4

Diffusion Processes

In this chapter diffusion processes are introduced. These are potential candidates for the modeling of asset prices, interest rates and other financial quantities. We cover examples on geometric Brownian motion, Ornstein-Uhlenbeck and square root processes.

4.1 Continuous Markov Processes

A Markov process that evolves in continuous time and has continuous trajectories is called a *continuous Markov process*. This type of process would appear to be well suited for the modeling of a range of financial quantities such as stock prices, exchange rates and interest rates. Unlike Markov chains, that have discontinuous paths, it allows us to model continuous random movements of stock prices. The typical trajectory of a transformed Wiener process, as given in Fig. 3.2.3, would seem to be a reasonable candidate for the representation of asset price dynamics, for example, the path of the S&P500 index that was displayed in Fig. 3.1.1.

The Wiener process evolves in continuous time and has continuous trajectories. That is, it has paths without any jumps. Since it has independent increments it is also a Markov process. However, the transformed Wiener process given in Sect. 3.2 can take negative values. To see this better we plot in Fig. 4.1.1 the Gaussian transition densities for the standard Wiener process for the time interval $[0.1, 3.0]$. The figure shows that for negative values to be obtained there is a positive probability. This observation also applies to a transformed Wiener process. It indicates that the Wiener process or a transformed Wiener process would not be suitable for the modeling of asset price dynamics.

E. Platen, D. Heath, *A Benchmark Approach to Quantitative Finance*,
Springer Finance,

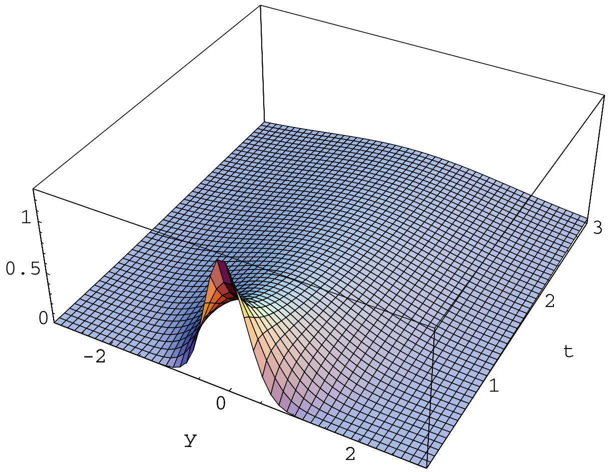

Fig. 4.1.1. Probability densities for the standard Wiener process

Black-Scholes Model

It is intuitively appealing to assume that asset prices can be modeled using some positive process which changes its value proportionally to its current value. On the basis of this assumption it makes sense to exponentially transform the Wiener process W to ensure positive asset price values. That is, we consider the random variable

$$X_t = \exp\{g\,t + b\,W_t\} \tag{4.1.1}$$

for $t \in [0, \infty)$. Here g denotes the *growth rate* and b is known as the *volatility* of the asset price process X. In Samuelson (1955, 1965a) this model was suggested for asset prices. Later, it was used in Merton (1973b) and Black & Scholes (1973) as a stock price model in their Nobel prize winning work on option pricing. The stochastic process given in (4.1.1) is called *geometric Brownian motion*. The corresponding asset price model is the *lognormal* or *Black-Scholes model*.

In Fig. 4.1.2 we show a path for geometric Brownian motion over a period of ten years with growth rate $g = 0.05$ and volatility $b = 0.2$. Note that the fluctuations become larger for larger values of the asset price.

To have flexibility in using different initial values in the lognormal model we define *geometric Brownian motion* more generally by the expression

$$X_t = X_{t_0}\,\exp\{g\,(t - t_0) + b\,(W_t - W_{t_0})\} \tag{4.1.2}$$

for $t \in [t_0, \infty)$ with initial asset price $X_{t_0} > 0$, growth rate g and volatility b, where W denotes a standard Wiener process.

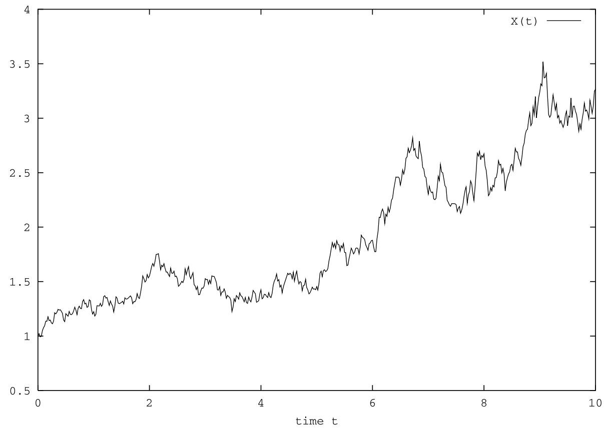

Fig. 4.1.2. A path of geometric Brownian motion

Markov Property

Geometric Brownian motion is an example of a *continuous Markov process*, a class of stochastic processes widely used for asset price modeling. Let us suppose that the share price of a stock is at present \$1 and follows a continuous Markov process. Then it is reasonable to assume that predictions of future stock price values should only depend on the present share price and be unaffected by the price one year, one month or one week ago. The only relevant information is that the price at present is \$1. Any predictions of future prices are uncertain, however, they can be expressed in terms of a probability distribution. The Markov property then implies that the probability distribution of the stock price at a particular future time depends only on the current stock price. This simplifies considerably the modeling, statistical inference and numerical analysis that typically arise.

The Markov property has a natural economic interpretation in the modeling of asset prices: The present price of a stock encapsulates all of the information contained in the knowledge of past prices. This does not exclude the possibility of using certain statistical properties of the stock price history to determine, that is calibrate, model parameters, for instance, the growth rate or the volatility of the lognormal model.

In what follows we shall suppose that for $k \in \{0, 1, \ldots\}$ every joint distribution $F_{X_{t_0}, X_{t_1}, \ldots, X_{t_k}}(x_0, x_1, \ldots, x_k)$ of the process $X = \{X_t,\ t \in [0, \infty)\}$ under consideration has a density $p(t_0, x_0;\ t_1, x_1;\ \ldots;\ t_k, x_k)$. This allows us to define the conditional probability distribution in the form

$$P\left(X_{t_{n+1}} < x_{n+1} \,|\, X_{t_0} = x_0, X_{t_1} = x_1, \ldots, X_{t_n} = x_n\right)$$
$$= \frac{\displaystyle\int_{-\infty}^{x_{n+1}} p(t_0, x_0; t_1, x_1; \ldots; t_n, x_n; t_{n+1}, y)\, dy}{\displaystyle\int_{-\infty}^{\infty} p(t_0, x_0; t_1, x_1; \ldots; t_n, x_n; t_{n+1}, y)\, dy} \tag{4.1.3}$$

for all time instants $0 \leq t_0 < t_1 < \ldots < t_n < t_{n+1} < \infty,\ n \in \{0, 1, \ldots\}$, and all states x_0, x_1, …, $x_{n+1} \in \Re$, provided the denominator is nonzero. Now, the *Markov property* can be formulated in the form

$$P\left(X_{t_{n+1}} < x_{n+1} \,|\, X_{t_0} = x_0, X_{t_1} = x_1, \ldots, X_{t_n} = x_n\right) \\ = P\left(X_{t_{n+1}} < x_{n+1} \,|\, X_{t_n} = x_n\right) \tag{4.1.4}$$

for all time instants $0 \leq t_0 < t_1 < \ldots < t_n < t_{n+1} < \infty,\ n \in \{0, 1, \ldots\}$ and all states x_0, x_1, …, $x_{n+1} \in \Re$ for which the conditional probabilities are defined.

For a continuous Markov process X we write its transition probability distribution in the form

$$P(s, x; t, (-\infty, y)) = P(X_t < y \,|\, X_s = x),$$

for $t \in [0, \infty)$, $s \in [0, t]$ and $x, y \in \Re$. If for s, x and t the probability distribution function $P(s, x; t, \cdot)$ has a probability density $p(s, x; t, \cdot)$, called the *transition density*, then it holds

$$P(s, x; t, (-\infty, y)) = \int_{-\infty}^{y} p(s, x; t, u)\, du \tag{4.1.5}$$

for all $y \in \Re$, $t \in [0, \infty)$ and $s \in [0, t]$.

Chapman-Kolmogorov Equation

The transition matrix equation (3.4.5) for continuous time Markov chains has a counterpart for the transition densities of continuous Markov processes. This continuous version is called the *Chapman-Kolmogorov equation* and has the form

$$p(s, x; t, y) = \int_{-\infty}^{\infty} p(s, x; \tau, z)\, p(\tau, z; t, y)\, dz \tag{4.1.6}$$

for $0 \leq s \leq \tau \leq t < \infty$ and x, $y \in \Re$, which follows directly from the Markov property. The Chapman-Kolmogorov equation is a fundamental relation that is used to derive important properties of continuous Markov processes.

4.2 Examples for Continuous Markov Processes

Let us discuss some examples of continuous Markov processes that, as we shall see later, are diffusion processes and play a role in financial modeling.

