $e^k,\ k \in \{1,2,\ldots,d\}$. We have to assume that the components

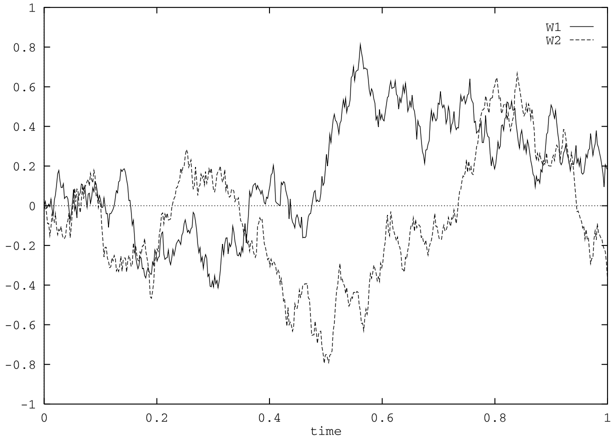

Fig. 6.2.1. Components of a two-dimensional standard Wiener process

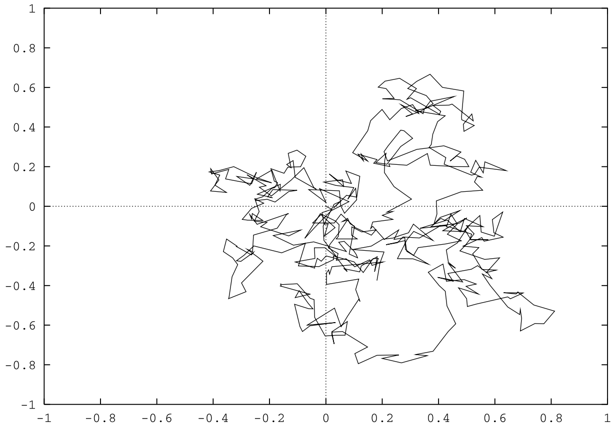

Fig. 6.2.2. Trace of a two-dimensional Wiener process

satisfy appropriate integrability and measurability conditions. These are similar to those we introduced for the one-dimensional case. For simplicity, we assume here that

$$\int_0^T |e_z^k|\, dz < \infty \tag{6.2.5}$$

almost surely for $k \in \{1, 2, \ldots, d\}$ and $\boldsymbol{F} : [0, T] \times \Omega \to \Re^{d \times m}$ to be a $d \times m$ matrix valued function with

$$\int_0^T \left(F_z^{i,j}\right)^2 \, dz < \infty \tag{6.2.6}$$

almost surely for $i \in \{1, 2, \ldots, d\}$, $j \in \{1, 2, \ldots, m\}$ and all $T \in (0, \infty)$, see Protter (2004). This allows us to introduce a d-dimensional stochastic vector process $\boldsymbol{X} = \{\boldsymbol{X}_t = (X_t^1, X_t^2, \ldots, X_t^d)^\top,\, t \in [0,\infty)\}$, where the kth component X^k is defined via the Itô integral equation

$$X_t^k - X_0^k = \int_0^t e_z^k \, dz + \sum_{j=1}^{m} \int_0^t F_z^{k,j} \, dW_z^j \tag{6.2.7}$$

for $t \in [0,\infty)$ and given $\mathcal{A}_0$-measurable initial value $X_0^k \in \Re$, $k \in \{1, 2, \ldots, d\}$.

Analogous to the scalar case we denote by $\boldsymbol{e}_t$ and $\boldsymbol{F}_t$ for a given time $t \in [0,\infty)$ the vector and matrix valued random variables, respectively. Then we write the vector valued stochastic integral equation in the form

$$\boldsymbol{X}_t - \boldsymbol{X}_0 = \int_0^t \boldsymbol{e}_z \, dz + \int_0^t \boldsymbol{F}_z \, d\boldsymbol{W}_z \tag{6.2.8}$$

for any $t \in [0,\infty)$ with initial value $\boldsymbol{X}_0 = (X_0^1, \ldots, X_0^d)^\top$. This can be expressed equivalently as the d-dimensional *vector Itô differential* given by

$$d\boldsymbol{X}_t = \boldsymbol{e}_t \, dt + \boldsymbol{F}_t \, d\boldsymbol{W}_t, \tag{6.2.9}$$

for $t \in [0,\infty)$ with initial value $\boldsymbol{X}_0 \in \Re^d$. Choosing the dimension $d = 1$, leads to the case of a scalar Itô differential with respect to several independent Wiener processes.

Multivariate Itô Formula

In the previous section it was noted that for the scalar case with one driving Wiener process the Itô formula involves the quadratic variation of this Wiener process. In the multivariate case, with a multi-dimensional driving Wiener process, it turns out that the covariations between different components of the vector stochastic differential appear in the following *multivariate Itô formula.*

Theorem 6.2.2. *Assume that the function* $u : [0,\infty) \times \Re^d \to \Re$ *has continuous partial derivatives* $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x^k}$ *and* $\frac{\partial^2 u}{\partial x^k \, \partial x^i}$ *for* $k, i \in \{1, 2, \ldots, d\}$ *and* $\boldsymbol{x} = (x^1, x^2, \ldots, x^d)^\top$. *Define a scalar stochastic process* $Y = \{Y_t,\, t \in [0,\infty)\}$ *by setting*

$$Y_t = u(t, X_t^1, X_t^2, \ldots, X_t^d), \tag{6.2.10}$$

for $t \in [0,\infty)$, *where the vector* $\boldsymbol{X}_t = (X_t^1, X_t^2, \ldots, X_t^d)^\top$ *satisfies the vector Itô differential* (6.2.9). *Then the Itô differential for* Y *is of the form*

$$\begin{aligned} dY_t &= du(t, X_t^1, X_t^2, \ldots, X_t^d) \\ &= \left\{ \frac{\partial u}{\partial t} + \sum_{k=1}^{d} e_t^k \, \frac{\partial u}{\partial x^k} + \frac{1}{2} \sum_{j=1}^{m} \sum_{i,k=1}^{d} F_t^{i,j} \, F_t^{k,j} \, \frac{\partial^2 u}{\partial x^i \, \partial x^k} \right\} dt \\ &\quad + \sum_{j=1}^{m} \sum_{i=1}^{d} F_t^{i,j} \, \frac{\partial u}{\partial x^i} \, dW_t^j, \end{aligned} \tag{6.2.11}$$

for $t \in [0, \infty)$ *with* $Y_0 = u(0, X_0^1, X_0^2, \ldots, X_0^d)$. *Here the partial derivatives in* (6.2.11) *are evaluated at* $(t, X_t^1, X_t^2, \ldots, X_t^d)$.

An informal derivation of this formula, similar to that for the scalar case presented earlier in (6.1.12), provides a quick and insightful way to understand where the various terms appearing in (6.2.11) come from. To see this easily one has simply to apply the corresponding Taylor expansion for the function u and use the rules

$$dW_t^i \, dW_t^j \approx \begin{cases} dt & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases} \tag{6.2.12}$$

and

$$dW_t^i \, dt \approx 0. \tag{6.2.13}$$

As previously explained, these rules of stochastic calculus yield terms in addition to those usually observed in the deterministic chain rule because of the effects of the covariations between the integrands and integrators involved.

Similar as in formula (6.1.14), by using the notion of covariation, we can write the multivariate Itô formula (6.2.11) in the form

$$du(t, X_t^1, X_t^2, \ldots, X_t^d) = \frac{\partial u}{\partial t} \, dt + \sum_{i=1}^{d} \frac{\partial u}{\partial x^i} \, dX_t^i + \frac{1}{2} \sum_{i,k=1}^{d} \frac{\partial^2 u}{\partial x^i \, \partial x^k} \, d[X^i, X^k]_t \tag{6.2.14}$$

for all $t \in [0, \infty)$. Here the partial derivatives of u on the right hand side of (6.2.14) are taken at $(t, X_t^1, X_t^2, \ldots, X_t^d)$.

