14

Markets with Event Risk

After having studied continuous financial markets, this chapter applies the benchmark approach to markets that exhibit jumps due to event risk. It generalizes several results previously obtained to the case of *jump diffusion markets* (JDMs).

14.1 Jump Diffusion Markets

This section extends the results of Sect. 10.1. It provides a unified framework for financial modeling, portfolio optimization, derivative pricing and risk measurement when security price processes exhibit intensity based jumps. These jumps allow for the modeling of event risk in finance, insurance and other areas. Conditions are formulated under which such a market does not permit arbitrage. The natural numeraire for pricing is again shown to be the GOP, which relates to the concept of real world pricing as previously explained. Nonnegative portfolios, when expressed in units of the GOP, turn out to be supermartingales again. An equivalent risk neutral probability measure needs not to exist in the JDMs considered. The approach presented avoids the problem of dealing with risk neutral intensities and similar complications and restrictions that apply under the standard risk neutral approach.

Continuous and Event Driven Uncertainty

We consider a market containing $d \in \mathcal{N}$ sources of trading uncertainty. *Continuous trading uncertainty* is represented by $m \in \{1, 2, \ldots, d\}$ independent standard Wiener processes $\tilde{W}^k = \{\tilde{W}^k_t,\, t \in [0, \infty)\},\ k \in \{1, 2, \ldots, m\}$. These are defined on a filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$. We also model events of certain types, for instance, corporate defaults, credit rating changes, operational failures or specified insured events, when these are reflected in the prices of traded securities. Events of the kth type are counted by the $\underline{\mathcal{A}}$-adapted kth

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counting process $p^k = \{p^k_t,\, t \in [0,\infty)\}$, whose *intensity* $h^k = \{h^k_t,\, t \in [0,\infty)\}$ is a given, predictable, strictly positive process with

$$h^k_t > 0 \tag{14.1.1}$$

and

$$\int_0^t h^k_s\, ds < \infty \tag{14.1.2}$$

almost surely for $t \in [0,\infty)$ and $k \in \{1,2,\ldots,d-m\}$. The kth counting process p^k leads to the kth *jump martingale* $q^k = \{q^k_t,\, t \in [0,\infty)\}$ with stochastic differential

$$dq^k_t = \left(dp^k_t - h^k_t\, dt\right)\left(h^k_t\right)^{-\frac{1}{2}} \tag{14.1.3}$$

for $k \in \{1,2,\ldots,d-m\}$ and $t \in [0,\infty)$. It is assumed that the above jump martingales do not jump at the same time. They represent the compensated, normalized sources of *event driven trading uncertainty.*

The evolution of trading uncertainty is modeled by the vector process of independent $(\underline{\mathcal{A}}, P)$-martingales $\boldsymbol{W} = \{\boldsymbol{W}_t = (\tilde{W}^1_t, \ldots, \tilde{W}^m_t, q^1_t, \ldots, q^{d-m}_t)^\top,\, t \in [0,\infty)\}$. Note that $W^1 = \tilde{W}^1, \ldots, W^m = \tilde{W}^m$ are Wiener processes, while $W^{m+1} = q^1, \ldots, W^d = q^{d-m}$ are compensated, normalized counting processes. The filtration $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t\in[0,\infty)}$ satisfies the usual conditions and $\mathcal{A}_0$ is the trivial σ-algebra, see Sect. 5.1. Note that the conditional variance of the increment of the kth source of event driven trading uncertainty over a time interval of length Δ equals

$$E\left(\left(q^k_{t+\Delta} - q^k_t\right)^2 \,\Big|\, \mathcal{A}_t\right) = \Delta \tag{14.1.4}$$

for all $t \in [0,\infty)$, $k \in \{1,2,\ldots,d-m\}$ and $\Delta \in [0,\infty)$. Note that in addition to trading uncertainties the market typically involves additional uncertainties that impact jump intensities, short rates, appreciation rates, volatilities and other financial quantities.

Primary Security Accounts

As previously explained, a primary security account is an investment account, consisting of only one kind of security. The jth risky primary security account value at time t is denoted by S^j_t, for $j \in \{1,2,\ldots,d\}$ and $t \in [0,\infty)$. These primary security accounts model the evolution of wealth due to the ownership of primary securities, with all dividends and income reinvested. The 0th primary security account $S^0 = \{S^0_t,\ t \in [0,\infty)\}$ is again the domestic riskless savings account, which continuously accrues at the short term interest rate r_t. In the market considered, the denominating security is the domestic currency.

Without loss of generality, we assume that the nonnegative jth primary security account value S^j_t satisfies the jump diffusion SDE

$$dS_t^j = S_{t-}^j \left(a_t^j \, dt + \sum_{k=1}^{d} b_t^{j,k} \, dW_t^k \right) \tag{14.1.5}$$

for $t \in [0, \infty)$ with initial value $S_0^j > 0$ and $j \in \{1, 2, \ldots, d\}$, see Sect. 7.6. Recall that S_{t-}^j denotes the value of the process S^j just before time t, which is defined as the left hand limit at time t, see (5.2.17). This SDE formally looks similar to the SDE (10.1.2). However, we have here also the jump martingales $W_t^k = q_t^{k-m}$ for $k \in \{m+1, \ldots, d\}$, $t \in [0, \infty)$.

We assume that the short rate process r, the appreciation rate processes a^j, the *generalized volatility processes* $b^{j,k}$ and the intensity processes h^k are almost surely finite and predictable, $j \in \{1, 2, \ldots, d\}$, $k \in \{1, 2, \ldots, d-m\}$. They are assumed to be such that a unique strong solution of the system of SDEs (14.1.5) exists, see Sect. 7.7. To ensure nonnegativity for each primary security account we need to make the following assumption.

Assumption 14.1.1. *The condition*

$$b_t^{j,k} \geq -\sqrt{h_t^{k-m}} \tag{14.1.6}$$

holds for all $t \in [0, \infty)$, $j \in \{1, 2, \ldots, d\}$ *and* $k \in \{m+1, \ldots, d\}$.

Taking into account (14.1.3), it can be seen from the SDE (14.1.5) that this assumption excludes jumps that would lead to negative values for S_t^j, see Sect. 7.6. To securitize the sources of trading uncertainty properly, we introduce the *generalized volatility matrix* $\boldsymbol{b}_t = [b_t^{j,k}]_{j,k=1}^d$ for all $t \in [0, \infty)$ and make the following assumption.

Assumption 14.1.2. *The generalized volatility matrix* $\boldsymbol{b}_t$ *is* invertible *for Lebesgue-almost-every* $t \in [0, \infty)$.

Assumption 14.1.2 generalizes Assumption 10.1.1 and allows us to introduce the *market price of risk* vector

$$\boldsymbol{\theta}_t = (\theta_t^1, \ldots, \theta_t^d)^\top = \boldsymbol{b}_t^{-1} \left[\boldsymbol{a}_t - r_t \, \mathbf{1} \right] \tag{14.1.7}$$

for $t \in [0, \infty)$. Here $\boldsymbol{a}_t = (a_t^1, \ldots, a_t^d)^\top$ is the *appreciation rate vector* and $\mathbf{1} = (1, \ldots, 1)^\top$ the *unit vector*. Using (14.1.7), we can rewrite the SDE (14.1.5) similarly to (10.1.7) in the form

$$dS_t^j = S_{t-}^j \left(r_t \, dt + \sum_{k=1}^{d} b_t^{j,k} \left(\theta_t^k \, dt + dW_t^k \right) \right) \tag{14.1.8}$$

for $t \in [0, \infty)$ and $j \in \{0, 1, \ldots, d\}$. For $k \in \{1, 2, \ldots, m\}$, the quantity θ_t^k denotes the *market price of risk* with respect to the kth Wiener process W^k. If $k \in \{m+1, \ldots, d\}$, then θ_t^k can be interpreted as the *market price of the* $(k-m)$*th event risk* with respect to the counting process p^{k-m}. As previously discussed, the market prices of risk play a central role, as they are invariants of the market and determine the risk premia that risky securities attract.

The vector process $\boldsymbol{S} = \{\boldsymbol{S}_t = (S_t^0, \ldots, S_t^d)^\top,\ t \in [0,\infty)\}$ characterizes the evolution of all primary security accounts. We say that a predictable stochastic process $\boldsymbol{\delta} = \{\boldsymbol{\delta}_t = (\delta_t^0, \ldots, \delta_t^d)^\top,\ t \in [0,\infty)\}$ is a strategy, see Sect. 10.1, if the Itô integral $I_{\delta,W}(t)$ of the corresponding gains from trade exists, see Sect. 5.3. As explained in Chap. 10, the jth component δ^j of $\boldsymbol{\delta}$ denotes the number of units of the jth primary security account held at time $t \in [0,\infty)$ in the portfolio S^δ, $j \in \{0,1,\ldots,d\}$. For a strategy $\boldsymbol{\delta}$ we denote by S_t^δ the value of the corresponding portfolio process at time t, when measured in units of the domestic currency. Thus, we set

$$S_t^\delta = \sum_{j=0}^{d} \delta_t^j\, S_t^j \tag{14.1.9}$$

for $t \in [0,\infty)$. As defined for a CFM, a strategy $\boldsymbol{\delta}$ and the corresponding portfolio process $S^\delta = \{S_t^\delta,\ t \in [0,\infty)\}$ are self-financing if

$$dS_t^\delta = \sum_{j=0}^{d} \delta_t^j\, dS_t^j \tag{14.1.10}$$

for all $t \in [0,\infty)$, see (10.1.10). We emphasize that $\boldsymbol{\delta}$ is assumed to be a predictable process and we consider only self-financing portfolios.

Growth Optimal Portfolio

As before, let us denote by $\mathcal{V}^+$ the set of strictly positive portfolio processes. For a given strategy $\boldsymbol{\delta}$ with strictly positive portfolio process $S^\delta \in \mathcal{V}^+$ denote by $\pi_{\delta,t}^j$ the fraction of wealth that is invested in the jth primary security account at time t, that is,

$$\pi_{\delta,t}^j = \delta_t^j\, \frac{S_t^j}{S_t^\delta} \tag{14.1.11}$$

for $t \in [0,\infty)$ and $j \in \{0,1,\ldots,d\}$, see (10.1.11). These fractions sum to one, see (10.1.13). In terms of the vector of fractions $\boldsymbol{\pi}_{\delta,t} = (\pi_{\delta,t}^1, \ldots, \pi_{\delta,t}^d)^\top$ we obtain from (14.1.10), (14.1.8) and (14.1.11) the SDE for S_t^δ

$$dS_t^\delta = S_{t-}^\delta \left\{ r_t\, dt + \boldsymbol{\pi}_{\delta,t-}^\top\, \boldsymbol{b}_t\, (\boldsymbol{\theta}_t\, dt + d\boldsymbol{W}_t) \right\} \tag{14.1.12}$$

for $t \in [0,\infty)$, where $d\boldsymbol{W}_t = (dW_t^1, \ldots, dW_t^m, dq_t^1, \ldots, dq_t^{m-d})^\top$. Note by (14.1.3) that a portfolio process S^δ remains strictly positive if and only if

$$\sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k} > -\sqrt{h_t^{k-m}} \tag{14.1.13}$$

almost surely for all $k \in \{m+1, \ldots, d\}$ and $t \in [0,\infty)$.

For a strictly positive portfolio $S^\delta \in \mathcal{V}^+$ we obtain for its logarithm, by application of Itô's formula, the SDE

$$d\ln(S_t^\delta) = g_t^\delta\, dt + \sum_{k=1}^{m}\sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k}\, dW_t^k + \sum_{k=m+1}^{d} \ln\left(1 + \sum_{j=1}^{d} \pi_{\delta,t-}^j \frac{b_t^{j,k}}{\sqrt{h_t^{k-m}}}\right) \sqrt{h_t^{k-m}}\, dW_t^k \quad (14.1.14)$$

for $t \in [0,\infty)$. Similarly to (10.2.2), the *growth rate* in this expression is

$$g_t^\delta = r_t + \sum_{k=1}^{m}\left[\sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k}\, \theta_t^k - \frac{1}{2}\left(\sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k}\right)^2\right] + \sum_{k=m+1}^{d}\left[\sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k}\left(\theta_t^k - \sqrt{h_t^{k-m}}\right) + \ln\left(1 + \sum_{j=1}^{d} \pi_{\delta,t}^j \frac{b_t^{j,k}}{\sqrt{h_t^{k-m}}}\right) h_t^{k-m}\right] \quad (14.1.15)$$

for $t \in [0,\infty)$, see Exercise 14.1. Note that for the first sum on the right hand side of (14.1.15) a unique maximum exists, because it is a quadratic form with respect to the fractions. Careful inspection of the terms in the second sum reveals that, in general, a unique maximum growth rate only exists if the market prices of event risks are less than the square roots of the corresponding jump intensities. This leads to the following assumption.

Assumption 14.1.3. *The intensities and market price of event risk components satisfy*

$$\sqrt{h_t^{k-m}} > \theta_t^k \quad (14.1.16)$$

for all $t \in [0,\infty)$ *and* $k \in \{m+1,\ldots,d\}$.

We shall see that Assumption 14.1.3 guarantees that there are no portfolios that explode for the given market, which would otherwise lead to some form of arbitrage. Furthermore, this condition allows us to introduce the predictable vector process $\boldsymbol{c}_t = (c_t^1,\ldots,c_t^d)^\top$ with components

$$c_t^k = \begin{cases} \theta_t^k & \text{for} \quad k \in \{1,2,\ldots,m\} \\ \frac{\theta_t^k}{1-\theta_t^k\,(h_t^{k-m})^{-\frac{1}{2}}} & \text{for} \quad k \in \{m+1,\ldots,d\} \end{cases} \quad (14.1.17)$$

for $t \in [0,\infty)$. Note that a very large jump intensity with $h_t^{k-m} \gg 1$ or $\frac{\theta_t^k}{\sqrt{h_t^{k-m}}} \ll 1$ causes the corresponding component c_t^k to approach the market price of jump risk θ_t^k asymptotically for given $t \in [0, \infty)$ and $k \in \{m+1, \ldots, d\}$. In this case the structure of the kth component $c_t^k \approx \theta_t^k$ is similar to those obtained with respect to Wiener processes.

