Boundary Element Methods for Eddy Current Computation

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Summary. This paper studies numerical methods for time-harmonic eddy current problems in the case of homogeneous, isotropic, and linear materials. It provides a survey of approaches that entirely rely on boundary integral equations and their conforming Galerkin discretization. Starting point are both **E**- and **H**-based strong formulation, for which issues of gauging and topological constraints on the existence of potentials are discussed.

Direct boundary integral equations and the so-called symmetric coupling of the integral equations corresponding to the conductor and the non-conducting regions are employed. They give rise to coupled variational problems that are elliptic in suitable trace spaces. This implies quasi-optimal convergence of conforming Galerkin boundary element methods, which make use of div_Γ-conforming trial spaces for surface currents.

1 Introduction

A great deal of electromagnetic field problems faced in an industrial context fall into the category of eddy current problems. This applies, for instance, for problems of inductive heating, magnetomechanical valves, and the computation of inductances of bulky conductors in power electronics.

The typical setting of eddy current problems involves a bounded conducting region Ω_c and its complement $\Omega_e := \mathbb{R}^3 \setminus \bar{\Omega}_c$, the non-conducting air region. Usually, Ω_e is supposed to have the electromagnetic properties of empty space ($\epsilon = \epsilon_0$, $\mu = \mu_0$), whereas Ω_c might be filled with some "complex" conducting material. In this paper we restrict our attention to the case of a simple, linear, homogeneous, and isotropic conductor characterized by a constant conductivity $\sigma > 0$ and permeability $\mu_c > 0$. This can be a reasonable approximation for a non-ferromagnetic material like aluminum.

In eddy current simulations the shape of the conductor is usually provided in some CAD format. Therefore, we can take for granted that the surface of Ω_c is piecewise smooth and consists of a few curved faces. In mathematical terms, Ω_c is a curvilinear Lipschitz polyhedron in the sense of [30, Sect. 1]. All the developments of this paper refer to such a geometric setting.

We restrict ourselves to time harmonic current excitation with angular frequency $\omega > 0$. Hence, thanks to the assumed linearity of all materials involved, temporal Fourier transform allows reduction to pure spatial boundary value problems for the unknown complex amplitudes (phasors) of the electromagnetic fields. Two common types of exciting alternating currents will be taken into account:

1. The total current in a loop of the conductor is prescribed (non-local inductive excitation, [42, Sect. 5]). Here, by "loop" we mean a connected component of Ω_c, whose first Betti number is equal to 1. Homeomorphic images of a torus are typical examples, see Fig. 1 (left)
2. A driving force on charge carriers is modelled by a compactly supported generator current $\mathbf{j}_s$, which has to be divergence free everywhere [42, Sect. 3]. The case $\operatorname{supp}\mathbf{j}_s \cap \Omega_c cl = \emptyset$ describes excitation through a stranded inductor coil or antenna (inductive coupling), whereas $\operatorname{supp}\mathbf{j}_s \cap \Omega_c \neq \emptyset$ models wires feeding a current into Ω_c (galvanic coupling, see Fig. 1, right).

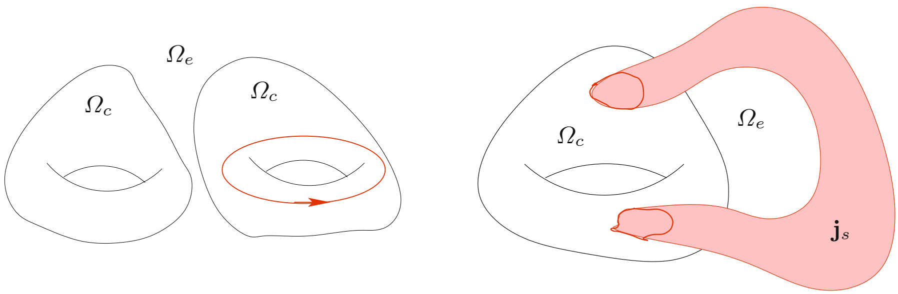

Fig. 1. Current excitations: prescribed total current in a conducting loop (left), generator current $\mathbf{j}_s$ (right). Note that $\mathbf{j}_s$ must be continued inside Ω_c in order to ensure $\operatorname{div}\mathbf{j}_s = 0$!

The goal of the numerical simulation may be the approximate computation of the total Ohmic losses in the conductors, and of the electromagnetic forces acting on the conductor. This entails discretizing the field equations and, in particular, coping with the *unbounded* part Ω_e of the generic computational domain $\mathbb{R}^3$. The standard approach is the finite element method [38], in which artificial homogeneous boundary conditions for the fields are imposed "sufficiently" far away from the conductor. This is justified by the decay properties of the fields, though it may be difficult to fix a viable cut-off distance a priori (see [5] for an adaptive procedure). After meshing the resulting bounded computational domain, the finite element discretization can proceed in the standard fashion. However, in case of a delicate shape of Ω_c, suitable finite element meshes may comprise a prohibitively large number of elements in Ω_e.

Boundary element methods (BEM) applied to the field equations steer clear of these difficulties, since they are based on integral equations posed on the surface $\Gamma := \partial\Omega$. These are only available for homogeneous equations with constant coefficients, but this is just the setting we take for granted (both in Ω_c and Ω_e). Consequently, boundary element methods that rely on a triangulation of Γ alone become an option and will be the focus of this presentation.

A central issue is how to couple the boundary integral equations associated with Ω_c and Ω_e. The basic coupling is provided by the transmission conditions for the electric and magnetic fields, more precisely, their tangential continuity. This still leaves many options, most of which lead to variational problems lacking useful structural properties.

The coupling challenge was first addressed in the context of linking domain based variational formulations with integral equations, a prerequisite for coupling finite elements (FEM) with boundary elements. In this context a breakthrough was achieved when M. Costabel in [27] introduced the so-called *symmetric coupling* by using the integral equations in the form of the Calderón projectors. This idea has been successfully extended to computational electromagnetism in [39, 40, 48].