6.3 Some Applications of the Itô Formula

Integration-by-Parts Formula

Let us consider two continuous processes $X^1 = \{X_t^1, \, t \in [0, \infty)\}$ and $X^2 = \{X_t^2, \, t \in [0, \infty)\}$ having an Itô differential and finite covariation. Suppose that the Itô differential of the product

$$Y_t = u(t, X_t^1, X_t^2) = X_t^1 \, X_t^2$$

is required. The Itô formula (6.2.14) can then be used to derive for the above product the following *integration-by-parts formula*

$$d(X_t^1 \, X_t^2) = X_t^1 \, dX_t^2 + X_t^2 \, dX_t^1 + d[X^1, X^2]_t. \tag{6.3.1}$$

We consider as an example three cases, where X^1 and X^2 are standard Wiener processes:

1. First assume that the two Wiener processes are the same, that is $X_t^1 = X_t^2 = W_t^1$, where W_t^1 is a standard Wiener process. This case was considered in (6.1.5). By rewriting this equation we obtain

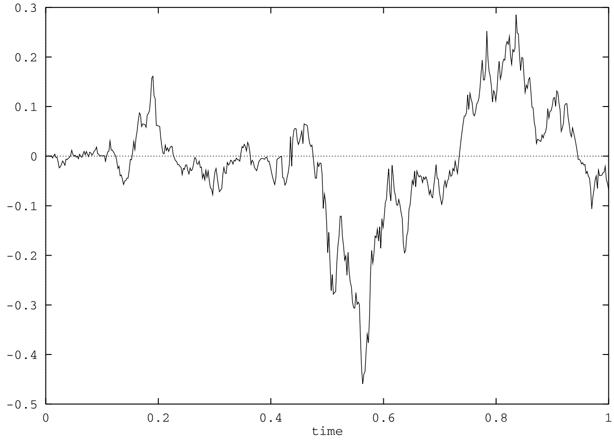

Fig. 6.3.1. Product of two independent Wiener processes

$$dY_t = d(W_t^1)^2 = 2\, W_t^1\, dW_t^1 + dt \tag{6.3.2}$$

for $t \in [0, \infty)$. Recall that Fig. 5.3.3 displayed a sample path for $\frac{1}{2}\, Y_t = \frac{1}{2}(W_t^1)^2$.

2. In the second case we assume that the two Wiener processes $X^1 = W^1$ and $X^2 = W^2$ are independent, that is W^1 and W^2 are two independent standard Wiener processes. This then leads by application of the integration-by-parts formula (6.3.1) to the Itô differential

$$dY_t = d(W_t^1\, W_t^2) = W_t^1\, dW_t^2 + W_t^2\, dW_t^1 \tag{6.3.3}$$

for $t \in [0, \infty)$. Note that there is no drift on the right hand side of (6.3.3) since the covariation between the two independent Wiener processes is zero. The formula (6.3.3) coincides with the classic integration by parts formula because there is zero covariation between W^1 and W^2. In Fig. 6.3.1 we use the same sample path of the two-dimensional standard Wiener process that was shown in Fig. 6.2.1 and Fig. 6.2.2 to generate a corresponding path for the product $Y_t = W_t^1\, W_t^2$.

3. The third case assumes that the two standard Wiener processes are correlated, that is we set $X_t^1 = \varrho\, W_t^1 + \sqrt{1-\varrho^2}\, W_t^2$ and $X_t^2 = W_t^1$, where W^1 and W^2 are independent standard Wiener processes and $\varrho \in [-1, 1]$ is the correlation coefficient, see (1.4.39). We then obtain by the formula (6.3.1) the Itô differential

$$\begin{aligned} dY_t &= d(X_t^1\, X_t^2) \\ &= X_t^1\, dW_t^1 + X_t^2 \left(\varrho\, dW_t^1 + \sqrt{1-\varrho^2}\, dW_t^2 \right) + \varrho\, dt \\ &= X_t^1\, dX_t^2 + X_t^2\, dX_t^1 + \varrho\, dt \end{aligned} \tag{6.3.4}$$

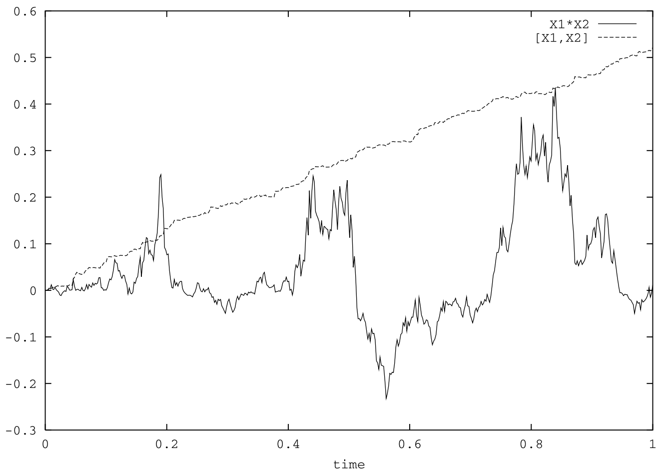

Fig. 6.3.2. Product of correlated Wiener processes and their approximate covariation

for $t \in [0, \infty)$. It is easy to see that both (6.3.2) and (6.3.3) are included in formula (6.3.4) for the choices of $\varrho = 1$ and $\varrho = 0$, respectively. Note that for the product of Wiener processes, the correlation coefficient ϱ appears as the drift coefficient in the resulting Itô differential.

Finally, to demonstrate the covariation of correlated Wiener processes, we show in Fig. 6.3.2 the approximate covariation $[X^1, X^2]_{h,t}$, see (5.2.15), together with the product $X_t^1 \, X_t^2$ for the correlation coefficient $\varrho = \frac{1}{2}$.

Example for Geometric Brownian Motion

Let us consider a one-dimensional Itô differential that uses two independent standard Wiener processes and is given by

$$dX_t = e_t^1 \, dt + F_t^{1,1} \, dW_t^1 + F_t^{1,2} \, dW_t^2 \tag{6.3.5}$$

with initial value $X_0 = 0$. The functional

$$Y_t = \exp\{X_t\} \tag{6.3.6}$$

has by the Itô formula (6.2.11) the Itô differential

$$\begin{aligned} dY_t &= d\left(\exp\{X_t\}\right) \\ &= Y_t \left(e_t^1 + \frac{1}{2}\left(\left(F_t^{1,1}\right)^2 + \left(F_t^{1,2}\right)^2 \right) \right) dt + Y_t \, F_t^{1,1} \, dW_t^1 + Y_t \, F_t^{1,2} \, dW_t^2 \end{aligned} \tag{6.3.7}$$

for $t \in [0, \infty)$, with initial value $Y_0 = 1$. Note that the process Y is a diffusion process. More precisely, it is a generalized version of geometric Brownian motion, introduced in (4.1.2). Here we have the drift coefficient

$$a(t,x) = x\left(e_t^1 + \frac{1}{2}\left(\left(F_t^{1,1}\right)^2 + \left(F_t^{1,2}\right)^2\right)\right), \tag{6.3.8}$$

that can be compared to (4.3.7). The diffusion coefficients corresponding to W^1 and W^2 are given by

$$b^1(t,x) = x\,F_t^{1,1} \tag{6.3.9}$$

and

$$b^2(t,x) = x\,F_t^{1,2}, \tag{6.3.10}$$

respectively. These diffusion coefficients generalize what was obtained in (4.3.8), where we had only one driving Wiener process.

For the above Itô differential both the drift and diffusion coefficients appear as products of the asset price with some constants, as was the case in (6.3.7). The constant associated with the drift coefficient is often called the *appreciation rate* or *expected rate of return*. Recall that the constant associated with a diffusion coefficient is the *volatility* component for this diffusion term. If appreciation rate and volatilities are constants, then the corresponding model is called the *Black-Scholes* (BS) *model*.

If one looks at the stochastic differential (6.3.7), then a certain feedback in the drift and diffusion term is modeled. We call an Itô differential that involves some feedback from the state variable, here Y_t, a *stochastic differential equation* (SDE). It will be our focus in the next chapter to present results on SDEs. However, within this chapter we continue to study the Itô formula applied to stochastic differentials which cover also SDEs.

Product of Two Geometric Brownian Motions

Since the Black-Scholes model plays such a central role in asset price modeling, we go in detail through a number of almost elementary applications of the Itô formula. Consider two asset price processes X^1 and X^2 that are defined as geometric Brownian motions by functionals of the type

$$X_t^i = \exp\left\{\mu^i\,t + \sigma^{i,1}\,W_t^1 + \sigma^{i,2}\,W_t^2\right\}$$

for $i \in \{1,2\}$ and $t \in [0,\infty)$, where W^1 and W^2 denote two independent standard Wiener processes.

By the Itô formula (6.2.11) we obtain, similarly to (6.3.7), the Itô differentials

$$dX_t^i = X_t^i\left(\mu^i + \frac{1}{2}\left((\sigma^{i,1})^2 + (\sigma^{i,2})^2\right)\right)dt + X_t^i\,\sigma^{i,1}\,dW_t^1 + X_t^i\,\sigma^{i,2}\,dW_t^2 \tag{6.3.11}$$

for $i \in \{1,2\}$ and $t \in [0,\infty)$.

We compute the Itô differential of the product $Y_t = X_t^1\,X_t^2$. Again, by application of the Itô formula (6.2.11) we obtain

$$
\begin{aligned}
dY_t &= d(X_t^1\, X_t^2) \\
&= Y_t \left(\mu^1 + \mu^2 + \frac{1}{2}\left(\sigma^{1,1} + \sigma^{2,1}\right)^2 + \frac{1}{2}\left(\sigma^{1,2} + \sigma^{2,2}\right)^2 \right) dt \\
&\quad + Y_t \left(\sigma^{1,1} + \sigma^{2,1}\right) dW_t^1 + Y_t \left(\sigma^{1,2} + \sigma^{2,2}\right) dW_t^2 \qquad (6.3.12)
\end{aligned}
$$

for $t \in [0, \infty)$. Consequently, the product of two geometric Brownian motions is a geometric Brownian motion, since the drift and diffusion coefficients in (6.3.12) appear as products of Y_t together with some constants. Note also that the appreciation rates and the volatilities of the product of two geometric Brownian motions are obtained by summing the appreciation rates and volatilities of their components.