We now define the fractions

$$\boldsymbol{\pi}_{\delta_*,t} = (\pi^1_{\delta_*,t}, \ldots, \pi^d_{\delta_*,t})^\top = \left(\boldsymbol{c}_t^\top \, \boldsymbol{b}_t^{-1}\right)^\top \tag{14.1.18}$$

of a particular portfolio $S^{\delta_*} \in \mathcal{V}^+$, which will be later identified as a GOP, $t \in [0, \infty)$. By (14.1.12) and (14.1.17) it follows that $S_t^{\delta_*}$ satisfies the SDE

$$\begin{aligned} dS_t^{\delta_*} &= S_{t-}^{\delta_*}\Big(r_t\, dt + \boldsymbol{c}_t^\top \,(\boldsymbol{\theta}_t\, dt + d\boldsymbol{W}_t)\Big) \\ &= S_{t-}^{\delta_*}\left(r_t\, dt + \sum_{k=1}^m \theta_t^k\,(\theta_t^k\, dt + dW_t^k)\right. \\ &\qquad \left. + \sum_{k=m+1}^d \frac{\theta_t^k}{1 - \theta_t^k\,(h_t^{k-m})^{-\frac{1}{2}}}\,(\theta_t^k\, dt + dW_t^k)\right) \end{aligned} \tag{14.1.19}$$

for $t \in [0, \infty)$, with $S_0^{\delta_*} > 0$. Inspection of (14.1.19) shows that Assumption 14.1.3 keeps the portfolio process S^{δ_*} strictly positive. Let us now define a GOP in the given market with intensity based jumps.

Definition 14.1.4. *In the given market a strictly positive portfolio process $S^{\underline{\delta}} \in \mathcal{V}^+$ that maximizes the growth rate g_t^δ, see (14.1.15), of strictly positive portfolio processes is called a* GOP, *that is, $g_t^\delta \leq g_t^{\underline{\delta}}$ almost surely for all $t \in [0, \infty)$ and $S^\delta \in \mathcal{V}^+$.*

This definition generalizes the Definition 10.2.1 of a GOP in a CFM. The proof of the following result is given at the end of this section, see also Platen (2004b).

Corollary 14.1.5. *Under Assumptions 14.1.1, 14.1.2 and 14.1.3 the portfolio process $S^{\delta_*} = \{S_t^{\delta_*},\, t \in [0, \infty)\}$, satisfying (14.1.19), is a GOP.*

By (14.1.15), (14.1.17) and (14.1.18) we obtain the optimal growth rate of the GOP in the form

$$g_t^{\delta_*} = r_t + \frac{1}{2}\sum_{k=1}^m (\theta_t^k)^2 - \sum_{k=m+1}^d h_t^{k-m}\left(\ln\left(1 + \frac{\theta_t^k}{\sqrt{h_t^{k-m}} - \theta_t^k}\right) + \frac{\theta_t^k}{\sqrt{h_t^{k-m}}}\right) \tag{14.1.20}$$

for $t \in [0, \infty)$. Note that the optimal growth rate is never less than the short rate. Furthermore, as long as $\frac{\theta_t^k}{\sqrt{h_t^{k-m}}} \ll 1$, that is, θ_t^k is significantly smaller than $\sqrt{h_t^{k-m}}$ for $k \in \{m+1, \ldots, d\}$, we approximately obtain

$$g_t^{\delta_*} \approx r_t + \frac{1}{2}\sum_{k=1}^{d}(\theta_t^k)^2 = r_t + \frac{|\boldsymbol{\theta}_t|^2}{2} \tag{14.1.21}$$

and

$$d\ln(S_t^{\delta_*}) \approx g_t^{\delta_*}\,dt + \sum_{k=1}^{d}\theta_t^k\,dW_t^k. \tag{14.1.22}$$

This SDE is analogous to the SDE for the logarithm of the GOP of a CFM in Sect. 10.2. Also by (14.1.19) we can derive the approximation

$$dS_t^{\delta_*} \approx S_t^{\delta_*}\left(r_t + \sum_{k=1}^{d}\theta_t^k\left(\theta_t^k\,dt + dW_t^k\right)\right), \tag{14.1.23}$$

which is similar to (10.2.8). Now, let us formally characterize the given jump diffusion market.

Definition 14.1.6. *We denote the above financial market by* $\mathcal{S}_{(d)}^{JD} = \{\boldsymbol{S}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{r}, \underline{\mathcal{A}}, P\}$ *and call it a* jump diffusion market *(JDM) when it has* $d \in \mathcal{N}$ *risky primary security accounts and satisfies Assumptions 14.1.1, 14.1.2 and 14.1.3.*

Supermartingale Property

As is the case for a CFM, we call prices, when expressed in units of S^{δ_*} benchmarked prices. By the Itô formula and relations (14.1.12) and (14.1.19), a benchmarked portfolio process $\hat{S}^\delta = \{\hat{S}_t^\delta,\ t \in [0,\infty)\}$, with

$$\hat{S}_t^\delta = \frac{S_t^\delta}{S_t^{\delta_*}} \tag{14.1.24}$$

for $t \in [0,\infty)$, satisfies the SDE

$$\begin{aligned} d\hat{S}_t^\delta &= \sum_{k=1}^{m}\left(\sum_{j=1}^{d}\delta_t^j\,\hat{S}_t^j\,b_t^{j,k} - \hat{S}_t^\delta\,\theta_t^k\right)dW_t^k \\ &\quad + \sum_{k=m+1}^{d}\left(\left(\sum_{j=1}^{d}\delta_t^j\,\hat{S}_{t-}^j\,b_t^{j,k}\right)\left(1 - \frac{\theta_t^k}{\sqrt{h_t^{k-m}}}\right) - \hat{S}_{t-}^\delta\,\theta_t^k\right)dW_t^k \end{aligned} \tag{14.1.25}$$

for $t \in [0,\infty)$.

The SDE (14.1.25) governs the dynamics of a benchmarked portfolio and generalizes the SDE (10.3.2). For example, by (14.2.6) and (14.2.5) the benchmarked savings account $\hat{S}_t^0$ satisfies the SDE

$$d\hat{S}_t^0 = -\hat{S}_{t-}^0\sum_{k=1}^{d}\theta_t^k\,dW_t^k \tag{14.1.26}$$

for $t \in [0,\infty)$.

Using previous notation, let us denote by $\mathcal{V}$ the set of all nonnegative portfolios in the given market. Note that the right hand side of (14.1.25) is driftless. Thus, for $S^\delta \in \mathcal{V}$ the nonnegative benchmarked portfolio $\hat{S}^\delta$ forms an $(\underline{\mathcal{A}}, P)$-local martingale when $\hat{S}^\delta$ is continuous, see Lemma 5.4.1. Also in the given JDM the driftless $\hat{S}^\delta$ is an $(\underline{\mathcal{A}}, P)$-local martingale, see Ansel & Stricker (1994). This provides by Lemma 5.2.3 for nonnegative $\hat{S}^\delta$ the important supermartingale property.

Theorem 14.1.7. *In a JDM any nonnegative benchmarked portfolio process $\hat{S}^\delta$ is an $(\underline{\mathcal{A}}, P)$-supermartingale, that is*

$$\hat{S}^\delta_t \geq E\left(\hat{S}^\delta_\tau \,\middle|\, \mathcal{A}_t\right) \tag{14.1.27}$$

for all bounded $\tau \in [0, \infty)$ and $t \in [0, \tau]$.

A proof of this theorem can be found for general semimartingale markets in Platen (2004a), or for jump diffusion markets driven by Poisson jump measures in Christensen & Platen (2005). We emphasize the fundamental fact that nonnegative benchmarked portfolios are supermartingales in general semimartingale markets as long as an almost surely finite GOP exists, see Platen (2004a). Based on the above supermartingale property of nonnegative benchmarked portfolios and the notion of arbitrage introduced in Definition 10.3.2, we can draw the following conclusion.

Corollary 14.1.8. *A JDM does not allow nonnegative portfolios that permit arbitrage.*

This result generalizes Corollary 10.3.3. Its proof is formally the same as the one given in Corollary 10.3.3. It is based on the fact that a nonnegative supermartingale that reaches zero remains afterwards at zero, see (10.5.4). This argument also can be used for a semimartingale market with a finite GOP to show that no nonnegative portfolio permits arbitrage, see Platen (2004a) and Christensen & Larsen (2007).

Real World Pricing

Recall now the notion of a fair security, see Definition 9.1.2, where its benchmarked price is an $(\underline{\mathcal{A}}, P)$-martingale. Generalizing Corollary 10.4.2 yields by Lemma 10.4.1 the following result.

Corollary 14.1.9. *Consider a JDM with a bounded stopping time $\tau \in (0, \infty)$ and a given future $\mathcal{A}_\tau$-measurable payoff H to be paid at τ with $E(\frac{H}{S^{\delta_*}_\tau} | \mathcal{A}_0) < \infty$. If there exists a fair nonnegative portfolio $S^\delta \in \mathcal{V}$ with $S^\delta_\tau = H$ almost surely, then this is the minimal nonnegative portfolio that replicates the payoff.*

This means that fair portfolios provide the best choice for an investor's tradable wealth. Otherwise, there exists a less expensive fair portfolio that achieves the same payoff H at time τ.

Let H denote an $\mathcal{A}_\tau$-measurable payoff, with $E(\frac{H}{S_\tau^{\delta_*}}) < \infty$, to be paid at a stopping time $\tau \in [0,\infty)$. The real world pricing formula (9.1.30) can also be applied in a JDM context for pricing the payoff H. Its fair price $U_H(t)$ at time $t \in [0,\tau]$ is then given by the real world pricing formula

$$U_H(t) = S_t^{\delta_*}\, E\left(\frac{H}{S_\tau^{\delta_*}} \,\middle|\, \mathcal{A}_t\right), \tag{14.1.28}$$

see (10.4.1). This formula will be used when pricing derivatives in a JDM. In the same way, as discussed in Sect. 10.4, real world pricing is equivalent to risk neutral pricing as long as the candidate Radon-Nikodym derivative value $\Lambda_\theta(t) = \frac{S_t^0}{S_t^{\delta_*}}\,\frac{S_0^{\delta_*}}{S_0^0}$ for the hypothetical risk neutral probability measure forms an $(\underline{\mathcal{A}}, P)$-martingale.

We remark that the actuarial pricing formula in the form (9.2.6) follows from the real world pricing formula (14.1.28) also for a JDM, when the payoff H paid at time T, is independent of the GOP value $S_T^{\delta_*}$. This is of particular importance in insurance, and also for measuring operational risk, as well as, for the pricing of weather derivatives and other payoffs that are not related to the fluctuations of the market index. We remark that even for semimartingale markets, the real world pricing formula is adequate for derivative pricing, as long as a finite GOP exists, see Platen (2004a) and Christensen & Platen (2005).

Forward Rate Equation

As in Sect. 10.4, a simple example of a derivative is the fair zero coupon bond. It pays one unit of the domestic currency at the given maturity date $T \in [0,\infty)$. By the real world pricing formula (14.1.28) the price $P(t,T)$ at time t for this derivative is given by the conditional expectation

$$P(t,T) = E\left(\frac{S_t^{\delta_*}}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t\right) \tag{14.1.29}$$

for $t \in [0,T]$, $T \in [0,\infty)$. This leads to the benchmarked fair zero coupon bond value $\hat{P}(t,T) = \frac{P(t,T)}{S_t^{\delta_*}}$, where we can assume, similarly to (10.4.9), that it satisfies an SDE of the form

$$d\hat{P}(t,T) = -\hat{P}(t-,T)\sum_{k=1}^{d} \sigma^k(t,T)\, dW_t^k \tag{14.1.30}$$

for $t \in [0,T]$, with predictable generalized volatility process $\sigma^k(\cdot,T) = \{\sigma^k(t,T),\, t \in [0,T]\}$ for $k \in \{1,2,\ldots,d\}$. By using a logarithmic transformation and an application of the Itô formula this becomes

$$\ln(\hat{P}(t,T)) = \ln(\hat{P}(0,T)) - \sum_{k=1}^{m} \left(\int_0^t \sigma^k(s,T)\, dW_s^k + \frac{1}{2} \int_0^t (\sigma^k(s,T))^2\, ds \right)$$
$$+ \sum_{k=m+1}^{d} \left(\int_0^t \sigma^k(s,T) \sqrt{h_s^{k-m}}\, ds + \int_0^t \ln\left(1 - \frac{\sigma^k(s,T)}{\sqrt{h_s^{k-m}}}\right) dp_s^k \right). \tag{14.1.31}$$

Hence, according to (10.4.12) the forward rate $f(t,T)$ at time $t \in [0,T]$ for the maturity $T \in [0,\infty)$ satisfies the equation

$$f(t,T) = -\frac{\partial}{\partial T} \ln(P(t,T)) = -\frac{\partial}{\partial T} \ln(\hat{P}(t,T)). \tag{14.1.32}$$

Consequently, by (14.1.31) we derive the *forward rate equation*

$$f(t,T) = f(0,T) + \sum_{k=1}^{m} \int_0^t \left(\frac{\partial}{\partial T} \sigma^k(s,T) \right) \left(\sigma^k(s,T)\, ds + dW_s^k \right)$$
$$+ \sum_{k=m+1}^{d} \int_0^t \frac{1}{1 - \frac{\sigma^k(s,T)}{\sqrt{h_s^{k-m}}}} \frac{\partial}{\partial T} \sigma^k(s,T) \left(\sigma^k(s,T)\, ds + dW_s^k \right) \tag{14.1.33}$$

for $t \in [0,T]$, see Exercise 14.2. This equation can also be found in Bruti-Liberati, Nikitopoulos-Sklibosios & Platen (2009). It is a generalization of (10.4.14) and the HJM equation (10.4.19). In the case when $\frac{\sigma^k(t,T)}{\sqrt{h_t^{k-m}}} \ll 1$ we obtain asymptotically the forward rate equation in the form of the CFM.

