11

Portfolio Optimization

This chapter derives and extends a range of classical results from portfolio optimization and derivative pricing in incomplete markets in the context of a CFM. First, we consider the question of how wealth should be optimally transferred into the future given the preferences of an investor. This is a central question in economics and finance and leads into the area of portfolio optimization. We shall advocate the GOP as the best long term investment. This is consistent with views formulated in Latané (1959), Breiman (1961), Hakansson (1971) and Thorp (1972).

For the case when the investment horizon is short it was pointed out in Samuelson (1963, 1969, 1979) that one should not use the GOP as the only investment. We shall show that the optimal portfolio of an investor, who maximizes an expected utility from discounted terminal wealth, can be separated into two funds, the savings account and the GOP. This generalizes earlier results in Tobin (1958b) and Sharpe (1964) to the continuous market case. Such an optimal portfolio, which invests only into the GOP and the savings account, turns out to be an efficient portfolio in a mean-variance sense, see Markowitz (1959). It has always the maximum Sharpe ratio in the sense of Sharpe (1964, 1966).

Furthermore, we generalize the intertemporal capital asset pricing model (ICAPM) derived in Merton (1973a) under very weak assumptions. Under the assumption that the fundamental relationships in the market are invariant under changes of currency denomination, it is demonstrated that the GOP matches the market portfolio.

Real world pricing emerges as the natural pricing concept when deriving for a nonreplicable payoff its utility indifference price. The resulting benchmarked prices are martingales, independent of the underlying utility of the investor. This provides a fundamental relationship between portfolio optimization and derivative pricing. The GOP is selected as numeraire and the real world probability measure is the pricing measure. The existence of an equivalent risk neutral probability measure is not required.

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11.1 Locally Optimal Portfolios

Within this section we aim to derive and generalize under weak assumptions classical results on portfolio selection, these include Sharpe ratio maximization, two fund separation, the Markowitz efficient frontier and the ICAPM.

Discounted Portfolios

Suppose investors select portfolios for the investment of their total tradable wealth, which perform better than other portfolios in a sense specified below. We aim to clarify when the MP approximates the GOP if all investors perform some form of portfolio optimization.

We assume that an investor adjusts for the time value of money by considering *discounted portfolios*, where the savings account is used for discounting. She or he can always invest in the locally riskless asset, which is the savings account, without facing short term fluctuations. When accepting short term fluctuations an investor expects a “better” portfolio performance than is provided by the savings account. Below we specify what we mean by “better” performance. Given a strictly positive portfolio $S^\delta \in \mathcal{V}^+$, its *discounted value*

$$\bar{S}_t^\delta = \frac{S_t^\delta}{S_t^0} \tag{11.1.1}$$

satisfies by (10.1.1), (10.1.14) and an application of the Itô formula the SDE

$$d\bar{S}_t^\delta = \sum_{k=1}^{d} \psi_{\delta,t}^k \left(\theta_t^k \, dt + dW_t^k\right) \tag{11.1.2}$$

with *kth diffusion coefficient*

$$\psi_{\delta,t}^k = \sum_{j=1}^{d} \delta_t^j \, \bar{S}_t^j \, b_t^{j,k} \tag{11.1.3}$$

for $k \in \{1, 2, \ldots, d\}$ and $t \in [0, \infty)$. Note that $\psi_{\delta,t}^k$ makes sense also in the case when $\bar{S}_t^\delta$ equals zero.

Obviously, by (11.1.2) and (11.1.3) the discounted portfolio process $\bar{S}^\delta$ has *discounted drift*

$$\alpha_t^\delta = \sum_{k=1}^{d} \psi_{\delta,t}^k \, \theta_t^k \tag{11.1.4}$$

at time $t \in [0, \infty)$, which measures its trend at time t. One can say that the discounted drift models the increase per unit of time of the *underlying value* of $\bar{S}^\delta$ at time t. This can be interpreted as the fundamental economic value of the portfolio, which would be visible if one were able to remove the speculative fluctuations

$$\bar{M}_t = \sum_{k=1}^{d} \int_0^t \psi_{\delta,s}^k \, dW_s^k$$

from the discounted portfolio value

$$\bar{S}_t^\delta = \bar{S}_0^\delta + \int_0^t \alpha_s^\delta \, ds + \bar{M}_t.$$

From an economic point of view $\alpha^\delta = \{\alpha_t^\delta, \, t \in [0,\infty)\}$ is a highly relevant parameter process, since it describes the average discounted wealth that underpins the market. It provides a natural link to the macro economy. We shall use the underlying value in Chap. 13 to derive a parsimonious market model.

The magnitude of the trading uncertainty of a discounted portfolio $\bar{S}^\delta$ at time $t \in [0,\infty)$ can be measured by its *aggregate diffusion coefficient*

$$\gamma_t^\delta = \sqrt{\sum_{k=1}^{d} \left(\psi_{\delta,t}^k\right)^2} \tag{11.1.5}$$

or equivalently by its *aggregate volatility*

$$b_t^\delta = \frac{\gamma_t^\delta}{\bar{S}_t^\delta} \tag{11.1.6}$$

for $\bar{S}_t^\delta > 0$. The square $(\gamma_t^\delta)^2$ of the aggregate diffusion coefficient measures the variance per unit of time of the fluctuating increments of $\bar{S}^\delta$.

Locally Optimal Portfolios

Let us identify the typical SDE of a family of portfolios that capture the objective of investors who locally in time on average prefer a larger discounted wealth increase for the same risk level. This means that these investors prefer a higher mean for the same variance. To characterize such a portfolio, which performs "better" than others in the above sense, we introduce the following definition, similar to those in Platen (2002, 2004a) and Christensen & Platen (2007).

Definition 11.1.1. *In a CFM $\mathcal{S}_{(d)}^C$ we call a strictly positive portfolio $S^{\tilde{\delta}} \in \mathcal{V}^+$* locally optimal*, if for all $t \in [0,\infty)$ and all strictly positive portfolios $S^\delta \in \mathcal{V}^+$ with given aggregate diffusion coefficient value*

$$\gamma_t^\delta = \gamma_t^{\tilde{\delta}} \tag{11.1.7}$$

it has the largest discounted drift, that is,

$$\alpha_t^\delta \le \alpha_t^{\tilde{\delta}} \tag{11.1.8}$$

almost surely.

This type of local optimality can be interpreted as a continuous time generalization of *mean-variance optimality* in the sense of Markowitz (1952, 1959). Indeed, we shall see later that a locally optimal portfolio can be shown to be an *efficient portfolio* in a generalized Markowitz mean-variance sense. A discounted, locally optimal portfolio exhibits at all times the largest trend in comparison with all other discounted strictly positive portfolios with the same aggregate diffusion coefficient and, thus, with the same risk level.

Sharpe Ratio

An important investment characteristic is the *Sharpe ratio* s_t^δ, see Sharpe (1964, 1966). It is defined for any strictly positive portfolio $S^\delta \in \mathcal{V}^+$ with positive aggregate volatility $b_t^\delta > 0$ at time t as the ratio of the *risk premium*

$$p_{S^\delta}(t) = \frac{\alpha_t^\delta}{\bar{S}_t^\delta} \tag{11.1.9}$$

over its aggregate volatility b_t^δ, see (11.1.6), that is,

$$s_t^\delta = \frac{p_{S^\delta}(t)}{b_t^\delta} = \frac{\alpha_t^\delta}{\gamma_t^\delta} \tag{11.1.10}$$

for $t \in [0, \infty)$, see (11.1.4)–(11.1.6). We observe that the Sharpe ratio equals the ratio of the discounted drift over the aggregate diffusion coefficient. Under the mean-variance approach of Markowitz, investors aim to maximize the Sharpe ratio, which in a CFM corresponds by Definition 11.1.1 to the selection of a locally optimal portfolio. Below we shall analyze Sharpe ratios of locally optimal portfolios. We show that these are greater or equal to the Sharpe ratios of other portfolios.

Portfolio Selection Theorem

In preparation for the Portfolio Selection Theorem, which we present below, let us introduce the *total market price of risk*

$$|\boldsymbol{\theta}_t| = \sqrt{\sum_{k=1}^{d} \left(\theta_t^k\right)^2} \tag{11.1.11}$$

at time $t \in [0, \infty)$, which is by (10.2.8) and (11.1.6) the aggregate volatility of the GOP. If the total market price of risk is zero, then all discounted drifts are zero and all strictly positive portfolios are, by Definition 11.1.1, locally optimal. To avoid such unrealistic dynamics we introduce the following assumption.

Assumption 11.1.2. *Assume in a CFM $\mathcal{S}_{(d)}^C$ for all $t \in [0, \infty)$ that the total market price of risk is strictly greater than zero and finite almost surely, with*

$$0 < |\boldsymbol{\theta}_t| < \infty, \tag{11.1.12}$$

and the fraction of the GOP wealth that is invested in the savings account does not equal one, that is,

$$\pi_{\delta_*,t}^0 \neq 1 \tag{11.1.13}$$

almost surely.

We now formulate a *Portfolio Selection Theorem*, see Platen (2002), which generalizes some classical results, for instance, given in Markowitz (1959), Sharpe (1964), Merton (1973a) and Khanna & Kulldorff (1999), to the case of a CFM.

Theorem 11.1.3. (Portfolio Selection Theorem) *Consider a CFM $\mathcal{S}_{(d)}^C$ satisfying Assumption 11.1.2. For any strictly positive portfolio $S^\delta \in \mathcal{V}^+$ with nonzero aggregate diffusion coefficient and aggregate volatility b_t^δ, see (11.1.6), its Sharpe ratio s_t^δ satisfies the inequality*

$$s_t^\delta \leq |\boldsymbol{\theta}_t| \tag{11.1.14}$$

for all $t \in [0, \infty)$, where equality arises when S^δ is locally optimal. Furthermore, the value $\bar{S}_t^\delta$ at time t of a discounted, locally optimal portfolio satisfies the SDE

$$d\bar{S}_t^\delta = \bar{S}_t^\delta \frac{b_t^\delta}{|\boldsymbol{\theta}_t|} \sum_{k=1}^d \theta_t^k \left(\theta_t^k\, dt + dW_t^k\right), \tag{11.1.15}$$

with fractions

$$\pi_{\delta,t}^j = \frac{b_t^\delta}{|\boldsymbol{\theta}_t|} \pi_{\delta_*,t}^j \tag{11.1.16}$$

for all $j \in \{1, 2, \ldots, d\}$ and $t \in [0, \infty)$. Each discounted portfolio that satisfies an SDE of the type (11.1.15) is a locally optimal portfolio.

The proof of this theorem is given at the end of this section and can be found in Platen (2002). It exploits the fact that at any time t the fractions of the family of discounted, locally optimal portfolios $\bar{S}^\delta$ can be parameterized by the aggregate volatility b_t^δ. Obviously, for $b_t^\delta = 0$ one obtains the savings account as locally optimal portfolio, whereas in the case $b_t^\delta = |\boldsymbol{\theta}_t|$ it is the GOP that arises.

Note that we would have obtained equivalent results if we searched for the family of portfolios that minimizes the aggregate diffusion coefficient for given discounted drift. Similarly, we could have minimized the aggregate volatility for given risk premium. Furthermore, we shall show in Sect. 11.3 that also expected utility maximization leads to locally optimal portfolios. This robustness of portfolio optimization in a CFM is very satisfying, because it demonstrates the equivalence of several seemingly different objectives.

Two Fund Separation and Fractional Kelly Strategies

By analyzing the structure of the fractions of a locally optimal portfolio, as given in (11.1.16), and applying (10.1.13) we obtain for the fraction of wealth held in the GOP the expression

$$\frac{b_t^\delta}{|\boldsymbol{\theta}_t|} = \frac{1-\pi_{\delta,t}^0}{1-\pi_{\delta_*,t}^0} \tag{11.1.17}$$

for $t \in [0,\infty)$. This leads directly to the following result.

Corollary 11.1.4. *Under the assumptions of Theorem 11.1.3, any locally optimal portfolio $S^\delta \in \mathcal{V}^+$ can be decomposed at time t into a fraction of wealth $\frac{b_t^\delta}{|\boldsymbol{\theta}_t|}$ that is invested in the GOP and a remaining fraction that is held in the savings account. In particular, one has*

$$\pi_{\delta,t}^0 = 1 - \frac{b_t^\delta}{|\boldsymbol{\theta}_t|}\left(1-\pi_{\delta_*,t}^0\right) \tag{11.1.18}$$

for all $t \in [0,\infty)$.