Powers of Geometric Brownian Motion

We have seen that products of two geometric Brownian motions are also geometric Brownian motions. We now show that the power of a geometric Brownian motion is also a geometric Brownian motion.

Let X denote a scalar geometric Brownian motion characterized by the Itô differential

$$dX_t = X_t\, a\, dt + X_t\, \sigma\, dW_t, \qquad (6.3.13)$$

for $t \in [0, \infty)$ with appreciation rate a, volatility σ and initial value $X_0 = x$, where W is a standard Wiener process. Then by application of the Itô formula (6.2.11) we obtain for any real valued exponent k and

$$Y_t = (X_t)^k \qquad (6.3.14)$$

the Itô differential

$$dY_t = d(X_t)^k = Y_t \left(k\, a + \frac{1}{2}\, k\, (k-1)\, \sigma^2 \right) dt + Y_t\, k\, \sigma\, dW_t \qquad (6.3.15)$$

for $t \in [0, \infty)$. This shows that Y is again a geometric Brownian motion, because the drift and diffusion coefficients in (6.3.15) are expressed as products of constants and Y_t.

Inverse of a Geometric Brownian Motion

An interesting phenomenon is observed when considering the dynamics of an inverse of a given geometric Brownian motion, which follows for the exponent $k = -1$ from equation (6.3.15). Taking the stochastic differentials (6.3.12) and (6.3.15) into account, it is clear that not only powers and products but also ratios of geometric Brownian motions are again geometric Brownian motions. These convenient properties certainly had some influence on the historical

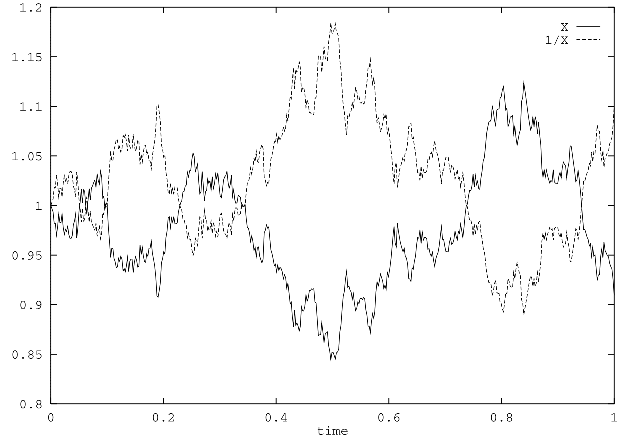

Fig. 6.3.3. A geometric Brownian motion and its inverse

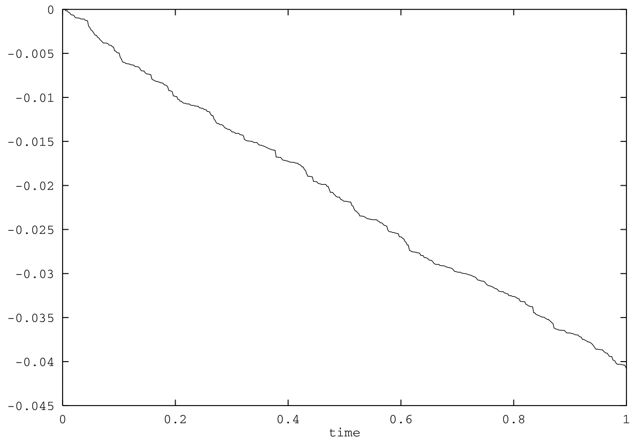

Fig. 6.3.4. Approximate covariation between $\frac{1}{X}$ and X

development of quantitative finance. In particular, they helped to make the BS model the standard market model.

Figure 6.3.3 shows a sample path of a geometric Brownian motion $X = \{X_t,\ t \in [0, \infty)\}$ with $X_0 = 1,\ a = 0,\ \sigma = 0.2$ together with its inverse $\frac{1}{X_t}$. As is apparent from (6.3.15), in this case the inverse X_t^{-1} has an appreciation rate equal to σ^2 and is negatively correlated to X_t. This negative correlation is visualized in Fig. 6.3.4, which displays the covariation $[X^{-1}, X]_t$ between X_t^{-1} and X_t. This covariation, see (5.4.5), is given by

$$[X^{-1}, X]_t = -\sigma^2\, t. \tag{6.3.16}$$

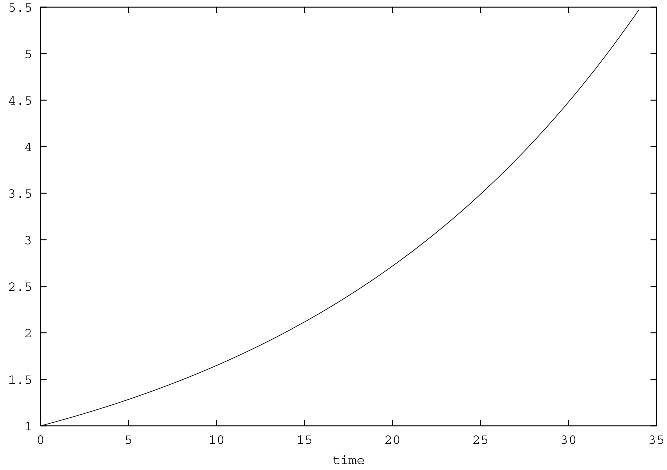

Fig. 6.3.5. Savings account

Black-Scholes Model for a Stock Market

Let us now model a stock market in continuous time with asset prices that follow geometric Brownian motions. The fluctuations of stock prices in the market are driven by continuous trading uncertainty, which is modeled by $d \in \mathcal{N}$ independent standard Wiener processes $W^1, W^2, \ldots, W^d$.

For simplicity, we consider a deterministic, constant short rate r. We assume that the interest is continuously accrued. To model the accumulation of interest we form the *savings account* S_t^0 at time t as the exponential

$$S_t^0 = \exp\left\{X_t^0\right\} \tag{6.3.17}$$

with

$$X_t^0 = r\,t \tag{6.3.18}$$

for $t \in [0, \infty)$. Obviously,

$$dX_t^0 = r\,dt$$

and, therefore, by the Itô formula when applied to the exponential function (6.3.17), we obtain the differential equation

$$dS_t^0 = S_t^0\,r\,dt \tag{6.3.19}$$

for $t \in [0, \infty)$ with $S_0^0 = 1$. In Fig. 6.3.5 we plot the resulting savings account for a period of $T = 34$ years when choosing a constant interest rate of $r = 0.05$. Note that an initial investment of one dollar in the savings account would have resulted over the given period in a value of about 5.5 dollars. The logarithm $X_t^0 = \ln(S_t^0)$ of the savings account is a linear increasing function, see (6.3.18), with slope equal to the short rate r.

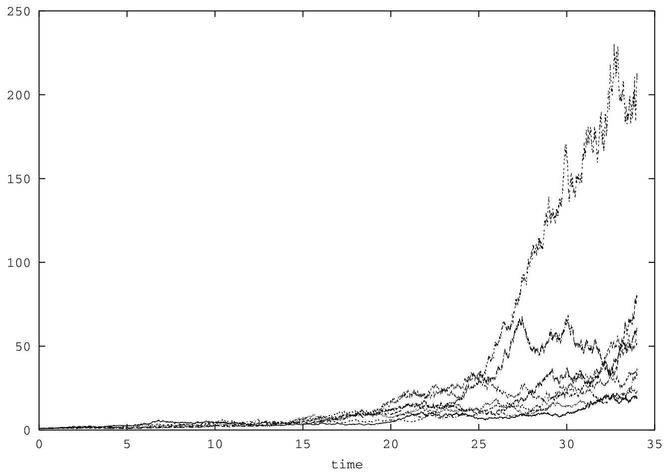

Fig. 6.3.6. Stock prices

Now we visualize in Fig. 6.3.6 eight cum dividend stock price processes over the period of 34 years. Here we reinvest all dividends. The jth stock price at time t is denoted by S_t^j for $t \in [0, \infty)$ and $j \in \{1, 2, \ldots, d\}$. For simplicity, we have chosen the volatility matrix to be of the form $\boldsymbol{b} = \sigma \boldsymbol{I}$, where $\sigma = 0.2$ is the volatility parameter and $\boldsymbol{I}$ the unit matrix. In this simple setting each stock evolves independently from all the others. For the simulated scenario we used the volatility parameter $\sigma = 0.2$, the short rate $r = 0.05$ and have set the growth rates to $g^j = 0.1$. As we shall see later, this is a rather poor stock market model since no correlations between the log-returns are modeling the example. Nevertheless such models have been used in practice. It is noticeable in Fig. 6.3.6 that extreme differences in stock prices over the 34 year period can occur. However, it is impossible to predict at any time which of the stocks will outperform the others in the future. They all have in our example the same appreciation rate and volatility.

