16

Solutions for Exercises

Solutions for Exercises of Chapter 1

1.1 For a random variable X with second moment we have

$$\begin{aligned}\mathrm{Var}(X) &= E((X - E(X))^2) = E(X^2 - 2\,X\,E(X) + (E(X))^2) \\ &= E(X^2) - 2(E(X))^2 + (E(X))^2 = E(X^2) - (E(X))^2.\end{aligned}$$

1.2 For a Poisson distributed random variable $X \sim P(\lambda)$ with intensity λ we have the mean

$$E(X) = \sum_{i=0}^{\infty} i\,p_i = \lambda\,e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!} = \lambda\,e^{-\lambda}\,e^{\lambda} = \lambda$$

and the second moment

$$\begin{aligned}E(X^2) &= \sum_{i=0}^{\infty} i^2\,p_i = \lambda\,e^{-\lambda} \sum_{i=1}^{\infty} i\,\frac{\lambda^{i-1}}{(i-1)!} \\ &= \lambda\,e^{-\lambda} \left(\sum_{i=1}^{\infty} \left(\frac{\lambda^{i-1}}{(i-1)!} + (i-1)\,\frac{\lambda^{i-1}}{(i-1)!} \right) \right) \\ &= \lambda\,e^{-\lambda} \left(\sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!} + \lambda \sum_{i=1}^{\infty} \frac{\lambda^{i-2}}{(i-2)!} \right) \\ &= \lambda\,e^{-\lambda}\,(e^{\lambda} + \lambda\,e^{\lambda}) = \lambda\,(1+\lambda).\end{aligned}$$

Thus, by the result from Exercise 1.1 we obtain the variance

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \lambda + \lambda^2 - \lambda^2 = \lambda.$$

E. Platen, D. Heath, *A Benchmark Approach to Quantitative Finance*,
Springer Finance,

1.3 We have for a uniformly distributed random variable $X \sim U(a,b)$ the mean

$$E(X) = \int_a^b \frac{x}{b-a}\,dx = \frac{1}{2(b-a)}\,(b^2-a^2) = \frac{a+b}{2}.$$

the second moment

$$E(X^2) = \int_a^b \frac{x^2}{b-a}\,dx = \frac{1}{3(b-a)}\,(b^3-a^3) = \frac{1}{3}\,(b^2+ab+a^2)$$

and, thus, the variance

$$\mathrm{Var}(X) = \frac{1}{3}\,(b^2+ab+b^2) - \frac{1}{4}\,(b+a)^2 = \frac{(b-a)^2}{12}.$$

1.4 For an exponentially distributed $X \sim Exp(\lambda)$ with intensity λ it follows

$$E(X) = \int_0^\infty x\,\lambda\,e^{-\lambda x}\,dx = \lim_{x\to\infty} \frac{1}{\lambda}\,(1-(\lambda\,x+1)\,e^{-\lambda x}) = \frac{1}{\lambda},$$

$$E(X^2) = \int_0^\infty x^2\,\lambda\,e^{-\lambda x}\,dx = \lim_{x\to\infty} \frac{1}{\lambda^2}\,(2-(\lambda^2\,x^2\,2\,\lambda\,x+2)\,e^{-\lambda x}) = \frac{2}{\lambda^2},$$

and therefore

$$\mathrm{Var}(X) = 2\,\lambda^{-2} - (\lambda^{-1})^2 = \lambda^{-2}.$$

1.5 For standard Gaussian $X \sim N(0,1)$ we have the mean

$$\begin{aligned} E(X) &= \int_{-\infty}^0 \frac{x}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}x^2}\,dx + \int_0^\infty \frac{x}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}x^2}\,dx \\ &= \sqrt{2\,\pi}\left(\lim_{h\to-\infty}\left(e^{-\frac{1}{2}h^2}-1\right) + \lim_{h\to\infty}\left(1-e^{-\frac{1}{2}h^2}\right)\right) \\ &= \sqrt{2\,\pi}\,(-1+1) = 0 \end{aligned}$$

and the variance

$$\begin{aligned} \mathrm{Var}(X) = E(X^2) &= \int_{-\infty}^0 \frac{x^2}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}x^2}\,dx + \int_0^\infty \frac{x^2}{\sqrt{2\,\pi}}\,e^{-\frac{1}{2}x^2}\,dx \\ &= \frac{1}{\sqrt{2\,\pi}}\lim_{h\to\infty}\left(-h\,e^{-\frac{1}{2}\,h^2}\right) + \frac{1}{\sqrt{2\,\pi}}\int_0^\infty e^{-\frac{1}{2}\,x^2}\,dx \\ &\quad + \frac{1}{\sqrt{2\,\pi}}\lim_{h\to\infty}\left(-h\,e^{-\frac{1}{2}\,h^2}\right) + \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^0 e^{-\frac{1}{2}\,x^2}\,dx \\ &= 0 + \frac{1}{2} + 0 + \frac{1}{2} = 1. \end{aligned}$$

1.6 For $X \sim N(0,1)$ standard Gaussian distributed and $k \in \mathcal{N}$ we have

$$E(X^{2k}) = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} x^{2k}\, e^{-\frac{1}{2}\,x^2}\, dx = \sqrt{\frac{2}{\sqrt{\pi}}} \int_0^{\infty} z^{2k}\, e^{-\frac{1}{2}z^2}\, dz$$

$$= 2^{\frac{2k-1}{2}} \sqrt{\frac{2}{\pi}} \int_0^{\infty} t^{\frac{2k-1}{2}}\, e^{-t}\, dt$$

$$= 2^{\frac{2k-1}{2}} \sqrt{\frac{2}{\pi}}\, \Gamma\left(k + \frac{1}{2}\right),$$

where $\Gamma(\cdot)$ is the gamma function, see (1.2.10). Thus we obtain

$$E(X^{2k}) = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1).$$

1.7 We show that

$$E(X) = \int_{-\infty}^{\infty} \frac{y-\mu}{\sigma}\, \frac{1}{\sqrt{2\,\pi}\,\sigma}\, e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}}\, dy = 0$$

and

$$E\left((X)^2\right) = \int_{-\infty}^{\infty} \left(\frac{y-\mu}{\sigma}\right)^2 \frac{1}{\sqrt{2\,\pi}\,\sigma}\, e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}}\, dy = 1$$

and notice that a linear transform of a Gaussian random variable is Gaussian.

1.8 The square Y^2 of a standard Gaussian random variable $Y \sim N(0,1)$ is $\chi^2(1)$, that is chi-square distributed with $n = 1$ degree of freedom. This means, it is $G(\frac{1}{2}, \frac{1}{2})$ gamma distributed.

1.9 We obtain by using the Gaussian density and the definition of an expectation that

$$E(Y) = E\left(\exp\{X\}\right) = \int_{-\infty}^{\infty} \exp\{x\}\, \frac{1}{\sqrt{2\,\pi}\,\sigma}\, \exp\left\{-\frac{(x-\mu)^2}{2\,\sigma^2}\right\} dx$$

$$= \exp\left\{\mu + \frac{\sigma^2}{2}\right\},$$

see also (1.3.76).

1.10 (*) We rely on the following property of the standard Gaussian density, which can be verified by completing the square in its exponent:

$$N'(x-\theta) = \exp\left\{-\frac{1}{2}\,\theta^2 + \theta\, x\right\} N'(x)$$

for all $x, \theta \in \Re$. It then follows by change of variable that

$$\begin{aligned} E(H(X+\theta)) &= \int_{-\infty}^{\infty} H(x+\theta)\, N'(x)\, dx \\ &= \int_{-\infty}^{\infty} H(\bar{x})\, N'(\bar{x}-\theta)\, d\bar{x} \\ &= \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\,\theta^2 + \theta\,\bar{x}\right\} H(\bar{x})\, N'(\bar{x})\, d\bar{x} \\ &= E\left(\exp\left\{-\frac{1}{2}\,\theta^2 + \theta\, X\right\} H(X)\right). \end{aligned}$$

1.11 (*) Assume that the inverse C of the covariance matrix $D = C^{-1}$ has the form

$$C = [c^{i,j}] = \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix}.$$

Its determinant is then $\det(C) = 12$. Furthermore, assume that $X_1 \sim N(0,1)$ and $X_2 \sim N(0, \frac{1}{3})$. We have then from (1.4.16) the joint density

$$\begin{aligned} p(x_1, x_2) &= \frac{\sqrt{\det(C)}}{2\,\pi} \exp\left\{-\frac{1}{2} \sum_{i,j=1}^{2} C^{i,j}\,(x_i - \mu_i)\,(x_j - \mu_j)\right\} \\ &= \frac{\sqrt{12}}{2\,\pi} \exp\left\{-\frac{1}{2}\,\left(4\,x_1^2 - 12\,x_1\,x_2 + 12\,x_2^2\right)\right\} \\ &\neq \frac{1}{\sqrt{2\,\pi}} \exp\left\{-\frac{x_1^2}{2}\right\} \frac{1}{\sqrt{\frac{2\,\pi}{3}}} \exp\left\{-\frac{1}{2}\,\frac{x_2^2}{3}\right\} \\ &= p(x_1) \cdot p(x_2), \end{aligned}$$

which shows that X_1 and X_2 are not independent. The random variables would be independent if C would be a diagonal matrix.

1.12 (*) Differentiating the function

$$F(x) = \frac{1}{\pi}\, \ln(\sqrt{1+x^2})$$

yields

$$F'(x) = x\,p(x) = x\,[\pi\,(1+x^2)]^{-1}.$$

We observe that both one sided improper integrals

$$\int_{-\infty}^{0} x\,p(x)\,dx \quad \text{and} \quad \int_{0}^{\infty} x\,p(x)\,dx$$

diverge. Therefore, the two sided improper integral $\int_{-\infty}^{\infty} x\, p(x)\, dx$ diverges.

1.13 (*) The conditional density for X with $f_X(x) = x$ with respect to the event $A = \{\omega \in [0, 0.5]\}$ is

$$f_X\left(x \,|\, A\right) = \begin{cases} 0 & \text{for} \quad x \notin [0, 0.5] \\ 8\,x & \text{for} \quad x \in [0, 0.5]. \end{cases}$$

Therefore, the conditional expectation amounts to

$$E\left(X \,|\, A\right) = \int_{-\infty}^{\infty} x\, f_X\left(x \,|\, A\right) dx = \int_0^{0.5} 8\, x^2\, dx = \frac{1}{3}.$$

Solutions for Exercises of Chapter 2

2.1 We can apply the strong Law of Large Numbers since

$$\sum_{i=1}^{\infty} \frac{Var(X_i)}{i^2} = K \sum_{i=1}^{\infty} (i)^{-2} = K\, \frac{\pi^2}{6} < \infty.$$

Therefore, it holds that

$$\mu \stackrel{\text{a.s.}}{=} \lim_{n\to\infty} \hat{\mu}_n.$$

2.2 By the Central Limit Theorem it follows that the sequence $\hat{Y}_n$ converges in distribution for $n \to \infty$ to a Gaussian random variable with mean zero and variance σ^2.

2.3 The $100(1-\alpha)\%$ confidence interval for $2Z$ uses its mean $2E(Z)$ and variance $4\text{Var}(Z)$ and is given in the form

$$\left(2\left(E(Z) - \sqrt{\text{Var}(Z)}\right) p_{1-\alpha}, 2\left(E(Z) + \sqrt{\text{Var}(Z)}\, p_{1-\alpha}\right)\right)$$

with $p_{1-\alpha} \approx 2.58$ for $\alpha = 99\%$.

2.4 We need to satisfy the relation

$$P\left(\frac{Z - E(Z)}{\sqrt{\text{Var}(Z)}}\right) < -z_\alpha$$

with

$$z_\alpha = \frac{\text{VaR}((1-\alpha)\%) + E(Z)}{\sqrt{\text{Var}(Z)}}.$$

The one sided confidence interval is of the form

$$(-\infty, z_\alpha),$$

where $z_\alpha = z_{0.01} \approx 2.35$ for $\alpha = 99\%$.

Solutions for Exercises of Chapter 3

3.1 Let W be a standard Wiener process and let $s \in [0, t]$. Then

$$\begin{aligned} C(s,t) &= E((W_t - E(W_t))(W_s - E(W_s))) = E(W_t W_s) \\ &= E((W_t - W_s + W_s)W_s) \\ &= E((W_t - W_s)W_s) + E(W_s^2) \\ &= E(W_t - W_s)\, E(W_s) + E(W_s^2) = 0 \cdot 0 + s = s \end{aligned}$$

since W_s and $W_t - W_s$ are independent for $s < t$. Analogously, $C(s,t) = t$ for $t < s$. Hence

$$C(s,t) = \min(s,t) = \frac{1}{2}\,(|s+t| - |s-t|).$$

3.2 The covariance of the Wiener process is not a function of $(t-s)$ only, so the Wiener process is not stationary. A similar argument applies for a random walk.

3.3 Relation (3.3.12) relates to a Bernoulli trial with n independent outcomes and $\frac{j-(k-n)}{2}$ successes (here upward moves) that occur with probability 0.5. The probability for such an event is given by the binomial distribution with

$$p_j(n) = \frac{n!}{\left(\frac{j-(k-n)}{2}\right)!\,\left(n - \frac{j-(k-n)}{2}\right)!} \left(\frac{1}{2}\right)^n .$$

3.4 The probability $q_j(n)$ for having j upwards moves in a non-symmetric random walk in n time steps is related to the binomial distribution with probability p for an upward move. Therefore it is

$$q_j(n) = \frac{n!}{j!\,(n-j)!}\,(p)^j\,(1-p)^{n-j}.$$

3.5 The stationary probability vector is $(0.5, 0.5)$ and, therefore, we have the mean $\mu = 0.5 \cdot 0.05 + 0.5 \cdot 0.06 = 0.055$, and the variance $v = 0.5(0.05 - 0.055)^2 + 0.5(0.06 - 0.055)^2 = 0.000025$.

3.6 The long term expected squared interest rate is computed by using the ergodicity and, thus, the stationary probability vector $(0.5, 0.5)$. It then follows

$$E\left((X_t)^2\right) = 0.5(0.05)^2 + 0.5(0.06)^2 = 0.00305.$$

3.7 We obtain by the formula (3.5.1) for the Poisson probabilities that

$$
\begin{aligned}
E(N_t) &= \sum_{k=1}^{\infty} \frac{k^2}{k!}\, e^{-\lambda t}\, (\lambda\, t)^k \\
&= \lambda\, t \left(\sum_{k=1}^{\infty} (k-1)\, e^{-\lambda t}\, \frac{(\lambda\, t)^{k-1}}{(k-1)!} + \sum_{k=1}^{\infty} e^{-\lambda t}\, \frac{(\lambda\, t)^{k-1}}{(k-1)!} \right) \\
&= \lambda\, t\, (\lambda\, t + 1).
\end{aligned}
$$

3.8 By the independence of the marks from the Poisson process it follows that

$$
E(Y_t) = E\left(\sum_{k=1}^{N_t} \xi_k \right) = E(N_t)\, E(\xi_k) = \frac{\lambda\, t}{2}.
$$

3.9 The probability for a compound Poisson process with intensity $\lambda > 0$ of having no jumps until time $t > 0$ equals the probability of the Poisson process N of having no jumps until that time. Thus, by (3.5.1) we obtain

$$
P(N_t = 0) = e^{-\lambda t}.
$$

3.10 (*) The given Lévy process is by (3.6.2) at time $t \in [0, T]$ of the form

$$
X_t = \alpha\, t + \beta\, W_t + \frac{1}{2} \left(p_\varphi \left(\left\{ \frac{1}{2} \right\}, [0, t] \right) - \lambda\, t \right).
$$

Therefore it follows by the formulas for the means of the Wiener and Poisson process that $E(X_t) = \alpha\, t$.

Similarly, we obtain from the formulas for the variance of the Wiener and Poisson process the variance of X_t as

$$
\begin{aligned}
E\left((X_t - \alpha\, t)^2 \right) &= E\left(\left(\beta\, W_t + \frac{1}{2} \left(p_\varphi \left(\left\{ \frac{1}{2} \right\}, [0, t] \right) - \lambda\, t \right) \right)^2 \right) \\
&= E\left((\beta\, W_t)^2 \right) + \frac{1}{4}\, E\left(\left(p_\varphi \left(\left\{ \frac{1}{2} \right\}, [0, t] \right) - \lambda\, t \right)^2 \right) \\
&= \beta^2\, (\varphi - \varphi(0)) + \frac{\lambda\, t}{4}.
\end{aligned}
$$

Solutions for Exercises of Chapter 4

4.1 The transition density of the standard Ornstein-Uhlenbeck process is a Gaussian one and of the form (4.2.3). For $t \to \infty$ it converges towards the standard Gaussian density. Thus, the process is stationary with mean 0 and variance 1.

4.2 According to (4.2.1) the transition density $p(s, x; t, y)$ of the standard Wiener process is Gaussian with mean x and variance $(t - s)$. Therefore, we obtain from (4.3.4) $a(s, x) = 0$ and from (4.3.5) $b(s, x) = 1$.

4.3 The standard Ornstein-Uhlenbeck process has the Gaussian transition density $p(s, x; t, y)$ given in (4.2.3) with mean $x \exp\{-(t-s)\}$ and variance $(1 - e^{-2(t-s)})$. Thus by (4.3.4) we have

$$a(s, x) = \lim_{t \downarrow s} \frac{1}{(t-s)} \left(x \exp\{-(t-s)\} - x\right) = -x$$

and by (4.3.5) it follows

$$\begin{aligned}
b^2(s, x) &= \lim_{t \downarrow s} \frac{1}{t-s} E\left((X_t - X_s)^2 \,\big|\, X_s = x\right) \\
&= \lim_{t \downarrow s} \frac{1}{t-s} \Big[E\Big(\left(X_t - X_s - E\left(X_t - X_s \,\big|\, X_s = x\right)\right)^2 \\
&\qquad \big| X_s = x\Big) + E\left(\left(X_t - X_s \,\big|\, X_s = x\right)\right)^2\Big] \\
&= \lim_{t \downarrow s} \frac{1}{t-s} \left[\left(1 - e^{-2(t-s)}\right) + x^2 \left(\exp\{-(t-s)\} - 1\right)^2\right] \\
&= 2.
\end{aligned}$$

Therefore we have $b(s, x) = \sqrt{2}$.