Theorem 11.1.3 can be interpreted as a *Two Fund Separation Theorem*, since only the two funds; the GOP and the savings account, are involved when forming locally optimal portfolios. Such an investment strategy is also known as a *fractional Kelly strategy*, see Kelly (1956), Latané (1959), Thorp (1972) and Hakansson & Ziemba (1995). When all wealth is invested in the GOP, then this corresponds to the *Kelly strategy*. Results on two fund separation go back to Tobin (1958b), Breiman (1960), Sharpe (1964), Merton (1973a), Khanna & Kulldorff (1999) and Nielsen & Vassalou (2004). An investor, who forms with her or his total tradable wealth a locally optimal portfolio, has according to Corollary 11.1.4 to choose the volatility b_t^δ of the portfolio and then invests the fraction of wealth $\frac{b_t^\delta}{|\boldsymbol{\theta}_t|}$ at time t in the GOP. The remainder of her or his wealth is held in the savings account. We emphasize that only these two funds are needed to form locally optimal portfolios. We shall see in the next section that two fund separation also arises if an investor aims to maximize expected utility from discounted terminal wealth.

Risk Aversion Coefficient

We can interpret

$$J_t^\delta = \frac{1-\pi_{\delta_*,t}^0}{1-\pi_{\delta,t}^0} = \frac{|\boldsymbol{\theta}_t|}{b_t^\delta} \tag{11.1.19}$$

as a *risk aversion coefficient* similar as in the sense of Pratt (1964) and Arrow (1965). The risk aversion coefficient for obtaining the GOP equals one, and when investing only in the savings account it equals infinity. The latter corresponds to being infinitely risk averse. According to (11.1.15), (11.1.19) and (11.1.17) a discounted locally optimal portfolio $\bar{S}^\delta_t$ satisfies then the SDE

$$d\bar{S}^\delta_t = \bar{S}^\delta_t \, \frac{1}{J^\delta_t} \, |\boldsymbol{\theta}_t| \, (|\boldsymbol{\theta}_t| \, dt + dW_t), \tag{11.1.20}$$

where

$$dW_t = \sum_{k=1}^{d} \frac{\theta^k_t}{|\boldsymbol{\theta}_t|} \, dW^k_t \tag{11.1.21}$$

for $t \in [0, \infty)$. From the SDEs (11.1.20) and (10.2.8) it follows that the fraction of wealth invested in the GOP is $\frac{1}{J^\delta_t}$. This fraction is, therefore, the fraction that characterizes at time t a fractional Kelly strategy.

Capital Market Line

Note that the *expected rate of return* or *appreciation rate* a^δ_t of a portfolio S^δ is at time t the sum of short rate and risk premium and, thus, given by the expression

$$a^\delta_t = r_t + p_{S^\delta}(t) \tag{11.1.22}$$

for $t \in [0, \infty)$.

One can visualize the relationship (11.1.22) by using (11.1.14) and (11.1.10) for the family of locally optimal portfolios by the *capital market line*, see Sharpe (1964). This line shows the expected return a^δ_t, given in (11.1.22), of a locally optimal portfolio S^δ in dependence on its aggregate volatility, see (11.1.6). That is, by (11.1.10) and (11.1.14) we obtain the fundamental linear relationship

$$a^\delta_t = r_t + |\boldsymbol{\theta}_t| \, b^\delta_t \tag{11.1.23}$$

for $t \in [0, \infty)$. Consequently, the slope of the capital market line equals the total market price of risk, which is, in general, a fluctuating stochastic process. The expected return for zero aggregate volatility is according to (11.1.23) the short rate. It follows from (11.1.19) that a portfolio process S^δ at the capital market line with volatility b^δ_t has at time t the fraction $\frac{1}{J^\delta_t} = \frac{b^\delta_t}{|\boldsymbol{\theta}_t|}$ invested in the GOP, which characterizes its fractional Kelly strategy.

Markowitz Efficient Frontier

For a locally optimal portfolio process S^δ it follows from the SDE (11.1.15) and (11.1.17) that at a given time t its aggregate volatility, see (11.1.6), equals

$$b^\delta_t = \frac{1 - \pi^0_{\delta,t}}{1 - \pi^0_{\delta_*,t}} \, |\boldsymbol{\theta}_t| \tag{11.1.24}$$

and its *risk premium* $p_{S^\delta}(t)$ is

$$p_{S^\delta}(t) = b_t^\delta \, |\boldsymbol{\theta}_t| \tag{11.1.25}$$

for $t \in [0, \infty)$.

Note that the risk premium, see (11.1.9), of a portfolio S^δ is the appreciation rate of the corresponding discounted portfolio $\bar{S}^\delta$. By analogy with the one period mean-variance approach in Markowitz (1959), one can introduce in a CFM a family of *efficient portfolios*, which is parameterized by the squared aggregate volatility. When using formula (11.1.22) for the expected rate of return this leads to the following definition:

Definition 11.1.5. *In a CFM satisfying Assumption 11.1.2, an* efficient portfolio $S^\delta \in \mathcal{V}^+$ *is one whose expected rate of return* a_t^δ*, as a function of its squared volatility* $(b_t^\delta)^2$*, lies on the* efficient frontier a_t^δ*, defined as*

$$a_t^\delta = r_t + \sqrt{(b_t^\delta)^2} \, |\boldsymbol{\theta}_t| \tag{11.1.26}$$

for all times $t \in [0, \infty)$.

By exploiting relations (11.1.25) and (11.1.26), the following result can be directly obtained.

Corollary 11.1.6. *Under the assumptions of Theorem 11.1.3 any locally optimal portfolio* $S^\delta \in \mathcal{V}^+$ *is also an efficient portfolio.*

The relationship (11.1.26) can be interpreted as a generalization of the Markowitz efficient frontier to the continuous time setting. It holds for locally optimal portfolios under rather weak assumptions. Due to the inequality (11.1.14) in the Portfolio Selection Theorem and relation (11.1.10) it is not possible to form a strictly positive portfolio that generates an expected rate of return above the efficient frontier.

Each optimal portfolio S^δ has an expected rate of return a_t^δ that is located at the efficient frontier given in (11.1.26). Note that the efficient frontier moves randomly up and down over time in dependence on the fluctuations of the short rate r_t. Its slope also changes over time according to the total market price of risk $|\boldsymbol{\theta}_t|$, which is, generally, stochastic. For a fixed time instant $t \in [0, \infty)$ the Fig. 11.1.1 shows the efficient frontier's dependence on the squared volatility $|b_t^\delta|^2$ of a locally optimal portfolio, where the parameter values $r_t = 0.05$ and $|\boldsymbol{\theta}_t|^2 = 0.04$ are chosen. This graph also includes the tangent of the efficient frontier with slope $\frac{1}{2}$ at the point $|b_t^\delta|^2 = |\boldsymbol{\theta}_t|^2$ that corresponds to the squared volatility of the GOP. The reason why the mean-variance approach holds generally in a CFM is that, due to the assumed continuity of asset prices the asset dynamics resembles, locally in time, that of a one period model with Gaussian log-returns.

Efficient Growth Rates

As we have seen in Theorem 10.5.1, the focus of the long term investor should be the growth rate of her or his portfolio of tradable wealth. For illustration,

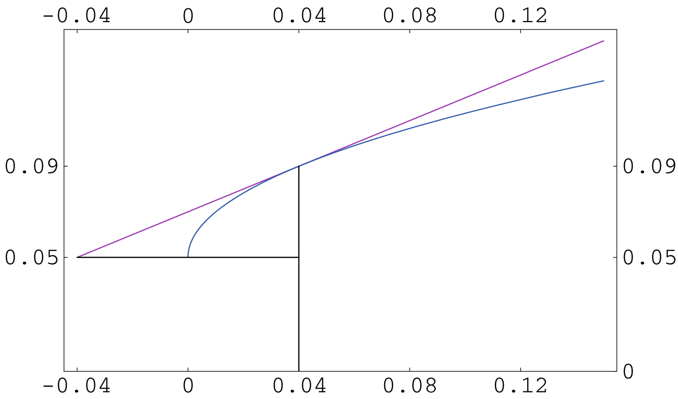

Fig. 11.1.1. Efficient frontier

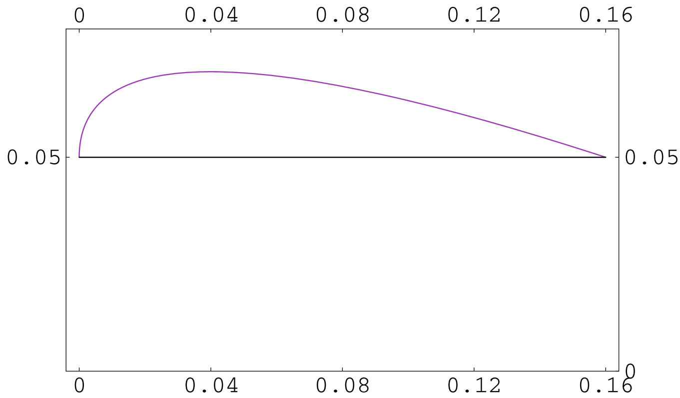

Fig. 11.1.2. Efficient growth rates

Fig. 11.1.2 shows for given $t \in [0, \infty)$ how the growth rate of a locally optimal portfolio S^δ depends on its squared volatility $|b_t^\delta|^2$, when using the same default parameters as in Fig. 11.1.1. One could call these growth rates the *efficient growth rates*. The corresponding frontier in dependence on the squared portfolio volatilities one can call the *efficient growth rate frontier*. The efficient growth rates satisfy the expression

$$g_t^\delta = r_t + \sqrt{|b_t^\delta|^2}\,|\boldsymbol{\theta}_t| - \frac{1}{2}\,|b_t^\delta|^2 = r_t + \frac{|\boldsymbol{\theta}_t|^2}{J_t^\delta}\left(1 - \frac{1}{2\,J_t^\delta}\right), \tag{11.1.27}$$

see (10.2.2), (11.1.16), (10.2.6) and (11.1.11). Note that for the value of the squared volatility $|b_t^\delta|^2 = |\boldsymbol{\theta}_t|^2$, that is $J_t^\delta = 1$, the efficient growth rates achieve their maximum, yielding the growth rate of the GOP

$$g_t^{\delta_*} = r_t + \frac{1}{2}\,|\boldsymbol{\theta}_t|^2. \tag{11.1.28}$$

For the volatility value $|b_t^\delta| = 2\,|\boldsymbol{\theta}_t|$ the efficient growth rate equals the short rate. As we have seen in Sect. 10.5, the GOP is the best performing portfolio under various criteria, in particular, for long term growth. By choosing a volatility value $|b_t^\delta| > |\boldsymbol{\theta}_t|$ one is, in principle, *overbetting*. This means that one faces larger fluctuations, which are more risky than those of the GOP due to a short position in the savings account. Such a fractional Kelly strategy does not perform as well as the Kelly strategy in the long term. Overbetting diminishes the long term growth rate. However, some investors may achieve by luck spectacular growth over some short period when overbetting but others may fail dramatically.

We have seen that the GOP is a central object in a CFM which facilitates the intertemporal generalization of the classical Markowitz-Tobin-Sharpe static mean-variance portfolio analysis, see Markowitz (1959), Tobin (1958a) and Sharpe (1964). Due to two fund separation the GOP is also a highly important benchmark for fund management. Two fund separation is equivalent to some kind of a fractional Kelly strategy. In Theorem 10.5.1 it was shown that the GOP almost surely outperforms pathwise any other portfolio after a sufficiently long time. Furthermore, Corollary 10.5.3 showed that even over any short time period it cannot be systematically outperformed by any other portfolio.

Lagrange Multipliers and Optimization (*)

As we shall see, the proof of the Portfolio Selection Theorem uses only standard multivariate calculus and a basic understanding of stochastic calculus. Before we give the proof of Theorem 11.1.3 let us mention a standard result on Lagrange multipliers and optimization.

Let $U : \Re^n \to \Re$ and $\boldsymbol{g} : \Re^k \to \Re^k$ be differentiable functions. Furthermore, assume that U is strictly concave and that $\boldsymbol{g}$ is convex. Under these assumptions we consider the problem of solving the optimization problem to find the maximum

$$U(\boldsymbol{x}_*) = \max_{\boldsymbol{x}\in\Re^n} U(\boldsymbol{x}) \tag{11.1.29}$$

such that

$$g^i(\boldsymbol{x}_*) = 0 \tag{11.1.30}$$

for all $i \in \{1, 2, \ldots, k\}$ and $\boldsymbol{x}_* \in \Re^n$. This problem is equivalent to finding a zero of the gradient of the corresponding Lagrangian

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = U(\boldsymbol{x}) - \boldsymbol{\lambda}^\top \boldsymbol{g}(\boldsymbol{x}) \tag{11.1.31}$$

for $\boldsymbol{x} = (x^1, x^2, \ldots, x^n)^\top \in \Re^n$ and $\boldsymbol{\lambda} = (\lambda^1, \lambda^2, \ldots, \lambda^k)^\top \in \Re^k$, see Luenberger (1969). More precisely, if the pair $(\boldsymbol{x}_*, \boldsymbol{\lambda}_*) \in \Re^n \times \Re^n$ solves the system of first order conditions

$$0 = \frac{\partial \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda})}{\partial x^i} = \frac{\partial U(\boldsymbol{x})}{\partial x^i} - \sum_{\ell=1}^{k} \lambda^\ell \, \frac{\partial g^\ell(\boldsymbol{x})}{\partial x^i} \tag{11.1.32}$$

for $i \in \{1, 2, \dots, n\}$ and

$$0 = \frac{\partial \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda})}{\partial \lambda^i} = g^i(\boldsymbol{x}) \tag{11.1.33}$$

for $i \in \{1, 2, \dots, k\}$, then $\boldsymbol{x}_*$ is the unique maximizer of the optimization problem.