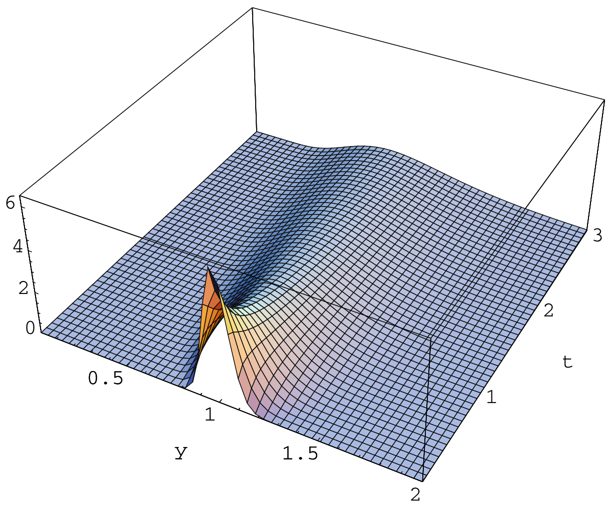

Fig. 4.2.1. Transition density for geometric Brownian motion

Wiener Process

An example of a continuous Markov process is given by the standard Wiener process defined in (3.2.6). The Wiener process obtains the Markov property from its independent increments. It has the Gaussian transition density

$$p(s,x;t,y) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left\{-\frac{(y-x)^2}{2(t-s)}\right\}, \tag{4.2.1}$$

for $t \in [0,\infty)$, $s \in [0,t]$ and $x, y \in \Re$. Figure 4.1.1 shows the transition density for a Wiener process that starts at time 0 with the initial value 0.

Geometric Brownian Motion

Geometric Brownian motion, see (4.1.2), is also a continuous Markov process. As we see later, it can be expressed as an exponential of a linearly transformed Wiener process, which gives it its Markov property. It has the transition density

$$p(s,x;t,y) = \frac{1}{\sqrt{2\,\pi(t-s)}\,b\,y} \exp\left\{-\frac{(\ln(y)-\ln(x)-g(t-s))^2}{2\,b^2(t-s)}\right\}, \tag{4.2.2}$$

for $t \in [0,\infty)$, $s \in [0,t]$ and $x, y \in (0,\infty)$. Figure 4.2.1 shows the transition density for a geometric Brownian motion with growth rate $g = 0.05$, volatility $b = 0.2$ and initial value $x = 1$ at time $s = 0$ for the period from 0.1 to 3 years.

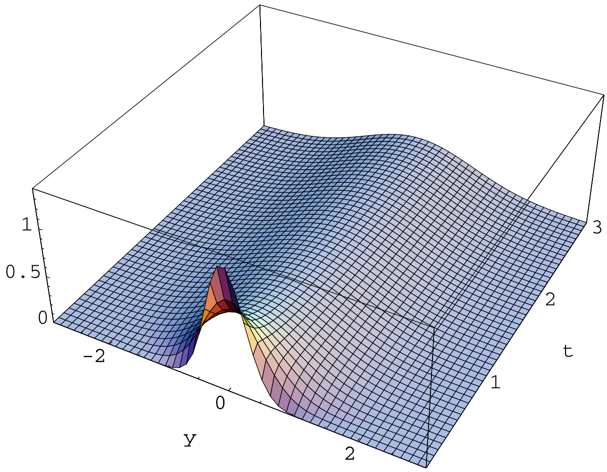

Fig. 4.2.2. Transition density of standard OU process starting at $(s, x) = (0, 0)$

Standard Ornstein-Uhlenbeck Process

Let us consider an example of another continuous Markov process which is also a Gaussian process. This is the *standard Ornstein-Uhlenbeck (OU) process* $X = \{X_t, \, t \in [0, \infty)\}$, where we start from an initial value X_0. Since it is a Gaussian process it can be characterized by the mean and the variance of its increments. More precisely, its Gaussian transition density is defined in the form

$$p(s, x; t, y) = \frac{1}{\sqrt{2\pi \left(1 - e^{-2(t-s)}\right)}} \exp\left\{ -\frac{\left(y - x e^{-(t-s)}\right)^2}{2\left(1 - e^{-2(t-s)}\right)} \right\}, \qquad (4.2.3)$$

for $t \in [0, \infty)$, $s \in [0, t]$ and $x, y \in \Re$, with mean $x\, e^{-(t-s)}$ and variance $1 - e^{-2(t-s)}$.

To illustrate the stochastic dynamic of this process we show in Fig. 4.2.2 the transition density of a standard OU process for the period from 0.1 to 3 years with initial value $x = 0$ at time $s = 0$. As can be seen from Fig. 4.2.2 that the transition densities for the standard OU process seem to stabilize after a period of about one year. In fact, as can be seen from (4.2.3) these transition densities asymptotically approach, as $t \to \infty$, a standard Gaussian density. This is in contrast, for example, to transition densities for the Wiener process, which do not converge to a *stationary density*, see (4.2.1) and Fig. 4.1.1. For illustration, we plot in Fig. 4.2.3 the transition density for a standard OU process that starts at the initial value $x = 2$ at time $t = 0$. Note how the transition density evolves towards a median that is close to 0.

In Fig. 4.2.4 a path of a standard OU process is shown. It can be observed that this trajectory fluctuates around some reference level. Indeed, as already indicated, the standard OU process has a stationary density. This can be

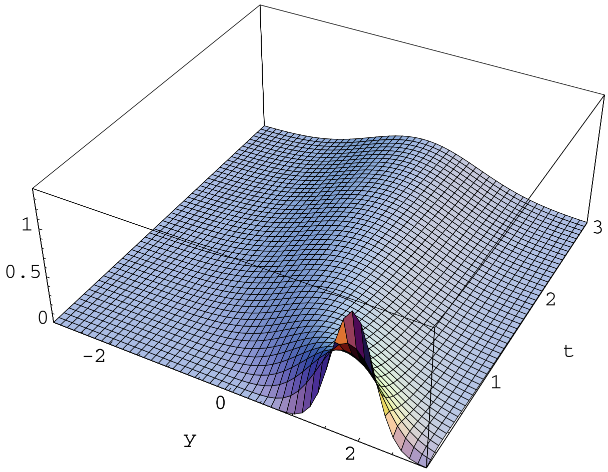

Fig. 4.2.3. Transition density of standard OU process starting at $(s, x) = (0, 2)$

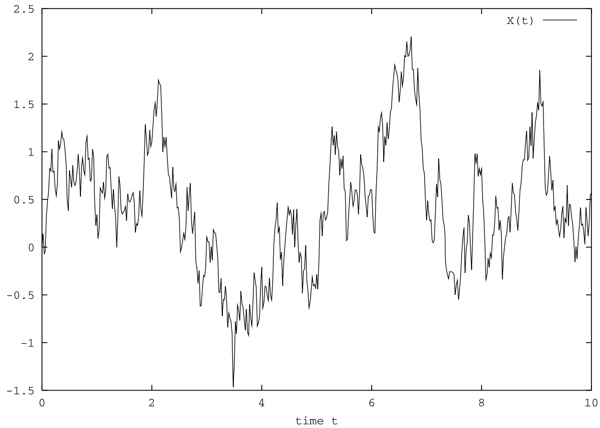

Fig. 4.2.4. Path of a standard Ornstein-Uhlenbeck process

seen from (4.2.3) when $t \to \infty$. Note also that the Gaussian property of the standard OU process means that even a scaled and shifted OU process may become negative.

More generally, as we shall describe later in Sect. 7.2, an *Ornstein-Uhlenbeck* (OU) *process* is a Gaussian process that is mean reverting to a reference level and its fluctuations can be more or less intense than that of a standard OU process. Such a model is suitable, for instance, for an inflation rate or a real interest rate. The fact that the OU process leads into an equilibrium dynamics is important for such modeling purposes.

Geometric Ornstein-Uhlenbeck Process

An asset price model that both has a stationary density and is positive is obtained by the *geometric Ornstein-Uhlenbeck process*. It is expressed as the

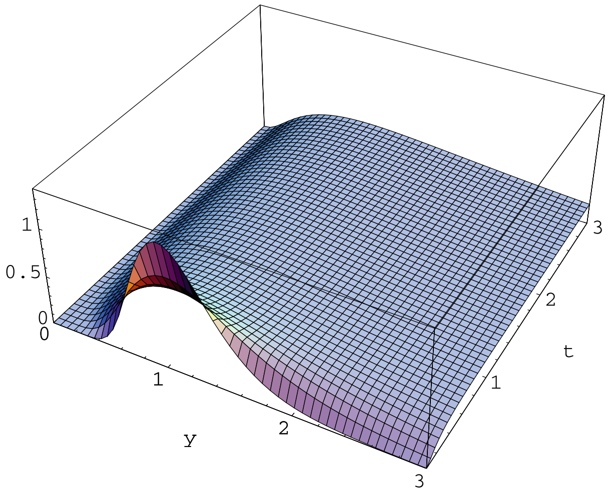

Fig. 4.2.5. Transition density of the geometric Ornstein-Uhlenbeck process

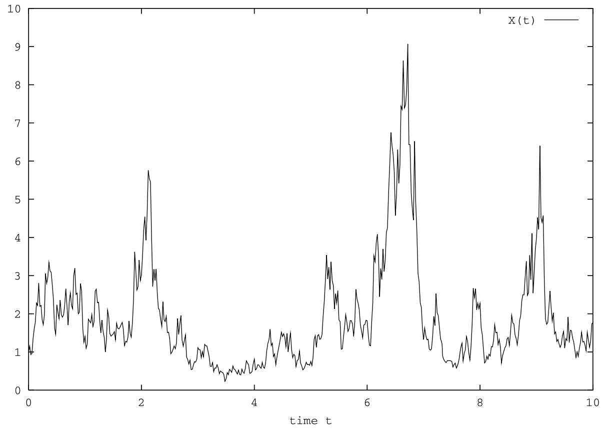

Fig. 4.2.6. Path of a geometric Ornstein-Uhlenbeck process

exponential of a standard OU process, that is, it has the *lognormal transition density*

$$p(s,x;t,y) = \frac{1}{y\sqrt{2\,\pi\,(1-e^{-2(t-s)})}}\,\exp\left\{-\frac{\left(\ln(y)-\ln(x)\,e^{-(t-s)}\right)^2}{2\left(1-e^{-2(t-s)}\right)}\right\}, \tag{4.2.4}$$

for $t \in [0,\infty)$, $s \in [0,t]$ and $x,y \in (0,\infty)$. In Fig. 4.2.5 we display the corresponding probability densities for the time period from 0.1 to 3 years with initial value $x = 1$ at time $s = 0$. In this case the transition density converges over time to a limiting lognormal density as stationary density, as can be seen from (4.2.4). Figure 4.2.6 shows a trajectory for the geometric OU process. We note that it stays positive and shows large fluctuations for large values.