4.4 For the Gaussian transition density (4.2.1) of the standard Wiener process it holds

$$\frac{\partial}{\partial y} p(s, x; t, y) = p(s, x; t, y) \left(-\frac{(y-x)}{(t-s)}\right)$$

$$\frac{\partial^2}{\partial y^2} p(s, x; t, y) = p(s, x; t, y) \frac{(y-x)^2}{(t-s)^2} - \frac{p(s, x; t, y)}{(t-s)}$$

and

$$\frac{\partial}{\partial t} p(s, x; t, y) = -\frac{1}{2(t-s)} p(s, x; t, y) + \frac{(y-x)^2}{2(t-s)^2} p(s, x; t, y).$$

Therefore

$$\frac{\partial}{\partial t} p(s, x; t, y) - \frac{1}{2} \frac{\partial^2 p(s, x; t, y)}{\partial y^2} = 0,$$

for (s, x) fixed, which provides the Kolmogorov forward equation (4.4.1) with boundary condition (4.4.3).

Similarly we have

$$\frac{\partial p(s,x;t,y)}{\partial s} = \frac{1}{2}\,\frac{p(s,x;t,y)}{(t-s)} - p(s,x;t,y)\left(\frac{(y-x)^2}{2(t-s)^2}\right)$$

and

$$\frac{\partial p(s,x;t,y)}{\partial x} = p(s,x;t,y)\,\frac{(y-x)}{(t-s)}$$

$$\frac{\partial^2 p(s,x;t,y)}{\partial x^2} = p(s,x;t,y)\,\frac{(y-x)^2}{(t-s)^2} - p(s,x;t,y)\,\frac{1}{(t-s)}.$$

Thus

$$\frac{\partial p(s,x;t,y)}{\partial s} + \frac{1}{2}\,\frac{\partial^2 p(s,x;t,y)}{\partial x^2} = 0$$

for (t, y) fixed, which represents the Kolmogorov backward equation (4.4.2) with boundary condition (4.4.3).

4.5 For the standard Ornstein-Uhlenbeck process we have the Kolmogorov forward equation, see (4.4.1),

$$\frac{\partial p(s,x;t,y)}{\partial t} - \frac{\partial}{\partial y}\,(y\,p(s,x;t,y)) - \frac{1}{2}\frac{\partial^2}{\partial y^2}\,(2\,p(s,x;t,y)) = 0$$

that is

$$\frac{\partial p(s,x;t,y)}{\partial t} - p(s,x;t,y) - y\,\frac{\partial}{\partial y}\,p(s,x;t,y) - \frac{\partial^2}{\partial y^2}\,p(s,x;t,y) = 0$$

with boundary condition (4.4.3).

4.6 The Kolmogorov backward equation for the standard Ornstein-Uhlenbeck process is, see (4.4.2),

$$\frac{\partial p(s,x;t,y)}{\partial s} - x\,\frac{\partial p(s,x;t,y)}{\partial x} + \frac{\partial^2 p(s,x;t,y)}{\partial x^2} = 0$$

with boundary condition (4.4.3). Taking the partial derivatives of the transition density (4.2.3) it follows that

$$\frac{\partial p(s,x;t,y)}{\partial s} = -\frac{1}{2}\,p(s,x;t,y)\,\frac{-2e^{-2(t-s)}}{1-e^{-2(t-s)}} + p(s,x;t,y)\cdot$$
$$\left(\frac{2\left(y-x\,e^{-(t-s)}\right)\,x\,e^{-(t-s)}}{2\left(1-e^{-2(t-s)}\right)} - \frac{\left(y-x\,e^{-(t-s)}\right)^2\,2e^{-2(t-s)}}{2\left(1-e^{-2(t-s)}\right)^2}\right)$$

$$\frac{\partial p(s,x;t,y)}{\partial x} = p(s,x;t,y)\,\frac{\left(y-x\,e^{-(t-s)}\right)\,e^{-(t-s)}}{1-e^{-2(t-s)}}$$

$$\frac{\partial^2 p(s,x;t,y)}{\partial x^2} = p(s,x;t,y)\left[\frac{\left(y-x\,e^{-(t-s)}\right)^2\,e^{-2(t-s)}}{\left(1-e^{-2(t-s)}\right)^2} - \frac{e^{-2(t-s)}}{1-e^{-2(t-s)}}\right].$$

Then we obtain by substituting these partial derivatives into the left hand side of the above Kolmogorov backward equation that

$$\begin{aligned}
p(s,x;t,y)\Bigg[&\frac{e^{-2(t-s)}}{(1-e^{-2(t-s)})} + \frac{y\,x\,e^{-(t-s)}}{(1-e^{-2(t-s)})} - \frac{x^2\,e^{-2(t-s)}}{(1-e^{-2(t-s)})}\\
&- \frac{y^2\,e^{-2(t-s)}}{(1-e^{-2(t-s)})^2} + \frac{2x\,y\,e^{-3(t-s)}}{(1-e^{-2(t-s)})^2} - \frac{x^2\,e^{-4(t-s)}}{(1-e^{-2(t-s)})^2}\\
&- \frac{x\,y\,e^{-(t-s)}}{1-e^{-2(t-s)}} + \frac{x^2\,e^{-2(t-s)}}{1-e^{-2(t-s)}} + \frac{y^2\,e^{-2(t-s)}}{(1-e^{-2(t-s)})^2}\\
&- \frac{2\,y\,x\,e^{-3(t-s)}}{(1-e^{-2(t-s)})^2} + \frac{x^2\,e^{-4(t-s)}}{(1-e^{-2(t-s)})^2} - \frac{e^{-2(t-s)}}{(1-e^{-2(t-s)})}\Bigg]\\
= 0.&
\end{aligned}$$

Obviously, for $t = s$ the transition density (4.2.3) equals the Dirac delta function (4.4.3).

4.7 The stationary density for the standard Ornstein-Uhlenbeck can be taken from formula (4.5.5) or for $(t-s) \to \infty$ from equation (4.2.3). It is with

$$\bar{p}(y) = \frac{1}{\sqrt{2\pi}}\,\exp\left\{-\frac{y^2}{2}\right\}$$

the density of a standard Gaussian random variable.

4.8 Geometric Brownian motion is not a stationary process because its transition density given in (4.2.2) does not converge for $(t-s) \to \infty$ to a stationary density.

4.9 The geometric Ornstein-Uhlenbeck process is a stationary process. Its stationary density is the log-normal probability density

$$\bar{p}(y) = \frac{1}{y\sqrt{2\pi}} \exp\left\{-\frac{(\ln(y))^2}{2}\right\}.$$

4.10 (*) Geometric Brownian motion is not an ergodic process because it does not have a stationary density.

4.11 (*) With the transition densities (4.2.1) of a standard Wiener process we can write

$$\begin{aligned}
&\int_{-\infty}^{\infty} p(s,x;r,z)\, p(r,z;t,y)\, dz \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{(r-s)(t-r)}} \exp\left\{-\frac{1}{2}\left(\frac{(z-x)^2}{r-s} + \frac{(y-z)^2}{t-r}\right)\right\} dz \\
&= \frac{1}{\sqrt{2\pi(t-s)}} \exp\left\{-\frac{(y-x)^2}{2(t-s)}\right\} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}u^2\right\} du \\
&= p(s,x;t,y) \cdot 1 = p(s,x;t,y),
\end{aligned}$$

where we used the substitution

$$u = u(z) = \left(z - \frac{x(t-r)+y(r-s)}{t-s}\right)\sqrt{\frac{t-s}{(r-s)(t-r)}}.$$

4.12 (*) The Ornstein-Uhlenbeck process is an ergodic process, because we have according to (4.5.11) the scale measure

$$s(x) = \exp\left\{\int_0^x y\, dy\right\} = \exp\left\{\frac{x^2}{2}\right\}$$

with the properties

$$\int_0^{\infty} s(x)\, dx = \int_{-\infty}^{0} s(x)\, dx = \int_0^{\infty} \exp\left\{\frac{x^2}{2}\right\} dx = \infty$$

and

$$\int_{-\infty}^{\infty} \frac{1}{2s(x)}\, dx = \int_{-\infty}^{\infty} \frac{1}{2} \exp\left\{-\frac{x^2}{2}\right\} dx = \sqrt{\frac{\pi}{2}} < \infty$$

that prove the conditions for ergodicity (4.5.12) and (4.5.13).

4.13 (*) Using (4.5.5) we have

$$\frac{d\bar{p}(y)}{dy} = 2\bar{p}(y)\frac{a(y)}{b^2(y)} - \bar{p}(y)\frac{1}{b^2(y)}\frac{db^2(y)}{dy} = \frac{\bar{p}(y)}{b^2(y)}\left(2a(y) - \frac{db^2(y)}{dy}\right).$$

Then it holds

$$Q(y) = a(y)\,\bar{p}(y) - \frac{1}{2}\,\frac{d}{dy}\,(b^2(y)\,\bar{p}(y))$$
$$= a(y)\,\bar{p}(y) - \frac{1}{2}\,\bar{p}(y)\,\frac{db^2(y)}{dy} - \frac{1}{2}\,b^2(y)\,\frac{d\bar{p}(y)}{dy}$$
$$= \bar{p}(y)\left[a(y) - \frac{1}{2}\,\frac{db^2(y)}{dy} - a(y) + \frac{1}{2}\,\frac{db^2(y)}{dy}\right] = 0$$

and it follows

$$\frac{dQ(y)}{dy} = 0$$

which proves (4.5.1).

4.14 (*) By (4.3.4) and (4.1.2) we obtain by using the Taylor expansion for the exponential

$$a(s,x) = \lim_{t \downarrow s} \frac{1}{t-s}\,E\left(X_t - X_s \,\middle|\, X_s = x\right)$$
$$= \lim_{t \downarrow s} E\left(\frac{X_s\,[\exp\{g(t-s) + b(W_t - W_s)\} - 1]}{t-s} \,\middle|\, X_s = x\right)$$
$$= x\,\lim_{t \downarrow s} E\left(\frac{g(t-s) + b(W_t - W_s)}{t-s} + \frac{1}{2}\,\frac{(g(t-s) + b(W_t - W_s))^2}{t-s} \,\middle|\, X_s = x\right)$$
$$= x\left(g + \frac{1}{2}\,b^2\right).$$

Solutions for Exercises of Chapter 5

5.1 Assuming a filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$ we have for $0 \leq s \leq t \leq T$ by the martingale property for Wiener processes that

$$E\left(Y_t \,\middle|\, \mathcal{A}_s\right) = E\left(\alpha_1 W_t^1 + \alpha_2 W_t^2 \,\middle|\, \mathcal{A}_s\right)$$
$$= \alpha_1\,E\left(W_t^1 \,\middle|\, \mathcal{A}_s\right) + \alpha_2\,E\left(W_t^2 \,\middle|\, \mathcal{A}_s\right)$$
$$= \alpha_1\,W_s^1 + \alpha_2\,W_s^2$$
$$= Y_s$$

which proves the martingale property (5.1.2).

5.2 We compute for $0 \le s \le t \le T < \infty$ the conditional expectation

$$\begin{aligned} E\left(Y_t \,\middle|\, \mathcal{A}_s\right) &= E\left(W_t^2 \,\middle|\, \mathcal{A}_s\right) \\ &= E\left((W_t - W_s)^2 + W_s^2 \,\middle|\, \mathcal{A}_s\right) \\ &= (t-s) + Y_s \\ &\ge Y_s, \end{aligned}$$

which shows that Y is a submartingale as defined in (5.1.7).

5.3 We obtain by the properties of the Wiener process for $0 \le t \le s \le T$

$$\begin{aligned} E\left(M_s \,\middle|\, \mathcal{A}_t\right) &= E\left((W_s - W_0)^2 - s \,\middle|\, \mathcal{A}_t\right) \\ &= E\left(((W_s - W_t) + (W_t - W_0))^2 - s \,\middle|\, \mathcal{A}_t\right) \\ &= E\left((W_s - W_t)^2 \,\middle|\, \mathcal{A}_t\right) + W_t^2 - s = W_t^2 - t = M_t, \end{aligned}$$

which shows that M is a martingale.

5.4 We consider for $0 \le s \le t \le T$ the conditional expectation

$$\begin{aligned} E\left(\bar{X}_t \,\middle|\, \mathcal{A}_s\right) &= E\left(\exp\left\{-\frac{1}{2}\,\sigma^2\,(\varphi - \varphi(0)) + \sigma\,W_t\right\} \,\middle|\, \mathcal{A}_s\right) \\ &= \exp\left\{-\frac{1}{2}\,\sigma^2\,s + \sigma\,W_s\right\} \\ &\quad \times E\left(\exp\left\{-\frac{1}{2}\,\sigma^2\,(t-s) + \sigma\,(W_t - W_s)\right\} \,\middle|\, \mathcal{A}_s\right) \\ &= \bar{X}_s, \end{aligned}$$

where we used the Laplace transform of the Gaussian increment $W_t - W_s$ of the Wiener process W in the form

$$E\left(\exp\{\sigma(W_t - W_s)\} \,\middle|\, \mathcal{A}_s\right) = \exp\left\{\frac{1}{2}\,\sigma^2(t-s)\right\}.$$

$\bar{X}$ is an $(\underline{\mathcal{A}}, P)$-martingale.

5.5 We have from the covariation property (5.4.5) of Itô integrals that

$$\left[\int_0 a\,du + \int_0 b\,dW_u\right]_t = \int_0^t b^2\,du = b^2\,(\varphi - \varphi(0)).$$

5.6 Similarly as in Exercise 5.5 we obtain

$$\left[\int_0 a\,du + \int_0 b\,dW_u, \int_0 1\,dW_u\right]_t = \int_0^t b\,du = b\,t.$$

5.7 By using Jensen's inequality, see (1.3.52), it follows for $0 \le t \le s \le T$ that

$$E\left(g(X_s) \,\middle|\, \mathcal{A}_t\right) \ge g\left(E\left(X_s \,\middle|\, \mathcal{A}_t\right)\right) = g(X_t),$$

which shows that $g(x)$ is a submartingale.

5.8 (*) For f being a deterministic step function corresponding to the partition $0 = t_1 < t_2 < \ldots < t_{n+1} = T$ with $f_t = f_j$ for $t \in [t_j, t_{j+1})$ for $j \in \{1, 2, \ldots, n\}$ we have for $0 \le s \le t \le T$ that

$$\begin{aligned}
E\left(I_{f,W}(t) \,\middle|\, \mathcal{A}_s\right) &= E\left(\int_0^t f_u \, dW_u \,\middle|\, \mathcal{A}_s\right) \\
&= E\left(\sum_{j=1}^{i_s-1} f_j (W_{t_{j+1}} - W_{t_j}) + f_{i_s}\left(W_s - W_{t_{i_s}}\right)\right. \\
&\quad + f_{i_s}\left(W_{t_{i_s+1}} - W_s\right) + \sum_{j=i_s+1}^{i_t-1} f_j\left(W_{t_{j+1}} - W_{t_j}\right) \\
&\quad \left. + f_{i_t}\left(W_t - W_{t_{i_t}}\right) \,\middle|\, \mathcal{A}_s\right),
\end{aligned}$$

where

$$i_t = \max\{k \in \{1, 2, \ldots\} : \, t_k \le t\}.$$

Thus, we obtain by the zero mean property of Wiener process increments that only the first two terms in the above expectation survive so that

$$E\left(I_{f,W}(t) \,\middle|\, \mathcal{A}_s\right) = I_{f,W}(s),$$

which proves the martingale property for $I_{f,W}(s)$.

5.9 (*) Using the notation and representation of the Itô integral of Exercise 5.8 we have for $0 \le s \le t \le T < \infty$ and deterministic step functions f and $\bar{f}$

$$\begin{aligned}
&E\left((I_{f,W}(t) - I_{f,W}(s))(I_{\bar{f},W}(t) - I_{\bar{f},W}(s)) \,\middle|\, \mathcal{A}_s\right) \\
&= E\left(\left[f_{i_s}(W_{t_{i_s+1}} - W_s) + \sum_{j=i_s+1}^{i_t+1} f_j\left(W_{t_{j+1}} - W_{t_j}\right) + f_{i_t}\left(W_t - W_{t_{i_t}}\right)\right]\right. \\
&\quad \left. \times \left[\bar{f}_{i_s}\left(W_{t_{i_s+1}} - W_s\right) + \sum_{j=i_s+1}^{i_t+1} \bar{f}_j\left(W_{t_{j+1}} - W_{t_j}\right) + \bar{f}_{i_t}\left(W_t - W_{t_{i_t}}\right)\right] \,\middle|\, \mathcal{A}_s\right).
\end{aligned}$$

Thus, it follows by the expectation properties for products of Wiener process increments that

$$E\left((I_{f,W}(t) - I_{f,W}(s))(I_{\bar{f},W}(t) - I_{\bar{f},W}(s)) \,\middle|\, \mathcal{A}_s\right) = \int_s^t E\left(f_u\,\bar{f}_u \,\middle|\, \mathcal{A}_s\right) du.$$

5.10 (*) Using the notation and representation of the Itô integrals of the Exercises 5.8 and 5.9 we have for $0 \leq s \leq t \leq T < \infty$, and deterministic step functions f and $\bar{f}$ and $\mathcal{A}_s$-measurable constants α and $\bar{\alpha}$ the equation

$$\begin{aligned}\int_s^t \left(\alpha\, f_u + \bar{\alpha}\,\bar{f}_u\right) dW_u =& \left(\alpha\, f_{t_{i_s}} + \bar{\alpha}\,\bar{f}_{t_{i_s}}\right)\,(W_{t_{i_s+1}} - W_s) \\ &+ \sum_{j=i_s+1}^{i_t-1} \left(\alpha\, f_{t_j} + \bar{\alpha}\,\bar{f}_{t_j}\right)(W_{t_{j+1}} - W_{t_j}) \\ &+ \left(\alpha\, f_{t_{i_t}} + \bar{\alpha}\,\bar{f}_{t_{i_t}}\right)\,(W_t - W_{t_{i_t}})\end{aligned}$$

$$\begin{aligned}=&\alpha\left[f_{t_{i_s}}(W_{t_{i_s+1}} - W_s) + \sum_{j=i_s+1}^{i_t-1} f_{t_j}(W_{t_{j+1}} - W_{t_j}) + f_{t_{i_t}}(W_t - W_{t_{i_t}})\right] \\ &+ \bar{\alpha}\left[\bar{f}_{t_{i_s}}(W_{t_{i_s+1}} - W_s) + \sum_{j=i_s+1}^{i_t-1} \bar{f}_{t_j}(W_{t_{j+1}} - W_{t_j}) + \bar{f}_{t_{i_t}}(W_t - W_{t_{i_t}})\right] \\ =&\alpha \int_s^t f_u\, dW_u + \bar{\alpha}\int_s^t \bar{f}_u\, dW_u,\end{aligned}$$

which proves the linearity property (5.4.2) for deterministic step functions.