One notes that the prices in Fig. 6.3.6 evolve quite differently. On average they seem to increase. We constructed these stock prices as exponentials of transformed Wiener processes $X^j = \{X_t^j,\, t \in [0, \infty)\}$, $j \in \{1, 2, \ldots, d\}$, where

$$X_t^j = g^j\, t + \sum_{k=1}^{d} b^{j,k}\, W_t^k \tag{6.3.20}$$

and the jth stock price is given as

$$S_t^j = \exp\{X_t^j\} \tag{6.3.21}$$

with $S_0^j > 0$.

The log-price X_t^j of the jth stock at time t, $j \in \{1, 2, \ldots, d\}$, is therefore modeled by the Itô differential

$$dX_t^j = g^j\, dt + \sum_{k=1}^{d} b^{j,k}\, dW_t^k \tag{6.3.22}$$

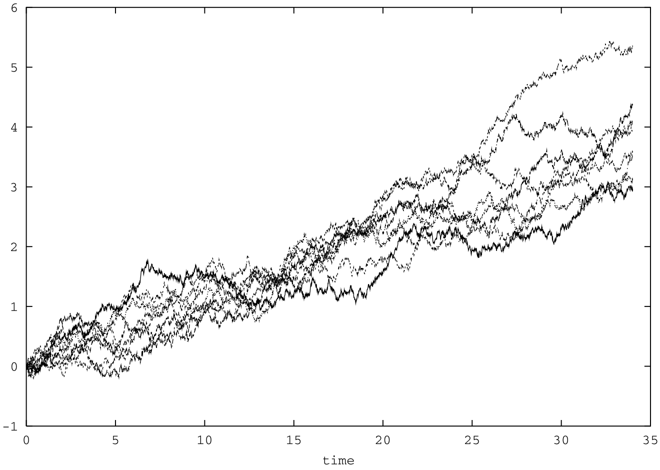

Fig. 6.3.7. Logarithms of stock prices

for $t \in [0, \infty)$ with initial value $X_0^j = \ln(S_0^j) \in \Re$, $j \in \{1, 2, \ldots, d\}$. The jth *growth rate* g^j and the j,kth volatilities $b^{j,k}$ are deterministic constants for $j, k \in \{1, 2, \ldots, d\}$. Here we have the jth growth rate

$$g^j = r + p^j - \frac{1}{2} \sum_{k=1}^{d} (b^{j,k})^2. \tag{6.3.23}$$

This leads by application of the Itô formula to the function (6.3.21) for the jth stock price to its Itô differential or SDE

$$dS_t^j = S_t^j \left((r + p^j)\, dt + \sum_{k=1}^{d} b^{j,k}\, dW_t^k \right) \tag{6.3.24}$$

for $t \in [0, \infty)$ and $j \in \{1, 2, \ldots, d\}$. The appreciation rate of the jth stock then equals the sum

$$a^j = r + p^j, \tag{6.3.25}$$

where p^j is the jth *risk premium* or jth *expected excess return*. The matrix $\boldsymbol{b} = [b^{j,k}]_{j,k=1}^d$ denotes the volatility matrix. In Fig. 6.3.7 we plot the logarithms X_t^i of the eight stock prices over time.

Covariation between a Wiener Process and a Functional (*)

Let g denote a twice continuously differentiable function and W a standard Wiener process. Then the covariation, see (5.2.16), between $g(W_t)$ and W_t is given by

$$[g(W), W]_t = \int_0^t g'(W_s)\, ds \tag{6.3.26}$$

for $t \in [0, \infty)$. This can be easily derived by application of both the Itô formula (6.1.12) together with the covariation property (5.4.5) of Itô integrals, which yields

$$d(g(W_t)) = \frac{1}{2}\, g''(W_t)\, dt + g'(W_t)\, dW_t \tag{6.3.27}$$

for $t \in [0, \infty)$. One can also formulate similar statements when the standard Wiener process is substituted by more general processes.

6.4 Extensions of the Itô Formula

Let us mention in this section some extensions of the Itô formula that will allow us to derive powerful results for models with jumps covering stochastic processes that are needed for modeling event driven uncertainty in finance and insurance.

Itô Formula for Jump Processes

The Itô formula can be easily generalized to the case of jump processes. Let us use again our standard notation for the *jump size*

$$\Delta Z_t = Z_t - Z_{t-} \tag{6.4.1}$$

at time $t \in [0, \infty)$ of a given process $Z = \{Z_t,\, t \in [0, \infty)\}$. Here Z_{t-} denotes, as usual, the left hand limit of the process Z at time t. Then the value X_t of a pure jump process $X = \{X_t,\, t \in [0, \infty)\}$ can be written at time $t \in [0, \infty)$ as

$$X_t = \sum_{s \in [0,t]} \Delta X_s \tag{6.4.2}$$

if this sum converges almost surely for all $t \in [0, \infty)$. This then allows us to formulate the Itô formula for the given pure jump process in such a simple form that does not need any extra proof.

Lemma 6.4.1. *For a pure jump process X and a measurable function $u : \Re \to \Re$ we have the Itô formula*

$$u(X_t) = u(X_0) + \sum_{s \in (0,t]} \Delta u(X_s) \tag{6.4.3}$$

for $t \in [0, T]$, where $\Delta u(X_t) = u(X_t) - u(X_{t-})$.

One notes that almost no assumptions are imposed on the function $u(\cdot)$ and the process X. What happens in (6.4.3) is that the jumps of X are simply transferred through the function u as they arise.

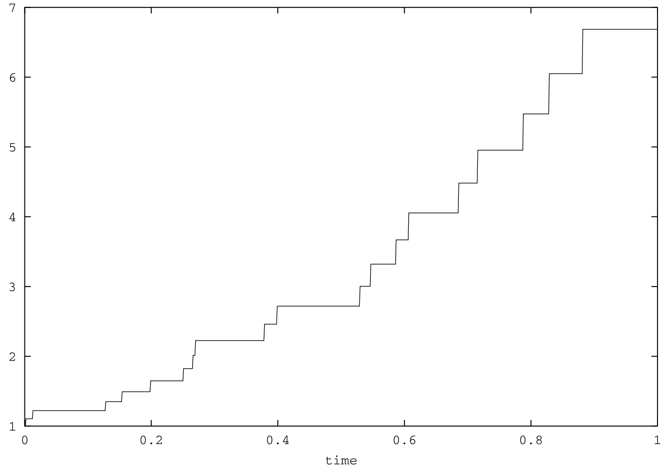

Fig. 6.4.1. Path of an exponential of a Poisson process

Exponential of a Poisson Process

Let us consider an example where a pure jump process plays a role. We denote by $N = \{N_t,\, t \in [0,\infty)\}$ a Poisson process with intensity λ as introduced in Sect. 3.5. A path of such a process for $\lambda = 20$ is shown in Fig. 3.5.1. Let us now apply the Itô formula (6.4.2) to obtain for N the differential of the exponential $u(N_t) = \exp\{c\,N_t\}$ with $c > 0$. Since N is a pure jump process that counts the arrival of events we have only to transform its jumps into the jumps of the exponential of N. Thus, at the kth jump time τ_k of N we have the identity

$$\exp\{c\,N_{\tau_k}\} = \exp\{c\,N_{\tau_k-}\} + \exp\{c\,N_{\tau_k-}\}\left(\frac{\exp\{c\,N_{\tau_k}\}}{\exp\{c\,N_{\tau_k-}\}} - 1\right). \quad (6.4.4)$$

In Fig. 3.5.1 we showed a trajectory of a Poisson process N. In Fig. 6.4.1 we plot now the corresponding exponential with $c = 0.1$.

By (6.4.4) we obtain the relationship

$$\begin{aligned}\exp\{c\,N_t\} &= \exp\{c\,N_0\} + \int_0^t \left(\exp\{c\,N_s\} - \exp\{c\,N_{s-}\}\right) dN_s \\ &= \exp\{c\,N_0\} + \int_0^t \exp\{c\,N_{s-}\}\left(\frac{\exp\{c\,N_s\}}{\exp\{c\,N_{s-}\}} - 1\right) dN_s. \quad (6.4.5)\end{aligned}$$

Equivalently, with the notation (6.4.1) we can write the corresponding Itô differential

$$\begin{aligned}d(\exp\{c\,N_t\}) &= \Delta(\exp\{c\,N_t\}) \\ &= \exp\{c\,N_{t-}\}\,(\psi_{\exp}(t-) - 1)\,\Delta N_t \quad (6.4.6)\end{aligned}$$

for $t \in [0,\infty)$ with *jump ratio*

$$\psi_{\exp}(\tau_k-) = \frac{\exp\{c\,N_{\tau_k}\}}{\exp\{c\,N_{\tau_k-}\}} = \exp\{c\,N_{\tau_k} - c\,N_{\tau_k-}\} = \exp\{c\} = e^c \qquad (6.4.7)$$

with τ_k as kth jump time. Note that the use of the notion of a jump ratio for the parametrization of the jump size is rather convenient.