This process was, for instance, interpreted in Föllmer & Schweizer (1993) as an asset price model. However, it is still somewhat restrictive in that it is not possible to model changes in the trend or volatility of the asset price. This will be conveniently achieved in the context of more general diffusion processes, which form a class of special continuous Markov processes and will be considered below.

4.3 Diffusion Processes

It is not surprising that the Wiener process serves as a prototype example of a diffusion process since it can model the diffusive motion of Brownian particles. As we attempt to show, diffusion processes form a powerful class of stochastic processes that can be applied to a range of financial modeling problems.

Characterization of Diffusion Processes

Definition 4.3.1. *A continuous time Markov process with* transition density $p(s,x;t,\,y)$ *is called a* diffusion process *if the following three limits exist for all* $\varepsilon > 0$*,* $s \in [0,\infty)$ *and* $x \in \Re$ *:*

$$\lim_{t\downarrow s} \frac{1}{t-s} \int_{|y-x|>\varepsilon} p(s,x;t,y)\,dy = 0, \tag{4.3.1}$$

$$\lim_{t\downarrow s} \frac{1}{t-s} \int_{|y-x|<\varepsilon} (y-x)p(s,x;t,y)\,dy = a(s,x) \tag{4.3.2}$$

and

$$\lim_{t\downarrow s} \frac{1}{t-s} \int_{|y-x|<\varepsilon} (y-x)^2 p(s,x;t,y)\,dy = b^2(s,x), \tag{4.3.3}$$

where a *and* b^2 *are integrable functions.*

The condition (4.3.1) prevents the diffusion process from having jumps. At time s and position x the quantity $a(s,x)$ in (4.3.2) is called the *drift coefficient* and $b(s,x)$ in (4.3.3) the *diffusion coefficient*. Condition (4.3.2) implies that the drift coefficient is given by the limit of the conditional expectation

$$a(s,x) = \lim_{t\downarrow s} \frac{1}{t-s} E\left(X_t - X_s \,\Big|\, X_s = x\right). \tag{4.3.4}$$

This means that the drift $a(s,x)$ is the instantaneous rate of change in the conditional mean of the diffusion process given that $X_s = x$.

Similarly, it follows from (4.3.3) that

$$b^2(s,x) = \lim_{t\downarrow s} \frac{1}{t-s} E\left((X_t - X_s)^2 \,\Big|\, X_s = x\right), \tag{4.3.5}$$

which denotes the limit of the second moment of the increments of the diffusion process normalized by the time $t-s$, given that $X_s = x$. Thus $b(s,x)$ measures the average size of the fluctuations of the diffusion process. In fact it can be shown that $b^2(s,x)$ is approximately the normalized variance of its increments $X_t - X_s$ as $t \to s$. Furthermore, it can be shown under fairly general conditions that for a given initial value X_0, drift $a(\cdot,\cdot)$ and diffusion coefficient $b(\cdot,\cdot)$ the diffusion process X is uniquely determined, for instance, in a mean square sense as will be discussed later.

Roughly speaking, the increment $X_t - X_s$ of a diffusion process over a small time interval of length $h = t - s$ can be interpreted approximately as a conditionally Gaussian random variable with mean $a(s, X_s)h$ and variance $b^2(s, X_s)h$. This can be expressed as

$$X_t - X_s \approx a(s, X_s)\, h + b(s, X_s)\, \sqrt{h}\, \xi, \tag{4.3.6}$$

where ξ is an independent standard Gaussian random variable. This equation is useful as a first approximation of increments of diffusion processes, to guide intuition and to indicate a relationship with the classical Taylor series expansion. Note however, in this simplified form no information is given about the corresponding error term.

Examples of One-Factor Asset Price Models

Let us list together with the already mentioned examples a few additional one-dimensional diffusion processes that have been applied in asset price modeling: The *linearly transformed Wiener process*, see Fig. 3.2.3, is an example of a diffusion process with drift $a(s,x) = 0$ and diffusion coefficient $b(s,x) = b$. As previously mentioned, it was used in Bachelier (1900) for stock price modeling. One of the disadvantages of this asset price model is given by the fact that it generates negative asset prices. In a very simplistic way it is sometimes argued that if one freezes the trajectory of this *Bachelier model* when it first hits zero, then one obtains a very basic asset price model. This model has many deficiencies. In particular, the asset price will hit zero with positive probability. This is usually not intended when modeling asset prices.

It can be shown that *geometric Brownian motion* or the *Black-Scholes* (BS) *model* as given in (4.1.2) is a diffusion process with drift

$$a(s,x) = x \left(g + \frac{1}{2}\, b^2 \right) \tag{4.3.7}$$

and diffusion coefficient

$$b(s,x) = x\, b. \tag{4.3.8}$$

As previously mentioned, the BS model was suggested in Samuelson (1955, 1965a) and used in Black & Scholes (1973). Despite the fact that this model became the standard financial market model, it does not generate a random, fluctuating volatility, which is usually observed in practice.

The *geometric Ornstein-Uhlenbeck* (GOU) *model*, see (4.2.4), can be shown to have drift coefficient

$$a(s, x) = x\,(1 - \ln(x)) \tag{4.3.9}$$

and diffusion coefficient

$$b(s, x) = \sqrt{2}\,x. \tag{4.3.10}$$

This asset price model permits an equilibrium type dynamics. It was used, as already mentioned, in Föllmer & Schweizer (1993), Platen & Rebolledo (1996) and Fleming & Sheu (1999). A disadvantage is again that it does not generate a fluctuating volatility.

The *constant elasticity of variance* (CEV) *model* introduced in Cox (1975), see also Schroder (1989), has drift

$$a(s, x) = x\,r \tag{4.3.11}$$

and diffusion coefficient

$$b(s, x) = \sigma\,x^{\alpha} \tag{4.3.12}$$

with constants r, σ and $\alpha \in (0, 1)$. It does generate a fluctuating volatility. The elasticity of the changes of the variance of log-returns can be shown to be constant due to the power structure of the diffusion coefficient. However, as shown in Delbaen & Shirakawa (2002), the model has a deficiency since the asset price will hit zero with positive probability in finite time, which is not what one usually intends to model.

The *minimal market model* (MMM) introduced in Platen (2001, 2002) has in its stylized version the drift

$$a(s, x) = \alpha_s \tag{4.3.13}$$

and the diffusion coefficient

$$b(s, x) = \sqrt{\alpha_s\,x} \tag{4.3.14}$$

with $\alpha_s = \alpha_0 \exp\{\eta\, s\}$, for initial trend $\alpha_0 > 0$ and net growth rate $\eta > 0$. This model generates a realistic, fluctuating volatility and does not hit zero. In particular, its volatility dynamics match closely that of observed index volatility and yields realistic option prices, as we shall see later.

Examples of One-Factor Short Rate Models

A large variety of short rate models has been developed that are formed by diffusion processes. In the following we shall mention several one-factor short rate models by specifying their drift and diffusion coefficients.

One of the simplest stochastic short rate models arises if the Wiener process is linearly transformed by assuming a deterministic drift coefficient $a(s, x) = a_s$ and a deterministic diffusion coefficient $b(s, x) = b_s$. This leads to the *Merton model*, see Merton (1973a), or to some specification of the continuous time version of the *Ho-Lee model*, see Ho & Lee (1986). Here the short rate does not remain positive, as one would expect.

A widely used short rate model is the *Vasicek model*, see Vasicek (1977), or the *extended Vasicek model*, which is an Ornstein-Uhlenbeck process with linear drift coefficient $a(s,x) = \gamma_s\,(\bar{x}_s - x)$ and deterministic diffusion coefficient $b(s,x) = b_s$. Also this model has a Gaussian transition density and, thus, allows negative interest rates.

In Black (1995) it was suggested that one considers the nonnegative value of a short rate like an option value, which only takes the positive part of an underlying quantity. This *Black model* results, when using an Ornstein-Uhlenbeck process $u = \{u_t,\ t \in [0,T]\}$ as underlying shadow short rate and a short rate of the form $x_s = (u_s)^+ = \max(0, u_s)$. Such type of short rate models, which allow the consideration of low interest rate regimes, have been studied, for instance, in Gorovoi & Linetsky (2004) and Miller & Platen (2005).

Cox et al. (1985) suggested the *CIR model*, which uses a square root process, see (4.4.6) below. Its drift coefficient $a(s,x) = \gamma_s\,(\bar{x}_s - x)$ is *affine*, which means that it is linear, and its diffusion coefficient is of the form $b(s,x) = b_s\,\sqrt{x}$. In the next section we shall describe the transition densities of the CIR model. This model has the desirable feature that it excludes negative interest rates. Furthermore, it yields an equilibrium dynamics. Unfortunately, when calibrated to market data, it shows a number of deficiencies which concern the possible shapes of the, so-called, *forward rate* or *yield curves*.

A translated or extended model of the CIR type is the *Pearson-Sun model*, see Pearson & Sun (1989), which assumes $a(s,x) = \gamma\,(\bar{x}_s - x)$ and $b(s,x) = \sqrt{b_1 + b_2\,x}$. Here the parameters are usually assumed to be constants which fulfill certain conditions, such as $\gamma(\bar{x} + \frac{b_1}{b_2}) > 0$. These ensure that the solution is contained in a certain region. Duffie & Kan (1994) generalized this model, which belongs to the affine class of diffusion processes, because the drift a and squared diffusion coefficient b^2 are affine. This model is therefore often called an *affine model*.