5.11 (*) Obviously, we have $E(X_t - X_0 \,|\, \mathcal{A}_0)$ for all $t \in [0,T]$. Since the Lévy process X has stationary independent increments it follows for $0 \leq s \leq t \leq T$ that

$$E\left(X_s \,\middle|\, \mathcal{A}_t\right) = E\left(X_s - X_t \,\middle|\, \mathcal{A}_t\right) + X_t = X_t.$$

This proves that X is a martingale.

Solutions for Exercises of Chapter 6

6.1 By the Itô formula it follows that

$$d(Y_t)^2 = (2Y_t\, a + b^2)\, dt + 2\, Y_t\, b\, dW_t.$$

6.2 By application of the Itô formula we obtain

$$dZ_t = Z_t \left(\mu + \frac{1}{2}\,\sigma^2 \right) dt + Z_t\,\sigma\,dW_t.$$

Applying again the Itô formula we obtain

$$\begin{aligned} d\ln(Z_t) &= \left(\mu + \frac{1}{2}\,\sigma^2 - \frac{1}{2}\,\sigma^2 \right) dt + \sigma\,dW_t \\ &= \mu\,dt + \sigma\,dW_t. \end{aligned}$$

6.3 It follows by the Itô formula that

$$\begin{aligned} d(Z_t)^2 &= 2\,Z_t^2 \left(\mu + \frac{1}{2}\,\sigma^2 \right) dt + Z_t^2\,\sigma^2\,dt + 2\,Z_t^2\,\sigma\,dW_t \\ &= 2\,Z_t^2 \left(\mu + \sigma^2 \right) dt + 2\,Z_t^2\,\sigma\,dW_t. \end{aligned}$$

6.4 We have by the Itô formula that

$$\begin{aligned} dZ_t^{-1} &= Z_t^{-1} \left(-\mu - \frac{1}{2}\,\sigma^2 + \sigma^2 \right) dt - Z_t^{-1}\,\sigma\,dW_t \\ &= Z_t^{-1} \left(-\mu + \frac{1}{2}\,\sigma^2 \right) dt - Z_t^{-1}\,\sigma\,dW_t. \end{aligned}$$

6.5 We obtain by the Itô formula

$$\begin{aligned} d(Y_t\,Z_t) &= \left(Z_t\,a + Y_t\,Z_t \left(\mu + \frac{1}{2}\,\sigma^2 \right) + b\,Z_t\,\sigma \right) dt \\ &\quad + Z_t\,b\,dW_t + Y_t\,Z_t\,\sigma\,dW_t \\ &= Z_t \left(a + Y_t \left(\mu + \frac{1}{2}\,\sigma^2 \right) + b\,\sigma \right) dt \\ &\quad + Z_t\,(b + Y_t\,\sigma)\,dW_t. \end{aligned}$$

6.6 We have by the Itô formula the stochastic differential

$$d(Y_t^1\,Y_t^2) = (Y_t^2\,a_1 + Y_t^1\,a_2)\,dt + Y_t^2\,b_1\,dW_t^1 + Y_t^1\,b_2\,dW_t^2.$$

6.7 The stochastic differential is obtained by the Itô formula and we obtain

$$\begin{aligned} dZ_t &= d\left(\exp\{Y_t^1\}\exp\{Y_t^2\}\right) \\ &= d\left(\exp\{Y_t^1 + Y_t^2\}\right) \\ &= Z_t\left(a_1 + a_2 + \frac{1}{2}(b_1^2 + b_2^2)\right) dt + Z_t\, b_1\, dW_t^1 + Z_t\, b_2\, dW_t^2. \end{aligned}$$

6.8 Applying the Itô formula we obtain

$$d(W_t)^2 = 2\, W_t\, dW_t + dt.$$

Now by the covariation property (5.4.5) of Itô integrals we have

$$\begin{aligned} [W,(W)^2]_t &= \left[\int_0 dW_s, 2\int_0 W_s\, dW_s + \int_0 ds\right]_t \\ &= \int_0^t 2\, W_s\, ds. \end{aligned}$$

6.9 (*) By the Itô formula we have

$$d(X_t)^2 = 2\, X_t\, \xi_t\, dW_t + (\xi_t)^2\, dt$$

and by the covariation property (5.4.5) of Itô integrals it follows

$$d[X]_t = (\xi_t)^2\, dt.$$

Therefore, it holds

$$\begin{aligned} dY_t &= d\left((X_t)^2 - [X]_t\right) \\ &= 2\, X_t\, \xi_t\, dW_t \end{aligned}$$

and Y_t is represented by an Itô integral. Thus, by the martingale property (5.4.3) of Itô integrals Y is a martingale.

6.10 (*) The stochastic differential of X is

$$dX_t = \sigma\, dW_t + \xi\, dp(t)$$

for $t \in [0,T]$. By the Itô formula (6.4.11) it follows that

$$d\exp\{X_t\} = \exp\{X_t\}\left(\sigma\, dW_t + \frac{1}{2}\sigma^2\, dt\right) + \exp\{X_{t-}\}\left(\exp\{\xi\} - 1\right) dp(t)$$

for $t \in [0,T]$.

6.11 (*) The stochastic differential of X is

$$dX_t = a\,dp^1(t) + b\,dp^2(t)$$

for $t \in [0,T]$. Using the Itô formula (6.4.11) we obtain

$$\begin{aligned} d\,\exp\{X_t\} &= \exp\{X_{t-}\}\,(\exp\{a\} - 1)\,dp^1(t) \\ &\quad + \exp\{X_{t-}\}\,(\exp\{b\} - 1)\,dp^2(t) \end{aligned}$$

for $t \in [0,T]$.

Solutions for Exercises of Chapter 7

7.1 We obtain from (7.2.6) or (7.3.5) the mean

$$\mu(t) = E(X_t) = \exp\{-(t - t_0)\}.$$

Furthermore we obtain from (7.2.6) or (7.3.9) the variance

$$\begin{aligned} v(t) &= E\left((X_t - E(X_t))^2\right) \\ &= E\left(\left(\int_{t_0}^t \sqrt{2}\,\exp\{-(t-s)\}\,dW_s\right)^2\right) \\ &= 2\int_{t_0}^t \exp\{-2(t-s)\}\,ds \\ &= 1 - \exp\{-2(t - t_0)\}. \end{aligned}$$

7.2 According to (7.3.5) we obtain for the mean

$$\mu(t) = \exp\{0.05\,t\}.$$

It follows for the variance

$$\begin{aligned} v(t) &= E\left((X_t - \mu(t))^2\right) \\ &= E(X_t^2) - (\mu(t))^2 \\ &= P(t) - (\mu(t))^2, \end{aligned}$$

where with (7.3.8) we obtain

$$dP(t) = (0.1 + 0.04)\,P(t)\,dt$$

with $P(0) = 1$ such that

$$v(t) = \exp\{0.14\,t\} - \exp\{0.1\,t\}$$

for $t \geq 0$.

7.3 We apply formula (7.4.5), where

$$\begin{aligned} X_t &= X_0\,\Psi_{t,0} \\ &= X_0\,\exp\left\{\left(-\frac{1}{2} - \frac{1}{2} - \frac{1}{2}\right) t + \sum_{l=1}^{2} \left(W_t^l - W_0^l\right)\right\} \\ &= X_0\,\exp\left\{-\frac{3}{2}\,t + W_t^1 + W_t^2\right\}. \end{aligned}$$

7.4 (*) By the Itô formula one obtains

$$dX_t = X_{t-}\left[\left(k\,a + \frac{k^2\,b^2}{2}\right) dt + k\,b\,dW_t + (\exp\{k\,c\} - 1)\,dN_t\right]$$

for $t \in [0, T]$ with $X_0 = 1$.

7.5 (*) In the linear SDE for X_t we take the expectation and obtain

$$d\mu(t) = \mu(t-)\left[k\,a + \frac{k^2\,b^2}{2} + \lambda\,(\exp\{k\,c\} - 1)\right] dt$$

for $t \in [0, T]$ with $\mu(0) = 1$.

7.6 (*) The explicit solution is of the form

$$X_t = \Psi_{t,0}\left(X_0 + \int_0^t (a_2 - b_1\,b_2)\,\Psi_{s,0}^{-1}\,ds + \int_0^t b_2\,\Psi_{s,0}^{-1}\,dW_s\right)$$

with

$$\Psi_{t,0} = \exp\left\{\int_0^t \left(a_1 - \frac{1}{2}\,b_1^2\right) ds + \int_0^t b_1\,dW_s\right\}$$

for $t \in [0, T]$. By application of the Itô formula we obtain

$$\begin{aligned} dX_t &= \Psi_{t,0}\left[(a_2 - b_1\,b_2)\,\Psi_{t,0}^{-1}\,dt + b_2\,\Psi_{t,0}^{-1}\,dW_t\right] \\ &\quad + \frac{X_t}{\Psi_{t,0}}\,d\Psi_{t,0} + d\left[\Psi_{\cdot,0}, \int_0^{\cdot} b_2\,\Psi_{s,0}^{-1}\,dW_s\right]_t. \end{aligned}$$

Noting by the Itô formula that

$$d\Psi_{t,0} = \Psi_{t,0}(a_1\,dt + b_1\,dW_t)$$

it follows

$$
\begin{aligned}
dX_t &= (a_2 - b_1\, b_2)\, dt + b_2\, dW_t \\
&\quad + X_t\,(a_1\, dt + b_1\, dW_t) + b_1\, b_2\, dt \\
&= (X_t\, a_1 + a_2)\, dt + (X_t\, b_1 + b_2)\, dW_t
\end{aligned}
$$

for $t \in [0, T]$.

Solutions for Exercises of Chapter 8

8.1 From (8.3.2) we obtain the discounted option price

$$
\bar{V}(t, \bar{S}_t) = \bar{S}_t\, N(d_1(t)) - K\, \exp\left\{ -\int_0^T r_s\, ds \right\} N(d_2(t))
$$

with

$$
d_1(t) = \left(\ln\left(\frac{\bar{S}_t}{K\,(B_T)^{-1}} \right) + \int_t^T \frac{1}{2}\, \sigma_s^2\, ds \right) \left(\int_t^T \sigma_s^2\, ds \right)^{-\frac{1}{2}}
$$

and

$$
d_2(t) = d_1(t) - \left(\int_t^T \sigma_s^2\, ds \right)^{\frac{1}{2}}.
$$

Then it is

$$
\frac{\partial \bar{V}(t, \bar{S}_t)}{\partial \bar{S}} = N(d_1(t)) + Q_t,
$$

where

$$
\begin{aligned}
Q_t &= \bar{S}_t\, N'(d_1(t))\, \frac{\partial d_1(t)}{\partial \bar{S}} - \frac{K}{B_T}\, N'(d_2(t))\, \frac{\partial d_2(t)}{\partial \bar{S}} \\
&= \frac{\partial d_1(t)}{\partial \bar{S}\, \sqrt{2\pi}} \left[\bar{S}_t\, \exp\left\{ -\frac{(d_1(t))^2}{2} \right\} - \frac{K}{B_T}\, \exp\left\{ -\frac{(d_2(t))^2}{2} \right\} \right] \\
&= \frac{\partial d_1(t)}{\partial \bar{S}\, \sqrt{2\pi}}\, \exp\left\{ -\frac{(d_1(t))^2}{2} \right\} \left[\bar{S}_t - \frac{K}{B_T}\, \exp\left\{ \ln\left(\frac{\bar{S}_t}{\frac{K}{B_T}} \right) \right. \right. \\
&\qquad \left. \left. - \frac{1}{2} \int_t^T \sigma_s^2\, ds + \frac{1}{2} \int_t^T \sigma_s^2\, ds \right\} \right] \\
&= 0.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
\frac{\partial^2 \bar{V}(t, \bar{S}_t)}{\partial \bar{S}^2} &= N'(d_1(t))\, \frac{\partial d_1(t)}{\partial \bar{S}} \\
&= N'(d_1(t))\, \bar{S}_t^{-1} \left(\int_t^T \sigma_s^2\, ds \right)^{-\frac{1}{2}}.
\end{aligned}
$$

We also obtain the time derivative

$$\begin{aligned}
\frac{\partial \bar{V}(t,\bar{S}_t)}{\partial t} &= \frac{1}{2}\,\sigma_t^2 \left(\int_t^T \sigma_s^2\,ds \right)^{-\frac{1}{2}} \Bigg(\bar{S}_t\,N'(d_1(t)) \\
&\qquad \times \left[\ln\left(\frac{\bar{S}_t\,B_T}{K} \right) \left(\int_t^T \sigma_s^2\,ds \right)^{-1} - \frac{1}{2} \right] \\
&\qquad - \frac{K}{B_T}\,N'(d_2(t)) \left[\ln\left(\frac{\bar{S}_t\,B_T}{K} \right) \left(\int_t^T \sigma_s^2\,ds \right)^{-1} + \frac{1}{2} \right] \Bigg) \\
&= \frac{1}{2}\,\sigma_t^2 \left(\int_t^T \sigma_s^2\,ds \right)^{-\frac{1}{2}} N'(d_1(t)) \Bigg(\ln\left(\frac{\bar{S}_t\,B_T}{K} \right) \left(\int_t^T \sigma_s^2\,ds \right)^{-1} \\
&\qquad \times \left[\bar{S}_t - \frac{K}{B_T}\,\exp\left\{ d_1(t) \left(\int_t^T \sigma_s^2\,ds \right)^{\frac{1}{2}} - \frac{1}{2} \int_t^T \sigma_s^2\,ds \right\} \right] \\
&\qquad -\frac{1}{2} \left[\bar{S}_t - \frac{K}{B_T}\,\exp\left\{ d_1(t) \left(\int_t^T \sigma_s^2\,ds \right)^{\frac{1}{2}} - \frac{1}{2} \int_t^T \sigma_s^2\,ds \right\} \right] \Bigg) \\
&= -\frac{1}{2}\,\sigma_t^2 \left(\int_t^T \sigma_s^2\,ds \right)^{-\frac{1}{2}} N'(d_1(t))\,\bar{S}_t.
\end{aligned}$$

We note that

$$\frac{\partial \bar{V}(t,\bar{S}_t)}{\partial t} + \frac{1}{2}\,\sigma_t^2\,\bar{S}_t^2\,\frac{\partial^2 \bar{V}(t,\bar{S}_t)}{\partial \bar{S}^2} = 0$$

and also that

$$\bar{V}(T,\bar{S}_T) = \left(\bar{S}_T - \frac{K}{B_T} \right)^+.$$

This shows that the discounted European call option price (8.3.2) satisfies the discounted BS-PDE (8.2.21) with terminal condition (8.2.22).

8.2 The hedge ratio is given by the expression

$$\begin{aligned}
\frac{\partial V(t,S_t)}{\partial S} &= \left(\frac{\partial}{\partial \bar{S}_t}\,V(t,S_t) \right) \frac{\partial \bar{S}_t}{\partial S_t} \\
&= \left(\frac{\partial}{\partial \bar{S}_t}\,\left(\bar{V}(t,\bar{S}_t)\,B_t \right) \right) \frac{1}{B_t} \\
&= \frac{\partial}{\partial \bar{S}_t} \bar{V}(t,\bar{S}_t).
\end{aligned}$$

Then it follows from our calculations in Exercise 8.1 that

$$\frac{\partial V(t,S_t)}{\partial S} = N(d_1(t))$$

which corresponds to (8.4.3).

8.3 The gamma for the European put option is given by the expression

$$\begin{aligned}\frac{\partial^2 V(t,S_t)}{\partial S^2} &= \frac{\partial}{\partial S} N(d_1(t)) \\ &= N'(d_1(t))\, S_t^{-1} \left(\int_t^T \sigma_s^2\, ds\right)^{-\frac{1}{2}},\end{aligned}$$

see (8.4.5) and (8.5.5), which is the same gamma as for the European call option.