GOP as Best Performing Portfolio

In Chap. 10 it was demonstrated by using various criteria that the GOP is the best performing portfolio for a CFM. Since the proofs of these results are based on the supermartingale property of nonnegative benchmarked portfolios, a similar set of proofs also applies for JDMs. Below, we generalize two of these results. First, let us formulate the property that in a JDM the GOP has the maximum long term growth rate, and, thus, almost surely, outperforms any other portfolio after a sufficiently long time.

Theorem 14.1.10. *In a JDM the GOP S^{δ_*} has almost surely the largest long term growth rate in comparison with that of any other strictly positive portfolio $S^\delta \in \mathcal{V}^+$, that is,*

$$\tilde{g}^{\delta_*} \overset{a.s.}{=} \limsup_{T\to\infty} \frac{1}{T} \ln\left(\frac{S_T^{\delta_*}}{S_0^{\delta_*}}\right) \geq \lim_{T\to\infty} \frac{1}{T} \ln\left(\frac{S_T^{\delta}}{S_0^{\delta}}\right) \overset{a.s.}{=} \tilde{g}^{\delta} \tag{14.1.34}$$

almost surely.

We now extend Corollary 10.5.3 by using the obvious extension of Definition 10.5.2 concerning the systematic outperformance of a portfolio.

Corollary 14.1.11. *In a JDM no strictly positive portfolio systematically outperforms the GOP in the sense of Definition 10.5.2.*

Thus, there also is no systematic way to beat the GOP in a JDM over any short or long term horizon. This is a fundamental property of the GOP and makes it very special for investment purposes. We emphasize that the proofs of the above theorem and corollary depend only on the supermartingale property of nonnegative benchmarked portfolios. As previously indicated, this supermartingale property holds for general semimartingale markets. Therefore, similar statements about the optimal performance of the GOP hold very generally, see Platen (2004a).

Proof of Corollary 14.1.5 (*)

Under the Assumption 14.1.3 it follows from the first order conditions for identifying the maximum growth rate (14.1.15) that the optimal generalized portfolio volatilities are described by $\boldsymbol{c}_t$ as given in (14.1.17). Note from (14.1.12) that the generalized volatility of a portfolio $S^\delta \in \mathcal{V}^+$ has at time t the form $\boldsymbol{\pi}_{\delta,t-}^\top \boldsymbol{b}_t$, which leads to the system of linear equations for the optimal fractions $\boldsymbol{\pi}_{\delta_*,t}$ for a GOP with

$$\boldsymbol{\pi}_{\delta_*,t-}^\top \, \boldsymbol{b}_t = \boldsymbol{c}_t. \tag{14.1.35}$$

By Assumption 14.1.2 the generalized volatility matrix $\boldsymbol{b}_t$ is invertible and the formula

$$\boldsymbol{\pi}_{\delta_*,t-}^\top = \boldsymbol{c}_t \, \boldsymbol{b}_t^{-1} \tag{14.1.36}$$

follows from (14.1.35) for the optimal fractions. This yields formula (14.1.18) for $t \in [0, \infty)$. These fractions are uniquely determined and so what is a GOP when its initial value is given. Consequently, the SDE (14.1.19) is, by (14.1.12), (14.1.18) and (14.1.17), the one that characterizes a GOP. □

14.2 Diversified Portfolios

This section considers diversified portfolios in a sequence of JDMs. It generalizes the Diversification Theorem of Sect. 10.6 to the case of markets with intensity based jumps.

Sequence of JDMs

We rely again on a filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$ with filtration $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t\in[0,\infty)}$, satisfying the usual conditions. Continuous trading uncertainty is represented by independent standard Wiener processes $\tilde{W}^k = \{\tilde{W}_t^k, t \in$

$[0, \infty)\}$ for $k \in \mathcal{N}$. Event driven trading uncertainty is modeled by counting processes $p^k = \{p^k_t,\, t \in [0, \infty)\}$ characterized by corresponding predictable, strictly positive intensity processes $h^k = \{h^k_t,\, t \in [0, \infty)\}$ for $k \in \mathcal{N}$. We define the kth jump martingale $q^k = \{q^k_t,\, t \in [0, \infty)\}$ as in (14.1.3), for $k \in \mathcal{N}$.

In what follows, we consider a sequence $(\mathcal{S}^{\mathrm{JD}}_{(d)})_{d \in \mathcal{N}}$ of JDMs indexed by the number $d \in \mathcal{N}$ of risky primary security accounts. For a given integer d, the corresponding JDM $\mathcal{S}^{\mathrm{JD}}_{(d)}$ comprises $d + 1$ primary security accounts, denoted by $S^0_{(d)}, S^1_{(d)}, \ldots, S^d_{(d)}$. These include a savings account $S^0_{(d)} = \{S^0_{(d)}(t),\, t \in [0, \infty)\}$, which is a locally riskless primary security account, whose value at time t is given by the exponential $S^0_{(d)}(t) = \exp\left\{\int_0^t r_s\, ds\right\}$ for $t \in [0, \infty)$. Here $r = \{r_t,\, t \in [0, \infty)\}$ denotes an adapted short rate process, which we assume, for simplicity, to be the same in each JDM. We include d nonnegative, risky primary security account processes $S^j_{(d)} = \{S^j_{(d)}(t),\, t \in [0, \infty)\}$, $j \in \{1, 2, \ldots, d\}$, each of which can be driven by the Wiener processes $\tilde{W}^1, \tilde{W}^2, \ldots, \tilde{W}^m$ and the jump martingales $q^1, q^2, \ldots, q^{d-m}$. Here $\mu \in [0, 1]$ is a fixed real number and $m = [\mu d]$ denotes the largest integer not exceeding μd. In the dth JDM we have the trading uncertainty driven by the d-dimensional vector process $\boldsymbol{W} = \{\boldsymbol{W}_t = (\tilde{W}^1_t, \ldots, \tilde{W}^m_t, q^1_t, \ldots, q^{d-m})^\top$, $t \in [0, \infty)\}$. Obviously, if μ equals one, then we have no jumps. This covers the case of a CFM, as was discussed in Sect. 10.1.

As previously noted, for fixed $d \in \mathcal{N}$ we call a predictable stochastic process $\boldsymbol{\delta} = \{\boldsymbol{\delta}_t = (\delta^0_t, \delta^1_t, \ldots, \delta^d_t)^\top,\, t \in [0, \infty)\}$ a strategy if for each $j \in \{0, 1, \ldots, d\}$ the Itô integral $\int_0^t \delta^j_s\, dS^j_{(d)}(s)$ exists. The corresponding portfolio value is then $S^\delta_{(d)}(t) = \sum_{j=0}^d \delta^j_t\, S^j_{(d)}(t)$ and satisfies the SDE

$$dS^\delta_{(d)}(t) = \sum_{j=0}^d \delta^j_t\, dS^j_{(d)}(t) \tag{14.2.1}$$

for $t \in [0, \infty)$. Note that in the dth JDM $\mathcal{S}^{\mathrm{JD}}_{(d)}$ a given strategy $\boldsymbol{\delta}$ depends typically on d. However, for simplicity we shall initially suppress this dependence and shall only mention it when later required.

The corresponding jth fraction of a strictly positive portfolio $S^\delta_{(d)}$ is given by the expression $\pi^j_{\delta,t} = \delta^j_t \frac{S^j_{(d)}(t)}{S^\delta_{(d)}(t)}$ for $t \in [0, \infty)$ and $j \in \{0, 1, \ldots, d\}$, as long as $S^\delta_{(d)}(t) > 0$.

As shown in Sect. 14.1, for each JDM $\mathcal{S}^{\mathrm{JD}}_{(d)}$ there exists a unique GOP $S^{\delta_*}_{(d)} = \{S^{\delta_*}_{(d)}(t),\, t \in [0, \infty)\}$ satisfying the SDE (14.1.19) when we fix the initial value, which we set, for simplicity, to

$$S^{\delta_*}_{(d)}(0) = 1. \tag{14.2.2}$$

Any portfolio $S^\delta_{(d)}$ in the dth JDM, when expressed in units of $S^{\delta_*}_{(d)}$, yields a corresponding benchmarked portfolio $\hat{S}^\delta_{(d)} = \{\hat{S}^\delta_{(d)}(t),\, t \in [0, \infty)\}$, defined by

$$\hat{S}^{\delta}_{(d)}(t) = \frac{S^{\delta}_{(d)}(t)}{S^{\delta_*}_{(d)}(t)} \tag{14.2.3}$$

at time $t \in [0, \infty)$. It forms a driftless SDE, see (14.1.25).

To obtain a more compact formulation of the SDE (14.1.25), let us define the (j,k)th *specific generalized volatility* $\sigma^{j,k}_{(d)}(t)$, see (10.6.3)–(10.6.4), by setting

$$\sigma^{0,k}_{(d)}(t) = \theta^k_t \tag{14.2.4}$$

for $j = 0$ and $k \in \{1, 2, \ldots, d\}$ and

$$\sigma^{j,k}_{(d)}(t) = \begin{cases} \theta^k_t - b^{j,k}_t & \text{for} \quad k \in \{1, 2, \ldots, m\} \\ \theta^k_t - b^{j,k}_t \left(1 - \frac{\theta^k_t}{\sqrt{h^{k-m}_t}}\right) & \text{for} \quad k \in \{m+1, \ldots, d\} \end{cases} \tag{14.2.5}$$

for $t \in [0, \infty)$ and $j \in \{1, 2, \ldots, d\}$. By using (14.2.5) and (14.2.4) one can rewrite the SDE (14.1.25) in the form

$$d\hat{S}^{\delta}_{(d)}(t) = -\sum_{k=1}^{d} \sum_{j=0}^{d} \delta^j_t \, \hat{S}^j_{(d)}(t-) \, \sigma^{j,k}_{(d)}(t) \, dW^k_t, \tag{14.2.6}$$

and for strictly positive $S^{\delta}_{(d)}(t)$ as

$$d\hat{S}^{\delta}_{(d)}(t) = -\hat{S}^{\delta}_{(d)}(t-) \sum_{k=1}^{d} \sum_{j=0}^{d} \pi^j_{\delta,t-} \, \sigma^{j,k}_{(d)}(t) \, dW^k_t \tag{14.2.7}$$

for $t \in [0, \infty)$.

The following assumption asks for the property that the specific generalized volatilities are finite in a certain sense.

Assumption 14.2.1. *For all $d \in \mathcal{N}$, $T \in [0, \infty)$ and $j \in \{0, 1, \ldots, d\}$ suppose that*

$$\int_0^T \sum_{k=1}^{d} \left(\sigma^{j,k}_{(d)}(t)\right)^2 dt \le \bar{K}_T < \infty \tag{14.2.8}$$

almost surely, where $\bar{K}_T < \infty$ denotes some finite $\mathcal{A}_T$-measurable random variable which does not depend on d. Furthermore, it is assumed that the inequality

$$\sigma^{j,k}_{(d)}(t) < \sqrt{h^{k-m}_t} \tag{14.2.9}$$

holds almost surely for all $t \in [0, \infty)$, $k \in \{m+1, m+2, \ldots, d\}$ and $j \in \{0, 1, \ldots, d\}$.

Sequences of Diversified Portfolios

Our aim is now to generalize the Diversification Theorem from Sect. 10.6 to the case of JDMs. Since for each $d \in \mathcal{N}$ the above model is a JDM, we can form a sequence of JDMs $(\mathcal{S}^{\mathrm{JD}}_{(d)})_{d\in\mathcal{N}}$, indexed by the number d of risky primary security accounts. As in Sect. 10.6, for such a sequence of financial market models we identify a class of sequences of portfolios that approximate the corresponding sequence of GOPs.

Let us extend the Definition 10.6.2 for a sequence of diversified portfolios (DPs).

Definition 14.2.2. *For a sequence of JDMs* $(\mathcal{S}^{JD}_{(d)})_{d\in\mathcal{N}}$ *we call a corresponding sequence* $(S^{\delta}_{(d)})_{d\in\mathcal{N}}$ *of strictly positive portfolio processes* $S^{\delta}_{(d)}$ a sequence of DPs *if some constants* $K_1, K_2 \in (0,\infty)$ *and* $K_3 \in \mathcal{N}$ *exist, independently of* d*, such that for* $d \in \{K_3, K_3+1, \ldots\}$ *the inequality*

$$\left|\pi^{j}_{\delta,t}\right| \leq \frac{K_2}{d^{\frac{1}{2}+K_1}} \tag{14.2.10}$$

holds almost surely for all $j \in \{0,1,\ldots,d\}$ *and* $t \in [0,\infty)$.