Marsh & Rosenfeld (1983) and also Dothan (1978) considered a short rate model with $a(s,x) = a_s x$ and $b(s,x) = b_s x$. This specification is known as the *lognormal model*. Here the short rate remains positive, however, it does not admit a stationary regime.

A generalized lognormal model, also called the *Black-Karasinski model*, see Black & Karasinski (1991), is obtained by setting $a(s,x) = x\,(a_s + g_s\,\ln(x))$ and $b(s,x) = b_s x$. This generates a geometric Ornstein-Uhlenbeck process, see Sect. 4.2. If $g_s = -\frac{b_s'}{b_s}$, then the above model is also called the continuous-time version of the *Black-Derman-Toy model*, see Black, Derman & Toy (1990). This type of model keeps interest rates positive and allows them to have an equilibrium.

Another model arises if one sets $a(s,x) = \gamma_s\,(\bar{x}_s - x)$ and $b(s,x) = b_s x$. In the case of constant parameters this formulation is known as the *Courtadon model*, see Courtadon (1982). The *Longstaff model*, see Longstaff (1989) is obtained by setting $a(s,x) = \gamma_s\,(\sqrt{\bar{x}_s} - \sqrt{x})$ and $b(s,x) = b_s\sqrt{x}$.

A rather general model is the *Hull-White model*, see Hull & White (1990). It has linear mean-reverting drift $a(s,x) = \gamma_s\,(\bar{x}_s - x)$ and diffusion coefficient $b(s,x) = b_s x^q$ for some choice of exponent $q \geq 0$. Obviously, this structure includes several of the above models. In the case $q = 0$ the Hull-White model is also called the extended Vasicek model, as already mentioned above.

The *Sandmann-Sondermann model*, see Sandmann & Sondermann (1994), was motivated by the aim to consider annual, continuously compounded interest rates. It has drift $a(s,x) = (1 - e^{-x})(a_s - \frac{1}{2}(1 - e^{-x})\,b_s^2)$ and diffusion coefficient $b(s,x) = (1 - e^{-x})c_s$.

An alternative short rate model was proposed in Platen (1999), which suggests a drift $a(s,x) = \gamma(x - a_s)(c_s - x)$ and a diffusion coefficient of the type $b(s,x) = b_s|x - c_s|^{\frac{3}{2}}$. The *Platen model* provides a reasonably accurate reflection of the short rate drift and diffusion coefficient as estimated from market data in Ait-Sahalia (1996).

As can be seen by these examples one can, in principle, choose quite general functions for the drift and diffusion coefficients to form meaningful diffusion models of asset prices, short rates and other financial quantities. These functions then characterize, together with the initial conditions, the dynamics of the diffusion process in an elegant and efficient way. This characterization is more compact than, for instance, that given by a transition matrix of a discrete or continuous time Markov chain. As we shall see, it also allows the exploitation of smoothness and other regularity properties of the transition densities for functionals of diffusions. We shall later aim to identify an optimal diffusion type dynamics of a financial market that takes advantage of these powerful mathematical features.

4.4 Kolmogorov Equations

In this section we describe some important results, which show that the transition densities for diffusion processes satisfy certain *partial differential equations* (PDEs).

Kolmogorov Equations

When the drift coefficient $a(\cdot)$ and diffusion coefficient $b(\cdot)$ of a diffusion process are appropriate functions, as will be discussed later, then its transition density $p(s,x;t,y)$ satisfies certain PDEs. These are the *Kolmogorov forward equation* or *Fokker-Planck equation*

$$\frac{\partial p(s,x;t,y)}{\partial t} + \frac{\partial}{\partial y}\left(a(t,y)\, p(s,x;t,y)\right) - \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(b^2(t,y)\, p(s,x;t,y)\right) = 0, \tag{4.4.1}$$

for (s,x) fixed, and the *Kolmogorov backward equation*

$$\frac{\partial p(s,x;t,y)}{\partial s} + a(s,x)\frac{\partial p(s,x;t,y)}{\partial x} + \frac{1}{2}\, b^2(s,x)\frac{\partial^2 p(s,x;t,y)}{\partial x^2} = 0, \tag{4.4.2}$$

for (t,y) fixed. Obviously, the *initial* or *terminal condition* for this PDE equals the Dirac delta function

$$p(s,x;s,y) = \delta(y-x) = \begin{cases} \infty & \text{for} \quad y = x \\ 0 & \text{for} \quad y \neq x, \end{cases} \tag{4.4.3}$$

where

$$\int_{-\infty}^{\infty} \delta(y-x)\, dy = 1 \tag{4.4.4}$$

for given x.

The first PDE (4.4.1) describes the forward evolution of the transition density with respect to the final time and state (t,y) and the second provides the backward evolution with respect to the initial time and position (s,x). The forward equation (4.4.1) is commonly called the *Fokker-Planck equation*. Both Kolmogorov equations follow from the Chapman-Kolmogorov equation (4.1.6) and the conditions (4.3.1)–(4.3.3). The Kolmogorov backward equation plays, in an extended form with other boundary conditions, an essential role in derivative pricing.

A few diffusion processes, for instance, those that arise from transformations of either Gaussian or square root processes have known transition densities. It is convenient to use such transformed diffusions with explicitly known transition densities to model financial quantities. As long as one is able to stay in such a framework the resulting quantitative methods are usually superior to numerical methods for solving PDEs. However, when the drift or diffusion coefficients become more complex or time dependent, then numerical methods have to be employed to approximate the solutions of the PDEs.

Transition Densities for the Square Root Process

Let us consider the *square root* (SR) *process* that appears in the CIR model mentioned in the previous section. Here we use the specification of the drift coefficient $a(s,x) = \gamma(\bar{x} - x)$ and the diffusion coefficient $b(s,x) = \beta\sqrt{x}$ for $s \geq 0$, $x > 0$, with constant reference level $\bar{x} > 0$, speed of adjustment $\gamma > 0$ and scaling parameter $\beta > 0$. A key feature of the SR process is that it is linear mean-reverting and can be shown to be nonnegative, see Borodin & Salminen (2002). For a value x above the reference level $\bar{x}$ the drift coefficient is negative and drives the process back to $\bar{x}$. For a value $x = 0$, the diffusion coefficient is zero and the drift coefficient is positive. Intuitively,

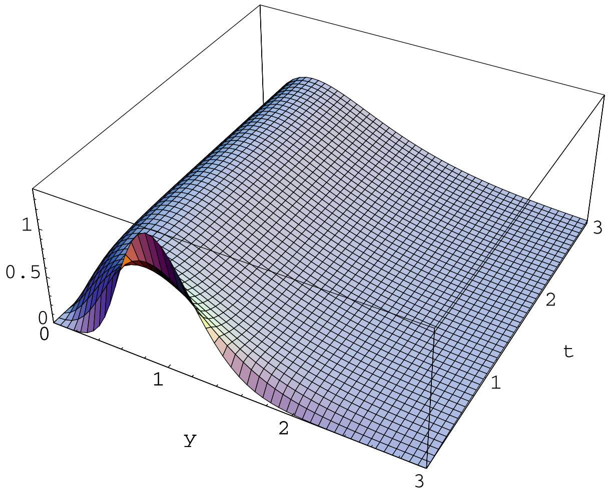

Fig. 4.4.1. Transition density of a square root process

the process is then driven back to its reference level $\bar{x}$. One can show that for $\frac{\gamma\,\bar{x}}{\beta^2} \geq \frac{1}{2}$ the SR process remains strictly positive, see Revuz & Yor (1999) or Borodin & Salminen (2002).

The quantity

$$n = \frac{4\,\gamma\,\bar{x}}{\beta^2} \tag{4.4.5}$$

is referred to as the *dimension* of the SR process. For the SR process with $\frac{\gamma\,\bar{x}}{\beta^2} \geq \frac{1}{2}$ the corresponding transition density $p(s,x;t,y)$ is available in analytic form. In fact, it is given by

$$p(s,x;t,y) = \frac{1}{2(\tau(t)-\tau(s))}\,\exp\left\{\gamma\,t - \frac{x\,\exp\{\gamma\,s\} + y\,\exp\{\gamma\,t\}}{2(\tau(t)-\tau(s))}\right\} \times \left(\frac{y\,\exp\{\gamma\,(t-s)\}}{x}\right)^{\frac{\nu}{2}} I_\nu\left(\frac{\sqrt{x\,y\,\exp\{\gamma\,(t+s)\}}}{\tau(t)-\tau(s)}\right) \tag{4.4.6}$$

with

$$\tau(t) = \frac{(\exp\{\gamma\,t\}-1)\,\beta^2}{4\,\gamma} \tag{4.4.7}$$

for $t \in [0,\infty)$, $s \in [0,t]$, $x > 0$, $y > 0$ and *modified Bessel function of the first kind* $I_\nu(z)$ with index

$$\nu = \frac{2}{\beta^2}\,\gamma\,\bar{x} - 1 = \frac{n}{2} - 1, \tag{4.4.8}$$

see (1.2.15). Here $\Gamma(\cdot)$ is the gamma function, see (1.2.10). One can show that the above transition density satisfies the Kolmogorov equations (4.4.1)–(4.4.4).