Itô Formula for Semimartingales (*)

After having seen that the inclusion of jumps does not create major problems for an Itô formula, the Itô formula can now be generalized to the case of semimartingales, see Definition 5.5.1. Assume that the vector process $\boldsymbol{X} = \{\boldsymbol{X}_t = (X_t^1, \ldots, X_t^\ell)^\top,\, t \in [0,\infty)\}$ has as its ith component the semimartingale X^i with the following decomposition

$$X_t^i = X_0^i + X_t^{i,c} + X_t^{i,d} \qquad (6.4.8)$$

for $t \in [0,\infty)$, $i \in \{1, 2, \ldots, \ell\}$. Here

$$X_t^{i,c} = A_t^{i,c} + M_t^{i,c} \qquad (6.4.9)$$

denotes the ith component of the continuous part of X_t^i and $X_t^{i,d}$ that of the pure jump part. This means, we have all jumps absorbed in the term

$$X_t^{i,d} = \sum_{s\in[0,t]} \Delta X_s^i \qquad (6.4.10)$$

for $t \in [0,\infty)$ and $i \in \{1, 2, \ldots, \ell\}$. Furthermore, $M^{i,c}$ denotes in (6.4.9) a continuous $(\underline{\mathcal{A}}, P)$-local martingale, see (5.2.26), and $A^{i,d}$ a continuous process of finite total variation, see (5.2.25).

Theorem 6.4.2. *For a twice continuously differentiable function* $u : [0,\infty) \times \Re^\ell \to \Re$, *with continuous first derivative with respect to time and second continuous derivatives with respect to the spatial variables, we have the Itô formula*

$$\begin{aligned} u(t, X_t^1, \ldots, X_t^\ell) &= u(0, X_0^1, \ldots, X_0^\ell) \\ &+ \int_0^t \frac{\partial}{\partial t} u(s, X_s^1, \ldots, X_s^\ell)\, ds + \sum_{i=1}^\ell \int_0^t \frac{\partial}{\partial x^i} u(s, X_s^1, \ldots, X_s^\ell)\, dX_s^{i,c} \\ &+ \frac{1}{2} \int_0^t \sum_{i,k=1}^\ell \frac{\partial^2}{\partial x^i\, \partial x^k} u(s, X_s^1, \ldots, X_s^\ell)\, d[M^{i,c}, M^{k,c}]_s \\ &+ \sum_{s\in(0,t]} \Delta u(s, X_s^1, \ldots, X_s^\ell) \end{aligned} \qquad (6.4.11)$$

for $t \in [0,\infty)$. *Here the jump size* Δu *of* u *at time* s *is defined as in* (5.5.8), *namely*

$$\Delta u(s, X_s^1, \ldots, X_s^\ell) = u(s, X_s^1, \ldots, X_s^\ell) - u(s-, X_{s-}^1, \ldots, X_{s-}^\ell). \qquad (6.4.12)$$

A proof of the general Itô formula (6.4.11) can be found, for instance, in Protter (2004). We remark that in (6.4.11) the jumps are simply transferred through the function u whenever they occur. The Itô formula is almost identical to that for diffusions if there were no jumps. The above general Itô formula can be essential for situations where continuous and event driven uncertainty arises in a model. Similarly to (6.2.14) we can write the Itô formula (6.4.11) in the form

$$du(t, X_t^1, \ldots, X_t^\ell) = \frac{\partial}{\partial t} u(t, X_t^1, \ldots, X_t^\ell)\, dt + \sum_{i=1}^{\ell} \frac{\partial}{\partial x^i} u(t, X_t^1, \ldots, X_t^\ell)\, dX_t^{i,c}$$

$$+\frac{1}{2} \sum_{i,k=1}^{\ell} \frac{\partial^2}{\partial x^i\, \partial x^k} u(t, X_t^1, \ldots, X_t^\ell)\, d[X^{i,c}, X^{k,c}]_t + \Delta u(t, X_t^1, \ldots, X_t^\ell) \tag{6.4.13}$$

for $t \in [0, \infty)$.

Exponential of Compensated Poisson Process (*)

Let us continue the example concerning the exponential of a Poisson process by considering the compensated Poisson process $q = \{q_t,\, t \in [0, \infty)\}$, which is a jump martingale, where

$$dq_t = dN_t - \lambda\, dt \tag{6.4.14}$$

for $t \in [0, \infty)$. By the Itô formula (6.4.11) we obtain for $u(q_t) = \exp\{c\, q_t\}$ the stochastic differential

$$\begin{aligned} d\left(\exp\{c\, q_t\}\right) &= -\exp\{c\, q_{t-}\}\, \lambda\, dt + \exp\{c\, q_{t-}\} \left(\psi_{\exp}(t-) - 1\right)\, dN_t \\ &= \exp\{c\, q_{t-}\}\, \lambda\, \left(\psi_{\exp}(t-) - 2\right)\, dt \\ &\quad + \exp\{c\, q_{t-}\} \left(\psi_{\exp}(t-) - 1\right)\, dq_t \\ &= \exp\{c\, q_{t-}\}\, \lambda\, (e^c - 2)\, dt + \exp\{c\, q_{t-}\}\, (e^c - 1)\, dq_t \end{aligned} \tag{6.4.15}$$

for $t \in [0, \infty)$. Here the jump ratio $\psi_{\exp}(t-) = e^c$ remains as in the case of the exponential of a Poisson process. Note that the last part of the sum on the right hand side of (6.4.15) is a martingale differential.

Exponential for Wiener Process with Jumps (*)

To provide another example for the above Itô formula (6.4.11) let us add to the dynamics of a Poisson process a Wiener process $W = \{W_t,\, t \in [0, \infty)\}$ and a trend. We consider now the exponential of the process $X = \{X_t,\, t \in [0, \infty)\}$ with Itô differential

$$dX_t = g\, dt + \sigma\, dW_t + c\, (dN_t - \lambda\, dt) \tag{6.4.16}$$

for $t \in [0, \infty)$. Here we use as additional parameters the growth rate g, the intensity λ and the volatility σ. The Itô formula (6.4.11) yields for the exponential

$$u(X_t) = \exp\{X_t\}$$

the stochastic differential

$$\begin{aligned} d\left(\exp\{X_t\}\right) &= \exp\{X_t\}\left(g + \frac{1}{2}\,\sigma^2 + \lambda\left(\psi_{\exp}(t) - 2\right)\right) dt + \exp\{X_t\}\,\sigma\,dW_t \\ &\quad + \exp\{X_{t-}\}\left(\psi_{\exp}(t-) - 1\right) dq_t \\ &= \exp\{X_{t-}\}\left(\left(g + \frac{1}{2}\,\sigma^2 + \lambda\left(e^c - 2\right)\right) dt + \sigma\,dW_t + \left(e^c - 1\right) dq_t\right) \end{aligned} \tag{6.4.17}$$

for $t \in [0, \infty)$. We observe that besides the jump terms all other terms are as in the earlier versions of the Itô differential for geometric Brownian motion, see (6.1.17). Therefore, we could call the above exponential a geometric Brownian motion with jumps. The jumps of X_t are transformed by the exponential function, analogous as described already by the identity (6.4.4).

This example indicates that the Itô formula is a powerful tool that allows us to determine the stochastic differential of a function of a given stochastic differential even when jumps are present. We emphasize that the jumps are directly transferred through the given function, which makes the jump part in (6.4.17) very simple to interpret. The above jump diffusion dynamics in (6.4.16) is a special case of the Merton model, see Merton (1976), which we shall study later.

Itô Formula for Poisson Jump Measure (*)

A particular case of the Itô formula (6.4.11) is obtained when only Wiener processes and Poisson jump measures are involved. Let us assume that $\boldsymbol{W} = \{\boldsymbol{W}_t = (W_t^1, \ldots, W_t^m)^\top, t \in [0, \infty)\}$ is an m-dimensional standard Wiener process and $p_{\varphi_r}^r(dv, dt)$ denotes a Poisson measure on $\mathcal{E} \times [0, \infty)$ with intensity measure

$$\nu_{\varphi_r}^r(dv, dt) = \varphi_r(dv)\,dt, \tag{6.4.18}$$

$r \in \{m+1, m+2, \ldots, \bar{\ell}\}$, as introduced in Sect. 3.5 and used in Sect. 5.5. Suppose that the ith component X_t^i at time t of the process $\boldsymbol{X}$ has the representation

$$X_t^i = X_0^i + \int_0^t a_s^i\,ds + \sum_{k=1}^m \int_0^t b_s^{i,k}\,dW_s^k + \sum_{r=m+1}^{\bar{\ell}} \int_0^t \int_{\mathcal{E}} c^{i,r}(v, s-)\,p_{\varphi_r}^r(dv, ds) \tag{6.4.19}$$

for $t \in [0, \infty)$ and $i \in \{1, 2, \ldots, \ell\}$, where a^i, $b^{i,j}$ and $c^{i,r}$ are appropriately chosen adapted processes and the mark space is given as $\mathcal{E} = \Re \backslash \{0\}$. Then the following version of the Itô formula follows from (6.4.11).