In the case when the vector $\boldsymbol{\lambda}_*$ of the Lagrangian multipliers consists only of nonnegative components, then $\boldsymbol{x}_*$ is also the unique maximizer of the optimization problem

$$U(\boldsymbol{x}_*) = \max_{\boldsymbol{x} \in \Re^n} U(\boldsymbol{x}) \tag{11.1.34}$$

such that

$$g^i(\boldsymbol{x}_*) \leq 0 \tag{11.1.35}$$

for all $i \in \{1, 2, \dots, k\}$.

Proof of Theorem 11.1.3 (*)

To prove the Portfolio Selection Theorem we follow essentially the proof given in Platen (2002). To identify a discounted, locally optimal portfolio, as described in Definition 11.1.1, we maximize locally in time the drift (11.1.4), subject to the constraint (11.1.7). For this purpose we use the Lagrange multiplier λ, as described in the above subsection, and consider the function

$$\mathcal{L}(\psi_\delta^1, \dots, \psi_\delta^d, \lambda) = \sum_{k=1}^{d} \psi_\delta^k \, \theta^k + \lambda \left(\left(\gamma^{\tilde{\delta}}\right)^2 - \sum_{k=1}^{d} \left(\psi_\delta^k\right)^2 \right) \tag{11.1.36}$$

by suppressing time dependence. For $\psi_\delta^1, \psi_\delta^2, \dots, \psi_\delta^d$ to provide a maximum for $\mathcal{L}(\psi_\delta^1, \dots, \psi_\delta^d, \lambda)$ it is necessary that the first-order conditions

$$\frac{\partial \mathcal{L}(\psi_\delta^1, \dots, \psi_\delta^d, \lambda)}{\partial \psi_\delta^k} = \theta^k - 2\,\lambda\,\psi_\delta^k = 0 \tag{11.1.37}$$

are satisfied for all $k \in \{1, 2, \dots, d\}$ as well as

$$\frac{\partial \mathcal{L}(\psi_\delta^1, \dots, \psi_\delta^d, \lambda)}{\partial \lambda} = \left(\gamma^{\tilde{\delta}}\right)^2 - \sum_{k=1}^{d} \left(\psi_\delta^k\right)^2 = 0. \tag{11.1.38}$$

Consequently, a locally optimal portfolio $S^{(\tilde{\delta})}$, which maximizes the discounted drift, must satisfy the relation

$$\psi_{\tilde{\delta}}^k = \frac{\theta^k}{2\,\lambda} \tag{11.1.39}$$

for all $k \in \{1, 2, \dots, d\}$. Furthermore, by (11.1.38) we must have

$$\sum_{k=1}^{d} \left(\psi_{\tilde{\delta}}^{k}\right)^{2} = \left(\gamma^{\tilde{\delta}}\right)^{2}. \tag{11.1.40}$$

We can now use the constraint (11.1.7), together with (11.1.40), (11.1.5) and (11.1.11), to obtain from (11.1.39) the relation

$$\left(\gamma^{\tilde{\delta}}\right)^{2} = \sum_{k=1}^{d} \left(\psi_{\tilde{\delta}}^{k}\right)^{2} = \frac{\sum_{k=1}^{d}(\theta^{k})^{2}}{4\,\lambda^{2}}. \tag{11.1.41}$$

By (11.1.12) we have $|\boldsymbol{\theta}| = \sqrt{\sum_{k=1}^{d}(\theta^{k})^{2}} > 0$ and obtain by (11.1.39) and (11.1.41) the equation

$$\psi_{\tilde{\delta}}^{k} = \frac{\gamma^{\tilde{\delta}}}{|\boldsymbol{\theta}|}\,\theta^{k} \tag{11.1.42}$$

for all $k \in \{1, 2, \ldots, d\}$. This yields at time t by (11.1.4) for a locally optimal portfolio $S^{\tilde{\delta}}$ the discounted drift

$$\alpha_{t}^{\tilde{\delta}} = \gamma_{t}^{\tilde{\delta}}\,\frac{|\boldsymbol{\theta}_{t}|^{2}}{|\boldsymbol{\theta}_{t}|} = \gamma_{t}^{\tilde{\delta}}\,|\boldsymbol{\theta}_{t}|. \tag{11.1.43}$$

This leads, by (11.1.10), to the equality in (11.1.14). Due to the above optimization the inequality in (11.1.14) follows for any strictly positive portfolio with nonzero aggregate diffusion coefficient.

Equation (11.1.42), when substituted into (11.1.2), provides by (11.1.5) the SDE

$$d\bar{S}_{t}^{\tilde{\delta}} = \gamma_{t}^{\tilde{\delta}} \sum_{k=1}^{d} \frac{\theta_{t}^{k}}{|\boldsymbol{\theta}_{t}|}\,(\theta_{t}^{k}\,dt + dW_{t}^{k}). \tag{11.1.44}$$

Using (11.1.6) this yields the SDE (11.1.15). Furthermore, it follows for $k \in \{1, 2, \ldots, d\}$ from (11.1.3), (10.1.12), (11.1.42) and (11.1.6) that

$$\psi_{\tilde{\delta},t}^{k} = \sum_{j=1}^{d} \tilde{\delta}_{t}^{j}\,\bar{S}_{t}^{j}\,b_{t}^{j,k} = \bar{S}_{t}^{(\tilde{\delta})} \sum_{j=1}^{d} \pi_{\tilde{\delta},t}^{j}\,b_{t}^{j,k} = \frac{\gamma_{t}^{\tilde{\delta}}}{|\boldsymbol{\theta}_{t}|}\,\theta_{t}^{k} = \bar{S}_{t}^{(\tilde{\delta})}\,b_{t}^{\tilde{\delta}}\,\frac{\theta_{t}^{k}}{|\boldsymbol{\theta}_{t}|}. \tag{11.1.45}$$

Using the invertibility of the volatility matrix one obtains, see Assumption 10.1.1, the fraction

$$\pi_{\tilde{\delta},t}^{j} = \frac{b_{t}^{\tilde{\delta}}}{|\boldsymbol{\theta}_{t}|} \sum_{k=1}^{d} \theta_{t}^{k}\,b_{t}^{-1\,j,k} \tag{11.1.46}$$

and, thus, by (10.2.6) the equation (11.1.16) for all $j \in \{1, 2, \ldots, d\}$ and $t \in [0, \infty)$. Using the SDE of the discounted GOP one notes that an SDE of the form (11.1.15) belongs to a discounted portfolio, which has for its given aggregate diffusion coefficient the maximum discounted drift. Thus, by Definition 11.1.1 the corresponding portfolio is a locally optimal portfolio. □

11.2 Market Portfolio and GOP

In Sect. 9.3 we already considered a version of the intertemporal capital asset pricing model (ICAPM). The capital asset pricing model (CAPM) was developed in one and multiperiod discrete time settings by Sharpe (1964), Lintner (1965) and Mossin (1966). Its continuous time analog, the ICAPM, was established for continuous markets in Merton (1973a) as an equilibrium model of exchange using utility maximization and equilibrium arguments. Most of the following results are established in Platen (2005c, 2006a, 2006b).

Intertemporal Capital Asset Pricing Model

By using a locally optimal portfolio as a reference portfolio, we shall now derive the ICAPM for a CFM. For this purpose let us consider a strictly positive, risky, locally optimal portfolio $S^{\underline{\delta}} \in \mathcal{V}^+$. Then by (11.1.9), (10.1.14), (10.2.1) and (11.1.15) the risk premium $p_{S^\delta}(t)$ of a strictly positive portfolio $S^\delta \in \mathcal{V}^+$ can be expressed as

$$p_{S^\delta}(t) = \sum_{k=1}^{d} \sum_{j=1}^{d} \pi^j_{\delta,t}\, b^{j,k}_t\, \theta^k_t = \frac{d[\ln(S^\delta), \ln(S^{\underline{\delta}})]_t}{dt}\, \frac{|\boldsymbol{\theta}_t|}{b^{\underline{\delta}}_t} \tag{11.2.1}$$

at time t. Here $[\ln(S^\delta), \ln(S^{\underline{\delta}})]_t$ denotes the covariation at time t of the stochastic processes $\ln(S^\delta)$ and $\ln(S^{\underline{\delta}})$, see Sect. 5.2. The time derivative of the covariation is the local, in time, analogue of the covariance of log-returns for continuous time processes.

For a strictly positive portfolio $S^\delta \in \mathcal{V}^+$ the *systematic risk parameter* $\beta_{S^\delta}(t)$, also called the *beta*, is defined as the ratio of the covariations

$$\beta_{S^\delta}(t) = \frac{\frac{d[\ln(S^\delta), \ln(S^{\underline{\delta}})]_t}{dt}}{\frac{d[\ln(S^{\underline{\delta}})]_t}{dt}}, \tag{11.2.2}$$

for $t \in [0, \infty)$, where $S^{\underline{\delta}}$ denotes again a strictly positive, risky, locally optimal portfolio. This allows us to deduce by (11.2.1) and (11.2.2) the core relationship of the ICAPM.

Theorem 11.2.1. *Under the assumptions of Theorem 11.1.3, for any strictly positive portfolio $S^\delta \in \mathcal{V}^+$ the portfolio beta with respect to a strictly positive, risky, locally optimal portfolio $S^{\underline{\delta}} \in \mathcal{V}^+$, with nonzero aggregate volatility, has the form*

$$\beta_{S^\delta}(t) = \frac{p_{S^\delta}(t)}{p_{S^{\underline{\delta}}}(t)} \tag{11.2.3}$$

for $t \in [0, \infty)$.

The above expression for the portfolio beta is exactly what the ICAPM suggests if the market portfolio (MP) is a locally optimal portfolio. In this case, Theorem 11.2.1 already proves the ICAPM in a general CFM setting. This raises the question: When is the MP a locally optimal portfolio?

Market Portfolio

Let us assume the existence of $n \in \mathcal{N}$ investors who hold all tradable wealth in the market, which is the total sum of all units of primary security accounts. The portfolio of tradable wealth of the ℓth investor is denoted by S^{δ_ℓ}, $\ell \in \{1, 2, \dots, n\}$. Due to the limited liability of investors $S^{\delta_\ell} \in \mathcal{V}$ is nonnegative. The total portfolio $S_t^{\delta_{\rm MP}}$ of the tradable wealth of all investors is then the MP, which is given by the sum

$$S_t^{\delta_{\rm MP}} = \sum_{\ell=1}^{n} S_t^{\delta_\ell} \tag{11.2.4}$$

at time $t \in [0, \infty)$. We have seen in the previous section that Sharpe ratio maximizing investors form locally optimal portfolios. We shall see in Sect. 11.3 that also expected utility maximizing investors form locally optimal portfolios. Therefore, it is natural to make the following assumption.

Assumption 11.2.2. *Each investor forms a nonnegative, locally optimal portfolio with her or his total tradable wealth.*

Since the sum of locally optimal portfolios is again a locally optimal portfolio we can prove the following result.

Theorem 11.2.3. *For a CFM, where each investor holds a locally optimal portfolio with respect to the domestic currency denomination, the MP is a locally optimal portfolio.*

Proof: The discounted MP $\bar{S}_t^{\delta_{\rm MP}} = \frac{S_t^{\delta_{\rm MP}}}{S_t^0}$ at time t is under the assumptions of the theorem by (11.1.15), (11.1.17) and (11.2.4) determined by the SDE

$$\begin{aligned} d\bar{S}_t^{\delta_{\rm MP}} &= \sum_{\ell=1}^{n} d\bar{S}_t^{\delta_\ell} = \sum_{\ell=1}^{n} \frac{\left(\bar{S}_t^{\delta_\ell} - \delta_\ell^0\right)}{\left(1 - \pi_{\delta_*,t}^0\right)} \sum_{k=1}^{d} \theta_t^k \left(\theta_t^k\, dt + dW_t^k\right) \\ &= \bar{S}_t^{\delta_{\rm MP}} \frac{\left(1 - \pi_{\delta_{\rm MP},t}^0\right)}{\left(1 - \pi_{\delta_*,t}^0\right)} \sum_{k=1}^{d} \theta_t^k \left(\theta_t^k\, dt + dW_t^k\right) \end{aligned} \tag{11.2.5}$$

for $t \in [0, \infty)$. This shows, by (11.1.15), that the MP $S_t^{\delta_{\rm MP}}$ has the SDE of a locally optimal portfolio. This proves by Theorem 11.1.3 that the MP is a locally optimal portfolio. □

It is straightforward to draw the following conclusion from Theorem 11.2.1 and Theorem 11.2.3.