Representations of Poincare-Steklov operators derived from Calderón projectors also guide the derivation of variational formulations involving only boundary integral equations on an interface [31, 63, 20], see [60, 50] for an application to domain decomposition. Here we aim to adapt these ideas to eddy current models. It turns out that surprising new aspects come into play, related to the issues of *gauging* and *topological obstructions*.

This paper deals with integral equations in variational form and their Galerkin discretization by means of boundary elements. We do not discuss "details" of implementation like computation of matrix entries [58, Ch. 5], matrix compression [58, Ch. 7], and boundary approximation, however important these topics are for a viable code. Instead we refer to the theses [54] and [10] for further information and numerical examples. We also gloss over the issue of how to construct fast iterative solvers for the resulting linear systems of equation. The reader may consult [39, Sect. 9] and [15, 23, 61].

2 Eddy Current Model

The behavior of an electromagnetic field is governed by Maxwell's equations. Instead of using these, in special situations simplified *quasistatic models* supply sufficiently accurate approximations to the true fields [33]. One of them is the eddy current model, representing a magneto-quasistatic approximation to Maxwell's equations in the sense that the electric field energy is neglected.

This model is reasonably accurate for *slowly varying* fields, for which the change in magnetic field energy is dominant [3, 33]. “Slowly varying”, means that

$$L\sqrt{\epsilon_0\,\mu_0}\,\omega \ll 1\,, \tag{1}$$

where L is the characteristic size of the region of interest: Ω_c has to be small compared to the wavelength of electromagnetic waves, which makes it possible to ignore wave propagation. There is a second condition for the validity of the eddy current approximation, requiring that the typical time-scale is long compared to the relaxation time for space charges, that is, the conductivity must be large enough so that

$$\omega\frac{\epsilon_0}{\sigma} \ll 1\,. \tag{2}$$

This implies that no space charges need to be taken into account. We point out that (1) and (2) provide a “rule of thumb”, but ignore the impact of geometry: in the presence of thin slots or gaps the eddy current approximation might become invalid locally [8, Ch. 8].

Formally, the eddy current model arises from Maxwell’s equations by dropping the displacement current $\mathbf{D}$. In the frequency domain the eddy current model for complex field amplitudes (for the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$) reads

$$\mathbf{curl}\,\mathbf{E} = -i\omega\mu\mathbf{H} \quad \text{in } \mathbb{R}^3\,, \quad \mathbf{curl}\,\mathbf{H} = \begin{cases} \sigma\mathbf{E} & \text{in } \Omega_c \\ \mathbf{j}_s & \text{in } \Omega_e \end{cases}\,. \tag{3}$$

According to the aforementioned assumptions, the permeability μ is constant $\equiv \mu_c$ in Ω_c and equal to μ_0 in the air region Ω_e. The conductivity σ is constant in Ω_c and set to zero in Ω_e. The first equation is called Faraday’s law, the second (reduced) Ampere’s law. These equations have to be supplemented by the decay conditions

$$\mathbf{H}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad \mathbf{E}(\mathbf{x}) = O(|\mathbf{x}|^{-1}) \quad \text{uniformly for } |\mathbf{x}| \to \infty\,. \tag{4}$$

Switching from the full Maxwell’s equations to the eddy current equations obviously involves a breach of the symmetry between electric and magnetic quantities. As a first consequence, we cannot expect a solution for $\mathbf{E}$ to be unique, because it can be altered by any gradient supported in Ω_e and will still satisfy the equations (3). The solution for $\mathbf{H}$ will not be affected. This reflects the fact that in a magnetoquasistatic model $\mathbf{E}$ is relegated to the role of a “fictitious quantity”. Imposing the constraints

$$\operatorname{div}\mathbf{E} = 0 \quad \text{in } \Omega_e \quad \text{and} \quad \int_{\Gamma_k} \mathbf{E}\cdot\mathbf{n}\,dS = 0\,, \tag{5}$$

where Γ_k, $k = 1,\dots,L$, are the connected components of Γ, will restore uniqueness of the solution for $\mathbf{E}$. Thus, one can single out a physically meaningful electric field in Ω_e [1]. However, this is rather a gauging procedure, i.e.

the selection of a representative from an equivalence class of meaningful fields [44], than part of the generic eddy current model. When devising a numerical scheme, we should target $\mathbf{H}$ as principal variable.

How can there be a role of the electric field in a magneto-quasistatic context? To understand this, recall that Faraday's law in strong form involves $\operatorname{div}(\mu\mathbf{H}) = 0$ everywhere. This makes it possible to introduce a magnetic vector potential $\mathbf{A}$ such that $\mathbf{curl}\,\mathbf{A} = \mu\mathbf{H}$ and to express $\mathbf{E}$ via a scalar potential Ψ as $\mathbf{E} = -\,\mathbf{grad}\,\Psi - i\mathbf{A}$. We have ample freedom to perform gauging and use it to set $\Psi = 0$. Thus, $\mathbf{E}$ turns out to be a scaled magnetic vector potential in disguise. I endorse this view as the proper reading of $\mathbf{E}$ in (3).

A second consequence of the magneto-quasistatic model reduction is the partial decoupling of electric and magnetic field in Ω_e. In fact, knowing $\mathbf{H}$ on Γ, we can solve a div-**curl** boundary value problem to obtain $\mathbf{H}$ and then, in light of (5), another div-**curl** problem will yield $\mathbf{E}$. Conversely, within the conductor, (3) permits the elimination of either $\mathbf{H}$ and $\mathbf{E}$, which leads to a second-order boundary value problem. The bottom line is, that in Ω_c and Ω_e we encounter elliptic systems of PDEs of different character. This will have profound consequences for the statement of transmission problems, see Sect. 5.