8.4 We obtain from (8.2.4) for the European put option the number of units δ_t^0 to be held at time t in the savings account in the form

$$\delta_t^0 = \frac{V(t,S_t)}{B_t} - \delta_t^1 \frac{S_t}{B_t}.$$

Therefore, it follows from (8.5.3) and (8.5.4) that

$$\begin{aligned}\delta_t^0 &= \bar{S}_t\,(N(d_1(t)) - 1) - \frac{K}{B_T}\,(N(d_2(t)) - 1) - (N(d_1(t)) - 1)\,\bar{S}_t \\ &= \frac{K}{B_T}\,(1 - N(d_2(t))).\end{aligned}$$

8.5 By using the notation $\bar{S} = \frac{S}{B_t}$ and $V(t,S) = \bar{V}(t,\bar{S})\,B_t$ we obtain with the partial derivatives $\frac{\partial S}{\partial \bar{S}} = B_t$

$$\frac{\partial \bar{V}(t,\bar{S})}{\partial \bar{S}} = \frac{1}{B_t}\,\frac{\partial V(t,S)}{\partial S}\,\frac{\partial S}{\partial \bar{S}} = \frac{\partial V(t,S)}{\partial S}$$

$$\frac{\partial^2 \bar{V}(t,\bar{S})}{\partial \bar{S}^2} = \frac{\partial V^2(t,S)}{\partial S^2}\,\frac{\partial S}{\partial \bar{S}} = \frac{\partial^2 V(t,S)}{\partial S^2}\,B_t$$

and

$$\begin{aligned}\frac{\partial \bar{V}(t,\bar{S})}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{B_t}\,V(t,\bar{S}\,B_t)\right) \\ &= -r\,\frac{1}{B_t}\,V(t,S) + \frac{1}{B_t}\,\frac{\partial V(t,S)}{\partial t} + \frac{1}{B_t}\,\frac{\partial V(t,\bar{S}\,B_t)}{\partial S}\,\bar{S}\,B_t\end{aligned}$$

by (8.2.21) the PDE

$$\begin{aligned} 0 &= \frac{\partial \bar{V}(t,\bar{S})}{\partial t} + \frac{1}{2}\,\sigma^2\,\bar{S}^2\,\frac{\partial^2 \bar{V}(t,\bar{S})}{\partial \bar{S}^2} \\ &= \frac{1}{B_t}\left(-r\,V(t,S) + \frac{\partial V(t,S)}{\partial t} + \frac{\partial V(t,S)}{\partial S}\,S\,r + \frac{1}{2}\,\sigma^2\,\bar{S}^2\,B_t^2\,\frac{\partial^2 V(t,S)}{\partial S^2}\right), \end{aligned}$$

which proves (8.2.23).

8.6 The discounted P&L process $\bar{C}$ for a European put option has according to (8.2.13) and (8.2.20) the form

$$\bar{C}_t = \bar{V}(t,\bar{S}_t) - \bar{V}(0,\bar{S}_0) - \int_0^t \frac{\partial \bar{V}(s,\bar{S}_s)}{\partial \bar{S}}\,d\bar{S}_s.$$

On the other hand, we have by the discounted BS-PDE for a European put option

$$d\bar{V}(t,\bar{S}_t) = \frac{\partial \bar{V}(t,\bar{S}_t)}{\partial \bar{S}}\,d\bar{S}_t$$

and it follows

$$\begin{aligned} d\bar{C}_t &= d\bar{V}(t,\bar{S}_t) - \frac{\partial \bar{V}(t,\bar{S}_t)}{\partial \bar{S}}\,d\bar{S}_t \\ &= 0. \end{aligned}$$

This means, the discounted P&L

$$\bar{C}_t = \bar{C}_0 = 0$$

equals the constant zero. Consequently, by (8.2.12) the P&L

$$C_t = \bar{C}_t\,B_t = 0$$

is zero for all $t \in [0,T]$.

8.7 We consider the square root process $Y = \{Y_t,\, t \in [0,\infty)\}$ of dimension $\delta > 2$ satisfying the SDE

$$dY_t = \left(\frac{\delta}{4}\,c^2 + b\,Y_t\right)dt + c\,\sqrt{Y_t}\,dW_t$$

for $t \in [0,\infty)$, $Y_0 > 0$, $c > 0$ and $b < 0$. The Itô integral

$$M_t = c\int_0^t \sqrt{Y_s}\,dW_s$$

forms a martingale due to Lemma 5.2.2 (iii), since the square root process, as a transformed time changed squared Bessel process, has moments of any positive order. Consequently,

$$dE(Y_t) = \left(\frac{\delta}{4}\, c^2 + b\, E(Y_t) \right) dt$$

for $t \in [0, \infty)$ with $E(Y_0) > 0$. Therefore, we obtain

$$E(Y_t) = E(Y_0)\, \exp\{b\, t\} + \frac{\delta\, c^2}{4\, b}\, (\exp\{b\, t\} - 1).$$

8.8 (*) Using the notation of Sect. 8.7 we have for $\delta > 2$, $\alpha \geq -\frac{\delta}{2}$ and $\varphi > \varphi(0)$ by (8.7.7) and (8.7.9) the αth moment in the form

$$\begin{aligned}
E(X_\varphi^\alpha) &= \int_0^\infty y^\alpha\, p_\delta(\varphi(0), x; \varphi, y)\, dy \\
&= \int_0^\infty y^\alpha\, \frac{1}{2\,(\varphi - \varphi(0))} \left(\frac{y}{x}\right)^{\frac{\delta}{4} - \frac{1}{2}} \exp\left\{ -\frac{x + y}{2\,(\varphi - \varphi(0))} \right\} \\
&\qquad \times \sum_{k=0}^\infty \frac{\left(\frac{\sqrt{x\, y}}{2\,(\varphi - \varphi(0))} \right)^{2k + \frac{\delta}{2} - 1}}{k!\, \Gamma\left(k + \frac{\delta}{2}\right)}\, dy \\
&= \sum_{k=0}^\infty \frac{\exp\left\{-\frac{x}{2\,(\varphi-\varphi(0))}\right\} x^k \left(\frac{1}{2\,(\varphi-\varphi(0))}\right)^{2k+\frac{\delta}{2}}}{k!\, \Gamma\left(k + \frac{\delta}{2}\right)} \int_0^\infty y^{\alpha + k + \frac{\delta}{2} - 1} \\
&\qquad \times \exp\left\{ -\frac{y}{2\,(\varphi - \varphi(0))} \right\} dy.
\end{aligned}$$

According to the gamma function (1.2.10) it holds for $\beta = \alpha + k + \frac{\delta}{2} > 0$ that

$$\int_0^\infty y^{\beta - 1}\, \exp\left\{ -\frac{y}{q} \right\} dy = \Gamma(\beta)\, (q)^\beta$$

and thus

$$\begin{aligned}
E(X_\varphi^\alpha) &= \sum_{k=0}^\infty \frac{\exp\left\{-\frac{x}{q}\right\} x^k \left(\frac{1}{q}\right)^{2k+\frac{\delta}{2}}}{k!\, \Gamma\left(k + \frac{\delta}{2}\right)}\, (q)^{\alpha + k + \frac{\delta}{2}}\, \Gamma\left(\alpha + k + \frac{\delta}{2}\right) \\
&= (q)^\alpha\, \exp\left\{ -\frac{x}{q} \right\} \sum_{k=0}^\infty \left(\frac{x}{q}\right)^k \frac{\Gamma\left(\alpha + k + \frac{\delta}{2}\right)}{k!\, \Gamma\left(k + \frac{\delta}{2}\right)}
\end{aligned}$$

with $q = 2\,(\varphi - \varphi(0))$, which shows the first equation in (8.7.16). For $k \geq 1$ and $\alpha \leq 0$ we have from the properties of the gamma function that $\Gamma\left(\alpha + k + \frac{\delta}{2}\right) \leq \Gamma\left(k + \frac{\delta}{2}\right)$ and thus the estimate

$$E(X_\varphi^\alpha) = (2\,(\varphi - \varphi(0)))^\alpha\, \exp\left\{ -\frac{x}{q} \right\} \left(\frac{\Gamma\left(\alpha + \frac{\delta}{2}\right)}{\Gamma\left(\frac{\delta}{2}\right)} + \exp\left\{ \frac{x}{q} \right\} \right) < \infty,$$

which provides also the second part of (8.7.16).

8.9 (*) Using (8.7.9) it follows

$$\begin{aligned}
&\int_0^\infty y^{1-\frac{n}{2}}\, p_n(\varphi(0),x;\varphi,y)\,dy \\
&= \int_0^\infty \frac{y^{1-\frac{n}{2}}}{2(\varphi-\varphi(0))}\left(\frac{y}{x}\right)^{\frac{n}{4}-\frac{1}{2}} \exp\left\{-\frac{x+y}{2(\varphi-\varphi(0))}\right\} I_{\frac{n}{2}-1}\left(\frac{\sqrt{x\,y}}{\varphi-\varphi(0)}\right) dy \\
&= \int_0^\infty \frac{x^{1-\frac{n}{2}}}{2(\varphi-\varphi(0))}\left(\frac{y}{x}\right)^{\frac{1}{2}-\frac{n}{4}} \exp\left\{-\frac{x+y}{2(\varphi-\varphi(0))}\right\} I_{\frac{n}{2}-1}\left(\frac{\sqrt{x\,y}}{\varphi-\varphi(0)}\right) dy \\
&= x^{1-\frac{n}{2}} \int_0^\infty p_{4-n}(\varphi(0),y;\varphi,x)\,dy.
\end{aligned}$$

8.10 (*) Combining (8.7.9) and (8.7.19) we obtain

$$\begin{aligned}
&\int_0^\infty p_{4-n}(\varphi(0),y;\varphi,x)\,dy \\
&= \int_0^\infty \frac{1}{2(\varphi-\varphi(0))}\left(\frac{y}{x}\right)^{\frac{1}{2}-\frac{n}{4}} \exp\left\{-\frac{x+y}{2(\varphi-\varphi(0))}\right\} I_{\frac{n}{2}-1}\left(\frac{\sqrt{x\,y}}{\varphi-\varphi(0)}\right) dy \\
&= \int_0^\infty \frac{1}{2(\varphi-\varphi(0))}\left(\frac{y}{x}\right)^{\frac{1}{2}-\frac{n}{4}} \exp\left\{-\frac{x+y}{2(\varphi-\varphi(0))}\right\} \sum_{k=0}^\infty \frac{\left(\frac{\sqrt{x\,y}}{2(\varphi-\varphi(0))}\right)^{2k+\frac{n}{2}-1}}{k\,!\,\Gamma\left(\frac{n}{2}+k\right)}\, dy \\
&= \sum_{k=0}^\infty \frac{x^{\frac{n}{2}-1+k}\exp\left\{-\frac{x}{2(\varphi-\varphi(0))}\right\}}{(2(\varphi-\varphi(0)))^{\frac{n}{2}+2k}\,k\,!\,\Gamma\left(\frac{n}{2}+k\right)} \int_0^\infty y^k \exp\left\{-\frac{y}{2(\varphi-\varphi(0))}\right\} dy \\
&= \sum_{k=0}^\infty \left(\frac{x}{2(\varphi-\varphi(0))}\right)^{\frac{n}{2}-1+k} \frac{\exp\left\{-\frac{x}{2(\varphi-\varphi(0))}\right\}}{\Gamma\left(\frac{n}{2}+k\right)} \\
&= \left(\frac{x}{2(\varphi-\varphi(0))}\right)^{\frac{n}{2}-1} \exp\left\{-\frac{x}{2(\varphi-\varphi(0))}\right\} \sum_{k=0}^\infty \left(\frac{x}{2(\varphi-\varphi(0))}\right)^k \frac{1}{\Gamma\left(\frac{n}{2}+k\right)}.
\end{aligned}$$

Using the series expansion

$$\Gamma(a)-\Gamma(a,z) = e^{-z}\,z^a \sum_{k=0}^\infty \frac{\Gamma(a)}{\Gamma(a+1+k)}\,z^k,$$

see Abramowitz & Stegun (1972), with $a=\frac{n}{2}-1$ and $z=\frac{x}{2(\varphi-\varphi(0))}$ the above equation becomes

$$\int_0^\infty p_{4-n}(\varphi(0),y;\varphi,x)\,dy = 1-\frac{\Gamma\left(\frac{n}{2}-1, \frac{x}{2(\varphi-\varphi(0))}\right)}{\Gamma\left(\frac{n}{2}-1\right)}.$$

Solutions for Exercises of Chapter 9

9.1 By the SDE (9.4.14) it follows that the discounted stock price $\bar{S}_t$ satisfies the SDE

$$d\bar{S}_t = \sigma\, \bar{S}_t\, dW_{\theta\, t},$$

where W_θ is a standard Wiener process under the risk neutral measure P_θ. Since this SDE is driftless $\bar{S}$ is an $(\underline{\mathcal{A}}, P_\theta)$-local martingale. Furthermore, because $\bar{S}$ is a geometric Brownian motion with bounded second moment, see (7.3.13)–(7.3.14), it follows that the diffusion coefficient $\sigma \bar{S}_t$ is square integrable for all $t \in [0, T]$. Consequently, by (5.4.1) $\sigma \bar{S}$ is from $\mathcal{L}_T^2$ and by the martingale property (5.4.3) of Itô integrals an $(\underline{\mathcal{A}}, P_\theta)$-martingale.

9.2 By application of the Itô formula it follows by (8.3.2) and the discounted BS-PDE as in Exercise 9.1 that

$$d\bar{V}(t, \bar{S}_t) = \frac{\partial \bar{V}(t, \bar{S}_t)}{\partial \bar{S}}\, \sigma\, \bar{S}_t\, (\theta_t\, dt + dW_t),$$

where the hedge ratio

$$\frac{\partial \bar{V}(t, \bar{S}_t)}{\partial \bar{S}} = \frac{\partial V(t, S_t)}{\partial S}$$

is by (8.4.3) bounded. With

$$dW_{\theta\, t} = \theta_t\, dt + dW_t$$

it follows by the Girsanov Theorem that W_θ is a P_θ-Wiener process. Since $\bar{S} \in \mathcal{L}_T^2$ it follows by the martingale property (5.4.3) of Itô integrals from the above SDE that the $(\underline{\mathcal{A}}, P_\theta)$-local martingale $\bar{V}$ is an $(\underline{\mathcal{A}}, P_\theta)$-martingale.

9.3 We obtain by the Itô formula (6.1.12) and the discounted BS-PDE (8.2.21) for the discounted put option price the SDE

$$\begin{aligned} d\left(\frac{p_{T,K}(t, S_t)}{B_t}\right) &= d\bar{V}(t, \bar{S}_t) \\ &= \left(\frac{\partial}{\partial t}\, \bar{V}(t, \bar{S}_t) + \frac{1}{2}\, \sigma^2\, \bar{S}_t^2\, \frac{\partial^2 \bar{V}(t, \bar{S}_t)}{\partial \bar{S}^2}\right) dt + \frac{\partial \bar{V}(t, \bar{S}_t)}{\partial \bar{S}}\, d\bar{S}_t \\ &= \frac{\partial \bar{V}(t, \bar{S}_t)}{\partial \bar{S}}\, d\bar{S}_t \end{aligned}$$

for $t \in [0, T]$. Therefore, by the Itô formula and (8.2.1)–(8.2.2) we obtain

$$
\begin{aligned}
dp_{T,K}(t,S_t) &= d(\bar{V}(t,\bar{S}_t)\,B_t) \\
&= p_{T,K}(t,S_t)\,r\,dt + B_t\,d\bar{V}(t,\bar{S}_t) \\
&= p_{T,K}(t,S_t)\,r\,dt + \left(\frac{\partial \bar{V}(t,\bar{S}_t)}{\partial \bar{S}}\right) B_t\,d\bar{S}_t \\
&= p_{T,K}(t,S_t)\,r\,dt + \left(\frac{\partial p_{T,K}(t,S_t)/B_t}{\partial S}\right) \frac{\partial S_t}{\partial \bar{S}}\,B_t\,d\bar{S}_t \\
&= p_{T,K}(t,S_t)\,r\,dt + \frac{\partial p_{T,K}(t,S_t)}{\partial S}\,B_t\,[(a-r)\,\bar{S}_t\,dt + \sigma\,\bar{S}_t\,dW_t] \\
&= \left(p_{T,K}(t,S_t)\,r + \frac{\partial p_{T,K}(t,S_t)}{\partial S}\,(a-r)\,S_t\right) dt \\
&\quad + \frac{\partial p_{T,K}(t,S_t)}{\partial S}\,\sigma\,S_t\,dW_t.
\end{aligned}
$$

We obtain with (9.4.1) and (9.1.16) for the real world dynamics of $p_{T,K}(t,S_t)$ the SDE

$$
\begin{aligned}
dp_{T,K}(t,S_t) &= p_{T,K}(t,S_t)\,r dt + \frac{\partial p_{T,K}(t,S_t)}{\partial S}\,\sigma\,S_t\,\left(dW_t + \frac{a-r}{\sigma}\,dt\right) \\
&= r\,p_{T,K}(t,S_t)\,dt + \frac{\partial p_{T,K}(t,S_t)}{\partial S}\,\sigma\,S_t\,(dW_t + \theta\,dt).
\end{aligned}
$$

Here W is a standard Wiener process under P. Since under the risk neutral measure P_θ the value

$$W_\theta(t) = W_t + \theta\,t$$

forms an $(\underline{\mathcal{A}}, P)$-Wiener process we obtain directly the risk neutral SDE

$$dp_{T,K}(t,S_t) = r\,p_{T,K}(t,S_t)\,dt + \frac{\partial p_{T,K}(t,S_t)}{\partial S}\,\sigma\,S_t\,dW_\theta(t)$$

for $t \in [0,T]$.