Corollary 11.2.4. *Under the assumptions of Theorem 11.2.3 the ICAPM relationship* (11.2.3) *holds when using the market portfolio as reference portfolio.*

This proves the ICAPM under the assumptions of Theorem 11.2.3. It is important to emphasize the fact that the derivation of this result does not require any assumptions about expected utility maximization, equilibrium or Markovianity, as typically imposed in the literature. Note also that no matter what locally optimal portfolio the investor holds, the ICAPM follows with the MP as reference portfolio.

Market Portfolio and GOP

It is reasonable to discuss the following invariance of a financial market model. By invariance we mean here the property of the market that relationships that hold for one currency denomination apply also for another currency denomination. This can be expressed by the following assumption.

Assumption 11.2.5. *The fundamental relationships in the market are invariant under a change of currency denomination.*

As the following theorem shows, this assumption has interesting consequences.

Theorem 11.2.6. *In a CFM where a strictly positive portfolio is locally optimal in at least two currency denominations this portfolio must be a GOP.*

This theorem will be derived at the end of this section. It allows us to draw interesting conclusions. If one assumes that the investors optimize their tradable wealth in two currency denominations by forming an MP that is a locally optimal portfolio in each of the two currencies, then by Theorem 11.2.6 the MP is the GOP. Of course, the investors will never exactly form an MP that is a perfect locally optimal portfolio in two currency denominations. However, the reality may come close to this situation. This then allows the conclusion that the MP may be not too far from the GOP.

In Sect. 10.6 we concluded under some regularity condition on the market that a portfolio approximates the GOP purely on the basis of the assumption that it is a diversified portfolio. The above optimal portfolio selection leads to a complementing result, as long as the sequence of CFMs $(\mathcal{S}^{\mathrm{C}}_{(d)})_{d\in\mathcal{N}}$ is regular and the corresponding sequence of MPs is that of diversified portfolios.

For the given world market the MP is, in principle, observable. For instance, a potential proxy is given by the daily *MSCI*, which essentially reflects the stock portfolio of the developed markets. For illustration, in Fig. 11.2.1 we show the MSCI in units of the US dollar savings account for the period from 1970 until 2003. We have alternatively studied the WSI and EWI in Sect. 10.6 as potential proxies of the GOP. We have already seen that the differences between all these proxies of the GOP are minor from a practical point of view.

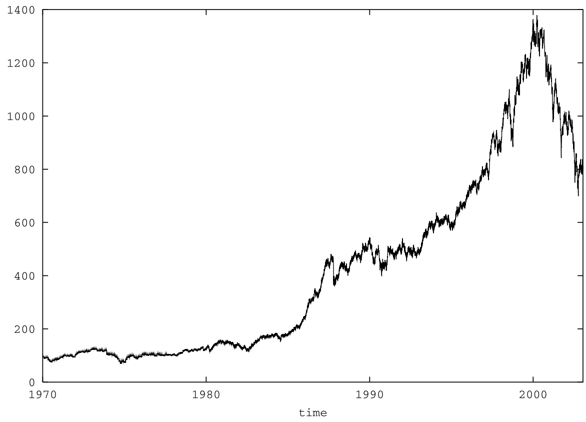

Fig. 11.2.1. Discounted MSCI

Proof of Theorem 11.2.6 (*)

Let us denote, the GOP at time t in the ith currency by $S_i^{\delta_*}(t)$ for $i \in \{0,1\}$. This satisfies by (10.2.8) the SDE

$$dS_i^{\delta_*}(t) = S_i^{\delta_*}(t)\left(r_t^i\,dt + \sum_{k=1}^{d}\theta_i^k(t)\left(\theta_i^k(t)\,dt + dW_t^k\right)\right) \tag{11.2.6}$$

for $t \in [0,\infty)$. Here r_t^i is the ith short rate for the ith currency denomination and $\theta_i^k(t)$ the market price of risk for the ith currency denomination with respect to the kth Wiener process. Furthermore, we denote by $S_i^j(t)$ the jth savings account at time t, denominated in the ith currency, $i,j \in \{0,1\}$.

A locally optimal portfolio $S_0^{\tilde{\delta}}(t)$ at time t, when denominated in units of the 0th currency, satisfies by Theorem 11.1.3, see (11.1.15), the SDE

$$dS_0^{\tilde{\delta}}(t) = S_0^{\tilde{\delta}}(t)\left(r_t^0\,dt + \frac{\left(1-\pi^0_{\tilde{\delta},t}\right)}{\left(1-\pi^0_{\delta_*,t}\right)}\sum_{k=1}^{d}\theta_0^k(t)\left(\theta_0^k(t)\,dt + dW_t^k\right)\right) \tag{11.2.7}$$

for $t \in [0,\infty)$.

The exchange rate $X_t^{1,0}$ from the 0th into the first currency at time t can be written as

$$X_t^{1,0} = \frac{S_1^{\delta_*}(t)}{S_0^{\delta_*}(t)}. \tag{11.2.8}$$

It satisfies by (11.2.6) and an application of the Itô formula the SDE

$$dX_t^{1,0} = X_t^{1,0}\left(\left(r_t^1 - r_t^0\right)dt + \sum_{k=1}^{d}\left(\theta_1^k(t) - \theta_0^k(t)\right)\left(\theta_1^k(t)\,dt + dW_t^k\right)\right) \tag{11.2.9}$$

for $t \in [0,\infty)$.

Denominating now the locally optimal portfolio $S^{\tilde{\delta}}$ in units of the first currency yields by the Itô formula, (11.2.6) and (11.2.9) the SDE

$$
\begin{aligned}
dS_1^{\tilde{\delta}}(t) &= d\left(S_0^{\tilde{\delta}}(t)\, X_t^{1,0}\right) \\
&= S_1^{\tilde{\delta}}(t)\left(r_t^1\, dt + \sum_{k=1}^d \left[\frac{\left(1-\pi^0_{\tilde{\delta},t}\right)}{\left(1-\pi^0_{\delta_*,t}\right)}\left(\theta_0^k(t)\right)^2 + \left(\theta_1^k(t)-\theta_0^k(t)\right)\theta_1^k(t)\right.\right. \\
&\qquad \left. + \frac{\left(1-\pi^0_{\tilde{\delta},t}\right)}{\left(1-\pi^0_{\delta_*,t}\right)}\theta_0^k(t)\left(\theta_1^k(t)-\theta_0^k(t)\right)\right] dt \\
&\qquad \left. + \sum_{k=1}^d \left[\frac{\left(1-\pi^0_{\tilde{\delta},t}\right)}{\left(1-\pi^0_{\delta_*,t}\right)}\theta_0^k(t) + \theta_1^k(t) - \theta_0^k(t)\right] dW_t^k\right) \\
&= S_1^{\tilde{\delta}}(t)\left(r_t^1\, dt + \sum_{k=1}^d \left(\theta_1^k(t) - \theta_0^k(t)\left(\frac{\pi^0_{\delta_*,t}-\pi^0_{\tilde{\delta},t}}{1-\pi^0_{\delta_*,t}}\right)\right)\left(\theta_1^k(t)\, dt + dW_t^k\right)\right).
\end{aligned}
$$

For $S_1^{\tilde{\delta}}(t)$ to satisfy the SDE of a locally optimal portfolio in the first currency denomination requires by (11.1.15) and (11.1.17) for $t \in [0,\infty)$ the equality

$$
\theta_1^k(t) - \theta_0^k(t)\left(\frac{\pi^0_{\delta_*,t}-\pi^0_{\tilde{\delta},t}}{1-\pi^0_{\delta_*,t}}\right) = \frac{\left(1-\pi^1_{\tilde{\delta},t}\right)}{\left(1-\pi^1_{\delta_*,t}\right)}\,\theta_1^k(t).
$$

To achieve this equality one needs to satisfy the equation

$$
\pi^0_{\delta_*,t} = \pi^0_{\tilde{\delta},t} \tag{11.2.10}
$$

for all $t \in [0,\infty)$. This demonstrates by (11.2.7) and (11.2.6) that $S^{\tilde{\delta}}$ is under the assumptions of Theorem 11.2.6 a GOP. □

11.3 Expected Utility Maximization

Utility functions, as introduced in von Neumann & Morgenstern (1953), have been widely used in portfolio optimization and economic modeling, see Merton (1973a). We study now the type of portfolio that an expected utility maximizer forms in a CFM. We shall show under appropriate assumptions that this will again be a locally optimal portfolio. As a consequence of Corollary 11.1.4 a two fund separation theorem holds also for expected utility maximization. Therefore, when some investors maximize expected utility, others maximize

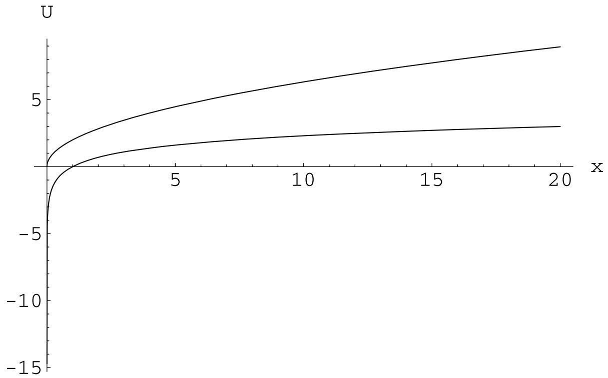

Fig. 11.3.1. Examples for power utility (upper graph) and log-utility (lower graph)

Sharpe ratios and the rest maximizes the growth rate for a given portfolio volatility, then the MP is still a locally optimal portfolio and, thus, a combination of the GOP and the savings account. As already mentioned, this can also be interpreted as a fractional Kelly strategy, see Hakansson & Ziemba (1995). Some of the following results appear in Platen (2006a, 2006c).

Utility Functions

A *utility function* is a real valued function $U(\cdot)$ which allocates a real number to any nonnegative level of wealth. Once a utility function is chosen, then all alternative wealth levels are ranked by evaluating their expected utility values. It turns out that the following class of utility functions can express the personal preferences of market participants.

Definition 11.3.1. *A* utility function $U : [0, \infty) \to [-\infty, \infty)$ *is a real valued, twice differentiable, strictly increasing and strictly concave function, where* $U'(0) = \infty$ *and* $U'(\infty) = 0$.

Examples of utility functions are given by the *power utility*

$$U(x) = \frac{1}{\gamma}\, x^{\gamma} \tag{11.3.1}$$

for $\gamma \neq 0$ and $\gamma < 1$ and the *log-utility*

$$U(x) = \ln(x) \tag{11.3.2}$$

for $x \in [0, \infty)$, where $\ln(0)$ is set to minus infinity. In Fig. 11.3.1 we show with the upper graph an example for a power utility function with $\gamma = \frac{1}{2}$, together with the log-utility displayed as the lower curve. The properties of a utility function given in Definition 11.3.1 have economic interpretations.

The strict monotonicity reflects the natural preference of an investor for more rather than less wealth. In this sense investors are *nonsatiable*. The concavity of $U(x)$ implies that $U'(x)$ is decreasing in x. This models the fact that a typical investor has some *risk aversion*, which may depend on her or his level of total tradable wealth. Note that the derivative U' of a utility function has an inverse function U'^{-1}, which will be of importance in our analysis below.

Expected Utility Maximization

We aim to identify the portfolio which an expected utility maximizer constructs. Let us consider a utility function $U : [0, \infty) \to [-\infty, \infty)$ and fix a terminal time horizon $T \in [0, \infty)$.

An investor can always compare her or his investment strategy δ with the one where all wealth is invested in the locally riskless security, that is, the savings account S^0. Therefore, we shall take the time value of money into account by discounting with the savings account S^0. This means, we shall consider an investor who maximizes expected utility from discounted terminal wealth. Furthermore, we assume that the investor maximizes only over fair portfolios, since according to Corollary 10.4.2, these are the portfolios that require the minimal initial investment to reach a desired future payoff. This payoff is in our case the utility of discounted terminal wealth. It is not rational to invest in an unfair portfolio, because there exists then a cheaper fair portfolio that provides exactly the same utility.