We finish this introduction to the eddy current model by explaining how to incorporate current excitation through *offset fields* $\mathbf{E}_s$ and $\mathbf{H}_s$. We demand

$$\begin{array}{cc} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E}_s = -i\omega\mu_0\mathbf{j}_s\ , & \mathbf{curl}\,\mathbf{H}_s = \mathbf{j}_s\ , \\ \operatorname{div}\mathbf{E}_s = 0\ , & \operatorname{div}\mathbf{H}_s = 0\ , \end{array} \quad \text{in } \Omega_e\ . \tag{6}$$

Such fields can be computed by evalutating the Newton potentials

$$\mathbf{E}_s(\mathbf{x}) = -i\omega\frac{\mu_0}{4\pi}\int_{\mathbb{R}^3}\frac{\mathbf{j}_s(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y}\ , \quad \mathbf{H}_s(\mathbf{x}) = \frac{1}{i\omega\mu_0}\,\mathbf{curl}\,\mathbf{E}_s\ , \tag{7}$$

provided that $\mathbf{j}_s$ has vanishing divergence everywhere in $\mathbb{R}^3$. In the case of thin wires represented by line currents, (7) amounts to the well-known Biot-Savart formula.

The requirements (6) impliy for the *reaction fields* $\mathbf{E}_r := \mathbf{E} - \mathbf{E}_s$, $\mathbf{H}_r := \mathbf{H} - \mathbf{H}_s$ that

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}_r = 0 \quad , \quad \mathbf{curl}\,\mathbf{H}_r = 0 \quad \text{in } \Omega_e\ . \tag{8}$$

In Ω_c we retain the original phasors $\mathbf{E}$, $\mathbf{H}$, often referrred to as *total fields*. By using offset fields the spatially distributed excitation $\mathbf{j}_s$ can be converted into an inhomogeneous jump condition across Γ for the fields. Spatial source terms are no longer present, which greatly facilitates the implementation of boundary element methods. The treatment of an excitation through a total loop current will be postponed until discretization is discussed in Sects. 7.2, 8.3.

3 Spaces and Traces

All developments in this paper will be consistently set in a variational framework. The Hilbert spaces, on which the variational approach rests, have a very concrete physical meaning as spaces of fields with finite energy. Let $\Omega \subset \mathbb{R}^3$ be a generic domain, not necessarily bounded. The natural Hilbert space for magnetic fields with finite total energy on Ω is

$$\boldsymbol{H}(\mathbf{curl}; \Omega) := \{\mathbf{V} \in \boldsymbol{L}^2(\Omega),\, \mathbf{curl}\,\mathbf{V} \in \boldsymbol{L}^2(\Omega)\}\ ,$$

equipped with the graph norm (cf. [36, Ch. 1]). In the context of the eddy current model the energy associated with the electric field is measured only by its **curl**. Of course, also the mean dissipated energy has to be finite, which entails square integrability over Ω_c, but in Ω_e the L^2-norm of the field need not be bounded. Therefore, weighted Beppo-Levi type spaces (cf. [35])

$$\boldsymbol{W}(\mathbf{curl}, \Omega) := \{\frac{\mathbf{V}(\mathbf{x})}{\sqrt{1+|\mathbf{x}|^2}} \in \boldsymbol{L}^2(\Omega),\, \mathbf{curl}\,\mathbf{V} \in \boldsymbol{L}^2(\Omega)\}$$

are the proper choice for $\mathbf{E}$. The property that their energy only depends on certain derivatives is characteristic for potentials. For them weighted spaces have to be used, for instance the standard Beppo Levi space (cf. [53, Sect. 2.5.4])

$$W^1(\Omega) := \{\frac{\Phi(\mathbf{x})}{\sqrt{1+|\mathbf{x}|^2}} \in \boldsymbol{L}^2(\Omega),\, \mathbf{grad}\,\Phi \in \boldsymbol{L}^2(\Omega)\}\ .$$

For each of the above spaces, the restrictions to Ω of smooth functions that are compactly supported in $\mathbb{R}^3$ form dense subsets.

Thanks to this density property we may wonder how to extend certain restrictions of smooth functions onto boundaries to continuous and surjective trace mappings. Now, assume that the boundary $\partial\Omega$ is compact and endowed with an exterior unit normal vectorfield $\mathbf{n} \in L^\infty(\partial\Omega)$. The pointwise restriction of functions in $C^\infty(\bar{\Omega})$ spawns the standard trace $\gamma : W^1(\Omega) \mapsto H^{\frac{1}{2}}(\partial\Omega)$. However, the relevant traces for electromagnetic fields are tangential traces of vectorfields. We can distinguish between the tangential components trace $\gamma_{\mathbf{t}}$ for $\mathbf{U} \in \boldsymbol{C}^\infty(\bar{\Omega})$ defined by $(\gamma_{\mathbf{t}}\mathbf{U})(\mathbf{x}) = \mathbf{n}(\mathbf{x}) \times (\mathbf{U}(\mathbf{x}) \times \mathbf{n}(\mathbf{x}))$ for almost all $\mathbf{x} \in \partial\Omega$, and the twisted tangential trace $(\gamma_\times \mathbf{U})(\mathbf{x}) := \mathbf{U}(\mathbf{x}) \times \mathbf{n}(\mathbf{x})$.

In eddy current computations we usually face non-smooth surfaces. This profoundly affects the smoothness of restrictions, in particular of tangential traces. Just keep in mind that even for smooth vectorfields their tangential traces will feature discontinuities at ridges and corners of $\partial\Omega$. Therefore it takes sophisticated techniques to devise meaningful tangential trace operators on the function spaces. For domains with piecewise smooth boundary they were developed in [16, 17, 12, 18]. These papers and, in particular [13], should be consulted as main references.