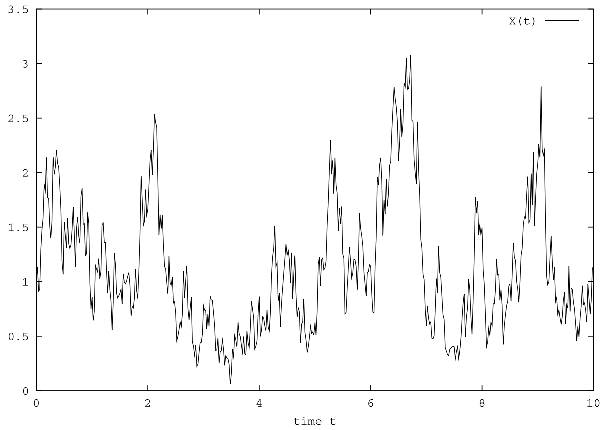

Fig. 4.4.2. Sample path of a square root process of dimension four

Figure 4.4.1 shows the transition density of an SR process for the period from 0.1 to 3.0 years, with initial value $X_0 = 1.0$, reference level $\bar{x} = 1.0$ and parameters $\gamma = 2$ and $\beta = \sqrt{2}$. By (4.4.5) this means that we consider an SR process of dimension $n = 4$. Figure 4.4.2 displays a sample path for the SR process.

Generalized Square Root Processes (*)

As we have seen above, for asset price modeling and short rate modeling but also for squared volatility modeling, positive diffusion processes, which potentially allow some equilibrium, have a great appeal. Therefore, we add the following explicit transition densities for generalized square root processes. Some of these have been recently derived in Craddock & Platen (2004) by symmetry group methods. Such transition densities can be potentially rather useful in quantitative finance.

Let us consider a *generalized square root process*, which is a diffusion process $X = \{X_t,\, t \in [0, \infty)\}$ with a square root function as diffusion coefficient of the form

$$b(t, x) = \sqrt{2\,x} \tag{4.4.9}$$

for all $t \geq 0$ and $x \in [0, \infty)$. Here the drift function $a(t, x) = a(x)$ is time homogeneous but otherwise rather flexible. This drift will be specified below for certain cases.

It is of interest to identify those drift functions $a(\cdot)$, where one has an analytic solution of the Kolmogorov backward PDE for the corresponding time homogeneous transition density $p(0, x; t, y)$, which can be written as

$$-\frac{\partial p(0, x; t, y)}{\partial t} + x\,\frac{\partial^2 p(0, x; t, y)}{\partial x^2} + a(x)\,\frac{\partial p(0, x; t, y)}{\partial x} = 0 \tag{4.4.10}$$

for $t \in (0, \infty)$ with

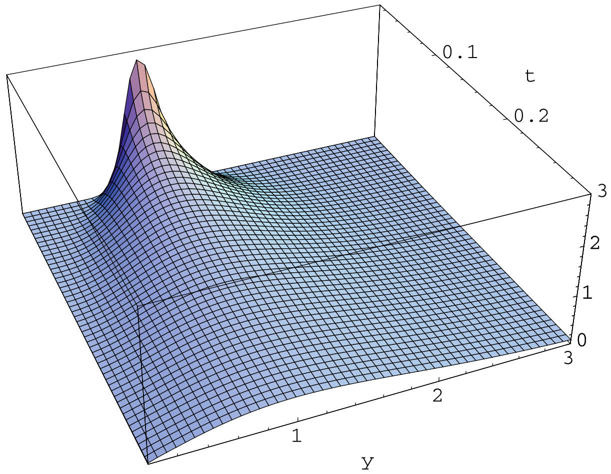

Fig. 4.4.3. Transition density for a squared Bessel process, case (i)

$$p(0, x, y) = \delta(x - y) \tag{4.4.11}$$

for $x, y \in (0, \infty)$. In Craddock & Platen (2004) for the following ten particular cases analytical solutions have been identified:

(i) When the drift function is a constant

$$a(x) = \alpha > 0, \tag{4.4.12}$$

then we have a, so-called, squared Bessel process of dimension $n = 2\alpha$ with transition density

$$p(0, x; t, y) = \frac{1}{t} \left(\frac{x}{y} \right)^{\frac{1-\alpha}{2}} I_{\alpha-1} \left(\frac{2\sqrt{x\,y}}{t} \right) \exp\left\{ -\frac{(x+y)}{t} \right\}. \tag{4.4.13}$$

Here $I_{\alpha-1}$ is again the modified Bessel function of the first kind with index $\alpha - 1$, see (1.2.15). In Fig. 4.4.3 we plot the transition density $p(0, x; t, y)$ for $x = 1$ and $\alpha = \frac{3}{2}$, that is, for a squared Bessel process of dimension $n = 3$.

(ii) When we set the drift function to

$$a(x) = \frac{\mu\, x}{1 + \frac{\mu}{2}\, x} \tag{4.4.14}$$

for $\mu > 0$, then we obtain the transition density

$$p(0, x; t, y) = \frac{\exp\left\{ -\frac{(x+y)}{t} \right\}}{\left(1 + \frac{\mu}{2}\, x\right) t} \left[\left(\sqrt{\frac{x}{y}} + \frac{\mu\sqrt{x\,y}}{2} \right) I_1 \left(\frac{2\sqrt{x\,y}}{t} \right) + t\,\delta(y) \right] \tag{4.4.15}$$

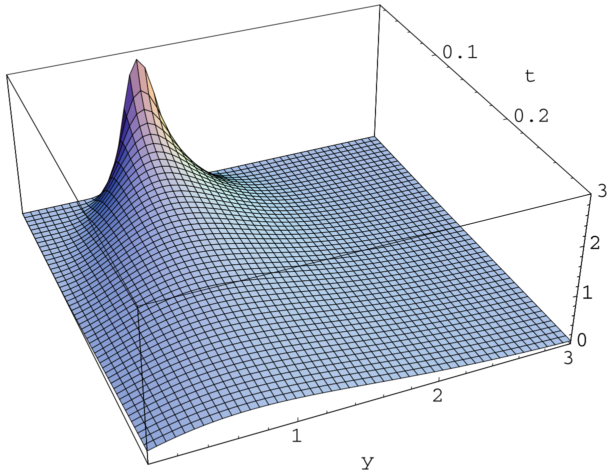

Fig. 4.4.4. Transition density for case (ii)

with $\delta(\cdot)$ denoting the Dirac delta function. For $y = 0$ one can interpret $\frac{\exp\{-\frac{x}{t}\}}{\left(1+\frac{\mu}{2}\,x\right)}$ as the probability of absorption at zero. In Fig. 4.4.4 we show the above transition density for $x = 1$ and $\mu = 1$.

(iii) In the case of the drift function

$$a(x) = \frac{1 + 3\sqrt{x}}{2\,(1 + \sqrt{x})}, \tag{4.4.16}$$

one obtains the transition density

$$p(0, x; t, y) = \frac{\cosh\left(\frac{2\sqrt{x\,y}}{t}\right)}{\sqrt{\pi\,y\,t}\,(1 + \sqrt{x})}\left(1 + \sqrt{y}\,\tanh\left(\frac{2\sqrt{x\,y}}{t}\right)\right) \times \exp\left\{-\frac{(x+y)}{t}\right\}. \tag{4.4.17}$$

In Fig. 4.4.5 we display the corresponding transition density for $x = 1$.

(iv) When we choose as drift function

$$a(x) = 1 + \mu\,\tanh\left(\mu + \frac{1}{2}\,\mu\,\ln(x)\right) \tag{4.4.18}$$

for $\mu = \frac{1}{2}\sqrt{\frac{5}{2}}$, then we obtain the transition density

$$p(0, x; t, y) = \left(\frac{x}{y}\right)^{\frac{\mu}{2}}\left[I_{-\mu}\left(\frac{2\sqrt{x\,y}}{t}\right) + e^{2\,\mu}\,y^{\mu}\,I_{\mu}\left(\frac{2\sqrt{x\,y}}{t}\right)\right] \times \frac{\exp\{-\frac{x+y}{t}\}}{(1 + \exp\{2\,\mu\}\,x^{\mu})\,t}. \tag{4.4.19}$$

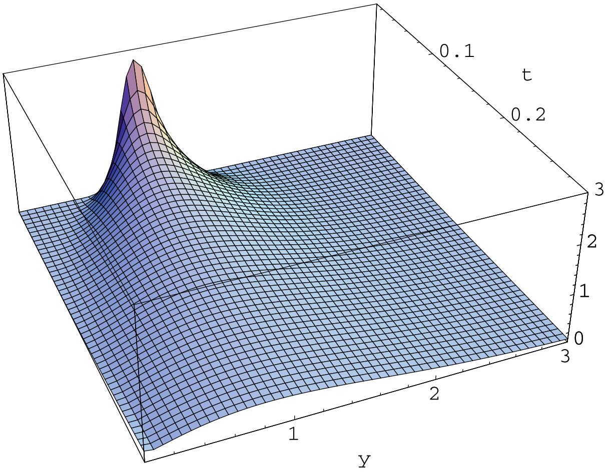

Fig. 4.4.5. Transition density for case (iii)

The shape of the density (4.4.19) for $x = 1$ looks quite similar to that in Fig. 4.4.5.

(v) For the drift function

$$a(x) = \frac{1}{2} + \sqrt{x}, \tag{4.4.20}$$

one obtains the transition density

$$p(0, x; t, y) = \cosh\left(\frac{(t + 2\sqrt{x})\sqrt{y}}{t}\right) \frac{\exp\{-\sqrt{x}\}}{\sqrt{\pi\, y\, t}} \exp\left\{-\frac{(x+y)}{t} - \frac{t}{4}\right\}. \tag{4.4.21}$$

Also the transition density (4.4.21) for $x = 1$ shows a lot of similarity with that in Fig. 4.4.5.