Note that in (14.2.10) the strategy δ depends on d. Consider for fixed $d \in \mathcal{N}$ the dth JDM $\mathcal{S}^{\mathrm{JD}}_{(d)}$ as an element of a given sequence of JDMs. By (14.2.7), when setting $\pi^{j}_{\delta,t} = 1$ and $\pi^{i}_{\delta,t} = 0$ for $i \neq j$, the jth benchmarked primary security account process $\hat{S}^{j}_{(d)} = \{\hat{S}^{j}_{(d)}(t),\ t \in [0,\infty)\}$, with

$$\hat{S}^{j}_{(d)}(t) = \frac{S^{j}_{(d)}(t)}{S^{\delta_*}_{(d)}(t)}, \tag{14.2.11}$$

satisfies the driftless SDE

$$d\hat{S}^{j}_{(d)}(t) = -\hat{S}^{j}_{(d)}(t-) \sum_{k=1}^{d} \sigma^{j,k}_{(d)}(t)\, dW^{k}_{t} \tag{14.2.12}$$

for $t \in [0,\infty)$ and $j \in \{0,1,\ldots,d\}$.

The (j,k)th specific generalized volatility $\sigma^{j,k}_{(d)}(t)$ of the benchmarked jth primary security account $\hat{S}^{j}_{(d)}(t)$ measures at time $t \in [0,\infty)$ the jth specific market risk with respect to the kth trading uncertainty W^k for $k \in \{1,2,\ldots,d\}$, $j \in \{0,1,\ldots,d\}$, see Platen & Stahl (2003) and Sect. 10.6. Similarly as for CFMs, we introduce for all $t \in [0,\infty)$, $d \in \mathcal{N}$ and $k \in \{1,2,\ldots,d\}$ the kth *total specific volatility* for the dth JDM $\mathcal{S}^{\mathrm{JD}}_{(d)}$ in the form

$$\hat{\sigma}^{k}_{(d)}(t) = \sum_{j=0}^{d} |\sigma^{j,k}_{(d)}(t)|. \tag{14.2.13}$$

Depending on k, the kth total specific volatility represents the sum of the absolute values of the specific generalized volatilities with respect to the kth trading uncertainty.

Similarly to Definition 10.6.3 the following regularity property of a sequence of markets ensures that each of the independent sources of trading uncertainty influences only a restricted range of benchmarked primary security accounts.

Definition 14.2.3. *A sequence of JDMs is called* regular *if there exists a constant $K_5 \in (0, \infty)$, independent of d, such that*

$$E\left(\left(\hat{\sigma}^k_{(d)}(t)\right)^2\right) \leq K_5 \tag{14.2.14}$$

for all $t \in [0, \infty)$, $d \in \mathcal{N}$ and $k \in \{1, 2, \ldots, d\}$.

Sequence of Approximate GOPs

As in the case of a CFM, we consider for given $d \in \mathcal{N}$ in the dth JDM $\mathcal{S}^{\text{JD}}_{(d)}$ a strictly positive portfolio process $S^\delta_{(d)}$ with strategy $\boldsymbol{\delta} = \{\boldsymbol{\delta}_t = (\delta^0_t, \delta^1_t, \ldots, \delta^d_t)^\top,\ t \in [0, \infty)\}$. We introduce again the *tracking rate* $R^\delta_{(d)}(t)$ at time t for the portfolio $S^\delta_{(d)}$ by setting

$$R^\delta_{(d)}(t) = \sum_{k=1}^{d} \left(\sum_{j=0}^{d} \pi^j_{\delta,t}\, \sigma^{j,k}_{(d)}(t)\right)^2 \tag{14.2.15}$$

for $t \in [0, \infty)$, see (10.6.22). By (14.2.7) one notes that the benchmarked portfolio $\hat{S}^\delta_{(d)}$ is constant with

$$\hat{S}^\delta_{(d)}(t) = \hat{S}^\delta_{(d)}(0) \tag{14.2.16}$$

almost surely, if and only if the tracking rate vanishes, that is,

$$R^\delta_{(d)}(t) = 0 \tag{14.2.17}$$

almost surely for all $t \in [0, \infty)$. Recall that by (14.2.2) $S^{\delta_*}_{(d)}(0) = 1$. In the case of a constant benchmarked portfolio $\hat{S}^\delta_{(d)}$, characterized by equation (14.2.16), the portfolio value $S^\delta_{(d)}(t)$ equals, by relation (14.2.3), a multiple of the GOP, that is,

$$S^\delta_{(d)}(t) = S^\delta_{(d)}(0)\, S^{\delta_*}_{(d)}(t) \tag{14.2.18}$$

almost surely for all $t \in [0, \infty)$. Therefore, a given portfolio process $S^\delta_{(d)}$ moves in step with the GOP if the tracking rate $R^\delta_{(d)}(t)$ remains small for all $t \in [0, \infty)$. Let us formalize this fact by extending Definition 10.6.4.

Definition 14.2.4. *For a sequence* $(\mathcal{S}^{JD}_{(d)})_{d\in\mathcal{N}}$ *of JDMs we call a sequence* $(S^\delta_{(d)})_{d\in\mathcal{N}}$ *of strictly positive portfolio processes a sequence of* approximate GOPs *if for all* $t \in [0,\infty)$ *the corresponding sequence of tracking rates vanishes in probability, see* (2.7.1)*. That is, we have*

$$\lim_{d\to\infty} R^\delta_{(d)}(t) \stackrel{P}{=} 0 \tag{14.2.19}$$

for all $t \in [0,\infty)$*.*

To obtain a moment based sufficient condition for the identification of a sequence of approximate GOPs, we introduce, for any given $d \in \mathcal{N}$ and strictly positive portfolio process $S^\delta_{(d)}$, the *expected tracking rate*

$$e^\delta_{(d)}(t) = E\left(R^\delta_{(d)}(t)\right) \tag{14.2.20}$$

at time $t \in [0,\infty)$. This leads to the following definition.

Definition 14.2.5. *For a sequence of JDMs* $(\mathcal{S}^{JD}_{(d)})_{d\in\mathcal{N}}$*, a sequence* $(S^\delta_{(d)})_{d\in\mathcal{N}}$ *of strictly positive portfolio processes is said to have a* vanishing expected tracking rate*, if their expected tracking rate converges to zero, that is,*

$$\lim_{d\to\infty} e^\delta_{(d)}(t) = 0 \tag{14.2.21}$$

for all $t \in [0,\infty)$*.*

Using Definition 14.2.5 and the Markov inequality (1.3.57), we obtain for given $\varepsilon > 0$ and any sequence $(S^\delta_{(d)})_{d\in\mathcal{N}}$ of strictly positive portfolios with vanishing expected tracking rate the asymptotic inequality

$$\lim_{d\to\infty} P\left(R^\delta_{(d)}(t) > \varepsilon\right) \leq \lim_{d\to\infty} \frac{1}{\varepsilon}\, e^\delta_{(d)}(t) = 0 \tag{14.2.22}$$

for all $t \in [0,\infty)$. Therefore, by Definition 14.2.4 and inequality (14.2.22) we obtain the following result.

Lemma 14.2.6. *For a sequence of JDMs, any sequence of strictly positive portfolios with vanishing expected tracking rate is a sequence of approximate GOPs.*

Diversification Theorem

Now, we state a crucial result of the benchmark approach. Using Definitions 14.2.2 and 14.2.3 the Lemma 14.2.6 allows us to extend the Diversification Theorem to the case of JDMs. Its proof is omitted since it is analogous to the one of Theorem 10.6.5 in Sect. 10.6 and can also be found in Platen (2005b).

Theorem 14.2.7. (Diversification Theorem for JDMs) *For a regular sequence of JDMs* $(\mathcal{S}^{JD}_{(d)})_{d\in\mathcal{N}}$*, each sequence* $(S^{\delta}_{(d)})_{d\in\mathcal{N}}$ *of DPs is a sequence of approximate GOPs. Moreover, for any* $d \in \{K_3, K_3+1, \ldots\}$ *and* $t \in [0,\infty)$*, the expected tracking rate of a given DP* $S^{\delta}_{(d)}$ *satisfies the inequality*

$$e^{\delta}_{(d)}(t) \leq \frac{(K_2)^2\, K_5}{d^{2K_1}}. \tag{14.2.23}$$

Here the constants K_1, K_2, K_3 *and* K_5 *are the same as in Definitions 14.2.2 and 14.2.3.*

The Diversification Theorem shows that for a regular sequence of JDMs any sequence of DPs approximates the GOP. This is highly relevant for the practical applicability of the benchmark approach, as previously discussed in Sect. 10.6. In particular, it allows one to approximate the GOP by a diversified market index without the need of an exact calculation of the fractions of the GOP. We emphasize that this result is model independent, which makes it very robust. The Diversification Theorem can be generalized under appropriate assumptions to the case of semimartingale markets, as will be shown in forthcoming work.

Diversification in an MMM Setting

In Sect. 10.6 we provided some examples for diversified portfolios in a Black-Scholes type CFM. These examples demonstrate that the asymptotic properties of approximate GOPs do not need extremely large numbers of primary security accounts to be practically relevant. In a JDM with only a few rare events this is not so easy to demonstrate unless one generates an extremely large number of primary security accounts. However, in practice, the number of primary security accounts is indeed very large and the default of a single stock, even a large one, does not significantly change the value of the market portfolio. Let us now illustrate the fundamental phenomenon of diversification by simulating diversified portfolios in an MMM type setting, for simplicity, without jumps.

We consider the following multi-asset stylized MMM, which we discussed in Sect. 13.2. This example also demonstrates how to construct efficiently a market model under the benchmark approach. Firstly, we introduce the savings account in the form

$$S^0_{(d)}(t) = \exp\{r\,t\} \tag{14.2.24}$$

with constant short rate $r > 0$ for $t \in [0,T]$, $d \in \mathcal{N}$. The discounted GOP drift is set in all denominations to

$$\alpha^{\delta_*}_t = \alpha_0\, \exp\{\eta\, t\} \tag{14.2.25}$$

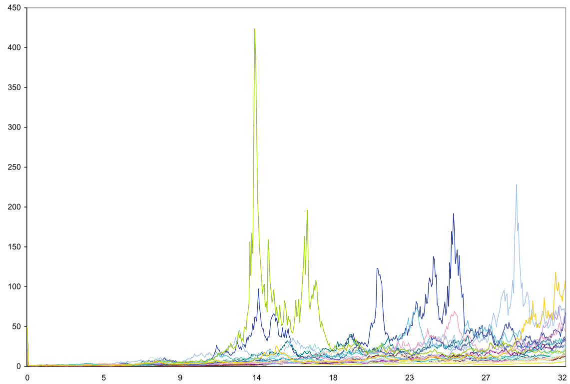

Fig. 14.2.1. Primary security accounts under the MMM

with net growth rate $\eta > 0$ and initial parameter $\alpha_0 > 0$. We model the jth benchmarked primary security account by the expression

$$\hat{S}^j_{(d)}(t) = \frac{1}{Y^j_t \, \alpha^{\delta_*}_t} \tag{14.2.26}$$

for all $j \in \{0, 1, \ldots, d\}$. In this context Y^j_t is the time t value of the SR process Y^j, which satisfies the SDE

$$dY^j_t = \left(1 - \eta \, Y^j_t\right) dt + \sqrt{Y^j_t} \, dW^j_t \tag{14.2.27}$$

for $t \in [0, T]$, where we set $Y^j_0 = \frac{1}{\eta}$ for $j \in \{0, 1, \ldots\}$. Also $W^0, W^1, \ldots$ are independent standard Wiener processes.

Now, with (14.2.11) the GOP is obtained as the ratio

$$S^{\delta_*}_{(d)}(t) = \frac{S^0_{(d)}(t)}{\hat{S}^0_{(d)}(t)}. \tag{14.2.28}$$

Hence, by (14.2.11) the value of the jth primary security account is given by

$$S^j_{(d)}(t) = \hat{S}^j_{(d)}(t) \, S^{\delta_*}_{(d)}(t) \tag{14.2.29}$$

for $t \in [0, \infty)$, $j \in \{1, 2, \ldots, d\}$ and $d \in \mathcal{N}$. By starting from the savings account and the benchmarked primary security accounts we have modeled all primary security accounts and the GOP in the denomination of the domestic currency.

We now simulate $d = 50$ primary security accounts $S^j_{(d)}$, $j \in \{0, 1, \ldots, d\}$, for a period of $T = 32$ years, where we set $r = \eta = \alpha_0 = 0.05$. We show in Fig. 14.2.1 the trajectories of the first twenty risky primary security accounts. One notes their typical increase but also a decline of some of the securities. It is

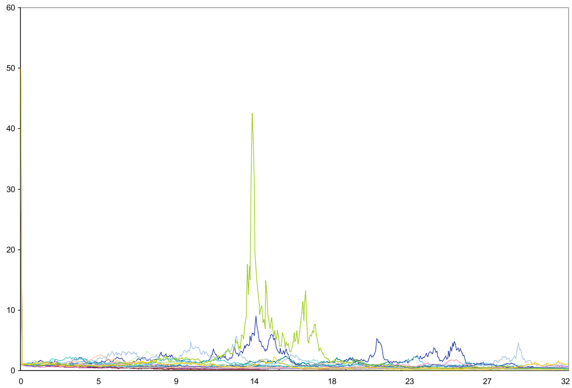

Fig. 14.2.2. Benchmarked primary security accounts

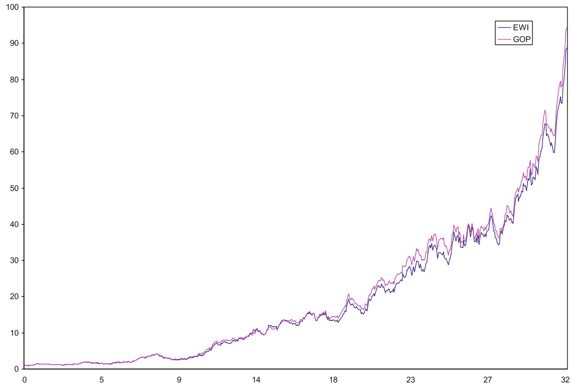

Fig. 14.2.3. GOP and EWI

noticeable that the primary security accounts have some common fluctuations. These are caused by the general market risk as captured by the GOP, which is shown in Fig. 14.2.3. In Fig. 14.2.2 we plot the corresponding benchmarked primary security accounts. These are strict supermartingales, as discussed previously in Chap. 13.