Before we tackle $\boldsymbol{W}(\mathbf{curl}, \Omega)$, we remind (see [16, Prop. 1.7]) that on piecewise smooth boundaries spaces $\boldsymbol{H}_{||}^{\frac{1}{2}}(\Gamma)$ and $\boldsymbol{H}_{\perp}^{\frac{1}{2}}(\Gamma)$ can be introduced so that the tangential traces become continuous and surjective operators $\gamma_{\mathbf{t}} : \boldsymbol{H}^1(\Omega) \mapsto \boldsymbol{H}_{||}^{\frac{1}{2}}(\Gamma)$, $\gamma_\times : \boldsymbol{H}^1(\Omega) \mapsto \boldsymbol{H}_{\perp}^{\frac{1}{2}}(\Gamma)$. Sloppily speaking, $\boldsymbol{H}_{||}^{\frac{1}{2}}(\Gamma)$

contains the tangential surface vectorfields that are in $\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$ for each smooth component of $\partial\Omega$ and feature a suitable "tangential continuity" across the edges. A corresponding "normal continuity" is satisfied by surface vectorfields in $\boldsymbol{H}_{\perp}^{\frac{1}{2}}(\Gamma)$. The associated dual spaces will be denoted by $\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)$ and $\boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\Gamma)$, respectively.

Armed with these spaces and the density of smooth functions, the integration by parts formula

$$\int_{\Omega} \mathbf{curl}\,\mathbf{V}\cdot\mathbf{U} - \mathbf{V}\cdot\mathbf{curl}\,\mathbf{U}\,\mathrm{d}\mathbf{x} = \int_{\partial\Omega} \gamma_{\times}\mathbf{U}\cdot\gamma_{\mathbf{t}}\mathbf{V}\,\mathrm{d}S \tag{9}$$

is the key to establishing trace theorems for $\boldsymbol{W}(\mathbf{curl},\Omega)$. Recall that the surface divergence operator div_{Γ} is the $\boldsymbol{L}^2(\partial\Omega)$-adjoint of the surface gradient $\mathbf{grad}_{\Gamma}$. By rotating tangential surface vectorfields by $\frac{\pi}{2}$, we get the same relationship between the scalar valued surface rotation curl_{Γ} and the tangent vector valued $\mathbf{curl}_{\Gamma}$. Using, first, $\mathbf{V}\in\boldsymbol{H}^1(\Omega)$, and, secondly, $\mathbf{V}\in\mathbf{grad}\,H^2(\Omega)$, we learn from (9) that

$$\begin{aligned}\gamma_{\mathbf{t}}\,:&\boldsymbol{H}(\mathbf{curl};\Omega)\mapsto\boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\mathbf{curl}_{\Gamma},\Gamma)\,,\\ \gamma_{\times}\,:&\boldsymbol{H}(\mathbf{curl};\Omega)\mapsto\boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma},\Gamma)\,,\end{aligned}$$

are continuous trace mappings. Here, we used the notations

$$\begin{aligned}\boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\mathbf{curl}_{\Gamma},\Gamma) &:= \{\mathbf{v}\in\boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\Gamma),\ \mathbf{curl}_{\Gamma}\,\mathbf{v}\in H^{-\frac{1}{2}}(\partial\Omega)\}\,,\\ \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma},\Gamma) &:= \{\lambda\in\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma),\ \mathrm{div}_{\Gamma}\,\lambda\in H^{-\frac{1}{2}}(\partial\Omega)\}\,,\end{aligned}$$

for spaces of tangential traces. Moreover, according to Thm. 2.7 and Thm. 2.8 in [16], they are surjective, too. Thus, we have found the right tangential trace spaces for $\boldsymbol{H}(\mathbf{curl};\Omega)$. By (9) the spaces $\boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma},\Gamma)$ and $\boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\mathbf{curl}_{\Gamma},\Gamma)$ can be seen to be dual to each other (see [17, Sect. 4]). The sesqui-linear duality pairing will be denoted by $\langle\cdot,\cdot\rangle_{\tau}$. Moreover, the rotation mapping $\mathsf{R}\mathbf{v}:=\mathbf{v}\times\mathbf{n}$ can be extended to an isometry between the two spaces.

Integration by parts permits us to introduce several important weakly defined traces: The weak normal trace $\gamma_{\mathbf{n}}$ is defined for vectorfields $\mathbf{U}\in\boldsymbol{H}(\mathrm{div};\Omega):=\{\mathbf{V}\in\boldsymbol{L}^2(\Omega),\ \mathrm{div}\,\mathbf{V}\in L^2(\Omega)\}$ by

$$\langle\gamma_{\mathbf{n}}\mathbf{U},\gamma\Phi\rangle_{1/2,\Gamma} = \int_{\Omega}\mathrm{div}\,\mathbf{U}\,\overline{\Phi}+\mathbf{U}\cdot\mathbf{grad}\,\overline{\Phi}\,\mathrm{d}\mathbf{x}\quad\forall\Phi\in H^1(\Omega)\,,$$

with $\langle\cdot,\cdot\rangle_{1/2,\Gamma}$ as duality pairing between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$. The mapping $\gamma_{\mathbf{n}}:\boldsymbol{H}(\mathrm{div};\Omega)\mapsto H^{-\frac{1}{2}}(\partial\Omega)$ is continuous and surjective, and an extension of the normal components trace $\gamma_{\mathbf{n}}\mathbf{U}(\mathbf{x}):=\mathbf{U}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x})$. Thus, the conormal trace $\partial_{\mathbf{n}}:=\gamma_{\mathbf{n}}\circ\mathbf{grad}$ is continuous and surjective from $H(\Delta,\Omega):=\{\Phi\in W^1(\Omega),\ \Delta\Phi\in\boldsymbol{L}^2(\Omega)\}$ onto $H^{-\frac{1}{2}}(\partial\Omega)$.