Corollary 6.4.3. *For a function $u : [0,\infty) \times \Re^\ell \to \Re$, which is assumed to be differentiable with respect to t and twice differentiable with respect to x, for the above process $\boldsymbol{X}$ the Itô formula has the form*

$$\begin{aligned} u(t, X_t^1, \ldots, X_t^\ell) = u(0, X_0^1, \ldots, X_0^\ell) + \int_0^t \Bigg(& \frac{\partial u(s, X_s^1, \ldots, X_s^\ell)}{\partial t} \\ & + \sum_{i=1}^{\ell} a_s^i \frac{\partial}{\partial x^i} u(s, X_s^1, \ldots, X_s^\ell) \\ & + \frac{1}{2} \sum_{i,j=1}^{\ell} \sum_{k=1}^{m} b_s^{i,k} b_s^{j,k} \frac{\partial^2 u(s, X_s^1, \ldots, X_s^\ell)}{\partial x^i \partial x^j} \Bigg) ds \\ & + \sum_{k=1}^{m} \sum_{i=1}^{\ell} \int_0^t b_s^{i,k} \frac{\partial u(s, X_s^1, \ldots, X_s^\ell)}{\partial x^i} dW_s^k \\ & + \sum_{r=m+1}^{\bar{\ell}} \int_0^t \int_{\mathcal{E}} \big(u(s, X_s^1, \ldots, X_s^\ell) \\ & \quad - u(s, X_{s-}^1, \ldots, X_{s-}^\ell) \big) \, p_{\varphi_r}^r(dv, ds) \end{aligned} \tag{6.4.20}$$

for $t \in [0, \infty)$.

By using (6.4.20) it is straightforward to handle problems which include Lévy processes as underlying factors.

6.5 Lévy's Theorem (*)

Identification of Martingales as Wiener Processes (*)

The Wiener process is a basic building block in financial modeling and plays a central role in stochastic calculus. A definition of the Wiener process is given via the properties (3.2.6). By (5.1.5) we saw that the Wiener process is a martingale and from (5.2.5) it followed that its quadratic variation equals time t. Note that the converse of this result can be shown, namely that a continuous martingale with a quadratic variation that equals time, is a Wiener process. *Lévy's Theorem* provides this important result, which we formulate below for multi-dimensional continuous martingales. Its derivation relies on an application of the multivariate Itô formula.

Theorem 6.5.1. (Lévy) *For $m \in \mathcal{N}$ let A be a given m-dimensional vector process $\boldsymbol{A} = \{\boldsymbol{A}_t = (A_t^1, A_t^2, \ldots, A_t^m)^\top, t \in [0,\infty)\}$ on a filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$. If each of the processes $A^i = \{A_t^i, t \in [0,\infty)\}$ is a* *continuous, square integrable* $(\underline{\mathcal{A}}, P)$*-martingale that starts at* 0 *at time* $t = 0$ *and their covariations are of the form*

$$[A^i, A^k]_t = \begin{cases} t & \textit{for} \quad i = k \\ 0 & \textit{for} \quad i \neq k \end{cases} \tag{6.5.1}$$

for $i, k \in \{1, 2, \ldots, m\}, \ t \in [0, \infty)$*, then the vector process* $\boldsymbol{A}$ *is an* m*-dimensional standard Wiener process on* $[0, \infty)$*. This means that each process* A^i *is a one-dimensional Wiener process that is independent of the other Wiener processes* A^k *for* $k \neq i$*.*

In particular, one can show that this result implies that a continuous process $X = \{X_t, \, t \in [0, \infty)\}$ is a one-dimensional Wiener process if and only if both the process X and the process $Y = \{Y_t = X_t^2 - t, \, t \in [0, \infty)\}$ are martingales. Furthermore, if one is able to construct for an observed vector process a transformation such that the transformed processes are square integrable continuous martingales with covariations of the form (6.5.1), then one has found the basic building blocks of the given dynamics in the form of a vector of independent Wiener processes. In this case one needs then only to take the inverse of that transformation to arrive at a realistic model. It is a challenge in financial modeling to construct a parsimonious market model with the above property.

Proof of Lévy's Theorem (*)

To indicate the proof of the above theorem we consider the characteristic function

$$\phi_{\boldsymbol{A}_t - \boldsymbol{A}_s}(\boldsymbol{\theta}) = E\left(\exp\left\{\imath \sum_{k=1}^{m} \theta^k \, (A_t^k - A_s^k)\right\} \bigg| \, \mathcal{A}_s\right) \tag{6.5.2}$$

for $\boldsymbol{\theta} \in \Re^m, \ t \in [0, \infty)$ and $s \in [0, t]$, with $\imath$ denoting the imaginary unit, see (1.3.77).

By application of a complex valued version of the Itô formula (6.4.11) for semimartingales we obtain

$$\exp\left\{\imath \sum_{k=1}^{m} \theta^k \, A_t^k\right\} - \exp\left\{\imath \sum_{k=1}^{m} \theta^k \, A_s^k\right\} = \sum_{k=1}^{m} \int_s^t \imath \, \theta^k \, \exp\left\{\imath \sum_{l=1}^{m} \theta^l \, A_u^l\right\} dA_u^k$$
$$+ \frac{1}{2} \sum_{k=1}^{m} \int_s^t \left(-(\theta^k)^2\right) \, \exp\left\{\imath \sum_{l=1}^{m} \theta^l \, A_u^l\right\} du. \tag{6.5.3}$$

We have introduced the Itô integral with respect to general integrators in (5.3.11). The martingale property for Itô integrals, which follows for integrators that are Wiener processes and integrands that are from $\mathcal{L}_T^2$, when considered on $[0, T]$ with $T \in (0, \infty)$, can be naturally extended to cover the wider class of square integrable martingale integrators with integrands that appear in (6.5.3), see Protter (2004). This means that the terms in the first sum on the right hand side of (6.5.3) are martingales and we have

$$E\left(\int_s^t \exp\left\{\imath \sum_{l=1}^m \theta^l\, A_u^l\right\} dA_u^k \,\middle|\, \mathcal{A}_s\right) = 0. \tag{6.5.4}$$

Let us now choose any event $\mathcal{F} \in \mathcal{A}_s$ and denote by $\mathbf{1}_{\mathcal{F}}$ the indicator function that equals one if $\mathcal{F}$ occurs. Then multiplying both sides of (6.5.3) by

$$\mathbf{1}_{\mathcal{F}} \exp\left\{-\imath \sum_{k=1}^m \theta^k\, A_s^k\right\}$$

and taking expectations yields

$$G(t) - P(\mathcal{F}) = -\frac{1}{2} \sum_{k=1}^m (\theta^k)^2 \int_s^t G(u)\, du,$$

where

$$G(u) = E\left(\exp\left\{\imath \sum_{k=1}^m \theta^k\,(A_u^k - A_s^k)\right\} \mathbf{1}_{\mathcal{F}}\right)$$

for $u \in [0, t]$. The solution to this ordinary integral equation is given by

$$G(t) = P(\mathcal{F}) \exp\left\{-\frac{1}{2} \sum_{k=1}^m (\theta^k)^2\,(t-s)\right\}.$$

Consequently, by the Bayes's formula for conditional means, see (1.1.13) or Karatzas & Shreve (1991), we obtain

$$E\left(\exp\left\{\imath \sum_{k=1}^m \theta^k\,(A_t^k - A_s^k)\right\} \,\middle|\, \mathcal{F}\right) = \frac{G(t)}{P(\mathcal{F})} = \exp\left\{-\frac{1}{2} \sum_{k=1}^m (\theta^k)^2\,(t-s)\right\}.$$

Clearly, this result holds for any $\mathcal{F} \in \mathcal{A}_s$. Therefore, we have shown that for all $\boldsymbol{\theta} \in \Re^m$, $t \in [0, \infty)$ and $s \in [0, t]$ the characteristic function of the vector increment $\boldsymbol{A}_t - \boldsymbol{A}_s$ is of the form

$$\phi_{\boldsymbol{A}_t - \boldsymbol{A}_s}(\boldsymbol{\theta}) = \exp\left\{-\frac{1}{2} \sum_{k=1}^m (\theta^k)^2\,(t-s)\right\}. \tag{6.5.5}$$

It is known, see (1.4.58), that this is the characteristic function of a vector of independent Gaussian distributed random variables, each with mean zero and variance $(t - s)$. Since the characteristic function of a random vector identifies uniquely the joint distribution of this random vector, we see by the Definition 6.2.1 that the process A is an m-dimensional standard Wiener process. $\square$

6.6 A Proof of the Itô Formula (*)

Since the Itô formula is extremely important in quantitative finance we highlight in the following the main steps of a classical proof of this fundamental tool. For simplicity, we consider the scalar, continuous process $X = \{X_t,\, t \in [0,\infty)\}$, given in (6.1.6), that is

$$X_t = X_0 + \int_0^t e_s\, ds + \int_0^t f_s\, dW_s \tag{6.6.1}$$

for $t \in [0,\infty)$ with initial value $X_0 = x_0$, standard Wiener process $W = \{W_t,\, t \in [0,\infty)\}$ and predictable processes e and f, where the second integral is an Itô integral. The proof of the multi-dimensional Itô formula stated in (6.2.11) is a straightforward generalization of what will be given below.