9.4 Using (9.4.3) we have

$$dS_t = r\,S_t\,dt + \sigma\,S_t\,(dW_t + \theta\,dt),$$

where W is a Wiener process under the real world probability measure P. By Itô's formula combined with (9.1.15) the SDE for the benchmarked security

$$\hat{S}_t = \frac{S_t}{D_t}$$

is given by

$$d\hat{S}_t = (\sigma - \theta)\,\hat{S}_t\,dW_t.$$

This SDE is that of a driftless geometric Brownian motion, which has by (6.3.2) the explicit solution

$$\hat{S}_t = \hat{S}_0 \exp\left\{-\frac{1}{2}(\sigma-\theta)^2\, t + (\sigma-\theta)\, W_t\right\}.$$

By the mean (7.3.13) and variance (7.3.14) of a geometric Brownian motion it follows that

$$E(((\sigma-\theta)\,\hat{S}_t)^2) < \infty$$

for $t \in [0,T]$ so that by (5.4.1) $(\sigma-\theta)\,\hat{S} \in \mathcal{L}_T^2$. Consequently, $\hat{S}$ is an $(\underline{\mathcal{A}}, P_\theta)$-martingale by the martingale property (5.4.3) of Itô integrals.

9.5 From (9.4.13), (8.3.4) and (8.1.1) we obtain

$$\begin{aligned}
c_{T,K}(t,S) &= \int_{-\infty}^{\infty} \exp\{-r(T-t)\} \\
&\quad \left(S \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}\,x\right\} - K\right)^+ N'(x)\,dx \\
&= \int_{-\infty}^{\infty} \left(S \exp\left\{-\frac{1}{2}\sigma^2\,(T-t) + \sigma\sqrt{T-t}\,x\right\}\right. \\
&\qquad \left. - K \exp\{-r(T-t)\}\right)^+ N'(x)\,dx \\
&= \int_{-d_2(t)}^{\infty} \left(S \exp\left\{-\frac{1}{2}\sigma^2\,(T-t) + \sigma\sqrt{T-t}\,x\right\}\right. \\
&\qquad \left. - K \exp\{-r(T-t)\}\right) N'(x)\,dx \\
&= \int_{-\infty}^{d_2(t)} \left(S \exp\left\{-\frac{1}{2}\sigma^2\,(T-t) - \sigma\sqrt{T-t}\,x\right\}\right. \\
&\qquad \left. - K \exp\{-r(T-t)\}\right) N'(x)\,dx \\
&= S \int_{-\infty}^{d_2(t)} \exp\left\{-\frac{1}{2}\sigma^2\,(T-t) - \sigma\sqrt{T-t}\,x\right\} \\
&\qquad \times \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx - K \exp\{-r(T-t)\}\, N(d_2(t)).
\end{aligned}$$

With the change of variables

$$z = x + \sigma \sqrt{T-t}$$

and (8.3.3)–(8.3.4) we finally obtain

$$\begin{aligned} c_{T,K}(t,S) = S \int_{-\infty}^{d_1(t)} \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}\sigma^2 (T-t) - z\,\sigma\sqrt{T-t} + \sigma^2 (T-t) \right. \\ \left. - \frac{z^2}{2} + z\,\sigma\sqrt{T-t} - \frac{\sigma^2}{2}(T-t) \right\} dz \\ - K \exp\{-r(T-t)\}\, N(d_2(t)) \\ = S\, N(d_1(t)) - K \exp\{-r(T-t)\}\, N(d_2(t)), \end{aligned}$$

which proves the Black-Scholes European call option pricing formula.

9.6 (*) By (9.4.8) the Radon-Nikodym derivative at time t for the standard BS model equals the expression

$$\Lambda_\theta(t) = \frac{\hat{S}^0_t}{\hat{S}^0_0}$$

for $t \in [0,T]$. By (9.1.21) and (9.1.20) we obtain

$$\begin{aligned} d\Lambda_\theta(t) &= -\theta \frac{\hat{S}^0_t}{\hat{S}^0_0}\, dW_t \\ &= -\theta\, \Lambda_\theta(t)\, dW_t \end{aligned}$$

for $t \in [0,T]$, where $\Lambda_\theta(0) = 1$.

9.7 (*) Under the BS model with savings account $B_t = \exp\{r\,t\}$ and risky security

$$S_t = S_0 \exp\left\{ \left(a - \frac{1}{2}\sigma^2 \right) t + \sigma\, W_t \right\}$$

we have the GOP in the form

$$S^{\delta_*}_t = S^{\delta_*}_0 \exp\left\{ r\,t + \frac{1}{2}\theta^2\, t + \theta\, W_t \right\}.$$

The fair zero coupon bond price $P(t,T)$ at time t, when T is the maturity date, is obtained by the real world pricing formula and the Laplace transform for Gaussian random variables

$$\begin{aligned}
P(t,T) &= S_t^{\delta_*}\, E\left(\frac{1}{S_T^{\delta_*}}\,\Big|\,\mathcal{A}_t\right)\\
&= E\left(\exp\left\{r\,(t-T) + \frac{1}{2}\,\theta^2\,(t-T) + \theta\,(W_t - W_T)\right\}\,\Big|\,\mathcal{A}_t\right)\\
&= \exp\{-r\,(T-t)\}\, E\left(\exp\left\{-\frac{1}{2}\,\theta^2\,(T-t) - \theta\,(W_T - W_t)\right\}\,\Big|\,\mathcal{A}_t\right)\\
&= \exp\{-r\,(T-t)\} = \exp\{-r\,T\}\, B_t.
\end{aligned}$$

The benchmarked zero coupon bond price, when normalized to one at time zero, has the form

$$\Lambda_{\theta_{P(\cdot,T)}}(t) = \frac{\hat{P}(t,T)}{\hat{P}(0,T)} = \frac{\exp\{-r\,T\}\,B_t}{S_t^{\delta_*}}\;\frac{S_0^{\delta_*}}{\exp\{-r\,T\}} = S_0^{\delta_*}\,\frac{B_t}{S_t^{\delta_*}}$$

and satisfies the SDE

$$d\Lambda_{\theta_{P(\cdot,T)}}(t) = \Lambda_{\theta_{P(\cdot,T)}}(t)\,(-\theta)\,dW_t$$

with market price of risk $\theta = \frac{a-r}{\sigma}$. The Radon-Nikodym derivative for the zero coupon bond $P(\cdot,T)$ as numeraire is by (9.6.21)–(9.6.23) of the form

$$\frac{dP_{\theta_{P(\cdot,T)}}}{dP} = \Lambda_{\theta_{P(\cdot,T)}}(T).$$

The drifted Wiener process $W_{\theta_{P(\cdot,T)}}$ with

$$dW_{\theta_{P(\cdot,T)}}(t) = dW_t + \theta_{P(\cdot,T)}(t)\,dt$$

and

$$\theta_{P(\cdot,T)}(t) = \theta$$

is a Wiener process under the probability measure $P_{\theta_{P(\cdot,T)}}$. Therefore, the corresponding numeraire pair is $(P(\cdot,T), P_{\theta_{P(\cdot,T)}}) = (P(\cdot,T), P_\theta)$. The risk neutral measure P_θ equals here the, so-called, *T-forward measure* $P_{\theta_{P(\cdot,T)}}$ is an important observation under the BS model.

9.8 (*) We obtain under the BS model the discounted price $\bar{V}(t,\bar{S}_t) = \frac{V(t,S_t)}{B_t}$ at time t of the payoff $f(S_T) = (S_T)^2$ by the risk neutral pricing formula (9.6.10) in the form

$$\bar{V}(t,\bar{S}_t) = E_\theta\left(\frac{(\bar{S}_T\,B_T)^2}{B_T}\,\Big|\,\mathcal{A}_t\right),$$

where

$$d\bar{S}_t = \bar{S}_t\,\sigma\,dW_{\theta\,t}$$

under the risk neutral probability measure P_θ for $t \in [0,T]$. We have here only a terminal payoff at time T. When using the Feynman-Kac formula (9.7.3), then we obtain by (9.7.4) the PDE

$$\frac{\partial \bar{V}(t,\bar{S})}{\partial t} + \frac{1}{2}\,\sigma^2\,\bar{S}^2\,\frac{\partial^2 \bar{V}(t,\bar{S})}{\partial \bar{S}^2} = 0$$

for $(t,\bar{S}) \in [0,T) \times (0,\infty)$ with terminal condition

$$\bar{V}(T,\bar{S}) = \frac{(\bar{S}_T)^2}{B_T}.$$

This PDE has, by using the explicit solution for geometric Brownian motion, the solution

$$\begin{aligned}\bar{V}(t,\bar{S}_t) &= \frac{1}{B_T}\,E_\theta\left((\bar{S}_t)^2\,\exp\left\{2\left[-\frac{1}{2}\,\sigma^2(T-t) + \sigma\,(W_{\theta T} - W_{\theta t})\right]\right\} \Bigm| \mathcal{A}_t\right)\\ &= \frac{(\bar{S}_t)^2}{B_T}\,\exp\{\sigma^2(T-t)\},\end{aligned}$$

since

$$\begin{aligned}\frac{\partial \bar{V}(t,\bar{S})}{\partial t} &= -\bar{V}(t,\bar{S})\,\sigma^2,\\ \frac{\partial \bar{V}(t,\bar{S})}{\partial \bar{S}} &= 2\,\frac{\bar{V}(t,\bar{S})}{\bar{S}},\\ \frac{\partial^2 \bar{V}(t,\bar{S})}{\partial \bar{S}^2} &= -2\,\frac{\bar{V}(t,\bar{S})}{\bar{S}^2} + \frac{2}{\bar{S}}\,\frac{\partial \bar{V}(t,\bar{S})}{\partial \bar{S}} = \frac{2}{\bar{S}^2}\,\bar{V}(t,\bar{S}).\end{aligned}$$

Therefore we have

$$\begin{aligned}\frac{\partial \bar{V}(t,\bar{S})}{\partial t} + \frac{1}{2}\,\sigma^2\,\bar{S}^2\,\frac{\partial^2 \bar{V}(t,\bar{S})}{\partial \bar{S}^2} &= -\bar{V}(t,\bar{S})\,\sigma^2 + \frac{1}{2}\,\sigma^2\,\bar{S}^2\,\frac{2}{\bar{S}^2}\,\bar{V}(t,\bar{S})\\ &= 0.\end{aligned}$$

Solutions for Exercises of Chapter 10

10.1 The growth rate g_t^δ of a portfolio is defined in (10.2.1) as the drift of the SDE of the logarithm of the portfolio S^δ. By application of the Itô formula to $\ln(S_t^\delta)$ one obtains the SDE

$$
\begin{aligned}
d\ln(S_t^\delta) &= \frac{1}{S_t^\delta}\, dS_t^\delta - \frac{1}{2(S_t^\delta)^2}\, d[S_t^\delta] \\
&= \left(r_t + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k}\, \theta_t^k - \frac{1}{2} \sum_{k=1}^{d} \left(\sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k}\, \theta_t^k \right)^2 \right) dt \\
&\quad + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k}\, dW_t^k.
\end{aligned}
$$

The drift of this SDE, which is the growth rate of S^δ, is then

$$
g_t^\delta = r_t + \sum_{k=1}^{d} \left(\sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k}\, \theta_t^k - \frac{1}{2} \left(\sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k}\, \theta_t^k \right)^2 \right).
$$

10.2 We apply for the benchmarked value $\hat{S}_t^\delta = \frac{S_t^\delta}{S_t^{\delta_*}}$ the integration by parts formula (6.3.1) and obtain the SDE

$$
\begin{aligned}
d\hat{S}_t^\delta &= S_t^\delta\, d\left(\frac{1}{S_t^{\delta_*}}\right) + \frac{1}{S_t^{\delta_*}}\, dS_t^\delta + d\left[\frac{1}{S^{\delta_*}}, S^\delta\right]_t \\
&= \frac{S_t^\delta}{S_t^{\delta_*}} \left(-r_t\, dt - \sum_{k=1}^{d} \theta_t^k\, dW_t^k \right) + \frac{S_t^\delta}{S_t^{\delta_*}} \left(r_t\, dt + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k} \left(\theta_t^k\, dt + dW_t^k \right) \right) \\
&\quad + \frac{S_t^\delta}{S_t^{\delta_*}} \sum_{k=1}^{d} \left(-\theta_t^k \sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k} \right) dt \\
&= \hat{S}_t^\delta \sum_{k=1}^{d} \left(\sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k} - \theta_t^k \right) dW_t^k.
\end{aligned}
$$

This SDE is driftless. Therefore, by Lemma 5.4.1 a square integrable $\hat{S}^\delta$ ensures that $\hat{S}^\delta$ is an $(\underline{\mathcal{A}}, P)$-local martingale. However, this is not sufficient to guarantee that $\hat{S}^\delta$ is, in general, an $(\underline{\mathcal{A}}, P)$-martingale. A counter example is the unfair portfolio (9.1.42) in Sect. 9.1. Since $\hat{S}^\delta$ is a nonnegative local martingale it is by Lemma 5.2.3 an $(\underline{\mathcal{A}}, P)$-supermartingale.

10.3 According to Definition 10.6.3 and (10.6.19) we need to show that

$$
E\left(\left(\hat{\sigma}_{(d)}^k(t) \right)^2 \right) = E\left(\left(\sum_{j=0}^{d} \left| \sigma_{(d)}^{j,k}(t) \right| \right)^2 \right)
$$

is bounded by a constant for all $k \in \mathcal{N}$. Due to (10.6.27) we have for $d \in \mathcal{N}$ and $k \in \{1, 2, \ldots, d\}$

$$\sum_{j=0}^{d} \left|\sigma_{(d)}^{j,k}(t)\right| \leq \sigma \left(1 + \frac{1}{\sqrt{d}}\right) \leq 2\,\sigma$$

and, therefore, $E\left(\left(\hat{\sigma}_{(d)}^{k}(t)\right)^2\right) \leq 4\,\sigma^2$. This demonstrates that the corresponding sequence of CFMs is regular.

Solutions for Exercises of Chapter 11

11.1 The expected log-utility $v^{\tilde{\delta}}$ follows by (11.3.3), (11.3.8), (11.3.11) and (10.2.8), as

$$\begin{aligned}
v^{\tilde{\delta}} &= E\left(U\left(U'^{-1}\left(\frac{\lambda}{\bar{S}_T^{\delta_*}}\right)\right) \,\Big|\, \mathcal{A}_0\right) = E\left(\ln\left(\bar{S}_T^{\delta_*}\right) - \ln(\lambda) \,\big|\, \mathcal{A}_0\right) \\
&= E\left(\ln\left(\bar{S}_T^{\delta_*}\right) - \ln\left(\bar{S}_0^{\delta_*}\right) + \ln(S_0) \,\big|\, \mathcal{A}_0\right) \\
&= \frac{1}{2}\, E\left(\int_0^T |\theta(s, \bar{S}_s^{\delta_*})|^2\, ds \,\big|\, \mathcal{A}_0\right) + \ln(S_0) \\
&= \frac{1}{2} \int_0^T E\left(|\theta(s, \bar{S}_s^{\delta_*})|^2 \,\big|\, \mathcal{A}_0\right) ds + \ln(S_0).
\end{aligned} \tag{S.1}$$

For the BS model with $\theta(s, \bar{S}_s^{\delta_*}) = \theta$ we obtain, therefore, $v^{\tilde{\delta}} = \frac{\theta^2}{2} T + \ln(S_0)$. In the case of other discounted GOP dynamics one has simply to calculate the conditional expectation in (S.1), which is possible for certain models.

11.2 Similarly as in the above exercise the expected power utility for $\gamma < 0$ under the BS model is obtained by (11.3.3), (11.3.8), (11.3.16) and (10.2.8) as

$$v^{\tilde{\delta}} = E\left(\frac{1}{\gamma}\left(\left(\lambda\, \hat{S}_T^0\right)^{\frac{1}{\gamma-1}}\, \hat{S}_T^0\right)^{\gamma} \,\Big|\, \mathcal{A}_0\right) = \frac{1}{\gamma}\,(S^0)^{\gamma}\, \left(S_0^{\delta_*}\right)^{-\gamma} \exp\left\{\frac{\theta^2}{2}\, T\, \frac{\gamma}{1-\gamma}\right\}.$$

11.3 (*) The benchmarked fair price $\hat{V}_t$ at time $t \in [0, T]$ of the payoff H paid at time $T \in (0, \infty)$ satisfies according to (11.5.9) the SDE

$$d\hat{V}_t = \sum_{k=1}^{m} x_H^k(t)\, dW_t^k \tag{S.2}$$

with

$$\hat{V}_0 = E\left(\frac{H}{S_T^{\delta_*}} \,\bigg|\, \mathcal{A}_0\right). \tag{S.3}$$

On the other hand, according to (10.2.8) the discounted GOP is characterized by the SDE

$$d\bar{S}_t^{\delta_*} = \bar{S}_t^{\delta_*} \sum_{k=1}^{d} \theta_t^k \,(\theta_t^k\, dt + dW_t^k) \tag{S.4}$$

for $t \in [0, T]$ with $\bar{S}_0^{\delta_*} > 0$.

The discounted payoff can now be expressed as

$$\bar{H} = \frac{H}{S_T^0} = \hat{H}\, \bar{S}_T^{\delta_*}. \tag{S.5}$$

By application of the Itô formula to the product $\bar{V}_t = \hat{V}_t \bar{S}_t^{\delta_*}$ we obtain by the Itô formula with (S.2) and (S.3) the SDE

$$\begin{aligned}
d\bar{V}_t &= \hat{V}_t\, d\bar{S}_t^{\delta_*} + \bar{S}_t^{\delta_*}\, d\hat{V}_t + d\left[\hat{V}, \bar{S}^{\delta_*}\right]_t \\
&= \bar{V}_t \sum_{k=1}^{d} \theta_t^k \,(\theta_t^k\, dt + dW_t^k) + \bar{S}_t^{\delta_*} \sum_{k=1}^{m} x_H^k(t)\, dW_t^k + \bar{S}_t^{\delta_*} \sum_{k=1}^{d} x_H^k(t)\, \theta_t^k\, dt \\
&= \bar{S}_t^{\delta_*} \sum_{k=1}^{d} \left(x_H^k(t) + \hat{V}_t\, \theta_t^k\right) (\theta_t^k\, dt + dW_t^k) + \bar{S}_t^{\delta_*} \sum_{k=d+1}^{m} x_H^k(t)\, dW_t^k.
\end{aligned}$$

This leads under a risk neutral probability measure to the martingale representation

$$\bar{H} = \bar{V}_0 + \sum_{k=1}^{d} \int_0^T \bar{S}_t^{\delta_*} \left(x_H^k(t) + \hat{V}_t\, \theta_t^k\right) dW_\theta{}^k(t) + \sum_{k=d+1}^{m} \int_0^T \bar{S}_t^{\delta_*}\, x_H^k(t)\, dW_t^k.$$

Here $\bar{V}_t = E_\theta(\bar{H} | \mathcal{A}_t)$ with E_θ denoting expectation under P_θ and

$$W_\theta{}^k(t) = \int_0^t \theta_t^k\, dt + W_t^k$$

for $k \in \{1, 2, \ldots, d\}$ forms a Wiener process under P_θ.