Figure 14.2.3 shows the equi-value weighted index (EWI) together with the GOP. One notes the closeness of the GOP and the EWI as predicted by the above Diversification Theorem. Figure 14.2.4 displays a market index, where its constituents represent simply one unit of each primary security account. Here one notes that the market index is initially a good proxy of the GOP. After an initial time period of about 13 years some extremely large stock values emerge, as can be seen in Fig. 14.2.1. The resulting large fractions of these stocks distort the performance of the market index. These fractions of the corresponding primary security accounts are simply too large to be acceptable as those of a DP and, thus, violate the conditions of the Diversification Theorem.

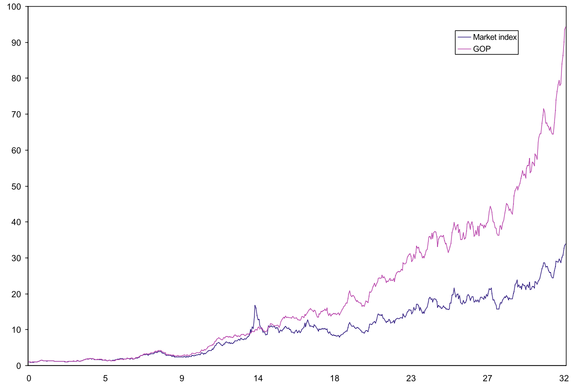

Fig. 14.2.4. GOP and market index

One can say that the market index is in our example, no longer interpretable as a DP after about 13 years because the fractions of a few excellent performing stocks are larger than the average fraction by magnitudes. The EWI does not suffer in this way and is in our example a good proxy for the GOP, as can be seen from Fig. 14.2.3. We emphasize that even for a market with only 50 risky primary security accounts a rather good approximation of the GOP by DPs like the EWI is obtained. Further experiments with other DPs reveal a similar behavior as shown in Fig. 14.2.3.

The Diversification Theorem identifies DPs as proxies for the GOP without any particular modeling assumptions on the market dynamics. This diversification phenomenon is, therefore, very robust. However, if the fractions of some primary security accounts become too large in a portfolio, then such a portfolio cannot be interpreted as a DP and it is unlikely to be a good proxy of the GOP.

14.3 Mean-Variance Portfolio Optimization

This section generalizes some of the results on mean-variance portfolio optimization that we presented in Chap. 11. It turns out that the kind of two fund separation of locally optimal portfolios into combinations of savings account and GOP, which we observed for a CFM, does not hold any longer in the same manner. Different classes of optimal portfolios arise in a JDM for different types of optimization objectives. For instance, Sharpe ratio maximization does not lead, in general, to portfolios that are a combination of the GOP and savings account.

Locally Optimal Portfolios

Our objective here is to try to generalize the results of Sect. 11.1 on locally optimal portfolios. Given a strictly positive portfolio S^δ, its discounted value $\bar{S}_t^\delta = \frac{S_t^\delta}{S_t^0}$ satisfies the SDE

$$d\bar{S}_t^\delta = \sum_{k=1}^{d} \psi_{\delta,t}^k \left(\theta_t^k \, dt + dW_t^k\right) \tag{14.3.1}$$

by (14.1.12) and an application of the Itô formula. Here

$$\psi_{\delta,t}^k = \sum_{j=1}^{d} \delta_t^j \, b_t^{j,k} \, \bar{S}_{t-}^\delta \tag{14.3.2}$$

is called the kth *generalized diffusion coefficient* at time t for $k \in \{1, 2, \ldots, d\}$ and $t \in [0, \infty)$. Obviously, by (14.3.1) and (14.3.2), the discounted portfolio process $\bar{S}^\delta$ has *discounted drift*

$$\alpha_t^\delta = \sum_{k=1}^{d} \psi_{\delta,t}^k \, \theta_t^k \tag{14.3.3}$$

for $t \in [0, T]$. This drift measures the portfolio's trend at time t. The fluctuations of a discounted portfolio $\bar{S}^\delta$ can be measured at time t by its *aggregate generalized diffusion coefficient*

$$\gamma_t^\delta = \sqrt{\sum_{k=1}^{d} \left(\psi_{\delta,t}^k\right)^2} \tag{14.3.4}$$

at time $t \in [0, \infty)$. Note that by relation (14.1.4) we have standardized the variances of the increments of the driving martingales $W^1, W^2, \ldots, W^d$ such that they equal the corresponding time increments, as is the case for standard Wiener processes.

For a given level of the aggregate generalized diffusion coefficient $\gamma_t^{\tilde{\delta}} > 0$, suppose that an investor aims to maximize the portfolio drift α_t^δ of a discounted portfolio $\bar{S}^\delta$. This objective can be interpreted as a possible generalization of mean-variance portfolio optimization in the sense of Markowitz (1959) to the case of a JDM. More precisely, let us identify the class of SDEs for the portfolios of investors who prefer locally optimal portfolios, defined in the following sense:

Definition 14.3.1. *A strictly positive portfolio process $S^{\tilde{\delta}} \in \mathcal{V}^+$ that maximizes the portfolio drift (14.3.3) among all strictly positive portfolio processes $S^\delta \in \mathcal{V}^+$ with a given aggregate generalized diffusion coefficient level $\gamma_t^{\tilde{\delta}}$ is called* locally optimal*, that is,*

$$\gamma_t^\delta = \gamma_t^{\tilde{\delta}} \qquad and \qquad \alpha_t^\delta \leq \alpha_t^{\tilde{\delta}} \tag{14.3.5}$$

almost surely for all $t \in [0, \infty)$.

This definition generalizes our Definition 11.1.1 to the case of JDMs.

Mean-Variance Portfolio Selection Theorem

For the following analysis we use the *total market price of risk*

$$|\boldsymbol{\theta}_t| = \sqrt{\sum_{k=1}^{d} \left(\theta_t^k\right)^2} \tag{14.3.6}$$

and the weighting factor

$$G(t) = \sum_{k=1}^{d} \sum_{j=1}^{d} \theta_t^k \, b_t^{-1\,j,k} \tag{14.3.7}$$

for $t \in [0, \infty)$. The following condition generalizes Assumption 11.1.2. It excludes the trivial situation of having the savings account as GOP.

Assumption 14.3.2. *In a JDM suppose that*

$$0 < |\boldsymbol{\theta}_t| < \infty \tag{14.3.8}$$

and

$$G(t) \neq 0 \tag{14.3.9}$$

almost surely for all $t \in [0, \infty)$.

Now, we can formulate a mean-variance portfolio selection theorem which generalizes the results of Theorem 11.1.3. It identifies the structure of the drift and generalized diffusion coefficients of the SDE of a discounted locally optimal portfolio.

Theorem 14.3.3. *Under Assumption 14.3.2, any discounted locally optimal portfolio* $\bar{S}^\delta$ *satisfies in a JDM the SDE*

$$d\bar{S}_t^\delta = \bar{S}_t^\delta \frac{\left(1 - \pi_{\delta,t}^0\right)}{G(t)} \sum_{k=1}^{d} \theta_t^k \left(\theta_t^k \, dt + dW_t^k\right), \tag{14.3.10}$$

with optimal fractions

$$\pi_{\delta,t}^j = \frac{\left(1 - \pi_{\delta,t}^0\right)}{G(t)} \sum_{k=1}^{d} \theta_t^k \, b_t^{-1\,j,k} \tag{14.3.11}$$

for all $t \in [0, \infty)$ *and* $j \in \{1, 2, \ldots, d\}$.

The proof of this theorem is analogous to that of Theorem 11.1.3. It is, therefore, omitted, but it can be found in Platen (2006b). According to Theorem 14.3.3, the family of discounted locally optimal portfolios is characterized by a single parameter process, namely the fraction of wealth $\pi^0_{\delta,t}$ held in the savings account at time t. However, we shall see that, in general, it is not the GOP which arises as the mutual risky portfolio in the resulting two fund separation.

Mutual Fund

Let us select a particular locally optimal portfolio $S^{\delta_{\rm MF}}$, which we call the *mutual fund* (MF), by choosing

$$\pi^0_{\delta_{\rm MF},t} = 1 - G(t) \tag{14.3.12}$$

for $t \in [0, \infty)$. By (14.3.10) the MF satisfies the SDE

$$dS^{\delta_{\rm MF}}_t = S^{\delta_{\rm MF}}_{t-} \left(r_t \, dt + \sum_{k=1}^{d} \theta^k_t \left(\theta^k_t \, dt + dW^k_t \right) \right) \tag{14.3.13}$$

for $t \in [0, \infty)$. Note that this SDE is very similar to that of a GOP in a CFM, see (10.2.8). However, in general, it is not the same SDE in the given JDM, as we shall see below.

By Theorem 14.3.3 it follows that any locally optimal portfolio S^δ can be obtained at any time by investing a fraction of wealth in the MF $S^{\delta_{\rm MF}}$ and holding the remaining fraction in the savings account. Therefore, Theorem 14.3.3 can be interpreted as a *mutual fund theorem*, see Merton (1973a). In this sense we have again two fund separation, see Corollary 11.1.4. The main difference here compared to the previous result obtained under a CFM is that the MF in a JDM, in general, does not coincide with the GOP. This can be seen when comparing the SDE (14.1.19) for the GOP and the SDE (14.3.13) for the MF. The MF coincides in a JDM with the GOP only if the market prices of event risk $\theta^{m+1}_t, \ldots, \theta^d_t$ are zero. Thus, mean-variance or Sharpe ratio maximization does, in general, not provide two fund separation into GOP and savings account. Further results in this direction can be found in Platen (2006b) and Christensen & Platen (2007).

For locally optimal portfolios the up and down movements of asset prices are weighted symmetrically by generalized diffusion coefficients. This is sufficient in a CFM for the purpose of identifying a superior asset allocation. For a practically useful portfolio selection in a JDM one needs to take into account the entire range of possible asset price jumps. Upward jumps are favorable for the investor, however, downward jumps can be disastrous. This asymmetric weighting of jumps can be conveniently modeled by utility functions.

The maximization of expected utility appears to be a useful objective in a JDM. In Sect. 11.3 we maximized expected utility from discounted terminal wealth for a CFM. The extension of this result to the case of a JDM is beyond the scope of this book.

14.4 Real World Pricing for Two Market Models

This section considers two examples of JDMs, a *Merton model* (MM) and a minimal market model with jumps (MMM). For both models real world pricing for some common payoffs is applied along the lines of results in Hulley, Miller & Platen (2005).

In the MM case, our aim is to illustrate how real world pricing retrieves the risk neutral prices for these instruments familiar from the literature. Of course, one could apply the standard risk neutral theory to obtain the pricing formulas under the MM, but this would defeat our purpose of illustrating real world pricing under the benchmark approach. In the case of the MMM, we wish to exhibit derivative pricing formulas where risk neutral pricing is not applicable and for what we believe is a more realistic market model.

Specifying a Continuous GOP

In a JDM $\mathcal{S}^{\mathrm{JD}}_{(d)}$ let us interpret the GOP as a large diversified portfolio that is expressed in units of, say, US dollars, $d \in \mathcal{N}$. One may think of a diversified market portfolio or market index. Then aggregating all the jumps in the underlying primary security accounts is assumed to produce noise which is approximately continuous. In other words, we would expect the jumps to be invisible to an observer of the GOP. According to the SDE (14.1.19), the only way to eliminate jumps from the GOP dynamics is by setting the market prices of event risk equal to zero. This is a key assumption that has been used in Merton (1976) for the MM. Of course, small jumps can be asymptotically modeled by some Wiener processes. Henceforth, the following simplifying assumption will be used.

Assumption 14.4.1. *The market prices of event risks are zero, that is*

$$\theta^k_t = 0, \tag{14.4.1}$$

for each $k \in \{m+1, \ldots, d\}$ *and all* $t \in [0, \infty)$.

Note that there is technically no problem to extend the following examples to the case of nonzero market prices of event risk. Substitution of (14.4.1) into (14.1.19) produces the following SDE for the GOP

$$dS^{\delta_*}_t = S^{\delta_*}_t \left(r_t\, dt + \sum_{k=1}^{m} \theta^k_t \left(\theta^k_t\, dt + dW^k_t \right) \right), \tag{14.4.2}$$

for all $t \in [0, \infty)$, with

$$S^{\delta_*}_0 = 1. \tag{14.4.3}$$

The solution to (14.4.2) is given by

$$S^{\delta_*}_t = \exp\left\{ \int_0^t \left(r_s + \frac{1}{2} \sum_{k=1}^{m} \left(\theta^k_s\right)^2 \right) ds + \sum_{k=1}^{m} \int_0^t \theta^k_s\, dW^k_s \right\}, \tag{14.4.4}$$

for all $t \in [0, T]$.