Against the backdrop of boundary value problems for the Laplacian $-\Delta$, the trace operator $\gamma : H^1(\Omega) \mapsto H^{\frac{1}{2}}(\Gamma)$ can be called "Dirichlet trace", whereas $\partial_{\mathbf{n}}$ provides the "Neumann trace". For $\Psi \in H(\Delta, \Omega)$ and $\Phi \in H^1(\Omega)$, they are linked by another Green's formula

$$\langle \partial_{\mathbf{n}} \Psi, \gamma \Phi \rangle_{1/2,\Gamma} = \int_\Omega \Delta\Psi\,\overline{\Phi} + \mathbf{grad}\,\Psi \cdot \mathbf{grad}\,\overline{\Phi}\,\mathrm{d}\mathbf{x}\;. \tag{10}$$

The eddy current equations prominently feature the operator $\mathbf{curl}\,\mathbf{curl}$ and we may wonder about suitable Dirichlet- and Neumann traces. Since the energy space associated with $\mathbf{curl}\,\mathbf{curl}$ is $\boldsymbol{H}(\mathbf{curl};\Omega)$, the previous discussion reveals that $\gamma_{\mathbf{t}}$ can be used as Dirichlet trace. In view of (10) a $\mathbf{curl}\,\mathbf{curl}$-counterpart γ_N of $\partial_{\mathbf{n}}$ can be defined for

$$\mathbf{U} \in \boldsymbol{W}(\mathbf{curl}^2, \Omega) := \{\mathbf{V} \in \boldsymbol{W}(\mathbf{curl}, \Omega),\ \mathbf{curl}\,\mathbf{curl}\,\mathbf{V} \in \boldsymbol{L}^2(\Omega)\}$$

by demanding that for all $\mathbf{V} \in \boldsymbol{H}(\mathbf{curl};\Omega)$

$$\langle \gamma_N \mathbf{U}, \gamma_{\mathbf{t}} \mathbf{V} \rangle_{\tau} = \int_\Omega \mathbf{curl}\,\mathbf{U} \cdot \mathbf{curl}\,\overline{\mathbf{V}} - \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} \cdot \overline{\mathbf{V}}\,d\mathbf{x}\;. \tag{11}$$

The trace γ_N furnishes a continuous and surjective mapping

$$\gamma_N : \boldsymbol{W}(\mathbf{curl}^2, \Omega) \mapsto \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \quad \text{(cf. [39, Lemma 3.3])},$$

which can be regarded as an extension of the restriction $(\gamma_N \mathbf{U})(\mathbf{x}) := \mathbf{curl}\,\mathbf{U}(\mathbf{x}) \times \mathbf{n}(\mathbf{x})$, $\mathbf{x} \in \partial\Omega$, for smooth $\mathbf{U}$.

We mention two commuting relationships between traces that are elementary for smooth functions and, by extension, carry over to the trace operators in Sobolev spaces:

$$\mathbf{grad}_\Gamma \circ \gamma = \gamma_{\mathbf{t}} \circ \mathbf{grad} \quad \text{on } W^1(\Omega)\;, \tag{12}$$

$$\gamma_{\mathbf{n}} \circ \mathbf{curl} = \mathrm{curl}_\Gamma \circ \gamma_{\mathbf{t}} = \mathrm{div}_\Gamma \circ \gamma_\times \quad \text{on } \boldsymbol{W}(\mathbf{curl}, \Omega)\;, \tag{13}$$

where equality is in the sense of the trace spaces $\boldsymbol{H}_\perp^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$, respectively.

Integration by parts also shows that a vectorfield in $\boldsymbol{C}^\infty(\Omega_c cl) \cap \boldsymbol{C}^\infty(\overline{\Omega}_e)$ must feature tangential continuity in order to be contained in $\boldsymbol{W}(\mathbf{curl}, \mathbb{R}^3)$. Thus, both $\mathbf{E}$ and $\mathbf{H}$ can only belong to $\boldsymbol{W}(\mathbf{curl}, \mathbb{R}^3)$, if the following *transmission conditions* hold across $\Gamma := \partial\Omega_c$

$$[\gamma_{\mathbf{t}} \mathbf{E}]_\Gamma = 0 \quad \text{and} \quad [\gamma_{\mathbf{t}} \mathbf{H}]_\Gamma = 0\;. \tag{14}$$

Here, the "jump" $[\cdot]_\Gamma$ designates the difference of the values of a trace from Ω_e ("exterior") and from Ω_c ("interior"). We also stick to the convention that exterior traces will be labeled by a superscript $+$, whereas traces from Ω_c bear a superscript $-$.

4 Topological Prerequisites

Topological considerations come into play, when one wants to represent irrotational vectorfields on manifolds through gradients of scalar potentials. This is only possible, if the first cohomology group of the manifold is trivial [59, Ch. 6]. Otherwise, *cuts* have to be used to take care of irrotational vectorfields that are no gradients [8, Sect. 8.3.4], see Fig. 2.

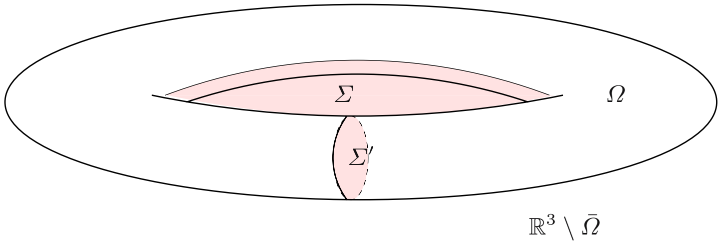

Fig. 2. Cut Σ' for the torus and cut Σ for its complement in $\mathbb{R}^3$.

Theorem 1. *For every domain $\Omega \subset \mathbb{R}^3$ with piecewise smooth boundary there exist piecewise smooth orientable embedded surfaces $\Sigma_1, \ldots, \Sigma_N \subset \bar{\Omega}$ (cuts), where N agrees with the first Betti number of Ω, such that*

- *the Σ_k, $k = 1, \ldots, N$, are mutually disjoint.*
- *the first cohomology group $H^1(\Omega', \mathbb{Z})$ of $\Omega' := \Omega \setminus (\Sigma_1 \cup \ldots \cup \Sigma_N)$ is trivial.*
- *Ω' is a generalized Lipschitz domain in the sense of [29], that is, when "seen from one side" its boundary $\partial\Omega'$ is Lipschitz continuous.*

Proof. The theorem is proved in [45]. □

In the sequel we are going to equip Ω_e with a set of cuts $\Sigma_1, \ldots, \Sigma_N$, according to Thm. 1. Each Σ_k has an orientation that translates into a crossing direction and thus we can distinguish between an "upper" surface Σ_k^+ and a "lower" surface Σ_k^-. Both surfaces are equipped with unit normal vectorfields $\mathbf{n}_k^+, \mathbf{n}_k^-$ pointing "away from Σ_k" into the interior of $\Omega' := \Omega_e \setminus (\Sigma_1 \cup \ldots \cup \Sigma_N)$. We fix $\mathbf{n}_{|\Sigma_k} := \mathbf{n}_k^+$ so that it agrees with the crossing direction.