Solutions for Exercises of Chapter 12

12.1 Using the time homogenous Fokker-Planck equation the stationary density of the ARCH diffusion model is of the form

$$
\begin{aligned}
\bar{p}(\theta^2) &= \frac{C}{\gamma^2\,\theta^4}\exp\left\{2\int_{\underline{\theta}^2}^{\theta^2}\frac{\kappa\,(\bar{\theta}^2-u)}{\gamma^2\,u^2}\,du\right\}\\
&= \frac{C}{\gamma^2\,\theta^4}\exp\left\{\frac{2\,\kappa\,\bar{\theta}^2}{\gamma^2}\int_{\underline{\theta}^2}^{\theta^2}\frac{1}{u^2}\,du-\frac{2\,\kappa}{\gamma^2}\int_{\underline{\theta}^2}^{\theta^2}\frac{1}{u}\,du\right\}\\
&= \frac{C}{\gamma^2\,\theta^4}\exp\left\{\frac{2\,\kappa}{\gamma^2}\left(\bar{\theta}^2\left(-\frac{1}{\theta^2}+\frac{1}{\underline{\theta}^2}\right)-\left(\ln(\theta^2)-\ln(\underline{\theta}^2)\right)\right)\right\}\\
&= C_1\,\exp\left\{-\frac{2\,\kappa\,\bar{\theta}^2}{\gamma^2}\,\frac{1}{\theta^2}\right\}\left(\frac{1}{\theta^2}\right)^{\frac{2\kappa}{\gamma^2}+2},
\end{aligned}
$$

which is an inverse gamma density with an appropriate constant $C_1 > 0$.

12.2 The stationary density for the squared volatility needs to satisfy the expression

$$
\begin{aligned}
\bar{p}(\theta^2) &= \frac{C}{\gamma^2\,\theta^6}\exp\left\{2\int_{\underline{\theta}^2}^{\theta^2}\frac{\kappa\,(\bar{\theta}^2-u)\,u}{\gamma^2\,u^3}\,du\right\}\\
&= \frac{C}{\gamma^2\,\theta^6}\exp\left\{\frac{2\,\kappa}{\gamma^2}\left(\bar{\theta}^2\int_{\underline{\theta}^2}^{\theta^2}\frac{1}{u^2}\,du-\int_{\underline{\theta}^2}^{\theta^2}\frac{1}{u}\,du\right)\right\}\\
&= \frac{C}{\gamma^2\,\theta^6}\exp\left\{\frac{2\,\kappa}{\gamma^2}\left(-\bar{\theta}^2\left(\frac{1}{\theta^2}-\frac{1}{\bar{\theta}^2}\right)-\left(\ln(\theta^2)-\ln(\underline{\theta}^2)\right)\right)\right\}\\
&= \frac{C}{\gamma^2\,\theta^6}\exp\left\{\frac{2\,\kappa}{\gamma^2}\left(-\bar{\theta}^2\left(\frac{1}{\theta^2}-\frac{1}{\underline{\theta}^2}\right)-\ln(\theta^2)+\ln(\underline{\theta}^2)\right)\right\}\\
&= C_1\,\exp\left\{-\frac{2\,\kappa}{\gamma^2}\,\bar{\theta}^2\,\frac{1}{\theta^2}\right\}\left(\frac{1}{\theta^2}\right)^{\frac{2\kappa}{\gamma^2}+3},
\end{aligned}
$$

which is an inverse gamma density.

12.3 For the Heston model we obtain the stationary density for the squared volatility

$$
\begin{aligned}
\bar{p}(\theta^2) &= \frac{C}{\gamma^2\,\theta^2}\exp\left\{2\int_{\underline{\theta}^2}^{\theta^2}\frac{\kappa\,(\bar{\theta}^2-u)}{\gamma^2\,u}\,du\right\}\\
&= \frac{C}{\gamma^2\,\theta^2}\exp\left\{\frac{2\,\kappa\,\bar{\theta}^2}{\gamma^2}\int_{\underline{\theta}^2}^{\theta^2}\frac{1}{u}\,du-\frac{2\,\kappa}{\gamma^2}\,(\theta^2-\underline{\theta}^2)\right\}\\
&= C_1\,\exp\left\{-\frac{2\,\kappa}{\gamma^2}\,\bar{\theta}^2\right\}(\theta^2)^{\frac{2\kappa\,\bar{\theta}^2}{\gamma^2}-1},
\end{aligned}
$$

which is a gamma density.

12.4 For the Scott model the volatility θ_t has the stationary density

$$\bar{p}(\theta) = \frac{C}{\gamma^2} \exp\left\{ 2 \int_{\underline{\theta}}^{\theta} \frac{\kappa\,(\bar{\theta} - u)}{\gamma^2}\, du \right\}$$

$$= \frac{C}{\gamma^2} \exp\left\{ \frac{2\,\kappa}{\gamma^2} \left(\bar{\theta}\,(\theta - \underline{\theta}) - \frac{1}{2}(\theta^2 - \underline{\theta}^2) \right) \right\},$$

which is a Gaussian density with mean $\bar{\theta}$ and variance $\frac{\gamma^2}{2\,\kappa}$. Thus θ_t^2 has a chi-square distribution with two degrees of freedom.

12.5 It follows by the Itô formula that

$$d\theta_t^2 = \theta_t^2 \left(\kappa\,\bar{\xi} + \frac{1}{2}\,\gamma^2 - \theta_t^2\,\kappa \right) dt + \theta_t^2\,\gamma\, dW_t.$$

Therefore, the stationary density satisfies the expression

$$\bar{p}(\theta^2) = \frac{C}{\gamma^2\,\theta^4} \exp\left\{ 2 \int_{\underline{\theta}^2}^{\theta^2} \frac{u\left(\kappa\,\bar{\xi} + \frac{1}{2}\,\gamma^2 - u\,\kappa\right)}{\gamma^2\,u^2}\, du \right\}$$

$$= \frac{C}{\gamma^2\,\theta^4} \exp\left\{ \frac{2}{\gamma^2} \left(\kappa\,\bar{\xi} + \frac{1}{2}\,\gamma^2 \right) \left(\ln(\theta^2) - \ln(\underline{\theta}^2) \right) - \frac{2\,\kappa}{\gamma^2}\,(\theta^2 - \underline{\theta}^2) \right\}$$

$$= C_1 \exp\left\{ -\frac{2\,\kappa}{\gamma^2}\,\theta^2 \right\} (\theta^2)^{\frac{2}{\gamma^2}\,(\kappa\,\bar{\xi} + \frac{1}{2}\,\gamma^2) - 2}.$$

It follows that the stationary density of the squared volatility is a gamma density.

12.6 (*) It follows by (12.2.8) and the Itô formula

$$dX_t^{-q} = -q\,X_t^{-(q+1)} \Big((2\,(1-a)\,r\,X_t + \psi^2\,(1-a)\,(3-2a))\,dt$$

$$+ 2\,\psi\,(1-a)\,\sqrt{X_t}\,dW_t \Big)$$

$$+ \frac{1}{2}\,q\,(q+1)\,X^{-(q+2)}\,4\,\psi^2\,(1-a)^2\,X_t\,dt$$

for $t \in [0, T]$, where X is a transformed squared Bessel process of dimension $\nu = \frac{3-2a}{1-a}$, see (12.2.9). Therefore, the benchmarked savings account satisfies by the Itô formula and (12.2.1) the SDE

$$
\begin{aligned}
d\hat{S}^0_t &= d\left(\frac{S^0_t}{S^{\delta_*}_t}\right) = d\left(\exp\{-r\,(\tau-t)\}\,X_t^{-q}\right) \\
&= S^0_t\,\Bigg(-q\,X_t^{-q}\Bigg[2\,(1-a)\,r \\
&\qquad + X_t^{-1}\,\psi^2\,\left((1-a)\,(3-2a) - \frac{q+1}{2}\,4\,(1-a)^2\right) - \frac{r}{9}\Bigg]\,dt \\
&\qquad - q\,X^{-q-\frac{1}{2}}\,2\,\psi\,(1-a)\,dW_t\Bigg)
\end{aligned}
$$

By noting that according to (12.2.11) one has

$$
q = \frac{1}{2\,(1-a)}
$$

we obtain

$$
d\hat{S}^0_t = -\hat{S}^0_t\,\frac{\psi}{\sqrt{X_t}}\,dW_t.
$$

Consequently, $\hat{S}^0$ is an $(\underline{\mathcal{A}}, P)$-local martingale. We have by the moments of Bessel processes (8.7.16) the finite expression

$$
E\left(\left(\hat{S}^0_t\right)^2\right) = \exp\{-2\,r\,(\tau-t)\}\,E(X_t^{-2q}) < \infty
$$

for $a < 1$ and $t \in [0, \tau]$. Thus, for $a < 1$ the process $\hat{S}^0$ is square integrable. Furthermore, the quadratic variation of $\hat{S}^0$ is

$$
[\hat{S}^0]_t = \exp\{-2\,r\,(\tau-t)\}\,\psi^2 \int_0^t X_s^{-2(q+\frac{1}{2})}\,ds
$$

and its expectation yields by (8.7.14) because of $\alpha = -2q-1 = -\frac{\nu}{2}$ an infinite value

$$
E\left([\hat{S}^0]_t\right) = \exp\{-2\,r\,(\tau-t)\}\,\psi^2 \int_0^t E\left(X_s^{-\frac{2-a}{1-a}}\right)\,ds = \infty.
$$

By (8.7.23) $\hat{S}^0$ is a strict local martingale and so is $\hat{P}^*_\tau$.

Solutions for Exercises of Chapter 13

13.1 The SDE for the discounted GOP is of the form

$$
d\bar{S}^{\delta_*}_t = \alpha^{\delta_*}_t\,dt + \sqrt{\bar{S}^{\delta_*}_t\,\alpha^{\delta_*}_t}\,dW_t.
$$

By the Itô formula we obtain for $\ln(\bar{S}^{\delta_*}_t)$ the SDE

$$d \ln \left(\bar{S}_t^{\delta_*} \right) = \frac{1}{2} \frac{\alpha_t^{\delta_*}}{\bar{S}_t^{\delta_*}} \, dt + \sqrt{\frac{\alpha_t^{\delta_*}}{\bar{S}_t^{\delta_*}}} \, dW_t,$$

which shows that the volatility of the discounted GOP equals $\sqrt{\frac{\alpha_t^{\delta_*}}{\bar{S}_t^{\delta_*}}}$.

13.2 By the Itô formula we obtain for $\sqrt{\bar{S}_t^{\delta_*}}$ the SDE

$$d\sqrt{\bar{S}_t^{\delta_*}} = \left(\frac{\alpha_t^{\delta_*}}{2\sqrt{\bar{S}_t^{\delta_*}}} - \frac{1}{2} \frac{1}{4} \frac{1}{\left(\bar{S}_t^{\delta_*}\right)^{\frac{3}{2}}} \bar{S}_t^{\delta_*} \, \alpha_t^{\delta_*} \right) dt + \frac{\sqrt{\bar{S}_t^{\delta_*} \, \alpha_t^{\delta_*}}}{2\sqrt{\bar{S}_t^{\delta_*}}} \, dW_t$$

$$= \frac{3}{8} \frac{\alpha_t^{\delta_*}}{\sqrt{\bar{S}_t^{\delta_*}}} \, dt + \frac{1}{2} \sqrt{\alpha_t^{\delta_*}} \, dW_t,$$

which confirms (13.1.12).

13.3 The differential equation for $\alpha_t^{\delta_*}$ is of the form

$$d\alpha_t^{\delta_*} = \eta_t \, \alpha_t^{\delta_*} \, dt.$$

Together with the SDE of the discounted GOP and the Itô formula it follows

$$dY_t = Y_t \left(\frac{\alpha_t}{\bar{S}_t^{\delta_*}} - \eta_t \right) dt + Y_t \sqrt{\frac{\alpha_t}{\bar{S}_t^{\delta_*}}} \, dW_t$$

$$= (1 - \eta_t \, Y_t) \, dt + \sqrt{Y_t} \, dW_t,$$

which confirms (13.2.5).

13.4 The squared volatility of the discounted GOP equals $|\theta_t|^2 = \frac{1}{Y_t}$, and, thus the inverse of the normalized GOP. This means, we obtain by the Itô formula the SDE

$$d|\theta_t|^2 = d\left(\frac{1}{Y_t}\right) = \left(-\left(\frac{1}{Y_t}\right)^2 (1 - \eta_t \, Y_t) + \frac{Y_t}{(Y_t)^3} \right) dt - \left(\frac{1}{Y_t}\right)^2 \sqrt{Y_t} \, dW_t$$

$$= \eta_t \frac{1}{Y_t} \, dt - \left(\frac{1}{Y_t}\right)^{\frac{3}{2}} dW_t = \eta_t \, |\theta_t|^2 \, dt - \left(|\theta_t|^2\right)^{\frac{3}{2}} \, dW_t,$$

which confirms (13.2.11).

Solutions for Exercises of Chapter 14

14.1 The SDE for a strictly positive portfolio S^δ is by (14.1.3) of the form

$$dS_t^\delta = S_{t-}^\delta \left(r_t\, dt + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k} \left(\theta_t^k\, dt + dW_t^k \right) \right).$$

By application of the Itô formula this leads for the logarithm of S_t^δ to the SDE

$$\begin{aligned} d\ln(S_t^\delta) &= r_t\, dt + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k}\, \theta_t^k\, dt - \frac{1}{2} \sum_{k=1}^{m} \left(\sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k} \right)^2 dt \\ &\quad + \sum_{k=1}^{m} \sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k}\, dW_t^k - \sum_{k=m+1}^{d} \sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k} \sqrt{h_t^k}\, dt \\ &\quad + \sum_{k=m+1}^{d} \ln \left(1 + \sum_{j=1}^{d} \pi_{\delta,t-}^j \frac{b_t^{j,k}}{\sqrt{h_t^k}} \right) dp_t^k \\ &= \left(r_t\, dt + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k}\, \theta_t^k - \frac{1}{2} \sum_{k=1}^{m} \left(\sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k} \right)^2 \right. \\ &\quad \left. + \sum_{k=m+1}^{d} h_t^k \left[\ln \left(1 + \sum_{j=1}^{d} \pi_{\delta,t}^j \frac{b_t^{j,k}}{\sqrt{h_t^k}} \right) - \sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k} \frac{1}{\sqrt{h_t^k}} \right] \right) dt \\ &\quad + \sum_{k=1}^{m} \pi_{\delta,t}^j\, b_t^{j,k}\, dW_t^k + \sum_{k=m+1}^{d} \ln \left(1 + \sum_{j=1}^{d} \pi_{\delta,t-}^j \frac{b_t^{j,k}}{\sqrt{h_t^k}} \right) \sqrt{h_t^k}\, dW_t^k. \end{aligned}$$

The drift of this SDE is the growth rate given in (14.1.15).

14.2 The forward rate at time t for maturity T has by (14.1.32) and (14.1.31) the form

$$
\begin{aligned}
f(t,T) &= -\frac{\partial}{\partial T}\ \ln(\hat{P}(t,T)) \\
&= -\frac{\partial}{\partial T}\left[\ln(\hat{P}(0,T)) - \sum_{k=1}^{m}\left(\int_0^t \sigma^k(s,T)\,dW_s^k + \frac{1}{2}\int_0^t (\sigma^k(s,T))^2\,ds\right)\right. \\
&\quad \left. + \sum_{k=m+1}^{d}\left(\int_0^t \sigma^k(s,T)\sqrt{h_s^{k-m}}\,ds + \int_0^t \ln\left(1 - \frac{\sigma^k(s,T)}{\sqrt{h_s^{k-m}}}\right)dp_s^k\right)\right] \\
&= f(0,T) + \sum_{k=1}^{m}\left(\int_0^t \frac{\partial}{\partial T}\,\sigma^k(s,T)\,dW_s^k + \frac{1}{2}\int_0^t \frac{\partial}{\partial T}\,(\sigma^k(s,T))^2\,ds\right) \\
&\quad + \sum_{k=m+1}^{d}\left(-\int_0^t \frac{\partial}{\partial T}\,\sigma^k(s,T)\sqrt{h_s^{k-m}}\,ds - \int_0^t \frac{\partial}{\partial T}\ \ln\left(1 - \frac{\sigma^k(s,T)}{\sqrt{h_s^{k-m}}}\right)dp_s^k\right) \\
&= f(0,T) + \sum_{k=1}^{m}\int_0^t \frac{\partial}{\partial T}\,\sigma^k(s,T)\,(\sigma^k(s,T)\,ds + dW_s^k) \\
&\quad + \sum_{k=m+1}^{d}\int_0^t \frac{1}{1 - \frac{\sigma^k(s,T)}{\sqrt{h_s^{k-m}}}}\,\frac{\partial}{\partial T}\,\sigma^k(s,T)\,(\sigma^k(s,T)\,ds + dW_s^k).
\end{aligned}
$$

14.3 (*) (Hardy Hulley) Fix $i, j \in \{0, 1, \ldots, d\}$ such that $i \neq j$, then the function

$$
\begin{aligned}
p_{s,t}^{i,j}(x_i, x_j; y_i, y_j) &= \frac{1}{2\pi y_i y_j |\sigma^i||\sigma^j|(t-s)\sqrt{1-(\varrho^{i,j})^2}} \\
&\quad \times \exp\left\{-\frac{1}{2(1-(\varrho^{i,j})^2)}\left[\left(\frac{\ln\left(\frac{y_i}{x_i}\right) + \frac{1}{2}|\sigma^i|^2(t-s)}{|\sigma^i|\sqrt{t-s}}\right)^2\right.\right. \\
&\quad - 2\varrho^{i,j}\,\frac{\left(\ln\left(\frac{y_i}{x_i}\right) + \frac{1}{2}|\sigma^i|^2(t-s)\right)\left(\ln\left(\frac{y_j}{x_j}\right) + \frac{1}{2}|\sigma^j|^2(t-s)\right)}{|\sigma^i||\sigma^j|(t-s)} \\
&\quad \left.\left. + \left(\frac{\ln\left(\frac{y_j}{x_j}\right) + \frac{1}{2}|\sigma^j|^2(t-s)}{|\sigma^j|\sqrt{t-s}}\right)^2\right]\right\},
\end{aligned}
\tag{S.6}
$$

for all $x_i, x_j, y_i, y_j \in (0, \infty)$, where $s, t \in [0, \infty)$ such that $s \leq t$, is the joint transition density of $\hat{S}^{i,c}$ and $\hat{S}^{j,c}$ over the time interval $[s, t]$. The parameter $\varrho^{i,j}$ in (S.6) is determined by

$$
\varrho^{i,j} = \sum_{k=1}^{m} \frac{\sigma^{i,k}\sigma^{j,k}}{|\sigma^i||\sigma^j|}. \tag{S.7}
$$

It follows from (14.4.8) that $\varrho^{i,j}$ is the correlation between the Brownian motions $\hat{W}^i$ and $\hat{W}^j$.