Theorem 6.6.1. *If we assume that* $u : [0,T] \times \Re \to \Re$ *is a function of time* $t \in [0,T]$ *and state variable* $x \in \Re$ *such that the partial derivatives* $\frac{\partial u(t,x)}{\partial t}$, $\frac{\partial u(t,x)}{\partial x}$ *and* $\frac{\partial^2 u(t,x)}{\partial x^2}$ *exist and are continuous for all* $(t,x) \in [0,T] \times \Re$ *and* $\sqrt{|e|},\, f \in \mathcal{L}_T^2$*, see* (5.4.1)*, then the* Itô formula *can be written in the form*

$$du(t,X_t) = \left(\frac{\partial u(t,X_t)}{\partial t} + e_t \frac{\partial u(t,X_t)}{\partial x} + \frac{1}{2} (f_t)^2 \frac{\partial^2 u(t,X_t)}{\partial x^2} \right) dt + f_t \frac{\partial u(t,X_t)}{\partial x}\, dW_t \tag{6.6.2}$$

for $t \in [0,T]$.

A Lemma (*)

Before we begin with the proof of the Itô formula given in Theorem 6.6.1 let us summarize some application of the Taylor series expansion and the Mean Value Theorem of classical calculus in a simple lemma.

Lemma 6.6.2. *Let the function* $u : [0,T] \times \Re \to \Re$ *be as in Theorem 6.6.1. Then for any* $t,\ t + \Delta t \in [0,T]$ *and* $x,\, x + \Delta x \in \Re$ *there exist constants* $\alpha, \beta \in [0,1]$ *such that*

$$u(t+\Delta t, x+\Delta x) - u(t,x) = \frac{\partial u(t+\alpha\Delta t, x)}{\partial t}\, \Delta t + \frac{\partial u(t,x)}{\partial x}\, \Delta x + \frac{1}{2} \frac{\partial^2 u(t, x+\beta\Delta x)}{\partial x^2}\, (\Delta x)^2.$$

Proof of Theorem 6.6.1 (*)

1. First assume that e and f are deterministic constants, that is, they do not depend on t. We choose a continuous sample-path of X and fix a subinterval $[s,t] \subseteq [0,T]$, for which we consider partitions of the form $s = t_1^{(n)} < t_2^{(n)} < \ldots < t_{n+1}^{(n)} = t$ with $\Delta t_j^{(n)} = t_{j+1}^{(n)} - t_j^{(n)}$ and $\delta^{(n)} = \max_{1\le j\le n} \Delta t_j^{(n)}$, where $\lim_{n\to\infty} \delta^{(n)} \overset{\text{a.s.}}{=} 0$. Then

$$u(t, X_t) - u(s, X_s) = \sum_{j=1}^{n} \Delta u_j^{(n)},$$

where

$$\Delta u_j^{(n)} = u\left(t_{j+1}^{(n)}, X_{t_{j+1}^n}\right) - u\left(t_j^{(n)}, X_{t_j^n}\right)$$

for $j \in \{1,2,\ldots,n\}$. Applying Lemma 6.6.2 on each subinterval $[t_j^{(n)}, t_{j+1}^{(n)}]$ for each $\omega \in \Omega$, we have $\alpha_j^{(n)}$, $\beta_j^{(n)} \in [0,1]$ such that

$$\begin{aligned}\Delta u_j^{(n)} = {} & \frac{\partial u}{\partial t}\left(t_j^{(n)} + \alpha_j^{(n)} \Delta t_j^{(n)}, X_{t_j^n}\right) \Delta t_j^{(n)} \\ & + \frac{\partial u}{\partial x}\left(t_j^{(n)}, X_{t_j^n}\right) \Delta X_j^{(n)} \\ & + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\left(t_j^{(n)}, X_{t_j^n} + \beta_j^{(n)} \Delta X_j^{(n)}\right)\left(\Delta X_j^{(n)}\right)^2, \end{aligned} \qquad (6.6.3)$$

almost surely, where $\Delta X_j^{(n)} = X_{t_{j+1}^n} - X_{t_j^n}$ for $j \in \{1,2,\ldots,n\}$. By the continuity of $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$, and the sample-path continuity of X, we have for each $j \in \{1,2,\ldots,n\}$

$$\lim_{n\to\infty} \frac{\partial u}{\partial t}\left(t_j^{(n)} + \alpha_j^{(n)} \Delta t_j^{(n)}, X_{t_j^n}\right) - \frac{\partial u}{\partial t}\left(t_j^{(n)}, X_{t_j^n}\right) \overset{\text{a.s.}}{=} 0, \qquad (6.6.4)$$

and

$$\lim_{n\to\infty} \frac{\partial^2 u}{\partial x^2}\left(t_j^{(n)}, X_{t_j^n} + \beta_j^{(n)} \Delta X_j^{(n)}\right) - \frac{\partial^2 u}{\partial x^2}\left(t_j^{(n)}, X_{t_j^n}\right) \overset{\text{a.s.}}{=} 0. \qquad (6.6.5)$$

Since e and f are independent of t, the increments of X are of the form

$$\Delta X_j^{(n)} = e \Delta t_j^{(n)} + f \Delta W_j^{(n)},$$

where $\Delta W_j^{(n)} = W_{t_{j+1}^n} - W_{t_j^n}$ for $j \in \{1,2,\ldots,n\}$. Consequently, it can be shown that the sum

$$\sum_{j=1}^{n} \left\{ \left(\Delta X_j^{(n)}\right)^2 - \left(f \Delta W_j^{(n)}\right)^2 \right\} = e^2 \sum_{j=1}^{n} \left(\Delta t_j^{(n)}\right)^2 + 2ef \sum_{j=1}^{n} \Delta W_j^{(n)} \Delta t_j^{(n)} \qquad (6.6.6)$$

tends to 0 in probability for $\delta^{(n)} \to 0$ as $n \to \infty$. By combining relations (6.6.3)–(6.6.6) we see that under convergence in probability

$$
\begin{aligned}
u(t, X_t) - u(s, X_s) &\stackrel{P}{=} \lim_{n\to\infty} \sum_{j=1}^{n} \Delta u_j^{(n)} \\
&\stackrel{P}{=} \lim_{n\to\infty} \sum_{j=1}^{n} \left\{ \frac{\partial u}{\partial t}\left(t_j^{(n)}, X_{t_j^n}\right) + e\,\frac{\partial u}{\partial x}\left(t_j^{(n)}, X_{t_j^n}\right) \right. \\
&\qquad \left. + \frac{1}{2}\, f^2 \frac{\partial^2 u}{\partial x^2}\left(t_j^{(n)}, X_{t_j^n}\right) \right\} \Delta t_j^{(n)} \\
&\quad + \lim_{n\to\infty} \sum_{j=1}^{n} f\, \frac{\partial u}{\partial x}\left(t_j^{(n)}, X_{t_j^n}\right) \Delta W_j^{(n)} \\
&\quad + \lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{2} f^2 \frac{\partial^2 u}{\partial x^2}\left(t_j^{(n)}, X_{t_j^n}\right) \left(\left(\Delta W_j^{(n)}\right)^2 - \Delta t_j^{(n)} \right).
\end{aligned}
\tag{6.6.7}
$$

The first two terms on the right hand side of (6.6.7) are the terms on the right hand side of (6.6.2). We shall show that the last term in (6.6.7) converges to zero in probability for $n \to \infty$. Let us write $\Gamma_j^{(n)} = (\Delta W_j^{(n)})^2 - \Delta t_j^{(n)}$ with $\mathbf{1}_{n,j}^{(N)}$ denoting the indicator function of the set

$$
A_{n,j}^{(N)} = \left\{ \omega \in \Omega : \left| X_{t_i^n} \right| \le N \quad \text{for} \quad i \in \{1, 2, \ldots, j\} \right\}
$$

for $j \in \{1, 2, \ldots, n\}$. For fixed n the random variables $\Gamma_j^{(n)}$ are independent with mean $E\left(\Gamma_j^{(n)}\right) = 0$ and variance $E\left(\left(\Gamma_j^{(n)}\right)^2\right) = 2\left(\Delta t_j^{(n)}\right)^2$ for $j \in \{1, 2, \ldots, n\}$. Using this result we obtain the estimate

$$
\begin{aligned}
\lim_{n\to\infty} E &\left(\left| \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x^2}\left(t_j^{(n)}, X_{t_j^n}\right) \mathbf{1}_{n,j}^{(N)}\, \Gamma_j^{(n)} \right|^2 \right) \\
&\stackrel{P}{=} \lim_{n\to\infty} \sum_{j=1}^{n} E\left(\left| \frac{\partial^2 u}{\partial x^2}\left(t_j^{(n)}, X_{t_j^n}\right) \mathbf{1}_{n,j}^{(N)}\, \Gamma_j^{(n)} \right|^2 \right) \\
&\stackrel{P}{\le} \lim_{n\to\infty} C_N \sum_{j=1}^{n} 2\left(\Delta t_j^{(n)}\right)^2 \\
&\stackrel{P}{\le} \lim_{n\to\infty} 2\, C_N |t - s|\, \delta^{(n)} \stackrel{P}{=} 0.
\end{aligned}
$$