Definition 11.3.2. *Define the set $\bar{\mathcal{V}}^+_{S_0}$ of strictly positive, savings account discounted, fair portfolios $\bar{S}^\delta$ with given initial value $\bar{S}^\delta_0 = S_0 > 0$.*

We maximize now the expected utility

$$v^{\tilde{\delta}} = \max_{\bar{S}^\delta \in \bar{\mathcal{V}}^+_{S_0}} v^\delta \tag{11.3.3}$$

with

$$v^\delta = E\left(U\left(\bar{S}^\delta_T\right) \,\middle|\, \mathcal{A}_0\right), \tag{11.3.4}$$

where the maximum is taken over the set $\bar{\mathcal{V}}^+_{S_0}$ and is assumed to exist.

Furthermore, to obtain a tractable solution of the expected utility maximization problem, we assume in this section, for simplicity, that the discounted GOP $\bar{S}^{\delta_*}$ itself is a strictly positive Markov process with

$$d\bar{S}^{\delta_*}_t = \bar{S}^{\delta_*}_t \, \theta(t, \bar{S}^{\delta_*}_t) \left(\theta(t, \bar{S}^{\delta_*}_t)\, dt + dW_t\right) \tag{11.3.5}$$

for $t \in [0, \infty)$ and given volatility function $\theta : [0, \infty) \times (0, \infty) \to (0, \infty)$. In Chap. 13 we shall demonstrate by deriving the minimal market model that this is an acceptable assumption. This assumption can be relaxed in many ways yielding slightly more complex but similar results.

The following theorem describes the structure of the optimal portfolio of the expected utility maximizer. Its derivation follows the, so-called, martingale approach in portfolio optimization as described, for instance, in Korn (1997), Karatzas & Shreve (1998) and Zhao & Ziemba (2003). The theorem is derived at the end of the section, see also Platen (2006c).

Theorem 11.3.3. *Consider a CFM that satisfies the Assumption 11.1.2 and has a Markovian, strictly positive discounted GOP $\bar{S}^{\delta_*}$, satisfying (11.3.5). Then the discounted, strictly positive, fair portfolio $\bar{S}^{\tilde{\delta}} \in \bar{\mathcal{V}}^+_{S_0}$, which maximizes the given utility function $U(\cdot)$, is a locally optimal portfolio in the sense of Definition 11.1.1 and satisfies the SDE*

$$d\bar{S}^{\tilde{\delta}}_t = \bar{S}^{\tilde{\delta}}_t \, \frac{1}{J^{\tilde{\delta}}_t} \, \theta(t, \bar{S}^{\delta_*}_t) \, \Big(\theta(t, \bar{S}^{\delta_*}_t) \, dt + dW_t \Big) \tag{11.3.6}$$

with risk aversion coefficient

$$J^{\tilde{\delta}}_t = \frac{1}{1 - \frac{\hat{S}^0_t}{\hat{u}(t, \hat{S}^0_t)} \, \frac{\partial \hat{u}(t, \hat{S}^0_t)}{\partial \hat{S}^0}}, \tag{11.3.7}$$

and benchmarked fair portfolio value

$$\hat{S}^{\tilde{\delta}}_t = \hat{u}(t, \hat{S}^0_t) = E\left(U'^{-1}\left(\lambda \, \hat{S}^0_T\right) \hat{S}^0_T \,\Big|\, \mathcal{A}_t \right) \tag{11.3.8}$$

at time $t \in [0, T]$. The constant λ follows from the match of the initial value

$$S_0 = \hat{u}(0, \hat{S}^0_0) \, S^{\delta_*}_0. \tag{11.3.9}$$

Note in (11.3.7) that the risk aversion coefficient is the inverse of the fraction of $S^{\tilde{\delta}}$ that is invested in the GOP. We notice from (11.3.8) that the problem of maximizing expected utility from discounted terminal wealth has been transformed into that of hedging a particular payoff of the type

$$H = U'^{-1}(\lambda \hat{S}^0_T) S^0_T.$$

This demonstrates that there is a deep link between expected utility maximization and hedging. We shall discuss hedging issues in more detail in the next section. Due to the Markovianity of $\bar{S}^{\delta_*}$ one can in the given case calculate $\hat{u}(\cdot, \cdot)$ and replicate the payoff H by a fair, locally optimal portfolio. More precisely, one can apply the Feynman-Kac formula, see Sect. 9.7, to obtain the function $\hat{u}(\cdot, \cdot)$ as the solution of a PDE. From $\hat{u}(\cdot, \cdot)$ one can then determine the fraction of wealth to be held in the GOP and the remaining fraction that has to be invested in the savings account. Note that if $\bar{S}^{\delta_*}$ is driven by $n \in \mathcal{N}$ tradable factors that form together a Markov process, then one obtains $n + 1$ fund separation for the resulting optimal portfolios. However, as we will see in Chap. 13 the MMM suggests in reality two fund separation.

Examples on Expected Utility Maximization

A disadvantage of the expected utility approach is that only in rare cases one can provide explicit results. To illustrate the above theorem we discuss two simple examples.

1. In the first example we consider the log-utility function $U(x) = \ln(x)$. Its derivative is $U'(x) = \frac{1}{x}$, which has the inverse $U'^{-1}(y) = \frac{1}{y}$. Since the second derivative $U''(x) = -\frac{1}{x^2}$ is negative, the utility function is concave, as required in Definition 11.3.1. We recall that maximizing expected logarithmic utility is equivalent to selecting the Kelly criterion for portfolio optimization, see Kelly (1956) and Hakansson & Ziemba (1995).

According to (11.3.8) we obtain for $t \in [0, \infty)$ the conditional expectation

$$\hat{u}(t, \hat{S}_t^0) = E\left(U'^{-1}\left(\lambda\, \hat{S}_T^0\right) \hat{S}_T^0 \,\Big|\, \mathcal{A}_t\right) = E\left(\frac{1}{\lambda\, \hat{S}_T^0}\, \hat{S}_T^0 \,\Big|\, \mathcal{A}_t\right) = \frac{1}{\lambda} \qquad (11.3.10)$$

for $t \in [0, T]$. By equation (11.3.9) we obtain the Lagrange multiplier

$$\lambda = \frac{\bar{S}_0^{\delta_*}}{S_0}. \qquad (11.3.11)$$

By formula (11.3.7) the risk aversion coefficient equals the constant

$$J_t^{\tilde{\delta}} = 1, \qquad (11.3.12)$$

which shows that the corresponding expected log-utility maximizing portfolio $S^{\tilde{\delta}}$ is a GOP, see (11.1.19). This allows us to interpret the GOP as the portfolio which maximizes expected log-utility. Therefore, we could have defined earlier the GOP as the strictly positive portfolio which maximizes expected log-utility from discounted terminal wealth. Indeed, this idea has been followed in Platen (2004a) in the case of other asset price dynamics, since such a definition is generally applicable beyond the setting of a CFM. Note that in the relationships of this example the particular dynamics of the GOP did not play any role. We obtain the expected log-utility in the form

$$v^{\tilde{\delta}} = E\left(\ln\left(\bar{S}_T^{\delta_*}\right) \,\Big|\, \mathcal{A}_0\right) = \ln(\lambda) + \ln(S_0) + \frac{1}{2}\int_0^T E\left(\left(\theta(s, \bar{S}_s^{\delta_*})\right)^2 \,\Big|\, \mathcal{A}_0\right) ds$$

if the local martingale part in the SDE for $\ln(\bar{S}_t^{\delta_*})$ forms a martingale, see Exercise 11.1.

2. Our second example uses the power utility $U(x) = \frac{1}{\gamma} x^\gamma$ for $\gamma < 1$ and $\gamma \neq 0$. Its derivative is $U'(x) = x^{\gamma-1}$ and the corresponding inverse has the form $U'^{-1}(y) = y^{\frac{1}{\gamma-1}}$. The second derivative $U''(x) = (\gamma-1)x^{\gamma-2}$ is negative, which makes $U(\cdot)$ a suitable concave function.

According to (11.3.8) we have

$$\hat{u}(t, \hat{S}_t^0) = E\left(\left(\frac{\lambda}{\bar{S}_T^{\delta_*}}\right)^{\frac{1}{\gamma-1}} \frac{1}{\bar{S}_T^{\delta_*}} \,\bigg|\, \mathcal{A}_t\right) = \lambda^{\frac{1}{\gamma-1}}\, E\left(\left(\bar{S}_T^{\delta_*}\right)^{\frac{\gamma}{1-\gamma}} \,\Big|\, \mathcal{A}_t\right). \tag{11.3.13}$$

If there are analytic formulas for the conditional moments of the discounted GOP $\bar{S}_T^{\delta_*}$, then one can write down an explicit expression for the value of $\hat{u}(t, \hat{S}_t^0)$. Since $\bar{S}^{\delta_*}$ is in Theorem 11.3.3 assumed to be Markovian, one can apply the Feynman-Kac formula, see Sect. 9.7, to obtain the function $\hat{u}(\cdot, \cdot)$.

For simplicity, let us consider here the case where $\bar{S}^{\delta_*}$ is a geometric Brownian motion with $\theta(t, \bar{S}_t^{\delta_*}) = \theta > 0$. Thus, we obtain from (11.3.13) the expression

$$\begin{aligned}\hat{u}(t, \hat{S}_t^0) &= \lambda^{\frac{1}{\gamma-1}} \left(\bar{S}_t^{\delta_*}\right)^{\frac{\gamma}{1-\gamma}} E\left(\exp\left\{\frac{\gamma}{1-\gamma}\left(\frac{\theta^2}{2}\,(T-t) + \theta\,(W_T - W_t)\right)\right\} \,\bigg|\, \mathcal{A}_t\right) \\ &= \lambda^{\frac{1}{\gamma-1}} \left(\bar{S}_t^{\delta_*}\right)^{\frac{\gamma}{1-\gamma}} \exp\left\{\frac{\theta^2}{2}\,\frac{\gamma}{(1-\gamma)^2}\,(T-t)\right\}\end{aligned} \tag{11.3.14}$$

for $t \in [0, T]$. By using (11.3.9) we obtain the Lagrange multiplier

$$\lambda = S_0^{\gamma-1} \left(\bar{S}_0^{\delta_*}\right)^{\gamma} \exp\left\{\frac{\theta^2}{2}\,\frac{\gamma}{1-\gamma}\,T\right\}. \tag{11.3.15}$$

Furthermore, from (11.3.14) by noting that $\hat{S}_t^0 = (\bar{S}_t^{\delta_*})^{-1}$ we obtain, see (10.3.1), the partial derivative

$$\frac{\partial \hat{u}(t, \hat{S}_t^0)}{\partial \hat{S}^0} = \frac{\hat{u}(t, \hat{S}_t^0)}{\hat{S}_t^0}\,\frac{\gamma}{\gamma-1}. \tag{11.3.16}$$

Therefore, by (11.3.7) for the power utility under the BS model we obtain the risk aversion coefficient

$$J_t^{\tilde{\delta}} = 1 - \gamma \tag{11.3.17}$$

and the expected utility

$$v^{\tilde{\delta}} = E\left(\frac{1}{\gamma}\left(\bar{S}_T^{\tilde{\delta}}\right)^{\gamma} \,\bigg|\, \mathcal{A}_0\right) = \frac{1}{\gamma}\,\exp\left\{\frac{\theta^2}{2}\,\frac{\gamma}{1-\gamma}\,T\right\}(S_0)^{\gamma},$$

see Exercise 11.2.

This recovers well-known results derived in Merton (1973a). One notes that as $\gamma \to 0$, the above risk aversion coefficient converges to one, which selects asymptotically the GOP as the expected utility maximizing portfolio. Note that for a power utility the particular dynamics of the discounted GOP are relevant. In this special case we have then also the constant fraction $\frac{1}{J_t^{\tilde{\delta}}} = \frac{1}{1-\gamma}$ of wealth invested in the GOP and the remainder in the savings account. This is again a fractional Kelly strategy.