To start with, we perform an auxiliary computation which allows us to price both instruments under consideration. Fix $t \in [0, \infty)$ and let g_t be a non-negative $\mathcal{A}_t$-measurable random variable. We will now evaluate the following expression:

$$\frac{1}{\hat{S}_t^{j,c}} E\left(\mathbf{1}_{\{\hat{S}_T^{j,c} \geq g_t \hat{S}_T^{i,c}\}} \hat{S}_T^{j,c} \,\Big|\, \mathcal{A}_t\right) = \int_0^\infty \int_{\alpha_t x}^\infty \frac{y}{\hat{S}_t^{j,c}} p_{t,T}^{i,j}\left(\hat{S}_t^{i,c}, \hat{S}_t^{j,c}; x, y\right) dy\, dx. \tag{S.8}$$

After the change of variables we obtain

$$\bar{x} = \frac{\ln\left(\frac{x}{\hat{S}_t^{i,c}}\right) + \frac{1}{2}|\sigma^i|^2(T-t)}{|\sigma^i|\sqrt{T-t}} \tag{S.9}$$

$$\bar{y} = \frac{\ln\left(\frac{y}{\hat{S}_t^{j,c}}\right) + \frac{1}{2}|\sigma^j|^2(T-t)}{|\sigma^j|\sqrt{T-t}}, \tag{S.10}$$

and (S.8) becomes

$$\begin{aligned} \frac{1}{2\pi\sqrt{1-(\varrho^{i,j})^2}} \int_{-\infty}^\infty \int_{d(\bar{x})}^\infty \exp\bigg\{-\frac{1}{2(1-(\varrho^{i,j})^2)} \bigg[\Big(-\bar{x} + \varrho^{i,j}|\sigma^j|\sqrt{T-t}\Big)^2 \\ - 2\varrho^{i,j}\Big(-\bar{x} + \varrho^{i,j}|\sigma^j|\sqrt{T-t}\Big)\Big(-\bar{y} + |\sigma^j|\sqrt{T-t}\Big) \\ + \Big(-\bar{y} + |\sigma^j|\sqrt{T-t}\Big)^2\bigg]\bigg\}\, d\bar{y}\, d\bar{x}, \end{aligned} \tag{S.11}$$

where

$$d(\bar{x}) = \frac{\ln\left(\frac{\alpha_t S_t^{i,c}}{S_t^{j,c}}\right) - \left(\frac{1}{2}|\sigma^i|^2 - \frac{1}{2}|\sigma^j|^2\right)(T-t)}{|\sigma^j|\sqrt{T-t}} + \frac{|\sigma^i|}{|\sigma^j|}\bar{x}, \tag{S.12}$$

for all $\bar{x} \in \Re$. Another transformation of variables,

$$\tilde{x} = -\bar{x} + \varrho^{i,j}|\sigma^j|\sqrt{T-t}; \tag{S.13}$$

$$\tilde{y} = -\bar{y} + |\sigma^j|\sqrt{T-t}, \tag{S.14}$$

allows us to express (S.11) as

$$\frac{1}{2\pi\sqrt{1-(\varrho^{i,j})^2}} \int_{-\infty}^\infty \int_{-\infty}^{d(\tilde{x})} \exp\bigg\{-\frac{1}{2(1-(\varrho^{i,j})^2)} \Big[\tilde{x}^2 - 2\varrho^{i,j}\tilde{x}\tilde{y} + \tilde{y}^2\Big]\bigg\}\, d\tilde{y}\, d\tilde{x}, \tag{S.15}$$

where

$$d(\tilde{x}) = \frac{\ln\left(\frac{S_t^{j,c}}{\alpha_t S_t^{i,c}}\right) + \frac{1}{2}\left(\hat{\sigma}^{i,j}\right)^2 (T-t)}{|\sigma^j|\sqrt{T-t}} + \frac{|\sigma^i|}{|\sigma^j|}\tilde{x} = a + b\tilde{x}, \tag{S.16}$$

for all $\tilde{x} \in \Re$, with

$$\hat{\sigma}^{i,j} = \sqrt{|\sigma^i|^2 - 2\varrho^{i,j}|\sigma^i||\sigma^j| + |\sigma^j|^2}. \tag{S.17}$$

After the transformation

$$\hat{y} = \tilde{y} - b\tilde{x}, \tag{S.18}$$

for all $\tilde{x} \in \Re$ and $\tilde{y} \in (-\infty, d(\tilde{x}))$, (S.15) becomes

$$\begin{aligned} \frac{1}{2\pi\sqrt{1-(\varrho^{i,j})^2}} \int_{-\infty}^{a} \int_{-\infty}^{\infty} \exp\Bigg\{ -\frac{1}{2(1-(\varrho^{i,j})^2)} \\ \times \left[\tilde{x}^2 - 2\varrho^j \tilde{x}(\hat{y} + b\tilde{x}) + (\hat{y} + b\tilde{x})^2\right] \Bigg\} \, d\tilde{x}\, d\hat{y}. \end{aligned} \tag{S.19}$$

Now, performing the change of variables

$$\hat{x} = \sqrt{\frac{1 - 2b\varrho^j + b^2}{1 - (\varrho^{i,j})^2}} \left(\tilde{x} + \frac{b - \varrho^{i,j}}{1 - 2b\varrho^{i,j} + b^2}\hat{y}\right), \tag{S.20}$$

for all $\tilde{x} \in \Re$, transforms (S.19) into

$$\frac{1}{\sqrt{1 - 2b\varrho^{i,j} + b^2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} \exp\left\{ -\frac{1}{2} \frac{\hat{y}^2}{1 - 2b\varrho^{i,j} + b^2} \right\} d\hat{y}. \tag{S.21}$$

Finally, we set

$$z = \frac{\hat{y}}{\sqrt{1 - 2b\varrho^{i,j} + b^2}}, \tag{S.22}$$

for all $\hat{y} \in (-\infty, a)$, so that (S.21) becomes

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\frac{a}{\sqrt{1-2b\varrho^{i,j}+b^2}}} \exp\left\{ -\frac{1}{2} z^2 \right\} dz &= N\left(\frac{a}{\sqrt{1 - 2b\varrho^{i,j} + b^2}} \right) \\ &= N\left(\frac{\ln\left(\frac{S_t^{j,c}}{g_t S_t^{i,c}}\right) + \frac{1}{2}\left(\hat{\sigma}^{i,j}\right)^2 (T-t)}{\hat{\sigma}^{i,j}\sqrt{T-t}} \right), \end{aligned} \tag{S.23}$$

where $N(\cdot)$ is the Gaussian distribution function.

Now, to obtain (14.4.33) perform the substitutions $g_t = g(n)$ and $i = 0$ in (S.23) and substitute the resulting expression into (14.4.31), while remembering that $S^{0,c} = S^0$. Finally, perform the substitutions $g_t = g(n)^{-1}$, $i = j$ and $j = 0$ in (S.23) and substitute the resulting expression into (14.4.37), to obtain (14.4.38). It is important to remember in this case that the symmetry of the Gaussian distribution gives $N(-d_2(n)) = 1 - N(d_2(n))$, for each $n \in \mathcal{N}$, while

$$\sum_{n=0}^{\infty} \exp\{-h^{k-m}(T-t)\} \frac{(h^{k-m}(T-t))^n}{n!} = 1, \tag{S.24}$$

since this expression is the total probability of a Poisson random variable with parameter $h^{k-m}(T-t)$.

14.4 (*) See above.

14.5 (*) (Hardy Hulley) Firstly, by the function

$$p_4(\varrho, x; \varphi, y) = \frac{1}{2(\varphi - \varrho)} \sqrt{\frac{y}{x}} \exp\left\{ -\frac{x+y}{2(\varphi - \varrho)} \right\} \times \sum_{n=0}^{\infty} \frac{1}{n!\, \Gamma(n+2)} \left(\frac{\sqrt{xy}}{2(\varphi - \varrho)} \right)^{2n+1}, \tag{S.25}$$

for all $x, y \in (0, \infty)$ and $\varrho, \varphi \in [0, \infty)$ such that $\varrho < \varphi$, is the transition density of a squared Bessel process of dimension four. Equation (S.25) is obtained from (8.7.9) with the help of the series expansion for the modified Bessel function of the second kind $I_1(\cdot)$, see Abramowitz & Stegun (1972). Note the presence in (S.25) of the gamma function $\Gamma(\cdot)$, defined in (1.2.10). It satisfies the following identity:

$$\Gamma(n) = (n-1)!\,, \tag{S.26}$$

for each $n \in \mathcal{N}$.

Now fix $i, j \in \{0, \ldots, d\}$ and $t \in [0, \infty)$ and let g_t be a positive $\mathcal{A}_t$-measurable random variable. Again, we perform an auxiliary computation that enables us to price both instruments under consideration. Noting that X^i and X^j, given by (14.4.21), are independent squared Bessel processes of dimension four, we have

$$
\begin{aligned}
&\frac{1}{\hat{S}_t^{j,\mathrm{c}}} E\left(\mathbf{1}_{\left\{\hat{S}_T^{j,\mathrm{c}} \geq g_t \hat{S}_T^{i,\mathrm{c}}\right\}} \hat{S}_T^{j,\mathrm{c}} \,\Big|\, \mathcal{A}_t\right) \\
&= X^j_{\varphi^j(t)} E\left(\mathbf{1}_{\left\{X^j_{\varphi^j(T)} \leq g_t^{-1} X^i_{\varphi^i(T)}\right\}} \frac{1}{X^j_{\varphi^j(T)}} \,\Big|\, \mathcal{A}_t\right) \\
&= \int_0^\infty \int_0^{g_t^{-1}x} \frac{X^j_{\varphi^j(t)}}{y} p_4\Big(\varphi^i(t), X^i_{\varphi^i(t)}; \varphi^i(T), x\Big) \\
&\qquad\qquad \times p_4\Big(\varphi^j(t), X^j_{\varphi^j(t)}; \varphi^j(T), y\Big)\, dy\, dx \\
&= \int_0^\infty \int_0^{g_t^{-1}x} \frac{1}{y} \exp\Big\{-\frac{1}{2}\lambda_t^j\Big\} \frac{1}{\varphi^i(T)-\varphi^i(t)} \exp\Big\{-\frac{1}{2}\lambda_t^i\Big\} \\
&\qquad \times \left[\sum_{l=0}^\infty \frac{1}{l!\,\Gamma(l+2)2^{l+1}} \left(\frac{1}{2}\lambda_t^j\right)^{l+1} \left(\frac{y}{\varphi^j(T)-\varphi^j(t)}\right)^{l+1}\right. \\
&\qquad\quad \left. \times \exp\Big\{-\frac{y}{2(\varphi^j(T)-\varphi^j(t))}\Big\}\right] \\
&\qquad \times \left[\sum_{q=0}^\infty \frac{1}{q!\,\Gamma(q+2)2^{q+2}} \left(\frac{1}{2}\lambda_t^i\right)^{q} \left(\frac{x}{\varphi^i(T)-\varphi^i(t)}\right)^{q+1}\right. \\
&\qquad\quad \left. \times \exp\Big\{-\frac{x}{2(\varphi^i(T)-\varphi^i(t))}\Big\}\right] dy\, dx \\
&= \exp\Big\{-\frac{1}{2}\big(\lambda_t^j + \lambda_t^i\big)\Big\} \sum_{l=0}^\infty \frac{\big(\frac{1}{2}\lambda_t^j\big)^{l+1}}{l!\,\Gamma(l+2)2^{l+1}} \sum_{q=0}^\infty \frac{\big(\frac{1}{2}\lambda_t^i\big)^{q}}{q!\,\Gamma(q+2)2^{q+2}} \\
&\qquad \times \int_0^\infty \int_0^{\bar{g}_t \bar{x}} \exp\Big\{-\frac{1}{2}(\bar{x}+\bar{y})\Big\} \bar{y}^l \bar{x}^{q+1}\, d\bar{y}\, d\bar{x}.
\end{aligned}
\tag{S.27}
$$

Here we have made the substitutions

$$\bar{x} := \frac{x}{\varphi^i(T)-\varphi^i(t)}; \tag{S.28}$$

$$\bar{y} := \frac{y}{\varphi^j(T)-\varphi^j(t)}. \tag{S.29}$$

The constant in the upper limit of the inner integral in (S.27) is thus given by

$$\bar{g}_t := \frac{\varphi^i(T)-\varphi^i(t)}{g_t}. \tag{S.30}$$

The random variables λ_t^i and λ_t^j are given by (14.4.24).

With the aid of another change of variables, namely

$$\tilde{y} := \frac{\bar{y}}{\bar{x}}, \tag{S.31}$$

(S.27) now becomes

$$\begin{aligned}
&\exp\Big\{-\frac{1}{2}\big(\lambda_t^j+\lambda_t^i\big)\Big\}\sum_{l=0}^{\infty}\frac{\big(\frac{1}{2}\lambda_t^j\big)^{l+1}}{l!\,\Gamma(l+2)2^{l+1}}\sum_{q=0}^{\infty}\frac{\big(\frac{1}{2}\lambda_t^i\big)^{q}}{q!\,\Gamma(q+2)2^{q+2}}\\
&\qquad\times\int_0^{\bar{g}_t}\int_0^{\infty}\exp\Big\{-\frac{1}{2}\bar{x}(1+\tilde{y})\Big\}\tilde{y}^l\bar{x}^{q+l+2}\,d\bar{x}\,d\tilde{y}\\
&\quad=\exp\Big\{-\frac{1}{2}\big(\lambda_t^j+\lambda_t^i\big)\Big\}\sum_{l=0}^{\infty}\frac{\big(\frac{1}{2}\lambda_t^j\big)^{l+1}}{l!\,\Gamma(l+2)2^{l+1}}\sum_{q=0}^{\infty}\frac{\big(\frac{1}{2}\lambda_t^i\big)^{q}}{q!\,\Gamma(q+2)2^{q+2}}\\
&\qquad\times\Gamma(q+l+3)2^{q+l+3}\int_0^{\bar{g}_t}\frac{\tilde{y}^l}{(1+\tilde{y})^{q+l+3}}\,d\tilde{y}\\
&\quad=\exp\Big\{-\frac{1}{2}\big(\lambda_t^j+\lambda_t^i\big)\Big\}\sum_{m=1}^{\infty}\frac{\big(\frac{1}{2}\lambda_t^j\big)^{m}}{m!}\sum_{q=0}^{\infty}\frac{\big(\frac{1}{2}\lambda_t^i\big)^{q}}{q!}\frac{\Gamma(q+m+2)}{\Gamma(m)\Gamma(q+2)}\\
&\qquad\times\int_0^{\bar{g}_t}\frac{\tilde{y}^{m-1}}{(1+\tilde{y})^{q+m+2}}\,d\tilde{y}\\
&\quad=\exp\Big\{-\frac{1}{2}\big(\lambda_t^j+\lambda_t^i\big)\Big\}\sum_{m=1}^{\infty}\frac{\big(\frac{1}{2}\lambda_t^j\big)^{m}}{m!}\sum_{q=0}^{\infty}\frac{\big(\frac{1}{2}\lambda_t^i\big)^{q}}{q!}\frac{\Gamma(q+m+2)}{\Gamma(m)\Gamma(q+2)}\\
&\qquad\times\frac{\bar{g}_t^m}{m}\,{}_2F_1\big(m,q+m+2;m+1;-\bar{g}_t\big)\\
&\quad=\sum_{m=1}^{\infty}\frac{\exp\{-\frac{1}{2}\lambda_t^j\}\big(\frac{1}{2}\lambda_t^j\big)^{m}}{m!}\sum_{q=0}^{\infty}\frac{\exp\{-\frac{1}{2}\lambda_t^i\}\big(\frac{1}{2}\lambda_t^i\big)^{q}}{q!}\frac{\Gamma(q+m+2)}{\Gamma(m+1)\Gamma(q+2)}\bar{g}_t^m\\
&\qquad\times{}_2F_1\big(m,q+m+2;m+1;-\bar{g}_t\big).
\end{aligned}\tag{S.32}$$

The first equality in (S.32) follows from the definition of the gamma function in (1.2.10). The second equality follows from some algebra and (S.26). The third equality was obtained with the help of Mathematica's symbolic integration facility. The final equality is another application of (S.26). The hypergeometric function ${}_2F_1(a,b;c;z)$ is described in Abramowitz & Stegun (1972). Now, note that

$$\begin{aligned}
&\frac{\exp\{-\frac{1}{2}\lambda_t^j\}\big(\frac{1}{2}\lambda_t^j\big)^{0}}{0!}\sum_{q=0}^{\infty}\frac{\exp\{-\frac{1}{2}\lambda_t^i\}\big(\frac{1}{2}\lambda_t^i\big)^{q}}{q!}\frac{\Gamma(q+0+2)}{\Gamma(0+1)\Gamma(q+2)}\bar{g}_t^0\\
&\qquad\qquad\times{}_2F_1\big(0,q+0+2;0+1;-\bar{g}_t\big)\\
&\quad=\exp\Big\{-\frac{1}{2}\lambda_t^j\Big\}\sum_{q=0}^{\infty}\frac{\exp\{-\frac{1}{2}\lambda_t^i\}\big(\frac{1}{2}\lambda_t^i\big)^{q}}{q!}\\
&\quad=\exp\Big\{-\frac{1}{2}\lambda_t^j\Big\}.
\end{aligned}\tag{S.33}$$

The first equality follows from (S.26) and the properties of the hypergeometric function. For the second equality, note that $\sum_{q=0}^{\infty} \frac{\exp\{-\frac{1}{2}\lambda_t^i\}\left(\frac{1}{2}\lambda_t^i\right)^q}{q!}$ is the total probability of a Poisson random variable with parameter $\frac{1}{2}\lambda_t^i$.