Benchmarked Primary Security Accounts

The SDEs for the benchmarked primary security accounts are derived from (14.2.7) by setting $\pi^j_{\delta,t} = 1$ for $i = j$ and $\pi^i_{\delta,t} = 0$ otherwise, yielding

$$d\hat{S}^j_t = -\hat{S}^j_{t-} \sum_{k=1}^{d} \sigma^{j,k}_t \, dW^k_t, \tag{14.4.5}$$

for all $j \in \{0, 1, \ldots, d\}$ and $t \in [0, \infty)$, with $\hat{S}^j_0 = S^j_0$. Here in our JDM we have set $\sigma^{j,k}_t = \sigma^{j,k}_{(d)}(t)$ for all $j \in \{0, 1, \ldots, d\}$, $k \in \{1, 2, \ldots, d\}$ and $t \in [0, \infty)$. Recall that $W^{m+1}, \ldots, W^d$ are compensated, normalized jump martingales with corresponding intensity processes $h^1, \ldots, h^{d-m}$, respectively. From (14.4.5), via the Itô formula we obtain, see Sect. 6.4, the explicit expression

$$\hat{S}^j_t = S^j_0 \exp\left\{ -\frac{1}{2} \int_0^t \sum_{k=1}^{m} \left(\sigma^{j,k}_s\right)^2 ds - \sum_{k=1}^{m} \int_0^t \sigma^{j,k}_s \, dW^k_s \right\}$$
$$\times \exp\left\{ \int_0^t \sum_{k=m+1}^{d} \sigma^{j,k}_s \sqrt{h^{k-m}_s} \, ds \right\} \prod_{k=m+1}^{d} \prod_{l=1}^{p^{k-m}_t} \left(1 - \frac{\sigma^{j,k}_{\tau^k_l -}}{\sqrt{h^{k-m}_{\tau^k_l -}}} \right) \tag{14.4.6}$$

for each $j \in \{0, 1, \ldots, d\}$ and all $t \in [0, \infty)$. Here $(\tau^k_l)_{l \in \mathcal{N}}$ denotes the sequence of jump times of the counting process p^k for the events of kth type, $k \in \{m+1, \ldots, d\}$.

Under the benchmark approach the benchmarked primary security accounts are the pivotal objects of study. The savings account together with the benchmarked primary security accounts are sufficient to specify the entire investment universe, see (14.2.28)–(14.2.29). For example, $S^{\delta_*}_t = \frac{S^0_t}{\hat{S}^0_t}$, for all $t \in [0, \infty)$, see (14.1.24), derives the GOP in terms of the savings account and the benchmarked savings account. Also, $S^j_t = \hat{S}^j_t S^{\delta_*}_t = \hat{S}^j_t \frac{S^0_t}{\hat{S}^0_t}$, for each $j \in \{1, \ldots, d\}$ and all $t \in [0, \infty)$, factors each primary security account in terms of the corresponding benchmarked primary security account, the savings account and the benchmarked savings account.

Before presenting the MM and the MMM we introduce some simplifying notation. Define the processes $|\sigma^j| = \{|\sigma^j_t|, t \in [0, \infty)\}$ for $j \in \{0, 1, \ldots, d\}$, by setting

$$|\sigma^j_t| = \sqrt{\sum_{k=1}^{m} \left(\sigma^{j,k}_t\right)^2}. \tag{14.4.7}$$

We also require the *aggregate continuous noise processes* $\hat{W}^j = \{\hat{W}^j_t, t \in [0, \infty)\}$ for $j \in \{0, 1, \ldots, d\}$, defined by

$$\hat{W}^j_t = \sum_{k=1}^{m} \int_0^t \frac{\sigma^{j,k}_s}{|\sigma^j_s|} \, dW^k_s. \tag{14.4.8}$$

By Lévy's Theorem for the characterization of the Wiener process, see Sect. 6.5, it follows that $\hat{W}^j$ is a Wiener process for each $j \in \{0, 1, \dots, d\}$. Note that these Wiener processes can be correlated. Furthermore, we require Assumption 14.1.2, such that the generalized volatility matrix $\boldsymbol{b}_t = [b_t^{j,k}]_{j,k=1}^d$ is for all $t \in [0, \infty)$ invertible. Recall by (14.2.5) that

$$b_t^{j,k} = \theta_t^k - \sigma_t^{j,k} \tag{14.4.9}$$

for $k \in \{1, 2, \dots, m\}$ and by (14.4.1) and (14.2.5) that

$$b_t^{j,k} = -\sigma_t^{j,k} \tag{14.4.10}$$

for $k \in \{m+1, \dots, d\}$, $j \in \{1, 2, \dots, d\}$ and $t \in [0, \infty)$.

In both models presented in this section we assume, for simplicity, that the parameters governing their jump behavior are constant. Thus, the counting processes p^k are, in fact, time homogenous Poisson processes with constant intensities, such that

$$h_t^k = h^k > 0 \tag{14.4.11}$$

for each $k \in \{1, 2, \dots, d-m\}$ and all $t \in [0, \infty)$. Also, the jump ratios $\sigma_t^{j,k}$ for the benchmarked primary security accounts are assumed to be constant, and so that

$$\sigma_t^{j,k} = \sigma^{j,k} \le \sqrt{h^{k-m}} \tag{14.4.12}$$

for all $j \in \{0, 1, \dots, d\}$, $k \in \{m+1, \dots, d\}$ and $t \in [0, \infty)$. Note that Assumption 14.4.1 on zero market prices of event risk ensures that (14.4.11) does not violate Assumption 14.1.3. Also, Assumption 14.4.1 and relation (14.2.5) ensure that (14.4.12) satisfies Assumption 14.1.1.

Using (14.4.7)–(14.4.12), we can rewrite the benchmarked jth primary security account in (14.4.6) as the product

$$\hat{S}_t^j = \hat{S}_t^{j,\mathrm{c}}\, S_t^{j,\mathrm{d}} \tag{14.4.13}$$

with continuous part

$$\hat{S}_t^{j,\mathrm{c}} = S_0^j \exp\left\{ -\frac{1}{2} \int_0^t |\sigma_s^j|^2\, ds - \int_0^t |\sigma_s^j|\, d\hat{W}_s^j \right\} \tag{14.4.14}$$

and compensated jump part

$$S_t^{j,\mathrm{d}} = \exp\left\{ \sum_{k=m+1}^d \sigma^{j,k} \sqrt{h^{k-m}}\, t \right\} \prod_{k=m+1}^d \left(1 - \frac{\sigma^{j,k}}{\sqrt{h^{k-m}}} \right)^{p_t^{k-m}} \tag{14.4.15}$$

for each $j \in \{0, 1, \dots, d\}$ and all $t \in [0, \infty)$. The two specific models for the benchmarked primary security accounts, which we now present, differ in terms of how the continuous processes (14.4.14) are modeled. The jump processes (14.4.15) are, for simplicity, chosen to be the same in both cases. Forthcoming work will model stochastic intensities in natural extensions of the MMM.

The Merton Model

The Merton model (MM) is the standard market model when including event risk with all parameters constant. We describe now a modification of the jump diffusion model introduced in Merton (1976), see Sect. 7.6. Each benchmarked primary security account can be expressed as the product of a driftless geometric Brownian motion and an independent jump martingale. Therefore, it is itself a martingale. The MM arises if one assumes that all parameter processes, that is, the short rate, the volatilities and the jump intensities, are constant. In addition to (14.4.11) and (14.4.12) we have then $r_t = r$ and $\sigma_t^{j,k} = \sigma^{j,k}$ for each $j \in \{0, 1, \ldots, d\}$, $k \in \{1, 2, \ldots, m\}$ and $t \in [0, \infty)$. In this case (14.4.14) can be written as

$$\hat{S}_t^{j,\mathrm{c}} = S_0^j \exp\left\{ -\frac{1}{2} \, |\sigma^j|^2 \, t - |\sigma^j| \, \hat{W}_t^j \right\} \tag{14.4.16}$$

for each $j \in \{0, 1, \ldots, d\}$ and all $t \in [0, \infty)$. In this special case, the benchmarked primary security accounts are the products of driftless geometric Brownian motions and compensated Poisson processes. The model is similar to that introduced in Samuelson (1965b), which was extended in Merton (1976) to include jumps. We refer to this model as the Merton model (MM). It is sometimes also called the Merton jump diffusion model.

By Assumption 14.4.1 and relations (14.4.13)–(14.4.15), the benchmarked savings account $\hat{S}^0$ exhibits no jumps. Furthermore, $\hat{S}^0$ satisfies Novikov's condition, see (9.5.12), and is, thus, a continuous martingale. Consequently, with this specification of the market, the benchmarked savings account is a Radon-Nikodym derivative process and an $(\underline{\mathcal{A}}, P)$-martingale. Therefore, Girsanov's theorem, see Sect. 9.5, is applicable, and so the standard risk neutral pricing approach can be used. While not advocating the MM as an accurate description of observed market behavior, its familiarity makes it useful for illustrating real world pricing under the benchmark approach.

A Minimal Market Model with Jumps

The *minimal market model* (MMM) is generalized here to a case with jumps. For simplicity, we suppose the parameters associated with the jump parts of the benchmarked primary security accounts to be constant. Their continuous parts are modeled as inverted time transformed squared Bessel processes of dimension four. Consequently, each benchmarked primary security account is the product of an inverted, time transformed squared Bessel process of dimension four and an independent jump martingale. Since inverted squared Bessel processes of dimension four are strict local martingales, see (8.7.21), the benchmarked savings account is not a martingale in the MMM, and hence a viable equivalent risk neutral probability measure does not exist. We advocate real world pricing for derivatives using the GOP as numeraire and the real world probability measure as pricing measure.

Without imposing significant constraints on the parameter processes, and working within the full generality of Sect. 14.1, we have shown in Sect. 13.2 that the discounted GOP follows a time transformed squared Bessel process of dimension four. Since the discounted GOP is given by $\frac{S_t^{\delta_*}}{S_t^0} = \frac{1}{\hat{S}_t^0}$ for all $t \in [0, \infty)$, it follows that the benchmarked savings account is an inverted time transformed squared Bessel process of dimension four. A version of the MMM for the continuous part of the benchmarked primary security accounts, see Sect. 13.2, is obtained by modeling the resulting time transformations as exponential functions. We provide here an outline of this model in the context of this section. For further details we refer to Chap. 13 or Hulley et al. (2005).

For each $j \in \{0, 1, \ldots, d\}$, let $\eta^j \in \Re$ and define the function $\alpha^j : \Re^+ \to \Re^+$ by setting

$$\alpha^j(t) = \alpha_0^j \exp\{\eta^j t\} \tag{14.4.17}$$

for all $t \in [0, \infty)$ with $\alpha_0^j > 0$. We refer to η^j again as the net growth rate of the jth primary security account, for $j \in \{0, 1, \ldots, d\}$. Next, we define the jth square root process $Y^j = \{Y_t^j, t \in [0, \infty)\}$ for $j \in \{0, 1, \ldots, d\}$, through the system of SDEs

$$dY_t^j = \left(1 - \eta^j Y_t^j\right) dt + \sqrt{Y_t^j}\, d\hat{W}_t^j \tag{14.4.18}$$

for each $j \in \{0, 1, \ldots, d\}$ and all $t \in [0, \infty)$, with $Y_0^j = \frac{1}{\alpha_0^j S_0^j}$. The continuous parts $\hat{S}_t^{j,c}$ of the benchmarked primary security accounts (14.4.14) are modeled in terms of these square root processes by setting

$$\hat{S}_t^{j,c} = \frac{1}{\alpha^j(t) Y_t^j} \tag{14.4.19}$$

for each $j \in \{0, 1, \ldots, d\}$ and all $t \in [0, \infty)$. Since (14.4.19) combined with (14.4.13) and (14.4.14) represents a version of the MMM for benchmarked primary security accounts we shall henceforth refer to it as such in this section.

As previously mentioned, between jumps the benchmarked primary security accounts are inverted time transformed squared Bessel processes of dimension four. The time transformations are deterministic in the given version of the MMM. More precisely, define the continuous strictly increasing functions $\varphi^j : \Re^+ \to \Re^+$ for $j \in \{0, 1, \ldots, d\}$ by setting

$$\varphi^j(t) = \varphi_0^j + \frac{1}{4} \int_0^t \alpha^j(s)\, ds \tag{14.4.20}$$

for each $j \in \{0, 1, \ldots, d\}$ and all $t \in [0, \infty)$ with $\varphi_0^j \in \Re^+$. Continuity and monotonicity imply that φ^j possesses an inverse $(\varphi^j)^{-1} : [\varphi_0^j, \infty) \to \Re^+$ for each $j \in \{0, 1, \ldots, d\}$. Now define the processes $X^j = \{X_\varphi^j, \varphi \in [\varphi_0^j, \infty)\}$ for each $j \in \{0, 1, \ldots, d\}$ by setting

$$X^j_{\varphi^j(t)} = \alpha^j(t) Y^j_t = \frac{1}{\hat{S}^{j,c}_t} \tag{14.4.21}$$

for each $j \in \{0, 1, \ldots, d\}$ and all $t \in [0, \infty)$. It then follows, see Sect. 8.7, that X^j is a squared Bessel process of dimension four, so that $\frac{1}{\hat{S}^{j,c}}$ is such time transformed squared Bessel process under the time transformation $(\varphi^j)^{-1}$ for each $j \in \{0, 1, \ldots, d\}$.

Under the MMM the benchmarked savings account is a strict local martingale, and hence a strict supermartingale, see Lemma 5.2.2 (i). This is also the candidate Radon-Nikodym derivative process employed by Girsanov's theorem to transform from the real world probability measure P to a hypothetical equivalent risk neutral probability measure, see Sects. 9.4 and 13.3. However, the fact that the candidate Radon-Nikodym derivative is not an $(\underline{\mathcal{A}}, P)$-martingale rules out this measure transformation. Consequently, risk neutral derivative pricing is impossible within the MMM, and we shall resort to the more general real world pricing under the benchmark approach. Chapter 13 showed that the MMM is attractive for a number of reasons. In particular, it follows from economic reasoning when using the discounted GOP drift as the main parameter process. The modest number of parameters employed makes it a practical tool.