The statement of Thm. 1 implies

$$\mathbf{V} \in \boldsymbol{H}(\mathbf{curl}; \Omega'),\ \mathbf{curl}\,\mathbf{V} = 0 \quad \Rightarrow \quad \exists \Phi \in H^1(\Omega') :\ \mathbf{V} = \mathbf{grad}\,\Phi\ .$$

It is even possible to characterize low dimensional spaces of vectorfields that fill the gap between $\mathrm{Ker}(\mathbf{curl}) \cap \boldsymbol{H}(\mathbf{curl}; \Omega_e)$ and $\mathbf{grad}\, H^1(\Omega_e)$. To that end, consider functions $\eta_k \in H^1(\Omega_e \setminus \Sigma_k)$, $k = 1, \ldots, N$, with $[\eta_k]_{\Sigma_k} = 1$. Here, $[\cdot]_S$ denotes the jump of some function across the externally oriented surface S, i.e. the difference of its value on the "+-side" and the "−-side".

Theorem 2. *Using the notations introduced above, we have the representation*

$$\mathrm{Ker}(\mathbf{curl}) \cap \boldsymbol{H}(\mathbf{curl}; \Omega_e) = \mathbf{grad}\, H^1(\Omega_e) + \mathrm{Span}\left\{\widetilde{\mathbf{grad}}\, \eta_1, \ldots, \widetilde{\mathbf{grad}}\, \eta_N\right\},$$

where $\widetilde{\mathbf{grad}}\, \eta_k \in \boldsymbol{L}^2(\Omega_e)$ *is the gradient of* η_k *on* $\Omega_e \setminus \Sigma_k$.

Proof. Compare Sect. 3 in [4]. □

From Thm. 2 we learn that

$$\mathrm{Ker}(\mathbf{curl}) \cap \boldsymbol{H}(\mathbf{curl}; \Omega_e) = \widetilde{\mathbf{grad}}\, H^1_\Sigma(\Omega_e), \tag{15}$$

with $H^1_\Sigma(\Omega_e) := \{\varphi \in H^1(\Omega'), [\varphi]_{\Sigma_k} = \text{const.}, 1 \leq k \leq N\}$.

Thm. 1 may also be applied to Ω_c yielding N cutting surfaces $\widehat{\Sigma}_1, \ldots, \widehat{\Sigma}_N$, since the first Betti numbers of Ω_c and Ω_e agree. The boundaries $\sigma_1, \ldots, \sigma_N$, $\widehat{\sigma}_1, \ldots, \widehat{\sigma}_N$ of Σ_k and $\widehat{\Sigma}_k$, $k = 1, \ldots, N$, respectively, represent a basis of the homology group $H_1(\Gamma, \mathbb{Z})$, see Fig. 3. In analogy to Thm. 2 we find that

$$\mathrm{Ker}(\mathrm{div}_\Gamma) \cap \boldsymbol{H}(\mathrm{div}; \Gamma) = \mathbf{curl}_\Gamma\, H^1(\Gamma) + \mathrm{Span}\left\{\mathbf{g}^1, \ldots, \mathbf{g}^N, \widehat{\mathbf{g}}^1, \ldots, \widehat{\mathbf{g}}^N\right\}, \tag{16}$$

where $\mathbf{g}^k$ is the vectorial surface rotation $\mathbf{curl}_\Gamma\, \varphi$ of some $\varphi \in H^1(\Gamma \setminus \sigma_k)$ that has a jump of constant height 1 across σ_i. The $\widehat{\mathbf{g}}^k$ are constructed analogously with respect to $\widehat{\sigma}_k$.

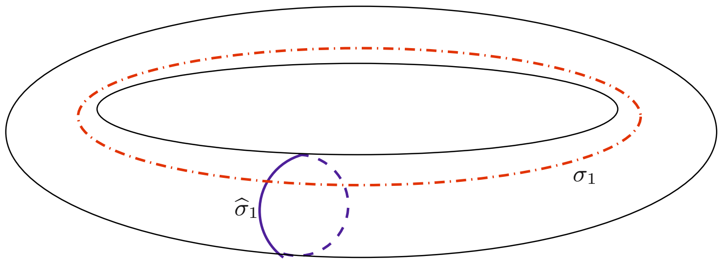

Fig. 3. Fundamental cycles σ_1 and $\widehat{\sigma}_1$ for the surface of the torus, a domain with first Betti number = 1.

We remark that if Γ is equipped with some non-degenerate triangulation Γ_h (rendering it a cellular complex) the boundaries of interior and exterior cutting surfaces can be chosen such that they agree with edge cycles of Γ_h. Further, it is possible to pick piecewise smooth Lipschitz surfaces as related cuts. Such a choice of cuts will be a tacit assumption, whenever a triangulation of Γ has been fixed.

Remark 1. Please be aware that it is not the purpose of cuts to render Ω' simply connected, i.e., to ensure that it has a trivial first *homotopy* group. This is easily seen in the case of knotted geometries.

5 Variational Formulations and Transmission Problems

Two fundamentally different approaches to a variational formulation of (3) are conceivable. They can be distinguished by which equation is preserved in strong form and which is taken into account only in weak form [7]. This distinction parallels the primal and dual variational principles known from second order elliptic boundary value problems [11, Ch. 1]. A discusion for the full Maxwell's equations in frequency domain is given in [38, Sect. 2.3].