(vi) In the case where the drift function is set to

$$a(x) = \frac{1}{2} + \sqrt{x}\, \tanh(\sqrt{x}), \tag{4.4.22}$$

we obtain the transition density

$$p(0, x; t, y) = \frac{\cosh\left(\frac{2\sqrt{x\, y}}{t}\right)}{\sqrt{\pi\, y\, t}} \frac{\cosh(\sqrt{y})}{\cosh(\sqrt{x})} \exp\left\{-\frac{(x+y)}{t} - \frac{t}{4}\right\}. \tag{4.4.23}$$

The above transition density (4.4.23) for $x = 1$ has also a similar shape as that in Fig. 4.4.5.

(vii) For the drift function

$$a(x) = \frac{1}{2} + \sqrt{x}\, \coth(\sqrt{x}) \tag{4.4.24}$$

the process has the transition density

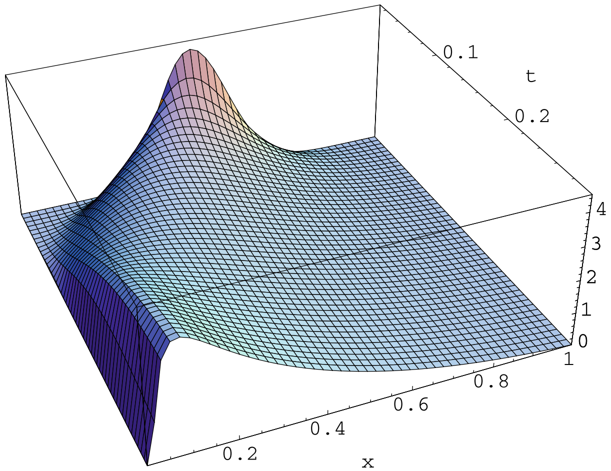

Fig. 4.4.6. Transition density for case (viii)

$$p(0,x;t,y) = \frac{\sinh\left(\frac{2\sqrt{x\,y}}{t}\right)}{\sqrt{\pi\,y\,t}}\,\frac{\sinh(\sqrt{y})}{\sinh(\sqrt{x})}\,\exp\left\{-\frac{(x+y)}{t}-\frac{t}{4}\right\}. \quad (4.4.25)$$

This transition density has for $x = 1$ some similarity with that shown in Fig. 4.4.3.

(viii) When we use as drift function

$$a(x) = 1 + \cot(\ln(\sqrt{x})) \quad (4.4.26)$$

for $x \in (\exp\{-2\pi\}, 1)$, then we obtain the real valued transition density

$$p(0,x;t,y) = \frac{\exp\{-\frac{(x+y)}{t}\}}{2\,\imath\,t\sin(\ln(\sqrt{x}))}\left(y^{\frac{\imath}{2}}\,I_\imath\left(\frac{2\sqrt{x\,y}}{t}\right) - y^{-\frac{\imath}{2}}\,I_{-\imath}\left(\frac{2\sqrt{x\,y}}{t}\right)\right), \quad (4.4.27)$$

where $\imath$ denotes the imaginary unit.

We plot in Fig. 4.4.6 the transition density (4.4.27) for $x = \frac{1}{2}$. Note that the process X lives on the bounded interval $(\exp\{-2\pi\}, 1)$.

(ix) If we choose the drift function

$$a(x) = x\,\coth\left(\frac{x}{2}\right), \quad (4.4.28)$$

then we obtain the transition density

$$p(0,x;t,y) = \frac{\sinh(\frac{y}{2})}{\sinh(\frac{x}{2})}\,\exp\left\{-\frac{(x+y)}{2\,\tanh(\frac{t}{2})}\right\} \times \left[\frac{\exp\{\frac{t}{2}\}}{\exp\{t\}-1}\sqrt{\frac{x}{y}}\,I_1\left(\frac{\sqrt{x\,y}}{\sinh(\frac{t}{2})}\right) + \delta(y)\right], \quad (4.4.29)$$

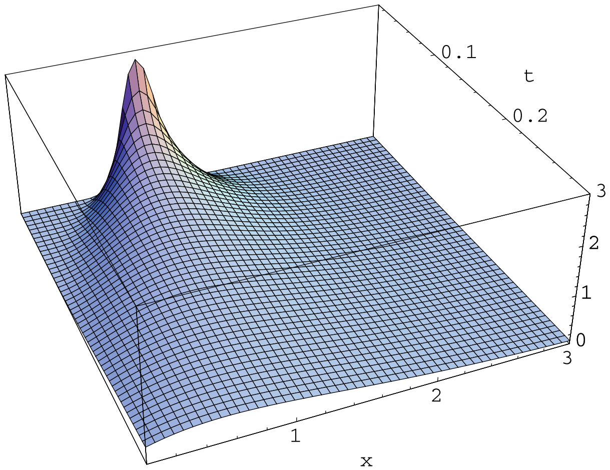

Fig. 4.4.7. Transition density for case (x)

where $\delta(\cdot)$ is again the Dirac delta function. In Fig. 4.4.3 we displayed a transition density of similar shape.

(x) Finally, let us set the drift function to

$$a(x) = x \tanh\left(\frac{x}{2}\right) \tag{4.4.30}$$

to obtain the transition density

$$p(0,x;t,y) = \frac{\cosh(\frac{y}{2})}{\cosh(\frac{x}{2})} \exp\left\{-\frac{(x+y)}{2\tanh(\frac{t}{2})}\right\} \times \left[\frac{\exp\{\frac{t}{2}\}}{\exp\{t\}-1}\sqrt{\frac{x}{y}}\, I_1\left(\frac{\sqrt{x\,y}}{\sinh(\frac{t}{2})}\right) + \delta(y)\right]. \tag{4.4.31}$$

We plot in Fig. 4.4.7 the transition density for $x = 1$.

All ten cases that we described above provide examples for generalized square root processes with diffusion coefficient function $b(x) = \sqrt{2\,x}$. In all these cases we have for the prescribed drift coefficient function an explicitly known transition density. This list of explicitly known transition densities provides valuable information for a quantitative analyst when a model needs to be designed with a square root diffusion coefficient. One can try to choose one of the above models to reflect the given dynamics. As we shall see later, by applications of stochastic calculus one can describe analytically the transition densities of a much wider class of diffusion processes that arise as twice differentiable functions of the above generalized square root processes.

4.5 Diffusions with Stationary Densities

Let us now consider diffusion processes that can model an equilibrium. Such stationary processes are important when the probabilistic features of a diffusion process do not change after a shift in time. In finance such processes are needed to model volatilities, short rates, credit spreads, inflation rates, market activity and other key quantities.

Stationary Density

When we use diffusion processes to provide models for financial quantities that can evolve into some equilibrium, then we restrict considerably the class of diffusion processes that we consider. For example, as previously noted, the standard and the geometric OU processes are diffusion processes with transition densities that converge over long periods of time towards corresponding stationary densities, see (4.2.3) and (4.2.4). The transition density of the standard OU process is shown in Fig. 4.2.2. In this figure we observe for increasing time the convergence of the transition density towards some stationary density, which in this case is the standard Gaussian density. Similarly, one notes in Fig. 4.2.5, the convergence of the transition density of the geometric OU process towards another stationary density, which is here the lognormal density. Also the SR process, see (4.2.4), has a stationary density.

More precisely, for a diffusion process that permits some equilibrium its *stationary density* $\bar{p}(y)$ is defined as the solution of the integral equation

$$\bar{p}(y) = \int_{-\infty}^{\infty} p(s, x; t, y)\, \bar{p}(x)\, dx$$

for $t \in [0, \infty)$, $s \in [0, t]$ and $y \in \Re$. This means, if one starts with the stationary density, then one obtains again the stationary density as the probability density of the process after any given time period. A *stationary diffusion process* is, therefore, obtained when the corresponding diffusion process starts with its stationary density. We shall not call a stationary diffusion process a diffusion process with stationary density that starts with a given fixed value. We rather say in this case that the process has a stationary density.

One can identify the stationary density $\bar{p}$ by noting that it satisfies the corresponding stationary, or time-independent, Kolmogorov forward equation, see (4.4.1). This *stationary Fokker-Planck equation* reduces to the ordinary differential equation (ODE)

$$\frac{d}{dy}\left(a(y)\, \bar{p}(y)\right) - \frac{1}{2}\frac{d^2}{dy^2}\left(b^2(y)\, \bar{p}(y)\right) = 0 \tag{4.5.1}$$

with drift $a(x) = a(s, x)$ and diffusion coefficient $b(x) = b(s, x)$. Consequently, it is necessary that equation (4.5.1) is satisfied to ensure that a diffusion has a stationary density. We assume in the following that a unique stationary density exists for the diffusion processes to be considered in the remainder of this section.

Note that since $\bar{p}$ is a probability density it must satisfy the relation

$$\int_{-\infty}^{\infty} \bar{p}(y)\, dy = 1. \tag{4.5.2}$$

Analytic Stationary Densities

Fortunately, one can identify for a large class of stationary diffusion processes the analytic form of their stationary density $\bar{p}(y)$. To do this, one notes from equation (4.5.1) when setting

$$H(y) = a(y)\bar{p}(y) - \frac{1}{2}\frac{d}{dy}\left(b^2(y)\,\bar{p}(y)\right)$$

that

$$\frac{d}{dy} H(y) = 0 \tag{4.5.3}$$

for $y \in \Re$ so that

$$H(y) = H = \text{ const.} \tag{4.5.4}$$

As $y \to \infty$ then $\bar{p}(y) \to 0$ and also $\frac{d\bar{p}(y)}{dy} \to 0$. This implies that $H = 0$ and one can therefore show that the stationary density is given by the explicit expression

$$\bar{p}(y) = \frac{C}{b^2(y)} \exp\left\{2\int_{y_0}^{y} \frac{a(u)}{b^2(u)}\, du\right\}. \tag{4.5.5}$$

This density satisfies the Fokker-Planck equation (4.5.1) for $y \in \Re$ with some fixed value $y_0 \in \Re$. Here y_0 is some appropriate point in the interval, where the process X is defined. The constant C can be obtained from the normalization condition (4.5.2). The formula (4.5.5) is useful in a number of applications since it allows one to obtain explicit analytic representations for the stationary density of diffusions. Moreover, if one observes from data the stationary density of a diffusion and has either its drift or its diffusion coefficient function given, then one can deduce the form of the missing diffusion or drift coefficient function, respectively.