Zero Coupon Bonds

We first consider a standard default-free zero coupon bond, paying one unit of the domestic currency at its maturity $T \in [0, \infty)$. According to the real world pricing formula (14.1.28), the value of the zero coupon bond at time t is given by

$$P(t,T) = S^{\delta_*}_t \, E\left(\frac{1}{S^{\delta_*}_T} \,\middle|\, \mathcal{A}_t \right) = \frac{1}{\hat{S}^0_t} \, E\left(\exp\left\{ - \int_t^T r_s \, ds \right\} \hat{S}^0_T \,\middle|\, \mathcal{A}_t \right) \tag{14.4.22}$$

for all $t \in [0, T]$. We shall now examine (14.4.22) under the two market models outlined above.

In the MM case, since $\hat{S}^0$ is an $(\underline{\mathcal{A}}, P)$-martingale we obtain

$$P(t,T) = \exp\{-r(T-t)\} \frac{1}{\hat{S}^0_t} \, E\left(\hat{S}^0_T \,\middle|\, \mathcal{A}_t \right) = \exp\{-r(T-t)\} \tag{14.4.23}$$

for all $t \in [0, T]$. In other words, we obtain the usual bond pricing formula determined by discounting at the short rate. This is fully in line with the results under risk neutral pricing, see Sect. 9.4.

To simplify the notation let us set in the MMM case

$$\lambda^j_t = \frac{1}{\hat{S}^j_t (\varphi^j(t) - \varphi^j(T))} \tag{14.4.24}$$

for $t \in [0,T]$ and $j \in \{0,1,\ldots,d\}$, where $\lambda^j_T = \infty$. It is argued in Miller & Platen (2005), with some empirical support, that the interest rate process and the discounted GOP can be assumed to be independent. If we accept this, and apply it in the MMM case to (14.4.22), while remembering that $\hat{S}^0_T = \hat{S}^{0,\mathrm{c}}_T$, we obtain

$$P(t,T) = E\left(\exp\left\{-\int_t^T r_s\,ds\right\}\,\middle|\,\mathcal{A}_t\right)\frac{1}{\hat{S}^0_t}\,E\left(\hat{S}^0_T\,\middle|\,\mathcal{A}_t\right)$$

$$= E\left(\exp\left\{-\int_t^T r_s\,ds\right\}\,\middle|\,\mathcal{A}_t\right)\left(1-\exp\left\{-\frac{1}{2}\lambda^0_t\right\}\right) \tag{14.4.25}$$

for all $t \in [0,T]$, from (8.7.23) and (14.4.24).

Forward Contracts

In this subsection we fix $j \in \{0,1,\ldots,d\}$, $T \in [0,\infty)$ and $t \in [0,T]$. Consider now a *forward contract*, see (10.4.26), with the delivery of one unit of the jth primary security account at the maturity date T, which is written at time $t \in [0,T]$. The value of the forward contract at the writing time t is defined to be zero. According to the real world pricing formula (14.1.28) the *forward price* $F^j(t,T)$ at time $t \in [0,T]$ for this contract is then determined by the relation

$$S^{\delta_*}_t\,E\left(\frac{F^j(t,T)-S^j_T}{S^{\delta_*}_T}\,\middle|\,\mathcal{A}_t\right) = 0. \tag{14.4.26}$$

By (14.4.22), solving this equation yields the forward price

$$F^j(t,T) = \frac{S^{\delta_*}_t\,E\left(\hat{S}^j_T\,\middle|\,\mathcal{A}_t\right)}{S^{\delta_*}_t\,E\left(\frac{1}{S^{\delta_*}_T}\,\middle|\,\mathcal{A}_t\right)} = \begin{cases}\frac{S^j_t}{P(t,T)}\,\frac{1}{\hat{S}^j_t}\,E\left(\hat{S}^j_T\,\middle|\,\mathcal{A}_t\right) & \text{if } S^j_t > 0\\ 0 & \text{if } S^j_t = 0\end{cases} \tag{14.4.27}$$

for all $t \in [0,T]$.

In the MM case, with reference to (14.4.16), the same argument, which established that the benchmarked savings account is a continuous martingale, also applies to the driftless geometric Brownian motion $\hat{S}^{j,\mathrm{c}}$, while the compensated Poisson process $\hat{S}^{j,\mathrm{d}}$ is a jump martingale. Consequently, $\hat{S}^j$ is the product of independent martingales, and hence itself an $(\underline{\mathcal{A}},P)$-martingale. Together with (14.4.23) this enables us to write the forward price (14.4.27) as

$$F^j(t,T) = S^j_t\,\exp\{r(T-t)\} \tag{14.4.28}$$

for all $t \in [0,T]$. Thus, in the MM case we recover the standard expression for the forward price, see, for instance, Musiela & Rutkowski (2005).

In the MMM case, according to (14.4.21), $\hat{S}^{j,\mathrm{c}}$ is an inverted time transformed squared Bessel process of dimension four, while $S^{j,\mathrm{d}}$ is an independent jump martingale, as before. Thus, we obtain

$$\frac{1}{\hat{S}^j_t} E\left(\hat{S}^j_T \,\Big|\, \mathcal{A}_t\right) = \frac{1}{\hat{S}^{j,c}_t} E\left(\hat{S}^{j,c}_T \,\Big|\, \mathcal{A}_t\right) \frac{1}{S^{j,d}_t} E\left(S^{j,d}_T \,\Big|\, \mathcal{A}_t\right) = 1 - \exp\left\{-\frac{1}{2}\,\lambda^j_t\right\} \tag{14.4.29}$$

for all $t \in [0,T]$, by (8.7.23) and (14.4.24). Putting (14.4.27) together with (14.4.25) and (14.4.29) gives for the forward price the formula

$$F^j(t,T) = S^j_t \, \frac{1 - \exp\left\{-\frac{1}{2}\,\lambda^j_t\right\}}{1 - \exp\left\{-\frac{1}{2}\,\lambda^0_t\right\}} \left(E\left(\exp\left\{-\int_t^T r_s\,ds\right\} \,\Big|\, \mathcal{A}_t\right)\right)^{-1} \tag{14.4.30}$$

for all $t \in [0,T]$. This demonstrates that the forward price of a primary security account is a tractable quantity under the MMM.

Asset-or-Nothing Binaries

Binary options may be regarded as basic building blocks for complex derivatives. This has been exploited in a recent approach to the valuation of exotic options, where a complex payoff is decomposed into a series of binaries, see Ingersoll (2000), Buchen (2004) and Buchen & Konstandatos (2005).

In this subsection we again fix $j \in \{0,1,\ldots,d\}$ and consider a derivative contract, with maturity T and strike $K \in \Re^+$, on the jth primary security account. We also fix $k \in \{m+1,\ldots,d\}$ and assume that $\sigma^{j,k} \neq 0$ and $\sigma^{j,l} = 0$, for each $l \in \{m+1,\ldots,d\}$ with $l \neq k$. In other words, we assume that the jth primary security account responds only to the $(k-m)$th jump process. This does not affect the generality of our calculations below, but it does result in more manageable expressions. In addition, we shall assume a constant interest rate throughout the rest of this section, so that $r_t = r$, for all $t \in [0,T]$. Although this is already the case for the MM, we now require it to obtain also a compact pricing formula under the MMM.

The derivative contract under consideration is an *asset-or-nothing binary* on the jth primary security account. At its maturity T it pays its holder one unit of the jth primary security account if this is greater than the strike K, and nothing otherwise. According to the real world pricing formula (14.1.28), its value is given by

$$\begin{aligned} A^{j,k}(t,T,K) &= S^{\delta_*}_t \, E\left(\mathbf{1}_{\{S^j_T \geq K\}} \, \frac{S^j_T}{S^{\delta_*}_T} \,\Bigg|\, \mathcal{A}_t\right) \\ &= \frac{S^j_t}{\hat{S}^j_t} \, E\left(\mathbf{1}_{\{\hat{S}^j_T \geq K (S^0_T)^{-1} \hat{S}^0_T\}} \, \hat{S}^j_T \,\Big|\, \mathcal{A}_t\right) \\ &= \frac{S^j_t}{\hat{S}^{j,c}_t} \, E\Bigg(\mathbf{1}_{\{\hat{S}^{j,c}_T \geq g(p^{k-m}_T - p^{k-m}_t)\hat{S}^0_T\}} \\ &\quad \times \exp\left\{\sigma^{j,k}\sqrt{h^{k-m}}\,(T-t)\right\}\left(1 - \frac{\sigma^{j,k}}{\sqrt{h^{k-m}}}\right)^{p^{k-m}_T - p^{k-m}_t} \hat{S}^{j,c}_T \,\Bigg|\, \mathcal{A}_t\Bigg) \end{aligned}$$

$$= \sum_{n=0}^{\infty} \exp\{-h^{k-m}(T-t)\} \frac{(h^k(T-t))^n}{n!} \exp\left\{\sigma^{j,k}\sqrt{h^{k-m}}(T-t)\right\}$$
$$\times \left(1 - \frac{\sigma^{j,k}}{\sqrt{h^{k-m}}}\right)^n \frac{S_t^j}{\hat{S}_t^{j,c}} E\left(\mathbf{1}_{\{\hat{S}_T^{j,c} \geq g(n)\hat{S}_T^0\}} \hat{S}_T^{j,c} \,\Big|\, \mathcal{A}_t\right) \quad (14.4.31)$$

for all $t \in [0,T]$, where

$$g(n) = \frac{K}{S_t^0\, S_t^{j,d}} \exp\left\{-\left(r + \sigma^{j,k}\sqrt{h^{k-m}}\right)(T-t)\right\}\left(1 - \frac{\sigma^{j,k}}{\sqrt{h^{k-m}}}\right)^{-n} \quad (14.4.32)$$

for all $n \in \mathcal{N}$.

In the MM case, (14.4.31) yields the following explicit formula:

$$A^{j,k}(t,T,K) = \sum_{n=0}^{\infty} \exp\left\{-h^{k-m}(T-t)\right\} \frac{(h^{k-m}(T-t))^n}{n!}$$
$$\times \exp\left\{\sigma^{j,k}\sqrt{h^{k-m}}(T-t)\right\}\left(1 - \frac{\sigma^{j,k}}{\sqrt{h^{k-m}}}\right)^n S_t^j\, N(d_1(n)) \quad (14.4.33)$$

for all $t \in [0,T]$, where

$$d_1(n) = \frac{\ln\left(\frac{S_t^j}{K}\right) + \left(r + \sigma^{j,k}\sqrt{h^{k-m}} + n\frac{\ln\left(1 - \frac{\sigma^{j,k}}{\sqrt{h^{k-m}}}\right)}{T-t} + \frac{1}{2}\left(\hat{\sigma}^{0,j}\right)^2\right)(T-t)}{\hat{\sigma}^{0,j}\sqrt{T-t}} \quad (14.4.34)$$

for each $n \in \mathcal{N}$. Here $N(\cdot)$ is the Gaussian distribution function. Deriving (14.4.33) is the subject of Exercise 14.3. In (14.4.34) we employ the following notation

$$\hat{\sigma}^{i,j} = \sqrt{|\sigma^i|^2 - 2\,\varrho^{i,j}\,|\sigma^i|\,|\sigma^j| + |\sigma^j|^2} \quad (14.4.35)$$

for $i, j \in \{0, 1, \ldots, d\}$, where $\varrho^{i,j}$ is the correlation between the Wiener processes $\hat{W}^i$ and $\hat{W}^j$.

For the MMM case, as we have just seen, calculating the price of a payoff written on a primary security account requires the evaluation of a double integral involving the transition density of a two-dimensional process. This is a consequence of choosing the GOP as numeraire. Closed form derivative pricing formulas can be obtained for the MM, but in the case of the MMM this is more difficult, because the joint transition densities of two squared Bessel processes are, in general, difficult to describe, see Bru (1991). A natural response to this is to solve the partial integro differential equation (PIDE) associated with the derivative price numerically by finite difference methods or Monte Carlo simulation as will be described in Chap. 15. However, to give the reader a feeling for the types of formulas that emerge from applying real world pricing in the MMM, we shall now assume, for simplicity, that the processes

$\hat{S}^0$ and $\hat{S}^{j,c}$ are independent, which is also a reasonable assumption in many practical situations. Combining (14.4.31) and (14.4.32), and remembering that $\hat{S}^0 = \hat{S}^{0,c}$, results in the formula

$$A^{j,k}(t,T,K) = \sum_{n=0}^{\infty} \exp\left\{-h^{k-m}(T-t)\right\} \frac{(h^{k-m}(T-t))^n}{n!} \times \exp\left\{\sigma^{j,k}\sqrt{h^{k-m}}(T-t)\right\}\left(1-\frac{\sigma^{j,k}}{\sqrt{h^{k-m}}}\right)^n \times S_t^j\left(G''_{0,4}\left(\frac{\varphi^0(T)-\varphi^0(t)}{g(n)};\lambda_t^j,\lambda_t^0\right) - \exp\left\{-\frac{1}{2}\lambda_t^j\right\}\right) \tag{14.4.36}$$

for all $t \in [0,T]$, $k \in \{m+1,\ldots,d\}$, see Exercise 14.5. Here $G''_{0,4}(x;\lambda,\lambda')$ equals the probability $P(\frac{Z}{Z'} \le x)$ for the ratio $\frac{Z}{Z'}$ of a non-central chi-square distributed random variable $Z \sim \chi^2(0,\lambda)$ with degrees of freedom zero and non-centrality parameter $\lambda > 0$, and a non-central chi-square distributed random variable $Z' \sim \chi^2(4,\lambda')$ with four degrees of freedom and noncentrality parameter λ'. By implementing this special function one obtains the pricing formula given in (14.4.36), see Johnson et al. (1995) and Hulley et al. (2005).