Thus, putting (S.32) and (S.33) together, we see that (S.27) can be expressed as

$$\begin{aligned}
&\left[\sum_{m=0}^{\infty} \frac{\exp\{-\frac{1}{2}\lambda_t^j\}\left(\frac{1}{2}\lambda_t^j\right)^m}{m!} \sum_{q=0}^{\infty} \frac{\exp\{-\frac{1}{2}\lambda_t^i\}\left(\frac{1}{2}\lambda_t^i\right)^q}{q!} \frac{\Gamma(q+m+2)}{\Gamma(m+1)\Gamma(q+2)} \bar{g}_t^m \right. \\
&\qquad \left. \times\, {}_2F_1\left(m, q+m+2; m+1; -\bar{g}_t\right)\right] - \exp\left\{-\frac{1}{2}\lambda_t^j\right\} \\
&= \left[\sum_{m=0}^{\infty} \frac{\exp\{-\frac{1}{2}\lambda_t^j\}\left(\frac{1}{2}\lambda_t^j\right)^m}{m!} \sum_{q=0}^{\infty} \frac{\exp\{-\frac{1}{2}\lambda_t^i\}\left(\frac{1}{2}\lambda_t^i\right)^q}{q!} \frac{\Gamma(q+m+2)}{\Gamma(m)\Gamma(q+2)} \right. \\
&\qquad \left. \times \int_0^{\bar{g}_t} \frac{\tilde{y}^{m-1}}{(1+\tilde{y})^{q+m+2}}\, d\tilde{y}\right] - \exp\left\{-\frac{1}{2}\lambda_t^j\right\} \\
&= P\left(\frac{\chi_0'^2(\lambda_t^j)}{\chi_4'^2(\lambda_t^i)} \leq \bar{g}_t\right) - \exp\left\{-\frac{1}{2}\lambda_t^j\right\},
\end{aligned} \tag{S.34}$$

according to Johnson et al. (1995), (30.49), p. 499, where $\chi_\nu'^2(\lambda)$ denotes a non-central chi-square distributed random variable with ν degrees of freedom and non-centrality λ. Following the lead of Johnson et al. (1995), we express the distribution function of the ratio of non-central chi-square random variables $\chi_{\nu_1}'^2(\lambda_1)/\chi_{\nu_2}'^2(\lambda_2)$ as $G''_{\nu_1,\nu_2}(\cdot\,; \lambda_1, \lambda_2)$, whence (S.34) becomes

$$G''_{0,4}\left(\frac{\varphi^i(T) - \varphi^i(t)}{g_t}; \lambda_t^j, \lambda_t^i\right) - \exp\left\{-\frac{1}{2}\lambda_t^j\right\}, \tag{S.35}$$

by (S.30).

Now to obtain (14.4.36) perform the substitutions $g_t = g(n)$ and $i = 0$ in (S.35) and substitute the resulting expression into (14.4.31). Finally, perform the substitutions $g_t = g(n)^{-1}$, $i = j$ and $j = 0$ in (S.35) and substitute the resulting expression into (14.4.37), to obtain (14.4.10).

14.6 (*) See above.

Solutions for Exercises of Chapter 15

15.1 By the Lyapunov inequality the variance $\mathrm{Var}((X_i)^{\frac{1}{2}})$ is bounded. That is,

$$\begin{aligned}
\mathrm{Var}\left((X_i)^{\frac{1}{2}}\right) &= E\left(\left((X_i)^{\frac{1}{2}} - E\left((X_i)^{\frac{1}{2}}\right)\right)^2\right) \leq E\left(\left((X_i)^{\frac{1}{2}}\right)^2\right) \\
&= E\left(|X_i|\right) \leq \sqrt{E\left((X_i)^2\right)} = \sqrt{\sigma^2 + \mu^2} < \infty.
\end{aligned}$$

Therefore, the Monte Carlo estimator $\hat{\varrho}_n = \frac{1}{n}\sum_{i=1}^n (X_i)^{\frac{1}{2}}$ is strongly consistent for estimating $\varrho = E((X_i)^{\frac{1}{2}})$ since

$$\begin{aligned}\operatorname{Var}(\hat{\varrho}_n) = E\left((\hat{\varrho}_n - \varrho)^2\right) = E\left(\left(\frac{1}{n}\sum_{i=1}^n \left((X_i)^{\frac{1}{2}} - \varrho\right)\right)^2\right) \\ = \frac{1}{n^2}\sum_{i=1}^n E\left(\left((X_i)^{\frac{1}{2}} - \varrho\right)^2\right) \\ = \frac{1}{n}\left(\frac{1}{n}\sum_{i=1}^n \operatorname{Var}\left((X_i)^{\frac{1}{2}}\right)\right) \\ = \frac{1}{n}\operatorname{Var}\left((X_i)^{\frac{1}{2}}\right) = \frac{1}{n}\sqrt{\sigma^2 + \mu^2}.\end{aligned}$$

Consequently, the variance of the estimator $\hat{\varrho}_n$ decreases proportionally to $\frac{1}{n}$ and $\operatorname{Var}(\hat{\varrho}_n)$ converges almost surely to zero.

15.2 By application of the Wagner-Platen expansion one obtains directly

$$X_{t_0+h} - X_{t_0} = a\,X_{t_0}\,dt + b\,X_{t_0}\,(W_{t_0+h} - W_{t_0}) + R.$$

15.3 The Euler scheme is given by

$$Y_{n+1} = Y_n + (\mu\,Y_n + \eta)\,\Delta + \gamma\,Y_n\,\Delta W,$$

where $\Delta W = W_{\tau_{n+1}} - W_{\tau_n}$. The Milstein scheme has the form

$$Y_{n+1} = Y_n + (\mu\,Y_n + \eta)\,\Delta + \gamma\,Y_n\,\Delta W + \frac{\gamma^2}{2}\,Y_n((\Delta W)^2 - \Delta).$$

15.4 Due to the additive noise of the Vasicek model the Euler and Milstein schemes are identical and of the form

$$Y_{n+1} = Y_n + \gamma(\bar{r} - Y_n)\,\Delta + \beta\,\Delta W,$$

where $\Delta W = W_{\tau_{n+1}} - W_{\tau_n}$.

15.5 The explicit strong order 1.0 scheme has the form

$$\begin{aligned}Y_{n+1} = Y_n + Y_n\,\mu\,\Delta + Y_n\,\sigma\,\Delta W \\ + \frac{\sigma\,Y_n}{2\sqrt{\Delta}}\left(\mu\,\Delta + \sigma\sqrt{\Delta}\right)((\Delta W)^2 - \Delta),\end{aligned}$$

where $\Delta W = W_{\tau_{n+1}} - W_{\tau_n}$.

15.6 (*) It follows that the diffusion coefficients for the first Wiener process W^1 are

$$b^{1,1} = 1 \qquad \text{and} \qquad b^{2,1} = 0$$

and that of the second Wiener process are

$$b^{1,2} = 0 \qquad \text{and} \qquad b^{2,2} = X_t^1.$$

Therefore it follows that

$$L^1\, b^{2,2} = b^{1,1}\, \frac{\partial}{\partial x^1}\, b^{2,2} + b^{2,1}\, \frac{\partial}{\partial x^2}\, b^{2,2} = 1$$

and

$$L^2\, b^{2,1} = b^{1,2}\, \frac{\partial}{\partial x^1}\, b^{2,1} + b^{2,2}\, \frac{\partial}{\partial x^2}\, b^{2,1} = 0.$$

Since the above values are not equal, the SDE is not commutative.

15.7 (*) The Milstein scheme applied to the given SDE is of the form

$$Y_{n+1}^1 = Y_n^1 + \Delta W^1$$
$$Y_{n+1}^2 = Y_n^2 + Y_n^1\, \Delta W^2 + I_{(1,2)}$$

with

$$I_{(1,2)} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_2} dW_{s_1}^1\, dW_{s_2}^2.$$

$$\Delta W^1 = W_{\tau_{n+1}}^1 - W_{\tau_n}^1$$
$$\Delta W^2 = W_{\tau_{n+1}}^2 - W_{\tau_n}^2$$

15.8 Due to symmetry one obtains

$$E\left(\Delta \hat{W}\right) = E\left(\left(\Delta \hat{W}\right)^3\right) = 0.$$

Furthermore, it follows

$$E\left(\left(\Delta \hat{W}\right)^2\right) = \frac{1}{2}\,\Delta + \frac{1}{2}\,\Delta = \Delta$$

and thus

$$E\left(\left(\Delta \hat{W}\right)^2\right) - \Delta = 0.$$

This proves that

$$\left|E\left(\Delta \hat{W}\right)\right| + \left|E\left(\left(\Delta \hat{W}\right)^3\right)\right| + \left|E\left(\left(\Delta \hat{W}\right)^2 - \Delta\right)\right| = 0 \leq K\,\Delta^2.$$

15.9 We have the expectation

$$E\left(1 + Z + \frac{1}{2}\,(Z)^2\right) = \frac{3}{2}$$

and it follows that

$$E(V_N^+) = E(V_N^-) = E(\hat{V}_N) = \frac{3}{2}.$$

Therefore, $\hat{V}_N$ is unbiased.

We calculate the variance of V_N^+, which is

$$\begin{aligned}
\mathrm{Var}(V_N^+) &= E\left(\left(\frac{1}{N}\sum_{k=1}^{N}\left(1 + Z(\omega_k) + \frac{1}{2}(Z(\omega_k))^2\right) - \frac{3}{2}\right)^2\right) \\
&= E\left(\left(\frac{1}{N}\sum_{k=1}^{N}\left(Z(\omega_k) + \frac{1}{2}\Big((Z(\omega_k))^2 - 1\Big)\right)\right)^2\right) \\
&= \frac{1}{N}\,E\left(\left(Z + \frac{1}{2}\Big((Z)^2 - 1\Big)\right)^2\right) \\
&= \frac{1}{N}\left(E\left((Z)^2\right) + \frac{1}{4}\Big(E\left((Z)^4\right) - 2\,E\left((Z)^2\right) + 1\Big)\right) \\
&= \frac{1}{N}\left(1 + \frac{1}{4}\,(3 - 2 + 1)\right) = \frac{3}{2\,N}.
\end{aligned}$$

Alternatively, we obtain

$$\begin{aligned}
\mathrm{Var}\left(\hat{V}_N\right) = E\Bigg(\Bigg(\frac{1}{2}\Bigg(&\frac{1}{N}\sum_{k=1}^{N}\left(1 + Z(\omega_k) + \frac{1}{2}\,(Z(\omega_k))^2\right) \\
&+ \frac{1}{N}\sum_{k=1}^{N}\left(1 - Z(\omega_k) + \frac{1}{2}\,(Z(\omega_k))^2\right)\Bigg) - \frac{3}{2}\Bigg)^2\Bigg) \\
= E&\left(\left(\frac{1}{N}\sum_{k=1}^{N}\frac{1}{2}\Big((Z(\omega_k))^2 - 1\Big)\right)^2\right) \\
= \frac{1}{N}\,&\frac{1}{4}\,E\left(\Big((Z)^2 - 1\Big)^2\right) = \frac{1}{4\,N}\left(E\left((Z)^4\right) - 2\,E\left((Z)^2\right) + 1\right) \\
= \frac{1}{4\,N}&\,(3 - 2 + 1) = \frac{1}{2\,N}.
\end{aligned}$$

This shows that the antithetic method provides a Monte Carlo estimate with a third of the variance of a raw Monte Carlo estimate.

15.10 For $E(V_N^*) = \gamma = 1$ we have $E(\tilde{V}_N) = \frac{2}{3}$, and $\tilde{V}_N$ is an unbiased estimator. The variance of $\tilde{V}_N$ is then obtained as

$$\begin{aligned}
\operatorname{Var}\left(\tilde{V}_N\right) &= E\left(\left(\frac{1}{N}\sum_{k=1}^{N}\left(1 + Z(\omega_k) + \frac{1}{2}(Z(\omega_k))^2\right)\right.\right. \\
&\qquad \left.\left. + \alpha\left(1 - \frac{1}{N}\sum_{k=1}^{N}(1 + Z(\omega_k))\right) - \frac{3}{2}\right)^2\right) \\
&= E\left(\left(\frac{1}{N}\sum_{k=1}^{N}\left(Z(\omega_k)\,(1-\alpha) + \frac{1}{2}\left((Z(\omega_k))^2 - 1\right)\right)\right)^2\right) \\
&= \frac{1}{N}\,E\left(\left(Z\,(1-\alpha) + \frac{1}{2}\left((Z)^2 - 1\right)\right)^2\right) \\
&= \frac{1}{N}\left(E\Big((Z)^2\,(1-\alpha)^2\Big) + \frac{1}{4}\Big(E\left((Z)^4\right) - 2\,E\left((Z)^2\right) + 1\Big)\right) \\
&= \frac{1}{N}\left((1-\alpha)^2 + \frac{1}{4}\,(3 - 2 + 1)\right) = \frac{1}{N}\left(\frac{1}{2} + (1-\alpha)^2\right).
\end{aligned}$$

It turns out that the minimum variance can be achieved for $\alpha = \alpha_{\min} = 1$, which yields $\operatorname{Var}(\tilde{V}_N) = \frac{1}{2\,N}$.

15.11 For the multi-period binomial tree we have the benchmarked return

$$u = \exp\{\sigma\,\sqrt{\Delta}\} - 1$$

with probability $p = \frac{-d}{u-d}$ and the benchmarked return

$$d = \exp\{-\sigma\,\sqrt{\Delta}\} - 1$$

with the remaining probability $1 - p$. The binomial European put price at time $t = 0$ is then given by the expression

$$
\begin{aligned}
S_0^{\delta_H} &= \hat{S}_0^{\delta_H} = E\left(\left(\hat{K} - \hat{S}_T\right)^+ \mid \mathcal{A}_0\right) \\
&= \sum_{k=0}^{n_T} \frac{n_T\,!}{k\,!\,(n_T - k)\,!}\, p^k\,(1-p)^{n_T - k}\left(\hat{K} - (1+u)^k (1+d)^{n_T-k}\,\hat{S}_0\right)^+ \\
&= \hat{K} \sum_{k=0}^{\bar{k}} \frac{n_T\,!}{k\,!\,(n_T - k)\,!}\, p^k\,(1-p)^{n_T - k} \\
&\quad - S_0 \sum_{k=0}^{\bar{k}} \frac{n_T\,!}{k\,!\,(n_T - k)\,!}\, p^k\,(1-p)^{n_T - k}\,(1+u)^k (1+d)^{n_T-k},
\end{aligned}
$$

where $\bar{k}$ denotes the first integer k for which $S_0\,(1+u)^k(1+d)^{n_T-k} < \hat{K}$.

15.12 Let us introduce for the Box-Muller random variables

$$
\begin{aligned}
Y_1 &= \cos(2\,\pi\,X_2)\,\sqrt{-2\,\ln X_1} \\
Y_2 &= \sin(2\,\pi\,X_2)\,\sqrt{-2\,\ln X_1},
\end{aligned}
$$

with $X_1, X_2 \sim U(0,1)$ uniformly distributed and independent, the functions $x_1(y_1, y_2) = \exp\{-\frac{1}{2}(y_1^2 + y_2^2)\}$ and $x_2(y_1, y_2) = \frac{1}{2\,\pi}\,\arctan(\frac{y_2}{y_1})$. If we denote by $p(x_1, x_2)$ the joint density of (X_1, X_2), the joint density $q(y_1, y_2)$ of (Y_1, Y_2) is given by

$$
\begin{aligned}
q(y_1, y_2) &= p(x_1(y_1, y_2),\, x_2(y_1, y_2))\,\left|\det\left[\frac{\partial x_i}{\partial y_i}\right]\right| \\
&= 1\,\frac{1}{2\pi}\,e^{-\frac{1}{2}}(y_1^2 + y_2^2) = \frac{1}{2\pi}\,e^{-\frac{1}{2}y_1^2}\,\frac{1}{2\pi}\,e^{-\frac{1}{2}y_2^2},
\end{aligned}
$$

which is the density of two independent Gaussian random variables.