Examples of Stationary Densities

Specifications for both the drift and diffusion coefficients are needed to determine the stationary density. For instance, in the case of the standard OU process, see (4.2.3), with $a(s,x) = a(x) = -x$ and $b(s,x) = b(x) = \sqrt{2}$ the stationary probability density is the standard Gaussian density

$$\bar{p}(y) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\} \tag{4.5.6}$$

for $y \in \Re$, see Fig. 1.2.3.

For the SR process, see (4.4.6), with $a(s, x) = a(x) = \gamma\,(1-x)$ and $b(s, x) = b(x) = \beta\,\sqrt{x}$ we obtain from (4.5.5) the stationary density

$$\bar{p}(y) = C\, y^{\frac{2\gamma}{\beta^2}-1} \exp\left\{\frac{-2\gamma}{\beta^2}\, y\right\} \tag{4.5.7}$$

for $y \in (0, \infty)$, where we assume $\frac{2\gamma}{\beta^2} > 1$. This is a gamma density, see (1.2.9), with $\alpha = p = \frac{2\gamma}{\beta^2}$.

An interesting class of diffusion processes with stationary density is obtained for a linear mean reverting drift

$$a(x) = \gamma(\bar{x} - x) \tag{4.5.8}$$

and a squared diffusion coefficient of the form

$$b^2(x) = 2(b_0 + b_1 x + b_2 x^2), \tag{4.5.9}$$

which is quadratic in $x \in \Re$. In this case, it can be shown that the corresponding stationary density $\bar{p}$ turns out to be a *Pearson type density* for an appropriate choice of constants γ, $\bar{x}$, b_0, b_1 and b_2. This class includes the normal, chi-square, gamma, Student t, uniform and exponential, but also the power exponential, beta, arcsin, Erlang and Pareto probability densities.

In Fig. 4.5.1 we show three stationary densities for specific choices of drift and diffusion coefficients. The stationary density for an Ornstein-Uhlenbeck process, labelled OU is obtained, using $\gamma = 2$ and $\bar{x} = 1$ in (4.5.8) and $b_1 = b_2 = 0$ and $b_0 = 1$ in (4.5.9). The stationary density of a square root process, labelled SR, is produced with the choices $\gamma = 2$ and $\bar{x} = 1$ in (4.5.8) and $b_0 = b_2 = 0$ and $b_1 = 1$ in (4.5.9). Finally, the stationary density of a geometric OU process, labelled GOU, is generated if we set in (4.5.5) $a(x) = x(1-\ln(x))$ and $b(x) = 2x^2$. We see in Fig. 4.5.1 the different shapes of stationary densities that can be obtained.

Ergodicity of a Diffusion Process (*)

In Sect. 3.4 we introduced the notation of ergodicity in the context of continuous time Markov chains. This property can be analogously defined for diffusion processes with stationary densities. A diffusion process $X = \{X_t, t \in [0, \infty)\}$ is called *ergodic* if it has a stationary density $\bar{p}$ and

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(X_t)\, dt = \int_{-\infty}^{\infty} f(x)\, \bar{p}(x)\, dx, \tag{4.5.10}$$

for all bounded measurable functions $f : \Re \to \Re$. That is, the limit as $T \to \infty$, of the random time average specified on the left hand side of relation (4.5.10) equals the spatial average with respect to $\bar{p}$, as given on the right hand side of (4.5.10). Ergodicity is an important property that allows us to describe and

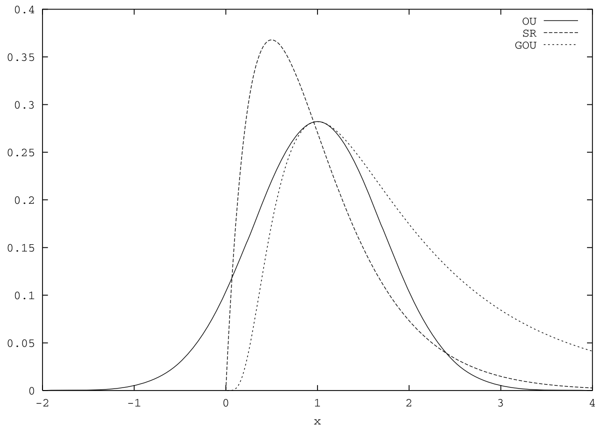

Fig. 4.5.1. Stationary density for OU, SR and GOU process

quantify functionals of equilibrium states of diffusion processes. It involves an expectation with respect to the stationary density. However, it does not require the diffusion process to be stationary. The process only needs to have a stationary density but it is not required to start with an initial value having the stationary density as its density.

Note that the widely used lognormal model, described in (4.1.2), does not yield an ergodic process, since it does not have a stationary density. For this reason its use and applicability, for instance, in long term short rate, volatility, credit spread or market activity modeling is limited. For instance, the geometric OU process discussed in (4.2.4) may be a better candidate for this type of modeling when aiming to use a diffusion coefficient that is multiplicative in the state variable.

Now we describe a result that permits us to identify a diffusion process with drift function $a(\cdot)$ and diffusion coefficient function $b(\cdot)$ as being ergodic. For this purpose we introduce the *scale measure* $s : \Re \to \Re^+$ given by

$$s(x) = \exp\left\{-2\int_{y_0}^{x} \frac{a(y)}{b^2(y)}\,dy\right\} \tag{4.5.11}$$

for $x \in \Re$ with y_0 as in (4.5.5). The following result can be found in Borodin & Salminen (2002).

Theorem 4.5.1. *A diffusion process with scale measure $s(\cdot)$ satisfying the following two properties:*

$$\int_{y_0}^{\infty} s(x)\,dx = \int_{-\infty}^{y_0} s(x)\,dx = \infty \tag{4.5.12}$$

and

$$\int_{-\infty}^{\infty} \frac{1}{s(x)\,b^2(x)}\,dx < \infty \tag{4.5.13}$$

is ergodic and its stationary density $\bar{p}$ is given by the expression (4.5.5).

Theorem 4.5.1 is formulated for diffusions with a state space that equals the set $\Re$ of all real numbers. In the case of diffusions that are confined to a smaller set of subintervals, the above conditions can be reformulated by including relevant boundary conditions.

Affine Diffusions (*)

Let us now introduce the important class of *affine diffusions*. An affine function is a linear function added to some constant. Here we have the affine drift function

$$a(x) = \theta_1 + \theta_2\, x \tag{4.5.14}$$

and the affine squared diffusion function

$$b^2(x) = \theta_3 + \theta_4\, x. \tag{4.5.15}$$

The parameter vector $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4)^\top \in \Re^4$ is chosen so that the diffusion process $X = \{X_t,\, t \in [0, \infty)\}$ has a stationary density. In particular, we set

$$\frac{\theta_2}{\theta_4} < 0 \tag{4.5.16}$$

and

$$\eta = \frac{2}{\theta_4}\left(\theta_1 - \frac{\theta_2\, \theta_3}{\theta_4}\right) > 1 \tag{4.5.17}$$

with

$$\theta_3 \geq 0 \quad \text{and} \quad \theta_4 \geq 0. \tag{4.5.18}$$

Then it can be shown that the process X is defined on the interval (y_0, ∞) with $y_0 = -\frac{\theta_3}{\theta_4}$, see Borodin & Salminen (2002). One obtains from (4.5.2) and (4.5.5) the stationary density for such an affine diffusion in the form

$$\bar{p}(x) = \frac{g(x)}{\int_{y_0}^{\infty} g(y)\, dy} \tag{4.5.19}$$

with

$$g(x) = \frac{\left(\frac{-2\,\theta_2}{\theta_4}\right)^{\eta} \left(x + \frac{\theta_3}{\theta_4}\right)^{\eta - 1} \exp\left\{\frac{2\,\theta_2}{\theta_4}\left(x + \frac{\theta_3}{\theta_4}\right)\right\}}{\Gamma(\eta)} \tag{4.5.20}$$

for $x \in (y_0, \infty)$, where $\Gamma(\cdot)$ denotes the gamma function, see (1.2.10). We plot in Fig. 4.5.2 the stationary density $\bar{p}(x)$ for $\theta_1 = -\theta_2 = 1$, $\theta_4 = 1 - \theta_3$ and $\theta_3 \in [0, 0.85]$.