Proof of Theorem 11.3.3 (*)

1. Since we only consider fair portfolios, we have a constrained optimization problem. Let us apply in the following the, so-called, martingale approach, see Karatzas & Shreve (1998). We express the constrained optimization problem (11.3.3) by using a Lagrange multiplier $\lambda \in \Re$, see Sect. 11.1, and maximize the functional

$$v^\delta = E\left(U\left(\bar{S}_T^\delta\right) \,\Big|\, \mathcal{A}_0\right) - \lambda\left(E\left(\frac{S_T^\delta}{S_T^{\delta_*}} \,\Big|\, \mathcal{A}_0\right) - \frac{S_0}{S_0^{\delta_*}}\right) \tag{11.3.18}$$

over the set $\bar{\mathcal{V}}_{S_0}^+$ of strictly positive, discounted, fair portfolios $\bar{S}^\delta$ starting with $\bar{S}_0^\delta = S_0$. Then (11.3.18) can be rewritten as

$$v^\delta = E\left(U\left(\bar{S}_T^\delta\right) - \lambda\left(\frac{\bar{S}_T^\delta}{\bar{S}_T^{\delta_*}} - \frac{S_0}{\bar{S}_0^{\delta_*}}\right) \,\Big|\, \mathcal{A}_0\right). \tag{11.3.19}$$

This means, we seek a discounted portfolio $\bar{S}^{\tilde{\delta}} \in \bar{\mathcal{V}}_{S_0}^+$ so that

$$\begin{aligned} v^{\tilde{\delta}} = \max_{\bar{S}^\delta \in \bar{\mathcal{V}}_{S_0}^+} v^\delta &\le E\left(\max_{\bar{S}^\delta \in \bar{\mathcal{V}}_{S_0}^+} \left\{U\left(\bar{S}_T^\delta\right) - \lambda\left(\frac{\bar{S}_T^\delta}{\bar{S}_T^{\delta_*}} - \frac{S_0}{\bar{S}_0^{\delta_*}}\right)\right\} \Big|\, \mathcal{A}_0\right) \\ &\le E\left(\max_{\bar{S}_T^\delta > 0} \left\{U\left(\bar{S}_T^\delta\right) - \lambda\left(\frac{\bar{S}_T^\delta}{\bar{S}_T^{\delta_*}} - \frac{S_0}{\bar{S}_0^{\delta_*}}\right)\right\} \Big|\, \mathcal{A}_0\right). \end{aligned} \tag{11.3.20}$$

First let us solve a static optimization problem. This is an optimization that maximizes in (11.3.20), the expression under the conditional expectation on the right hand side of the last inequality, with respect to $\bar{S}_T^\delta$. One can read off the corresponding first order condition

$$U'\left(\bar{S}_T^\delta\right) - \frac{\lambda}{\bar{S}_T^{\delta_*}} = 0, \tag{11.3.21}$$

which for $\lambda > 0$ characterizes a maximum since U is concave, $U'(0) = \infty$ and $U'(\infty) = 0$. Note that due to the strict concavity of U its derivative U' has an inverse function U'^{-1}. By applying the inverse function U'^{-1} of U' it follows from (11.3.21) that the value

$$\bar{S}_T^\delta = U'^{-1}\left(\frac{\lambda}{\bar{S}_T^{\delta_*}}\right) \tag{11.3.22}$$

is the candidate for the optimal value of the discounted portfolio at time T that an expected utility maximizer should replicate. It is very important to realize that this candidate value turns out to be a function of the discounted GOP value. In principle, we face now a hedging problem that replicates via $\bar{S}^\delta$ the payoff given in (11.3.22).

2. Since $U'^{-1} : [0,\infty] \to [0,\infty]$, it makes only sense to consider in the following strictly positive values of λ. Since S^δ is assumed to be a fair portfolio one needs by (11.3.22) to choose the constant $\lambda \in (0,\infty)$ such that

$$\frac{S_0}{S_0^{\delta_*}} = \frac{S_0^\delta}{S_0^{\delta_*}} = E\left(\frac{S_T^\delta}{S_T^{\delta_*}} \,\Big|\, \mathcal{A}_0\right) = E\left(\frac{\bar{S}_T^\delta}{\bar{S}_T^{\delta_*}} \,\Big|\, \mathcal{A}_0\right) = E\left(U'^{-1}\left(\frac{\lambda}{\bar{S}_T^{\delta_*}}\right)\frac{1}{\bar{S}_T^{\delta_*}} \,\Big|\, \mathcal{A}_0\right). \tag{11.3.23}$$

Due to the properties of U given in Definition 11.3.1, it follows that there exists a $\lambda \in (0,\infty)$ such that (11.3.23) holds. Note that for very small $\lambda > 0$ one obtains extremely large payoffs $U'^{-1}(\frac{\lambda}{\bar{S}_T^{\delta_*}})\frac{1}{\bar{S}_T^{\delta_*}}$. With (11.3.23) we have identified a candidate value for an expected utility maximizing portfolio.

3. We now show that there is a strategy $\tilde{\delta}$ that replicates with its benchmarked portfolio value $\hat{S}_T^{\tilde{\delta}}$ the payoff $U'^{-1}(\frac{\lambda}{\bar{S}_T^{\delta_*}})\frac{1}{\bar{S}_T^{\delta_*}}$ in (11.3.22), such that $\bar{S}^{\tilde{\delta}} \in \bar{\mathcal{V}}_{S^0}^+$. Since the benchmarked savings account $\hat{S}_t^0 = (\bar{S}_t^{\delta_*})^{-1}$ forms a Markov process we obtain the $(\underline{\mathcal{A}}, P)$-martingale $\hat{u}(\cdot, \hat{S}_\cdot^0) = \{\hat{u}(t, \hat{S}_t^0),\, t \in [0,T]\}$ with

$$\hat{u}(t,\hat{S}_t^0) = \hat{S}_t^{\tilde{\delta}} = E\left(U'^{-1}\left(\frac{\lambda}{\bar{S}_T^{\delta_*}}\right)\frac{1}{\bar{S}_T^{\delta_*}} \,\Big|\, \mathcal{A}_t\right) = E\left(U'^{-1}\left(\lambda\,\hat{S}_T^0\right)\hat{S}_T^0 \,\Big|\, \mathcal{A}_t\right) \tag{11.3.24}$$

for $t \in [0,T]$. Here $\hat{u}(t,\hat{S}_t^0)$ is a function of t and $\hat{S}_t^0$ only, which can be identified via the Feynman-Kac formula (9.7.3). By application of the Itô formula and using the martingale property of $\hat{u}(\cdot,\hat{S}_\cdot^0)$ we obtain

$$d\hat{u}(t,\hat{S}_t^0) = \frac{\partial \hat{u}(t,\hat{S}_t^0)}{\partial \hat{S}^0}\, d\hat{S}_t^0$$

for $t \in [0,\infty)$. Hence, one can form the locally optimal portfolio $S^{\tilde{\delta}}$ by investing at time t in $\hat{u}(t,\hat{S}_t^0) - \hat{S}_t^0 \frac{\partial \hat{u}(t,\hat{S}_t^0)}{\partial \hat{S}^0}$ units of the GOP and investing the remaining wealth in $\frac{\partial \hat{u}(t,\hat{S}_t^0)}{\partial \hat{S}^0}$ units of the savings account. Note that we have

$$\bar{S}_t^{\tilde{\delta}} = \frac{\hat{S}_t^{\tilde{\delta}}}{\hat{S}_t^0} = \frac{\hat{u}(t,\hat{S}_t^0)}{\hat{S}_t^0}.$$

Consequently, the discounted, locally optimal portfolio $\bar{S}_t^{\tilde{\delta}}$ satisfies the SDE

$$
\begin{aligned}
d\bar{S}_t^{\tilde{\delta}} &= \hat{u}(t,\hat{S}_t^0)\, d\bar{S}_t^{\delta_*} + \bar{S}_t^{\delta_*}\, d\hat{u}(t,\hat{S}_t^0) + d[\bar{S}^{\delta_*},\hat{u}]_t \\
&= \left(\hat{u}(t,\hat{S}_t^0) - \hat{S}_t^0\, \frac{\partial \hat{u}(t,\hat{S}_t^0)}{\partial \hat{S}^0}\right) d\bar{S}_t^{\delta_*} \\
&= \bar{S}_t^{\tilde{\delta}} \left(\hat{u}(t,\hat{S}_t^0) - \hat{S}_t^0\, \frac{\partial \hat{u}(t,\hat{S}_t^0)}{\partial \hat{S}^0}\right) \frac{\bar{S}_t^{\delta_*}}{\bar{S}_t^{\tilde{\delta}}}\, \theta(t,\bar{S}_t^{\delta_*}) \left(\theta(t,\bar{S}_t^{\delta_*})\, dt + dW_t\right) \\
&= \bar{S}_t^{\tilde{\delta}} \left(J_t^{\tilde{\delta}}\right)^{-1} \theta(t,\bar{S}_t^{\delta_*}) \left(\theta(t,\bar{S}_t^{\delta_*})\, dt + dW_t\right),
\end{aligned}
$$

where $\hat{u}(t,\hat{S}_t^0) = \frac{\bar{S}_t^{\tilde{\delta}}}{\bar{S}_t^{\delta_*}}$, with risk aversion coefficient

$$
J_t^{\tilde{\delta}} = \frac{\hat{u}(t,\hat{S}_t^0)}{\hat{u}(t,\hat{S}_t^0) - \hat{S}_t^0\, \frac{\partial \hat{u}(t,\hat{S}_t^0)}{\partial \hat{S}^0}} = \left(1 - \frac{\hat{S}_t^0}{\hat{u}(t,\hat{S}_t^0)}\, \frac{\partial \hat{u}(t,\hat{S}_t^0)}{\partial \hat{S}^0}\right)^{-1}
$$

for $t \in [0,\infty)$.

4. It follows from (11.3.24) that $\hat{S}^{\tilde{\delta}}$ is a martingale. Furthermore, we note by the nonnegativity of U'^{-1} that $S^{\tilde{\delta}}$ is nonnegative. The solution that has been obtained must be shown to belong to the set $\bar{\mathcal{V}}_{S_0}^+$. For this purpose it suffices to show that equality holds in (11.3.20). This is achieved by observing that for positive λ, satisfying (11.3.9), one has

$$
\begin{aligned}
&E\left(\max_{\bar{S}_T^{\delta}>0} \left\{U\left(\bar{S}_T^{\delta}\right) - \lambda\left(\frac{\bar{S}_T^{\delta}}{\bar{S}_T^{\delta_*}} - \frac{S_0}{\bar{S}_0^{\delta_*}}\right)\right\} \Bigg| \mathcal{A}_0\right) \\
&\quad = E\left(U\left(U'^{-1}\left(\frac{\lambda}{\bar{S}_T^{\delta_*}}\right)\right) - \lambda\left(\frac{U'^{-1}\left(\frac{\lambda}{\bar{S}_T^{\delta_*}}\right)}{\bar{S}_T^{\delta_*}} - \frac{S_0}{\bar{S}_0^{\delta_*}}\right) \Bigg| \mathcal{A}_0\right) \\
&\quad = E\left(U\left(\bar{S}_T^{\tilde{\delta}}\right) - \lambda\left(\frac{\bar{S}_T^{\tilde{\delta}}}{\bar{S}_T^{\delta_*}} - \frac{S_0}{\bar{S}_0^{\delta_*}}\right) \Bigg| \mathcal{A}_0\right) \\
&\quad = E\left(U\left(\bar{S}_T^{\tilde{\delta}}\right) \Big| \mathcal{A}_0\right) = v^{\tilde{\delta}}. \qquad \square
\end{aligned}
$$

11.4 Pricing Nonreplicable Payoffs

This section addresses the problem of pricing nonreplicable payoffs. These are payoffs that cannot be replicated by a fair portfolio of primary security accounts. By utility indifference pricing we shall show that the concept of real world pricing naturally applies to the pricing of nonreplicable payoffs.

Utility Indifference Price

In the following we shall continue to use our notation of the previous sections, in particular Sect. 11.3, which considered expected utility maximization in the framework of a CFM. Let us assume that the investor uses the utility function U with time horizon $T \in (0, \infty)$, as defined in Definition 11.3.1. The investor has the total tradable wealth $S_t^{\tilde{\delta}}$ accumulated at time $t \in [0, \infty)$, which she or he invests according to an expected utility maximizing strategy $\tilde{\delta}$, see Sect. 11.1.

We consider now the problem that the investor has to price a random, discounted, nonnegative payoff $\bar{H}$ that is $\mathcal{A}_T$-measurable and delivered at the same time T which determines the time horizon for the expected utility function. We allow $\bar{H}$ to be nonreplicable. This means that the discounted payoff $\bar{H}$ or parts of it cannot be replicated by a fair portfolio of primary security accounts. Let us assume that the total face value of the discounted payoff that the investor wants to purchase is vanishing small, that is, it amounts to $\varepsilon \bar{H}$ where $\varepsilon \ll 1$ is a very small real number.

We aim to identify a consistent price for the above payoff at time $t = 0$ from the viewpoint of the expected utility maximizer. For this purpose we apply the concept of *utility indifference pricing*. This is a classical economic concept that has been generating renewed interest in continues time finance due to the important work in Davis (1997). The utility indifference price is the price at which the investor is indifferent between buying the contract that provides the discounted payoff $\varepsilon \bar{H}$, or not accepting the price when taking her or his expected utility maximization objective into account.