The first approach involves Faraday's law in strong form. It is used to replace $\mathbf{H}$ in Ampere's law and the latter is multiplied by a test function in $\boldsymbol{H}(\mathbf{curl};\mathbb{R}^3)$ and subjected to integration by parts according to (9). This results in the following "$\mathbf{E}$-based" variational problem (cf. [7, Sect. 3], [56], and [57]): Seek $\mathbf{E} \in \boldsymbol{W}(\mathbf{curl},\mathbb{R}^3)$ such that for all $\mathbf{V} \in \boldsymbol{W}(\mathbf{curl},\mathbb{R}^3)$

$$\left(\tfrac{1}{\mu}\,\mathbf{curl}\,\mathbf{E},\mathbf{curl}\,\mathbf{V}\right)_{\boldsymbol{L}^2(\mathbb{R}^3)} + i\omega\,(\sigma\mathbf{E},\mathbf{V})_{\boldsymbol{L}^2(\Omega_c)} = -i\omega\,(\mathbf{j}_s,\mathbf{V})_{\boldsymbol{L}^2(\mathbb{R}^3)} \ . \tag{17}$$

Theorem 3. *The variational problem* (17) *has a unique solution for* $\mathbf{H} := -\frac{1}{i\omega\mu}\,\mathbf{curl}\,\mathbf{E} \in \boldsymbol{H}(\mathbf{curl};\mathbb{R}^3)$*. If it is posed on the constrained space*

$$\mathcal{W} := \{\mathbf{V} \in \boldsymbol{W}(\mathbf{curl},\mathbb{R}^3),\ \operatorname{div}\mathbf{V} = 0 \ \text{in}\ \Omega_e,\ \int_{\Gamma_k} \gamma_{\mathbf{n}}\mathbf{V}\,dS = 0,\ k=1,\dots,L\}\ ,$$

a unique solution $\mathbf{E} \in \mathcal{W}$ *exists. Here* Γ_k*,* $k=1,\dots,L$*, stand for the connected components of* Γ*.*

Proof. The reader is referred to [3, Sect. 3] and [39, Sect. 2]. □

A crucial observation is that (17) is equivalent to a *transmission problem.* To state it, we first appeal to the transmission conditions (14). Secondly, testing (17) with fields compactly supported in Ω_c or Ω_e, and making use of the offset fields from (6), we get

$$\begin{gathered} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} + i\omega\mu_c\sigma\mathbf{E} = 0 \quad \text{in}\ \Omega_c\ , \\ \operatorname{div}\mathbf{E}_r = 0 \quad , \quad \mathbf{curl}\,\mathbf{curl}\,\mathbf{E}_r = 0 \quad \text{in}\ \Omega_e\ , \\ \gamma_{\mathbf{t}}^{+}\mathbf{E}_r - \gamma_{\mathbf{t}}^{-}\mathbf{E} = -\gamma_{\mathbf{t}}^{+}\mathbf{E}_s \quad , \quad \frac{1}{\mu_0}\gamma_N^{+}\mathbf{E}_r - \frac{1}{\mu_c}\gamma_N^{-}\mathbf{E} = -\frac{1}{\mu_0}\gamma_N^{+}\mathbf{E}_s\ . \end{gathered} \tag{18}$$

Here we have skimped on the full "gauge conditions" (5), that is $\mathbf{E}_r \in \mathcal{W}$, for the reaction field $\mathbf{E}_r$ in Ω_e.

The second option for a variational formulation is to keep Ampere's law strongly, leading to "$\mathbf{H}$-based" formulations. Then, we have to use the trial space $\mathbf{H}_s + \mathcal{V}$ with

$$\mathcal{V} := \{\mathbf{V} \in \boldsymbol{H}(\mathbf{curl};\mathbb{R}^3),\ \mathbf{curl}\,\mathbf{V} = 0 \ \text{in}\ \Omega_e\}$$

for $\mathbf{H}$. Remember that the offset field $\mathbf{H}_s$ is to satisfy $\mathbf{curl}\,\mathbf{H}_s = \mathbf{j}_s$ and $\operatorname{div}\mathbf{H}_s = 0$ in Ω_e. Now, testing the first equation of (3) with a compactly supported $\mathbf{V} \in \mathcal{V}$, employing integration by parts on a ball with sufficiently large radius, and using the second equation inside Ω_c, we obtain: Seek $\mathbf{H} \in \mathcal{V} + \mathbf{H}_s$ such that

$$\left(\sigma^{-1}\, \mathbf{curl}\, \mathbf{H}, \mathbf{curl}\, \mathbf{V}\right)_{\boldsymbol{L}^2(\Omega_c)} + i\omega\, (\mu \mathbf{H}, \mathbf{V})_{\boldsymbol{L}^2(\mathbb{R}^3)} = 0 \quad \forall \mathbf{V} \in \mathcal{V}\,. \tag{19}$$

For a more detailed presentation of the considerations leading to (19) the reader is referred to [9], [7, Sect. 2], and [8, Ch. 8]. Existence and uniqueness of solutions of (19) immediately follow from the Lax-Milgram lemma.

Straight from (19) we infer $\operatorname{div}(\mu_r \mathbf{H}) = 0$ in all of $\mathbb{R}^3$. This involves the normal continuity of $\mu_r \mathbf{H}$ across Γ. We are led to a transmission problem for the total magnetic field inside Ω_c and the reaction field $\mathbf{H}_r$ outside:

$$\begin{aligned} \mathbf{curl}\, \sigma^{-1}\, \mathbf{curl}\, \mathbf{H} + i\omega\mu_c \mathbf{H} = 0 \quad \text{in } \Omega_c\,, \\ \mathbf{curl}\, \mathbf{H} = 0 \quad , \quad \operatorname{div} \mathbf{H} = 0 \quad \text{in } \Omega_e\,, \\ \mu_0 \gamma_{\mathbf{n}}^+ \mathbf{H}_r - \mu_c \gamma_{\mathbf{n}}^- \mathbf{H} = -\mu_0 \gamma_{\mathbf{n}}^+ \mathbf{H}_s \quad , \quad \gamma_{\mathbf{t}}^+ \mathbf{H}_r - \gamma_{\mathbf{t}}^- \mathbf{H} = \gamma_{\mathbf{t}} \mathbf{H}_s \quad \text{on } \Gamma\,. \end{aligned} \tag{20}$$