Bond-or-Nothing Binaries

In this subsection we price a *bond-or-nothing binary*, which pays the strike $K \in \Re^+$ at maturity T, when the jth primary security account at time T is not less than K, where $j \in \{0,1,\ldots,d\}$ is still fixed. As before, let us assume that the jth primary security account only responds to the kth jump martingale W^k, where $k \in \{m+1,\ldots,d\}$ is fixed. We shall again require a constant interest rate for the MMM as well as the MM.

Since at its maturity the bond-or-nothing binary under consideration pays its holder the strike amount K if the value of the jth primary security account is in excess of this, and nothing otherwise, the real world pricing formula (14.1.28), yields

$$B^{j,k}(t,T,K) = S_t^{\delta_*} E\left(\mathbf{1}_{\{S_T^j \ge K\}} \frac{K}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t\right)$$

$$= K\,P(t,T) - K S_t^{\delta_*} E\left(\mathbf{1}_{\{S_T^j < K\}} \frac{1}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t\right)$$

$$= K\,P(t,T) - K \frac{S_t^0}{\hat{S}_t^0} E\left(\mathbf{1}_{\left\{\hat{S}_T^0 > K^{-1} S_T^0 \hat{S}_T^j\right\}} \frac{\hat{S}_T^0}{S_T^0} \,\middle|\, \mathcal{A}_t\right)$$

$$= K\,P(t,T) - K \exp\{-r\,(T-t)\} \frac{1}{\hat{S}_t^0} E\left(\mathbf{1}_{\left\{\hat{S}_T^0 > g(p_T^{k-m} - p_t^{k-m})^{-1} \hat{S}_T^{j,c}\right\}} \hat{S}_T^0 \,\middle|\, \mathcal{A}_t\right)$$

$$= K\,P(t,T) - K \exp\{-r\,(T-t)\} \sum_{n=0}^{\infty} \exp\{-h^{k-m}\,(T-t)\} \frac{(h^{k-m}(T-t))^n}{n!} \times \frac{1}{\hat{S}^0_t}\, E\left(\mathbf{1}_{\{\hat{S}^0_T > g(n)^{-1}\hat{S}^{j,c}_T\}} \hat{S}^0_T \,\Big|\, \mathcal{A}_t\right) \quad (14.4.37)$$

for all $t \in [0,T]$, where $g(n)$ is given by (14.4.32), for each $n \in \mathcal{N}$.

In the MM case, (14.4.37) yields the following explicit formula:

$$\begin{aligned} B^{j,k}(t,T,K) &= K \exp\{-r\,(T-t)\} \\ &\quad \times \left(1 - \sum_{n=0}^{\infty} \exp\{-h^{k-m}\,(T-t)\} \frac{(h^{k-m}\,(T-t))^n}{n!} N(-d_2(n))\right) \\ &= \sum_{n=0}^{\infty} \exp\{-h^{k-m}\,(T-t)\} \frac{(h^{k-m}\,(T-t))^n}{n!} K \exp\{-r(T-t)\}\, N(d_2(n)) \end{aligned} \quad (14.4.38)$$

for all $t \in [0,T]$, where

$$\begin{aligned} d_2(n) &= \frac{\ln\left(\frac{S^j_t}{K}\right) + \left(r + \sigma^{j,k}\sqrt{h^{k-m}} + n\frac{\ln(1-\frac{\sigma^{j,k}}{\sqrt{h^{k-m}}})}{T-t} - \frac{1}{2}\left(\hat{\sigma}^{0,j}\right)^2\right)(T-t)}{\hat{\sigma}^{0,j}\sqrt{T-t}} \\ &= d_1(n) - \hat{\sigma}^{0,j}\sqrt{T-t} \end{aligned} \quad (14.4.39)$$

for each $n \in \mathcal{N}$, see Hulley et al. (2005). Again $\hat{\sigma}^{0,j}$ is given by (14.4.35). Deriving (14.4.38) is the subject to Exercise 14.4.

For the MMM case, subject to the assumption that $\hat{S}^0_T$ and $\hat{S}^{j,c}_T$ are independent, we can combine (14.4.37), (14.4.32) and (14.4.25), to obtain

$$\begin{aligned} B^{j,k}(t,T,K) &= K \exp\{-r(T-t)\}\left(1 - \exp\left\{-\frac{1}{2}\lambda^0_t\right\}\right) \\ &\quad - \sum_{n=0}^{\infty} \exp\{-h^{k-m}\,(T-t)\} \frac{(h^{k-m}\,(T-t))^n}{n!} K \exp\{-r(T-t)\} \\ &\quad \times \left(G''_{0,4}\left((\varphi^j(T) - \varphi^j(t))\,g(n); \lambda^0_t, \lambda^j_t\right) - \exp\left\{-\frac{1}{2}\lambda^0_t\right\}\right) \\ &= \sum_{n=0}^{\infty} \exp\{-h^{k-m}(T-t)\} \frac{(h^{k-m}\,(T-t))^n}{n!} \\ &\quad \times K \exp\{-r(T-t)\}\left(1 - G''_{0,4}\left((\varphi^j(T) - \varphi^j(t))g(n); \lambda^0_t, \lambda^j_t\right)\right) \end{aligned} \quad (14.4.40)$$

for all $t \in [0,T]$, see Hulley et al. (2005). For the second equality in (14.4.40), we have once again used the fact that

$$\sum_{n=0}^{\infty} \exp\{-h^{k-m}(T-t)\} \frac{(h^{k-m}(T-t))^n}{n!}$$

is the total probability of a Poisson random variable with parameter $h^{k-m}(T-t)$. Deriving (14.4.40) is the subject of Exercise 14.6.

European Call Options

In this subsection we fix $j \in \{0,1,\ldots,d\}$ again and consider a European call option with maturity T and strike $K \in \Re^+$ on the jth primary security account. As before, we make the simplifying assumption that the jth primary security account is only sensitive to the $(k-m)$th jump process, for some fixed $k \in \{m+1,\ldots,d\}$. We also continue to use a constant interest rate for both market models. According to the real world pricing formula (14.1.28) the European call option price is given by

$$\begin{aligned} c^{j,k}_{T,K}(t) &= S^{\delta_*}_t E\left(\frac{\left(S^j_T - K\right)^+}{S^{\delta_*}_T} \,\bigg|\, \mathcal{A}_t\right) = S^{\delta_*}_t E\left(\mathbf{1}_{\{S^j_T \geq K\}} \frac{S^j_T - K}{S^{\delta_*}_T} \,\bigg|\, \mathcal{A}_t\right) \\ &= A^{j,k}(t,T,K) - B^{j,k}(t,T,K) \end{aligned} \tag{14.4.41}$$

for all $t \in [0,T]$.

For the MM case, combining (14.4.33) and (14.4.38) gives

$$\begin{aligned} c^{j,k}_{T,K}(t) =& \sum_{n=0}^{\infty} \exp\{-h^{k-m}(T-t)\} \frac{(h^{k-m}(T-t))^n}{n!} \Bigg(\exp\left\{\sigma^{j,k}\sqrt{h^{k-m}}\,(T-t)\right\} \\ &\times \left(1 - \frac{\sigma^{j,k}}{\sqrt{h^{k-m}}}\right)^n S^j_t\, N(d_1(n)) - K\exp\{-r(T-t)\} N(d_2(n))\Bigg) \end{aligned} \tag{14.4.42}$$

for all $t \in [0,T]$, where $d_1(n)$ and $d_2(n)$ are given by (14.4.34) and (14.4.39), respectively, for each $n \in \mathcal{N}$.

It is easily seen that (14.4.42) corresponds to the original pricing formula for a European call on a stock whose price follows a jump diffusion, as given in Merton (1976). The only difference is that there the jump ratios are taken to be independent log-normally distributed, while in our case they are constant. Furthermore, this formula can be used to price an option to exchange the jth primary security account for the ith primary security account. In that case, the option pricing formula obtained instead of (14.4.42) is a generalization of that given in Margrabe (1978).

In the MMM case the European call option pricing formula is obtained by subtracting (14.4.40) from (14.4.36), according to (14.4.41), yielding

$$
\begin{aligned}
c^{j,k}_{T,K}(t) = &\sum_{n=0}^{\infty} \exp\{-h^{k-m}(T-t)\} \frac{(h^{k-m}(T-t))^n}{n!} \Bigg[\exp\Big\{ \sigma^{j,k} \sqrt{h^{k-m}}\,(T-t) \Big\} \\
&\times \left(1 - \frac{\sigma^{j,k}}{\sqrt{h^{k-m}}}\right)^n S^j_t \left(G''_{0,4}\left(\frac{\varphi^0(T)-\varphi^0(t)}{g(n)}; \lambda^j_t, \lambda^0_t \right) - \exp\Big\{ -\frac{1}{2}\lambda^j_t \Big\} \right) \\
&- K \exp\{-r(T-t)\} \left(1 - G''_{0,4}\left(\left(\varphi^j(T) - \varphi^j(t)\right) g(n); \lambda^0_t, \lambda^j_t \right) \right) \Bigg] \quad (14.4.43)
\end{aligned}
$$

for all $t \in [0,T]$, where $g(n)$ is given by (14.4.32), for each $n \in \mathcal{N}$ and λ^j_t in (14.4.24).

Defaultable Zero Coupon Bonds

We have incorporated default risk in our modeling. This allows us to study the pricing of credit derivatives. Here we consider the canonical example of such a contract, namely a defaultable zero coupon bond with maturity T. To keep the analysis simple, fix $k \in \{m+1,\dots,d\}$ and assume that the bond under consideration defaults at the first jump time τ^{k-m}_1 of p^{k-m}, provided that this time is not greater than T. In other words, default occurs if and only if $\tau^{k-m}_1 \le T$, in which case τ^{k-m}_1 is the default time. As a further simplification, we assume zero recovery upon default. According to the real world pricing formula (14.1.28), the price of this instrument is given by

$$
\begin{aligned}
\tilde{P}^{k-m}(t,T) &= S^{\delta_*}_t E\left(\frac{\mathbf{1}_{\{\tau^{k-m}_1 > T\}}}{S^{\delta_*}_T} \,\Big|\, \mathcal{A}_t \right) = S^{\delta_*}_t E\left(\frac{1}{S^{\delta_*}_T} \,\Big|\, \mathcal{A}_t \right) E\left(\mathbf{1}_{\{\tau^{k-m}_1 > T\}} \,\Big|\, \mathcal{A}_t \right) \\
&= P(t,T) P(p^{k-m}_T = 0 \,|\, \mathcal{A}_t) \qquad (14.4.44)
\end{aligned}
$$

for all $t \in [0,T]$. Note that the second equality above follows from the independence of the GOP and the underlying Poisson process, see (14.4.2).

Equation (14.4.44) shows that the price of the defaultable bond can be expressed as the product of the price of the corresponding default-free bond and the conditional probability of survival. In our setup the latter may be further evaluated as

$$
\begin{aligned}
P\left(p^{k-m}_T = 0 \,\big|\, \mathcal{A}_t\right) &= E\left(\mathbf{1}_{\{p^{k-m}_t = 0\}}\, \mathbf{1}_{\{p^{k-m}_T - p^{k-m}_t = 0\}} \,\Big|\, \mathcal{A}_t \right) \\
&= \mathbf{1}_{\{p^{k-m}_t = 0\}}\, P\left(p^{k-m}_T - p^{k-m}_t = 0 \,\Big|\, \mathcal{A}_t \right) \\
&= \mathbf{1}_{\{p^{k-m}_t = 0\}}\, E\left(\exp\left\{ - \int_t^T h^{k-m}_s \, ds \right\} \,\Big|\, \mathcal{A}_t \right) \qquad (14.4.45)
\end{aligned}
$$

for all $t \in [0,T]$.

One has to combine (14.4.44) and (14.4.45) with (14.4.23) to obtain an explicit pricing formula for the defaultable bond under consideration in the MM. Similarly, one can combine (14.4.44) and (14.4.45) with (14.4.25) to obtain the pricing formula for this instrument under the MMM.

Note that the expression obtained by combining (14.4.44) and (14.4.45) is similar to the familiar formula for the price of a defaultable zero coupon bond in a simple reduced form model for credit risk, see Schönbucher (2003). However, the difference is that for this standard case expectations are computed in the literature typically with respect to an equivalent risk neutral probability measure. In particular, the survival probability is usually a risk neutral probability. In (14.4.44) and (14.4.45), however, only the real world probability measure is in evidence. The crucial advantage of the benchmark approach in such a situation is that one avoids the undesirable dichotomy of distinguishing between real world default probabilities, as determined by historical data and credit rating agencies, and hypothetical risk neutral default probabilities, as determined by observed credit spreads. Note that substantial effort has been expended on the problem of trying to reconcile real world and risk neutral probabilities of default, see, for instance, Albanese & Chen (2005). This problem is, fortunately, avoided by using the benchmark approach with real world pricing since the real world probability measure is the pricing measure.

The above two market models highlight some aspects of the benchmark approach in derivative pricing for jump diffusion markets. This methodology can be applied generally and yields for many derivative and insurance instruments explicit formulas for the MMM and its extensions.

14.5 Exercises for Chapter 14

14.1. Calculate the growth rate of a strictly positive portfolio.

14.2. Derive the forward rate equation (14.1.33) from the benchmarked zero coupon bond SDE (14.1.30).

14.3. (*) Calculate for the Merton model, given in Sect. 14.4, the price of an asset-or-nothing binary from formula (14.4.31).

14.4. (*) Calculate for the Merton model, as in Sect. 14.4, the price of a bond-or-nothing binary from formula (14.4.37).

14.5. (*) Derive for the MMM, given in Sect. 14.4, the price of an asset-or-nothing binary from formula (14.4.31).

14.6. (*) Derive for the MMM, as in Sect. 14.4, the pricing formula of a bond-or-nothing binary from formula (14.4.37).