Consider now a contract that can be purchased for a hypothetical price V at time $t = 0$ and which delivers the discounted payoff $\bar{H}$ at maturity $T \in (0, \infty)$. Assume that the investor buys a vanishing fraction $\varepsilon \ll 1$ of the contract at time $t = 0$ for the amount εV. This corresponds to the price V at time $t = 0$ per total contract. She or he continues to invest the bulk of the wealth with her or his locally optimal strategy $\tilde{\delta}$, determined by the expected utility maximization for the utility function $U(\cdot)$ with time horizon T. Similarly to (11.3.3)–(11.3.4) we introduce the expected utility function

$$v^{\tilde{\delta}}_{\varepsilon,V} = E\left(U\left((S_0 - \varepsilon\, V)\, \frac{\bar{S}^{\tilde{\delta}}_T}{S_0} + \varepsilon \bar{H} \right) \Bigg| \mathcal{A}_0 \right) \tag{11.4.1}$$

for $\varepsilon \geq 0$. Here $S_0 - \varepsilon V$ is invested at time $t = 0$ in a portfolio which starts at one and follows the locally optimal strategy $\tilde{\delta}$. At the delivery date T the discounted payoff $\varepsilon \bar{H}$ is added to the discounted payoff $(S_0 - \varepsilon V)\frac{\bar{S}^{\tilde{\delta}}_T}{S_0}$ of the investment in the locally optimal portfolio. Note that the purchasing price εV is at time $t = 0$ subtracted from the locally optimal portfolio value. This allows us to formulate the following definition of a utility indifference price.

Definition 11.4.1. *In the above framework the value V is called the* utility indifference price *for the discounted payoff $\bar{H}$ if*

$$\lim_{\varepsilon \to 0} \frac{v^{\tilde{\delta}}_{\varepsilon,V} - v^{\tilde{\delta}}_{0,V}}{\varepsilon} = 0 \tag{11.4.2}$$

almost surely.

This means that the maximized expected utility of the investor changes only by a small amount for prices that are in the neighborhood of the utility indifference price. To see the structure of the resulting expected utility more clearly, let us derive from (11.4.1), by a Taylor expansion, the representation

$$\begin{aligned} v^{\tilde{\delta}}_{\varepsilon,V} &\approx E\left(U\left(\bar{S}^{\tilde{\delta}}_T\right) + U'\left(\bar{S}^{\tilde{\delta}}_T\right) \varepsilon \left(\bar{H} - V \frac{\bar{S}^{\tilde{\delta}}_T}{S_0}\right) \Big| \mathcal{A}_0 \right) \\ &= v^{\tilde{\delta}}_{0,V} + \varepsilon\, E\left(U'\left(\bar{S}^{\tilde{\delta}}_T\right) \left(\bar{H} - V \frac{\bar{S}^{\tilde{\delta}}_T}{S_0}\right) \Big| \mathcal{A}_0 \right). \end{aligned} \tag{11.4.3}$$

Here we neglect higher order terms in ε, assuming appropriate conditions. This expansion allows us to identify the utility indifference price. It is clear that for particular dynamics and specific utility functions, as well as payoffs, one needs to check whether the above expansion applies.

Utility Indifference Pricing Formula

When appropriate conditions are imposed, one can derive for a given utility function, discounted payoff $\bar{H}$ and prescribed market dynamics a corresponding utility indifference price. What is needed in such a derivation are sufficient integrability and smoothness properties. For instance, for a BS model and power utility such properties are guaranteed. For the utility indifference price we derive its general formula heuristically by indicating the crucial steps for its derivation without formulating any assumptions. However, this can be done for particular classes of models, utilities and payoffs. The general result that we shall obtain below will always be the same.

We obtain from the expansion (11.4.3) the relation

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(v^{\tilde{\delta}}_{\varepsilon,V} - v^{\tilde{\delta}}_{0,V} \right) = E\left(U'\left(\bar{S}^{\tilde{\delta}}_T\right) \left(\bar{H} - V \frac{\bar{S}^{\tilde{\delta}}_T}{S_0}\right) \Big| \mathcal{A}_0 \right). \tag{11.4.4}$$

We emphasize that $S^{\tilde{\delta}}$ is here the locally optimal portfolio that maximizes the given expected utility when ε is set to zero. From equation (11.4.4) and Definition 11.4.1 we obtain then the *utility indifference pricing formula* in the form

$$V = \frac{E\left(U'\left(\bar{S}^{\tilde{\delta}}_T\right) \bar{H} \,\big|\, \mathcal{A}_0 \right)}{E\left(U'\left(\bar{S}^{\tilde{\delta}}_T\right) \frac{\bar{S}^{\tilde{\delta}}_T}{S_0} \,\big|\, \mathcal{A}_0 \right)}. \tag{11.4.5}$$

This formula holds rather generally. It allows us to determine the utility indifference price for a given utility and given discounted payoff $\bar{H}$. We emphasize that the payoff is possibly not replicable. If it were replicable, then the minimal price for replicating this payoff is the fair price, which is given by the real world pricing formula.

Real World Pricing of Nonreplicable Payoffs

For a general payoff H and a general utility function $U(\cdot)$ we obtain under the assumptions of Theorem 11.3.3 by (11.3.8) that

$$\bar{S}_T^{\tilde{\delta}} = U'^{-1}\left(\frac{\lambda}{\bar{S}_T^{\delta_*}}\right). \tag{11.4.6}$$

It follows from formula (11.4.5) that in a surprisingly simple way U' and U'^{-1} offset each other in the following calculation

$$V = \frac{E\left(U'\left(U'^{-1}\left(\frac{\lambda}{\bar{S}_T^{\delta_*}}\right)\right)\bar{H}\,\Big|\,\mathcal{A}_0\right)}{E\left(U'\left(U'^{-1}\left(\frac{\lambda}{\bar{S}_T^{\delta_*}}\right)\right)\frac{\bar{S}_T^{\tilde{\delta}}}{S_0}\,\Big|\,\mathcal{A}_0\right)} = \frac{E\left(\frac{\lambda}{\bar{S}_T^{\delta_*}}\,\bar{H}\,\Big|\,\mathcal{A}_0\right)}{E\left(\frac{\lambda}{\bar{S}_T^{\delta_*}}\,\frac{\bar{S}_T^{\tilde{\delta}}}{S_0}\,\Big|\,\mathcal{A}_0\right)}.$$

Therefore, we obtain with (11.3.8) for V the expression

$$V = \frac{E\left(\frac{H}{S_T^{\delta_*}}\,\Big|\,\mathcal{A}_0\right)}{\frac{1}{S_0}\,E\left(\hat{S}_T^{\tilde{\delta}}\,\big|\,\mathcal{A}_0\right)} = \frac{E\left(\frac{H}{S_T^{\delta_*}}\,\Big|\,\mathcal{A}_0\right)}{\frac{1}{S_0}\,\hat{u}(0,\hat{S}_0^0)} = \frac{E\left(\frac{H}{S_T^{\delta_*}}\,\Big|\,\mathcal{A}_0\right)}{\frac{S_0}{S_0\,S_0^{\delta_*}}}. \tag{11.4.7}$$

For the utility indifference price this yields by (11.3.9) the relation

$$V = S_0^{\delta_*}\,E\left(\frac{H}{S_T^{\delta_*}}\,\Big|\,\mathcal{A}_0\right). \tag{11.4.8}$$

We observe that this is the real world pricing formula (9.1.30). This means that under utility indifference pricing payoffs, which are not replicable by a fair portfolio of primary security accounts, are priced according to the real world pricing formula. Most importantly, we see that the utility indifference price does *not* depend on the utility function of the investor.

This is a very satisfying result not only from the theoretical but also from the practical point of view. It extends real world pricing naturally to the case of general nonreplicable payoffs. From a practical viewpoint it gives the buyer and the seller an acceptable price for any nonreplicable payoff.

11.5 Hedging

One important feature of a market is the possibility to hedge future uncertainties. In this section we study the hedging of uncertain payoffs.

Hedge Portfolios

In the following we consider a CFM $\mathcal{S}^{\mathrm{C}}_{(d)}$, as defined in Sect. 10.1, and discuss the problem of hedging. Let $\tau \in (0, \infty)$ be a bounded stopping time and H a nonnegative payoff that is paid at τ. By generalizing (8.2.8), we say that a portfolio S^δ *replicates* a nonnegative payoff H_τ paid at a stopping time τ if

$$S^\delta_\tau = H \tag{11.5.1}$$

almost surely. Note that a general payoff can always be decomposed into its nonnegative and its negative part and considering nonnegative payoffs is therefore no restriction. As previously, a nonnegative payoff is replicable if there exists a nonnegative, replicating fair portfolio. We shall demonstrate later that there may exist several self-financing portfolios in a CFM that replicate a given nonnegative payoff. In the case of nonnegative replicating portfolios it follows from Corollary 10.4.2 that for a nonnegative payoff H the replicating, fair portfolio S^{δ_H} is the minimal portfolio that replicates H. Note that this portfolio process is uniquely determined as a value process. However, there may be different securities that can be used for hedging.

Tradable Martingale Representation

For a nonnegative replicable payoff the real world pricing formula provides the minimal nonnegative price process. From an economic point of view it is in a competitive market the correct price process. We shall determine below the strategy of the fair portfolio which hedges a given replicable nonnegative payoff. Recall that the benchmarked fair price process forms an $(\underline{\mathcal{A}}, P)$-martingale. It is of primary interest to find a representation for this martingale process. There are various methods that can be used to find the martingale representation of a benchmarked nonnegative payoff.

For instance, under the standard BS model, which we used for illustration in Chap. 9, we obtained in (9.1.31) a corresponding martingale representation for the benchmarked European call option payoff. It was derived from the real world pricing formula together with an application of the Itô formula to the benchmarked pricing function.

More generally, in the case when the market dynamics can be expressed via a set of Markovian factor processes, then one can apply the Feynman-Kac formula, see Sect. 9.7. This yields the benchmarked, fair pricing function of a corresponding replicable payoff. The corresponding benchmarked, fair price process forms then a martingale and similarly to (9.1.31), a real world martingale representation. We shall not present in this section any particular example for such a martingale representation. However, the following two chapters will discuss several such examples.

In a CFM not all Wiener processes which drive volatility processes and short rates need to represent trading uncertainty. Therefore, in general, not all payoffs are replicable. Under rather general assumptions one can usually establish a real world martingale representation for reasonable payoffs in a CFM. The particular structure of a martingale representation depends strongly on the model dynamics and can become quite complex for certain payoffs. As indicated, in a Markovian setting one can explicitly derive, via the Feynman-Kac formula, martingale representations for nonreplicable payoffs.

The following definition of a *tradable martingale representation* allows us to formulate general results on pricing and hedging of particular payoffs without specifying the dynamics of the CFM.

Definition 11.5.1. *We say that a given $\mathcal{A}_\tau$-measurable, nonnegative payoff H, which matures at a bounded stopping time τ, has a* tradable martingale representation *if there exists a predictable vector process $\boldsymbol{x}_H = \{\boldsymbol{x}_H(t) = (x_H^1(t), \ldots, x_H^d(t))^\top,\ t \in [0, \tau]\}$, where*

$$\int_0^\tau \sum_{k=1}^d \left(x_H^k(s)\right)^2 ds < \infty \tag{11.5.2}$$

almost surely such that

$$\frac{H}{S_\tau^{\delta_*}} = \hat{U}_H(t) + \sum_{k=1}^d \int_t^\tau x_H^k(s)\, dW_s^k \tag{11.5.3}$$

almost surely with

$$\hat{U}_H(t) = E\left(\frac{H}{S_\tau^{\delta_*}} \,\middle|\, \mathcal{A}_t\right) < \infty \tag{11.5.4}$$

for all $t \in [0, \tau]$.

Note that the above tradable martingale representation (11.5.3) is expressed with respect to trading uncertainty, that is with respect to the Wiener processes $W^1, \ldots, W^d$. There are, in general, other sources of uncertainty in the market that are not securitized and therefore not tradable. Consequently, there exist, in general, nonnegative payoffs which do not have a tradable martingale representation. We shall see below that such payoffs are not fully replicable.

Hedging Strategy

By using the above notion of a tradable martingale representation we prove the following result on the hedging of derivatives. In the corresponding proof, which is given at the end of this section, we use the SDE (10.3.2) of a benchmarked portfolio together with (11.5.3).