If we denote by E_∞ the expectation under the corresponding stationary distribution, then we have the *stationary mean*

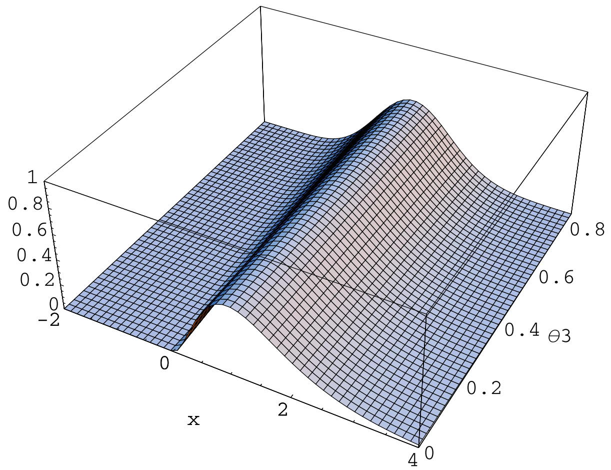

Fig. 4.5.2. Stationary density for $\theta_1 = -\theta_2 = 1$, $\theta_4 = 1 - \theta_3$ and $\theta_3 \in [0, 0.85]$

$$E_\infty(X_\infty) = \int_{-\infty}^{\infty} x\,\bar{p}(x)\,dx = -\frac{\theta_1}{\theta_2} \tag{4.5.21}$$

and the *stationary second moment*

$$E_\infty((X_\infty)^2) = \frac{(2\,\theta_1 + \theta_4)\,\theta_1 - \theta_3\,\theta_2}{2\,(\theta_2)^2}. \tag{4.5.22}$$

Obviously, in the case $\theta_4 = 0$ we obtain an OU process, see (4.2.3), with Gaussian stationary density. For the case when θ_3 equals zero we have an SR process, see (4.4.6), of dimension

$$n = 4\,\frac{\theta_1}{\theta_4} > 2, \tag{4.5.23}$$

which has the gamma density, see (1.2.9), as stationary density. The OU and the SR process are ergodic diffusions, which have explicit expressions for their transition densities. This makes these two ergodic affine diffusion processes attractive for a wide range of applications in finance. We remark that at the end of Sect. 4.4 additional diffusion processes are mentioned that also have explicit transition densities and could be linked to ergodic diffusions.

4.6 Multi-Dimensional Diffusion Processes (*)

Vector Diffusion (*)

In financial and insurance markets one observes a large number of quantities concurrently, including equity prices, exchange rates, market indices, volatilities, credit spreads and short rates. These quantities influence each other and can be modeled as a vector stochastic process because interactions need to be considered. For this type of modeling one can use a d-dimensional diffusion process

$$\boldsymbol{X} = \left\{ \boldsymbol{X}_t = \left(X_t^1, X_t^2, \ldots, X_t^d\right)^\top, \, t \in [0, \infty) \right\}$$

that generalizes the one-dimensional diffusion process introduced in the previous section. We call such a continuous time process with continuous paths a *vector diffusion*. Here superscripts index the components of the vector.

The transition density for the vector Markov process $\boldsymbol{X}$ to move from the state $\boldsymbol{x} \in \Re^d$ at time s to the state $\boldsymbol{y} \in \Re^d$ at the later time t is denoted by $p(s, \boldsymbol{x}; t, \boldsymbol{y})$. The continuous time Markov property for this vector process can be restated in a similar manner as given in (4.1.3) and we require the following limits to exist for any $\varepsilon > 0$, $s \geq 0$ and $\boldsymbol{x} \in \Re^d$, see (4.3.1)–(4.3.3):

$$\lim_{t \downarrow s} \frac{1}{t - s} \int_{|\boldsymbol{y} - \boldsymbol{x}| > \varepsilon} p(s, \boldsymbol{x}; t, \boldsymbol{y}) \, d\boldsymbol{y} = 0, \tag{4.6.1}$$

$$\lim_{t \downarrow s} \frac{1}{t - s} \int_{|\boldsymbol{y} - \boldsymbol{x}| \leq \varepsilon} (\boldsymbol{y} - \boldsymbol{x}) \, p(s, \boldsymbol{x}; t, \boldsymbol{y}) \, d\boldsymbol{y} = \boldsymbol{a}(s, \boldsymbol{x}) \tag{4.6.2}$$

and

$$\lim_{t \downarrow s} \frac{1}{t - s} \int_{|\boldsymbol{y} - \boldsymbol{x}| \leq \varepsilon} (\boldsymbol{y} - \boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x})^\top p(s, \boldsymbol{x}; t, \boldsymbol{y}) \, d\boldsymbol{y} = \boldsymbol{S}^\top(s, \boldsymbol{x}) \, \boldsymbol{S}(s, \boldsymbol{x}). \tag{4.6.3}$$

Here $\boldsymbol{a}$ is a d-dimensional vector valued function and $\boldsymbol{D} = [d^{i,j}]_{i,j=1}^d = \boldsymbol{S}^\top \boldsymbol{S}$ is a symmetric $d \times d$-matrix valued function. Each component of these functions must satisfy appropriate measurability and integrability conditions, see Stroock & Varadhan (1982). We used above the Euclidean norm $|\cdot|$, see (1.4.63), and interpret the vectors as column vectors, for example, $(\boldsymbol{y} - \boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x})^\top$ is a $d \times d$-matrix with (i, j)th component $(y_i - x_i)(y_j - x_j)$. The *drift vector* $\boldsymbol{a}$ and the *covariance matrix* $\boldsymbol{D} = \boldsymbol{S}^\top \boldsymbol{S}$ have similar interpretations to their one-dimensional counterparts in the previous section. However, we note that the components of $\boldsymbol{D}$ are the conditional covariances or variances of the increments of corresponding components of the vector diffusion, that is

$$d^{i,j}(s, x) = \lim_{t \downarrow s} \frac{1}{t - s} E\left(\left(X_t^i - X_s^i\right) \left(X_t^j - X_s^j\right) \Big| X_s = x \right),$$

where $d^{i,j}(s, x) = d^{j,i}(s, x)$. They indicate which components of the vector diffusion are correlated.

Kolmogorov Equations (*)

For vector diffusions the transition densities satisfy the multi-dimensional *Kolmogorov forward equation*, also known as Fokker-Planck equation, given by

$$\frac{\partial p(s,\boldsymbol{x};t,\boldsymbol{y})}{\partial t} + \sum_{i=1}^{d} \frac{\partial}{\partial y_i}\left(a^i(t,\boldsymbol{y})p(s,\boldsymbol{x};t,\boldsymbol{y})\right)$$
$$- \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2}{\partial y_i \partial y_j}\left(d^{i,j}(t,\boldsymbol{y})p(s,\boldsymbol{x};t,\boldsymbol{y})\right) = 0 \qquad (4.6.4)$$

for $(s,\boldsymbol{x}) \in (0,\infty)\times\Re^d$ fixed and $(t,\boldsymbol{y}) \in (s,\infty)\times\Re^d$ with the initial condition

$$\lim_{t\downarrow s} p(s,\boldsymbol{x};t,\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y})$$

for $\boldsymbol{x},\boldsymbol{y} \in \Re^d$. Here $\delta(\boldsymbol{z})$ denotes again the *Dirac delta function* but now on $\Re^d$, which defines a measure that has a mass of one concentrated at the point $(0,\ldots,0)^\top \in \Re^d$.

We can write the parabolic partial differential equation (4.6.4) more compactly in operator form as

$$\frac{\partial p(s,\boldsymbol{x};t,\boldsymbol{y})}{\partial t} - \mathcal{L}^* p(s,\boldsymbol{x};t,\boldsymbol{y}) = 0$$

for $(s,\boldsymbol{x}) \in [0,\infty)\times\Re^d$ fixed and $(t,\boldsymbol{y}) \in (s,\infty)\times\Re^d$. Here $\mathcal{L}^*$ is the formal adjoint of the operator $\mathcal{L}^0$ defined as

$$\mathcal{L}^0 u(s,\boldsymbol{x}) = \sum_{i=1}^{d} a^i(s,\boldsymbol{x})\frac{\partial u(s,\boldsymbol{x})}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} d^{i,j}(s,\boldsymbol{x})\frac{\partial^2 u(s,\boldsymbol{x})}{\partial x_i \partial x_j} \qquad (4.6.5)$$

for $(s,\boldsymbol{x}) \in (0,\infty)\times\Re^d$. The *Kolmogorov backward equation*, which as previously mentioned plays a central role in derivative pricing, is given by

$$\frac{\partial u(s,\boldsymbol{x})}{\partial s} + \mathcal{L}^0\, u(s,\boldsymbol{x}) = 0 \qquad (4.6.6)$$

for $(s,\boldsymbol{x}) \in (0,t)\times\Re^d$ with $u(s,\boldsymbol{x}) = p(s,\boldsymbol{x};t,\boldsymbol{y})$ for fixed $t \in [0,\infty)$ and $\boldsymbol{y} \in \Re^d$.

To model and analyze the quantities in a financial market purely via corresponding partial differential equations is rather complex. A more elegant and also more general framework for modeling stochastic dynamics is provided when using stochastic calculus, which will be introduced in the following chapters.

4.7 Exercises for Chapter 4

4.1. Verify that the standard Ornstein-Uhlenbeck process is a diffusion process with stationary density and identify its mean and its variance.

4.2. Identify the drift and diffusion coefficient for the standard Wiener process as a specific diffusion process.

4.3. Compute the drift and diffusion coefficients for the standard Ornstein-Uhlenbeck process.

4.4. Prove that the transition density of the standard Wiener process solves the Kolmogorov forward equation and the Kolmogorov backward equation.

4.5. Formulate the Kolmogorov forward equation for the transition density of the standard Ornstein-Uhlenbeck process.

4.6. Verify that the transition density of the standard Ornstein-Uhlenbeck process satisfies the corresponding Kolmogorov backward equation.

4.7. Determine the stationary density for the standard Ornstein-Uhlenbeck process.

4.8. Does geometric Brownian motion have a stationary density?

4.9. Verify whether the geometric Ornstein-Uhlenbeck process has a stationary density.

4.10. (*) Is the geometric Brownian motion an ergodic process?

4.11. (*) Prove that the transition density $p(s, x; t, y)$ of the standard Wiener process satisfies the Chapman-Kolmogorov equation.

4.12. (*) Is the standard Ornstein-Uhlenbeck process an ergodic process?

4.13. (*) Show that the stationary density $\bar{p}$ of a one dimensional diffusion process solves the time-independent Kolmogorov forward equation.

4.14. (*) Show that a geometric Brownian motion with growth rate g and volatility b has the drift $a(s, x) = x(g + \frac{1}{2}b^2)$.