Here we have used the upper bound

$$C_N = \max_{\substack{s \le z \le t \\ |x| \le N}} \left| \frac{\partial^2 u}{\partial x^2}(z, x) \right|^2 < \infty.$$

As mentioned in Sect. 2.1, for an event D its *complement* denoted by D^c is given by $D^c = \{\omega \in \Omega : \omega \notin D\}$. Since

$$\bigcup_{j=1}^{n} \left(A_{n,j}^{(N)} \right)^c \subseteq B^{(N)} = \left\{ \omega \in \Omega : \sup_{s \le z \le t} |X_z| > N \right\},$$

so that $\lim_{N \to \infty} P\left(B^{(N)}\right) = 0$, then $\lim_{N \to \infty} P(A_{n,j}^{(N)}) = 1$. Combining these two results it can be shown that the last term in (6.6.7) converges to zero in probability $n \to \infty$. For e and f, which do not depend on t, the proof is thus complete. We can show that a similar result holds for random step functions e and f since these remain constant within partition subintervals, when conditioned on the sigma-algebra of the last discretization point.

2. For general e and f with $\sqrt{|e|}$, $f \in \mathcal{L}_T^2$ we can construct sequences of step functions $(\sqrt{|e^{(n)}|})$, $(f^{(n)})$ in $\mathcal{L}_T^2$ such that the integrals

$$\lim_{n \to \infty} \int_s^t \left| e_z^{(n)} - e_z \right| dz \stackrel{P}{=} 0$$

and

$$\lim_{n \to \infty} \int_s^t \left| f_z^{(n)} - f_z \right|^2 dz \stackrel{P}{=} 0$$

converge in probability to zero. This is because p-mean convergence for $p = 1$ implies convergence in probability, see (2.7.6). Then we can show that the sequence defined by

$$X_r^{(n)} = X_s + \int_s^r e_z^{(n)} \, dz + \int_s^r f_z^{(n)} \, dW_z$$

converges in probability to X_r as $n \to \infty$ for each $r \in [0, t]$ and $s \in [0, r]$, that is $\lim_{n \to \infty} X_r^{(n)} \stackrel{P}{=} X_r$. Since the Itô formula has been shown for step functions, then

$$\begin{aligned} u\left(t, X_t^{(n)}\right) - u\left(s, X_s^{(n)}\right) = \int_s^t \Bigg(& \frac{\partial u}{\partial t}\left(z, X_z^{(n)}\right) + e_z^{(n)} \frac{\partial u}{\partial x}\left(z, X_z^{(n)}\right) \\ & + \frac{1}{2}\left(f_z^{(n)}\right)^2 \frac{\partial^2 u}{\partial x^2}\left(z, X_z^{(n)}\right) \Bigg) dz \\ & + \int_s^t f_z^{(n)} \frac{\partial u}{\partial x}(z, X_z^{(n)}) \, dW_z, \end{aligned} \tag{6.6.8}$$

almost surely for each n. Now, from the convergence of $X_z^{(n)}$ to X_z in probability as $n \to \infty$ for $z \in [s,t]$ it follows convergence in probability for the left hand side of (6.6.8), that is

$$\lim_{n\to\infty} (u(t, X_t^{(n)}) - u(s, X_s^{(n)})) \overset{P}{=} u(t, X_t) - u(s, X_s). \tag{6.6.9}$$

Using similar arguments as given in the first part of this proof it can be shown that under convergence in probability

$$\lim_{n\to\infty} \int_s^t \left(\frac{\partial u}{\partial t}\left(z, X_z^{(n)}\right) + e_z^{(n)} \frac{\partial u}{\partial x}\left(z, X_z^{(n)}\right) + \frac{1}{2}\left(f_z^{(n)}\right)^2 \frac{\partial^2 u}{\partial x^2}\left(z, X_z^{(n)}\right)\right) dz$$
$$\overset{P}{=} \int_s^t \left(\frac{\partial u}{\partial t}(z, X_z) + e_z \frac{\partial u}{\partial x}(z, X_z) + \frac{1}{2}(f_z)^2 \frac{\partial^2 u}{\partial x^2}(z, X_z)\right) dz \tag{6.6.10}$$

and

$$\lim_{n\to\infty} \int_s^t f_z^{(n)} \frac{\partial u}{\partial x}\left(z, X_z^{(n)}\right) dW_z \overset{P}{=} \int_s^t f_z \frac{\partial u}{\partial x}(z, X_z)\, dW_z. \tag{6.6.11}$$

As explained at the end of Sect. 2.1, by taking subsequences the above convergences in probability can be considered to hold a.s. Thus, we see by passing to the limit on both sides of equation (6.6.8) for $n \to \infty$ it follows that equation (6.6.2) holds a.s. The processes on the two sides of equation (6.6.2) are continuous and, thus, indistinguishable, see (3.1.6). Note that the integrals appearing on the right hand side of (5.4.1) are well defined as limits in probability. This means that these integrals can be interpreted in a wider sense, namely as limits in probability, when either $\sqrt{\left|\frac{\partial u(\cdot, X(\cdot))}{\partial t}\right|}$, $\sqrt{\left|e(\cdot)\, \frac{\partial u(\cdot, X(\cdot))}{\partial x}\right|}$, $\sqrt{f^2(\cdot)\, \left|\frac{\partial^2 u(\cdot, X(\cdot))}{\partial x^2}\right|}$ or $\left|f(\cdot)\, \frac{\partial u(\cdot, X(\cdot))}{\partial x}\right|$ are not elements of the space $\mathcal{L}_T^2$. In cases where the integrals in (5.4.1) exist in the mean square sense, as described in Sect. 5.3, these limits coincide almost surely with the limits in probability. □

6.7 Exercises for Chapter 6

6.1. Derive the Itô differential for $(Y_t)^2$ if $Y = \{Y_t = a\, t + b\, W_t,\ t \in [0, \infty)\}$ denotes a transformed Wiener process, where W is a standard Wiener process.

6.2. Determine for a geometric Brownian motion $Z_t = Z_0 \exp\{\mu\, t + \sigma\, W_t\}$ the Itô differential for Z_t and $\ln(Z_t)$ by the use of the Itô formula, where W is a standard Wiener process.

6.3. What is the Itô differential for the square $(Z_t)^2$ of the geometric Brownian motion in Exercise 6.2?

6.4. Derive the Itô differential for the inverse $(Z_t)^{-1}$ of the geometric Brownian motion in Exercise 6.2.

6.5. Compute the Itô differential of the product $Y_t\, Z_t$ of the transformed Wiener process Y in Exercise 6.1 and the geometric Brownian motion Z_t in Exercise 6.2.

6.6. Consider two transformed Wiener processes with $Y_t^1 = a_1\, t + b_1\, W_t^1$ and $Y_t^2 = a_2\, t + b_2\, W_t^2$, where W^1 and W^2 are two independent standard Wiener processes. What is the Itô differential for $Y_t^1\, Y_t^2$?

6.7. Assume the same transformed Wiener processes as in Exercise 6.6 and compute the Itô differential for the expression $\exp\{Y_t^1\}\, \exp\{Y_t^2\}$.

6.8. Calculate the covariation between a standard Wiener process and its square.

6.9. (*) Assume $\xi : [0, \infty) \to \Re$ is a given deterministic function of time and that X is given by an Itô integral, such that

$$X_t = \int_0^t \xi(s)\, dW_s$$

for $t \in [0, \infty)$, where W is a standard Wiener process. Show that $Y = \{Y_t = X_t^2 - [X]_t,\, t \in [0, \infty)\}$ is a martingale.

6.10. (*) For a process $X = \{X_t,\, t \in [0, \infty)\}$ with $X_t = \sigma\, W_t + \xi\, N_t$, where W is a standard Wiener process and N a Poisson process with intensity $\lambda > 0$, characterize the stochastic differential of its exponential when $\sigma, \xi > 0$.

6.11. (*) For the sum $X_t = a\, N_t^1 + b\, N_t^2$, where N^1 and N^2 are two independent Poisson processes with intensity $\lambda > 0$, compute the stochastic differential of the exponential.