Theorem 11.5.2. *For a nonnegative payoff H with a tradable martingale representation there exists a replicating, fair portfolio S^{δ_H}, which satisfies at time $t \in [0, \tau]$ the real world pricing formula*

$$S_t^{\delta_H} = S_t^{\delta_*} \hat{U}_H(t) \tag{11.5.5}$$

with $\hat{U}_H(t)$ given in (11.5.4). This portfolio has the vector of fractions

$$\boldsymbol{\pi}_{\delta_H}(t) = \left(\boldsymbol{b}_{\delta_H}(t)^\top \boldsymbol{b}_t^{-1}\right)^\top, \tag{11.5.6}$$

where the vector $\boldsymbol{b}_{\delta_H}(t) = (b^1_{\delta_H}(t), \ldots, b^d_{\delta_H}(t))^\top$ of portfolio volatilities has kth component

$$b^k_{\delta_H}(t) = \sum_{j=1}^d \frac{\delta^j_H(t)\, \hat{S}^j_t}{\hat{U}_H(t)}\, b^{j,k}_t = \frac{x^k_H(t)}{\hat{U}_H(t)} + \theta^k_t \tag{11.5.7}$$

for $t \in [0, \tau]$ and $k \in \{1, 2, \ldots, d\}$.

Theorem 11.5.2 states that a nonnegative payoff with tradable martingale representation can be replicated. It also characterizes the minimal hedge portfolio. We emphasize here again that for a CFM, which is built as a Markovian factor model, one can obtain by the Feynman-Kac formula for each integrable benchmarked payoff a corresponding martingale representation. This makes it advisable to prefer Markovian factor models if one aims to construct computationally tractable CFMs.

Martingale Representation Theorem (*)

By the following result we shall see that payoffs can be decomposed into the sum of their hedgable part and their unhedgable part. Let us mention a *Martingale Representation Theorem*, for a proof see Karatzas & Shreve (1991), which is convenient for establishing a real world martingale representation for payoffs in a wide range of CFMs.

Theorem 11.5.3. (Martingale Representation Theorem) *For $T \in [0, \infty)$ assume that in a CFM $\mathcal{S}^C_{(d)}$ with given filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$ the filtration $\underline{\mathcal{A}}$ is the augmentation under P of the natural filtration $\mathcal{A}^W$ generated by the vector $\boldsymbol{W} = \{\boldsymbol{W}_t = (W^1_t, \ldots, W^m_t)^\top,\ t \in [0, T]\}$ of Wiener processes, $m \in \{d, d+1, \ldots\}$. Then for any square integrable benchmarked fair price process $\hat{V}_t = \{\hat{V}_t = \frac{V_t}{S_t^{\delta_*}},\ t \in [0, T]\}$ there exists a predictable, measurable process $\boldsymbol{x}_{V_T} = \{\boldsymbol{x}_{V_T}(t) = (x^1_{V_T}(t), \ldots, x^d_{V_T}(t))^\top,\ t \in [0, T]\}$ such that*

$$E\left(\int_0^T \left(x^k_{V_T}(s)\right)^2 ds\right) < \infty \tag{11.5.8}$$

for $k \in \{1, 2, \ldots, d\}$ and

$$\hat{V}_t = \hat{V}_0 + \sum_{k=1}^{m} \int_0^t x^k_{V_T}(s)\, dW^k_s \tag{11.5.9}$$

for $t \in [0,T]$, *where* $\hat{V}$ *is almost surely continuous. Furthermore, if* $\tilde{x}^k = \{\tilde{x}^k(t),\, t \in [0,T],\, k \in \{1,2,\dots,d\}$, *are any other predictable measurable processes satisfying* (11.5.8) *and* (11.5.9), *then*

$$\int_0^T \sum_{k=1}^{m} \left| x^k_{V_T}(s) - \tilde{x}^k(s) \right|^2 ds = 0 \tag{11.5.10}$$

almost surely.

Real World Martingale Decomposition (*)

Note that under the assumptions of the above theorem one has for any square integrable, benchmarked payoff $\hat{H} = \frac{H}{S^{\delta_*}_T}$ paid at time T, the unique representation (11.5.9), where

$$\hat{H} = \hat{V}_T = \hat{V}_0 + \sum_{k=1}^{m} \int_0^t x^k_H(s)\, dW^k_s. \tag{11.5.11}$$

It is essential to realize that Theorem 11.5.3 assumes that only the m Wiener processes $W^1,\dots,W^m$ generate the total uncertainty in the model. This is why we have chosen in Theorem 11.5.3 the filtration $\underline{\mathcal{A}}$ to be the augmentation of the natural filtration $\mathcal{A}^W$. The Wiener processes $W^1,\dots,W^d$ model the trading uncertainty.

The *real world martingale decomposition* of the nonnegative, square integrable benchmarked payoff $\hat{H}$ is then given by the sum

$$\hat{H} = \hat{H}_h + \hat{H}_u. \tag{11.5.12}$$

It consists of its *hedgable part* $\hat{H}_h = \hat{U}_{H_h}(T)$, which we obtain at time t as

$$\hat{U}_{H_h}(t) = \hat{U}^{(0)}_{H_h} + \sum_{k=1}^{d} \int_0^t x^k_H(s)\, dW^k_s \tag{11.5.13}$$

and its *unhedgable part* $\hat{H}_u = \hat{U}_{H_u}(T)$, which at time $t \in [0,T]$ is

$$\hat{U}_{H_u}(t) = \sum_{k=d+1}^{m} \int_0^t x^k_H(s)\, dW^k_s. \tag{11.5.14}$$

The hedgable part $\hat{H}_h$ can be replicated according to Theorem 11.5.2. We use in (11.5.13) for the nonnegative payoff H its benchmarked fair price $\hat{U}_{H_h}(0)$ at time $t = 0$, that is,

$$\hat{U}_{H_h}(0) = E\left(\hat{H} \,\middle|\, \mathcal{A}_0\right). \tag{11.5.15}$$

Then the benchmarked value

$$\hat{V}_t = \hat{U}_{H_h}(t) + \hat{U}_{H_u}(t) \tag{11.5.16}$$

corresponds to a fair process since it forms an $(\underline{\mathcal{A}}, P)$-martingale. As we have seen in Sect. 5.1, this martingale minimizes the expected least squares error of the benchmarked hedge. The choice of the real world pricing formula for the unhedgable part appears, therefore, as a projection in a least square sense. More precisely, the benchmarked fair price $\hat{V}_0$ can be interpreted as the projection of the benchmarked payoff into the space of $\mathcal{A}_0$-measurable, tradable portfolio values. Note that the benchmarked fair price $\hat{U}_{H_u}(0)$ of the unhedgable part is zero at time $t = 0$.

This means, when applying real world pricing for a payoff one is leaving its unhedgable part totally untouched. This is reasonable because any extra trading would create unnecessary uncertainty and potential costs. The benchmarked unhedgable part has according to (11.5.14) zero conditional expectation

$$E\left(\hat{U}_{H_u}(T) \,\middle|\, \mathcal{A}_0\right) = 0. \tag{11.5.17}$$

In summary, we obtain from (11.5.12)–(11.5.17) for the benchmarked payoff $\hat{H}$ payable at time T the real world martingale decomposition

$$\hat{H} = \hat{U}_{H_h}(0) + \sum_{k=1}^{d} \int_0^T x_H^k(s)\, dW_s^k + \sum_{k=d+1}^{m} \int_0^T x_H^k(s)\, dW_s^k. \tag{11.5.18}$$

Let us indicate that by pooling a wide variety of independent unhedgable parts of payoffs, under appropriate integrability conditions, the Law of Large Numbers, see Sect. 2.1, makes their impact vanishing. For instance, the books of large investment banks and insurance companies pool substantial unhedgable payoffs and benefit from this effect.

Föllmer-Schweizer Decomposition (*)

In the case when an equivalent risk neutral probability measure exists in a CFM, real world pricing coincides with the risk neutral pricing obtained under the, so-called, *minimal equivalent martingale measure* of Föllmer and Schweizer, see Föllmer & Schweizer (1991), Hofmann, Platen & Schweizer (1992) and Heath, Platen & Schweizer (2001).

It has been shown in Föllmer & Schweizer (1991), by assuming the existence of an equivalent risk neutral probability measure P_θ, that the hedging of a payoff is linked to the existence of a corresponding martingale representation under P_θ for the discounted payoff. This important representation is known as the *Föllmer-Schweizer decomposition*, see Schweizer (1995). A similar decomposition exists for the benchmarked payoff in a general CFM, where one does not require the existence of an equivalent risk neutral probability measure.

To formulate explicitly the Föllmer-Schweizer decomposition we consider a CFM as assumed in Theorem 11.5.3 and multiply both sides of the representation (11.5.16) by the discounted GOP value and apply then the Itô formula. This provides the decomposition

$$\begin{aligned}\bar{H} &= \hat{H}\,\bar{S}_T^{\delta_*}\\ &= \hat{U}_{H_h}(0)\,\bar{S}_0^{\delta_*} + \sum_{k=1}^{d}\int_0^T \bar{S}_t^{\delta_*}\left(x_H^k(t) + \left(\hat{U}_{H_h}(t) + \hat{U}_{H_u}(t)\right)\theta_t^k\right)\\ &\qquad \times \left(\theta_t^k\,dt + dW_t^k\right) + \sum_{k=d+1}^{m}\int_0^T \bar{S}_t^{\delta_*}\,x_H^k(t)\,dW_t^k, \qquad (11.5.19)\end{aligned}$$

which is a Föllmer-Schweizer decomposition for the discounted payoff $\bar{H}$, see Schweizer (1995) and Exercise 11.3.

Note that in the case when a risk neutral probability measure P_θ exists, then the second term in the sum on the right hand side of (11.5.19) is a martingale under P_θ. The third term is then a martingale under P and under P_θ. The, so-called, *minimal equivalent martingale measure*, see Schweizer (1995), changes only the drift of the Wiener processes that model trading uncertainty. The other sources of uncertainty remain unchanged.

Complete Market (*)

In the literature one is often using the notion of a *complete market*, which we introduce now.

Definition 11.5.4. *A CFM where all integrable, benchmarked nonnegative payoffs have a tradable martingale representation in the sense of Definition 11.5.1, is called a* complete CFM. *Any other CFM we call* incomplete.

Note that in some literature a market is called complete when a unique equivalent risk neutral probability measure exists, see Harrison & Kreps (1979) and Harrison & Pliska (1981, 1983). As we have seen in Sect. 10.3, there is no economic necessity to insist on the existence of an equivalent risk neutral probability measure. Therefore, we defined here the completeness of a market in a more practical way. By Theorem 11.5.2 we obtain now directly the following result.

Corollary 11.5.5. *In a complete CFM all integrable, nonnegative payoffs can be perfectly replicated with the hedge portfolio characterized by relation* (11.5.6)*. The price for setting up this replicating portfolio at some time $t \in [0, \tau)$ is obtained by the real world pricing formula* (9.1.30)*. This price is the minimal price that permits the replication of a given payoff.*

This result emphasizes the fact that in a complete market all integrable payoffs can be replicated. An equivalent risk neutral probability measure is not required for the existence of a complete market. This is very important from a practical point of view when hedging derivatives for advanced models, as we shall see later. We have seen in our previous discussion, if the CFM is incomplete, then one can still perfectly replicate the hedgable part of a benchmarked payoff. Under real world pricing one leaves the unhedgable part as it is.

Proof of Theorem 11.5.2 (*)

For a given payoff H, paid at a bounded stopping time $\tau \geq 0$, with tradable martingale representation we use the martingale representation (11.5.3). This leads us for a benchmarked hedge portfolio $\hat{S}^{\delta_H}$, see (10.3.2), to the replication condition

$$\frac{H}{S^{\delta_*}_\tau} - \hat{U}_H(t) = \sum_{k=1}^{d} \int_t^\tau x^k_H(s)\, dW^k_s$$
$$= \sum_{k=1}^{d} \int_t^\tau \hat{S}^{\delta_H}_s \left(b^k_{\delta_H}(s) - \theta^k_s\right) dW^k_s = \hat{S}^{\delta_H}_\tau - \hat{S}^{\delta_H}_t \quad (11.5.20)$$

for $t \in [0, \tau]$. The formulas (10.3.2), (11.5.7) and (10.1.12) provide by direct comparison of the integrands in (11.5.20) the equation

$$\left(\boldsymbol{\pi}^\top_{\delta_H}(t)\, \boldsymbol{b}_t\right)^\top = \boldsymbol{b}_{\delta_H}(t)$$

for $t \in [0, \tau]$. By the invertibility of $\boldsymbol{b}_t$, see Assumption 10.1.1, this proves (11.5.6), and thus with (11.5.1) equation (11.5.5). □

11.6 Exercises for Chapter 11

11.1. Calculate the maximum expected log-utility for the BS model.

11.2. Compute the maximum expected power utility for the BS model.

11.3. (*) In the case when a risk neutral probability measure exists, derive by using the setup of Theorem 11.5.3 a representation for a discounted payoff $\bar{H} = \frac{H}{S^0_T}$, which is paid at time $T \in (0, \infty)$.