However, if Ω_c has non-vanishing first Betti number, then there is no unique solution of (5) [55, 34]. To see this please notice that thanks to Thm. 2 the path integrals $f_k(\mathbf{H}) := \int_{\widehat{\sigma}_k} \mathbf{H} \cdot d\mathbf{s}$ supply continuous functionals on $\mathcal{V}$. They do not vanish, because plugging in an extension to $\boldsymbol{W}(\mathbf{curl}, \mathbb{R}^3)$ of $\widetilde{\mathbf{grad}}\, \eta_k$ results in 1. Next, consider the variational problem (19) posed over $\mathcal{V}$, but with f_k as non-homogeneous right hand side. A unique non-zero solution $\mathbf{H}_k \in \mathcal{V}$ exists. From $f_k(\mathbf{grad}\, \Phi) = 0$ for all $\Phi \in W^1(\mathbb{R})$ we conclude that still $\operatorname{div}(\mu_r \mathbf{H}_k) = 0$. Hence, $\mathbf{H}_k$ satisfies all the transmission conditions of (5). Testing with smooth vectorfields that are compactly supported in Ω_c establishes the first equation of (5). Summing up, (5) may have non-zero solutions, even if $\mathbf{H}_s = 0$.

These considerations refute the equivalence of (5) and (19). The bottom line is that in general the **H**-based model does not allow a formulation as transmission problem, unless some extra coupling conditions that, however, fail to involve traces on Γ only, are taken into account. These additional conditions are formulated and investigated in [1] (see also [47]). They turn out to be an integral version of Faraday's law with respect to cuts.

A third class of variational formulations, the hybrid methods, combines primal and dual variational principles, one kind applied in Ω_c the other in Ω_e. An extensive discussion with finite elements in mind is given in [2]. The first option is to work "**H**-based" inside Ω_c and "**E**-based" in Ω_e: these formulations can be nicely combined into a transmission problem

$$\begin{aligned} \mathbf{curl}\, \mathbf{curl}\, \mathbf{H} + i\omega\mu_c\sigma \mathbf{H} = 0 \quad \text{in } \Omega_c\,, \\ \mathbf{curl}\, \mathbf{curl}\, \mathbf{E}_r = 0 \quad , \quad \operatorname{div} \mathbf{E}_r = 0 \quad \text{in } \Omega_e\,, \\ \gamma_\times^+ \mathbf{E}_r - \frac{1}{\sigma} \gamma_N^- \mathbf{H} = -\gamma_\times^+ \mathbf{E}_s \quad , \quad -\frac{1}{i\omega\mu_0} \gamma_N^+ \mathbf{E} - \gamma_\times^- \mathbf{H} = \gamma_{\mathbf{t}}^+ \mathbf{H}_s \quad \text{on } \Gamma\,. \end{aligned} \tag{21}$$

Alternatively, Faraday's law can be used in strong form in Ω_c, and Ampere's law is tested with $\mathbf{V} \in \boldsymbol{H}(\mathbf{curl}; \mathbb{R}^3)$, but integration by parts is performed on Ω_c only. Therefore, boundary terms have to be retained in the variational equation

$$\left(\frac{1}{\mu_c}\,\mathbf{curl}\,\mathbf{E}, \mathbf{curl}\,\mathbf{V}\right)_{\boldsymbol{L}^2(\Omega_c)} + i\omega\,(\sigma\mathbf{E}, \mathbf{V})_{\boldsymbol{L}^2(\Omega_c)} - \left\langle \frac{1}{\mu_c}\gamma_N^-\mathbf{E}, \gamma_{\mathbf{t}}^-\mathbf{V} \right\rangle_\tau = 0$$

for $\mathbf{V} \in \boldsymbol{H}(\mathbf{curl}; \Omega_c)$. In Ω_e Ampere's law is incorporated strongly by zeroing in on $\mathbf{H} \in \mathbf{H}_s + \mathcal{V}$. Faraday's law is tested with compactly supported irrotational fields only, and subsequently we integrate by parts. We end up with

$$\left\langle \gamma_\times^+\mathbf{V}, \gamma_{\mathbf{t}}^+\mathbf{E} \right\rangle_\tau + i\omega\,(\mu_0\mathbf{H}, \mathbf{V})_{\boldsymbol{L}^2(\Omega_e)} = 0 \quad \forall \mathbf{V} \in \mathcal{V}\,.$$

Both variational problems are linked through the transmission conditions, which enable us to replace $\frac{1}{\mu_c}\gamma_N^-\mathbf{E}$ by $-i\omega\gamma_\times^+\mathbf{H}$ in the boundary terms. This results in the variational problem [48]: Seek $\mathbf{E} \in \boldsymbol{H}(\mathbf{curl}; \Omega_c)$, $\mathbf{H} \in \mathbf{H}_s + \mathcal{V}$ such that for all $\mathbf{W} \in \boldsymbol{H}(\mathbf{curl}; \Omega_c)$, $\mathbf{V} \in \mathcal{V}$

$$\begin{aligned} \left(\tfrac{1}{\mu_c}\,\mathbf{curl}\,\mathbf{E}, \mathbf{curl}\,\mathbf{W}\right)_{\boldsymbol{L}^2(\Omega_c)} + i\omega\,(\sigma\mathbf{E}, \mathbf{W})_{\boldsymbol{L}^2(\Omega_c)} + \; i\omega\,\langle \gamma_\times\mathbf{H}, \gamma_{\mathbf{t}}\mathbf{W}\rangle_\tau &= 0\,, \\ i\omega\,\langle \gamma_\times\mathbf{V}, \gamma_{\mathbf{t}}\mathbf{E}\rangle_\tau \qquad - \omega^2\overline{(\mu_0\mathbf{H}, \mathbf{V})_{\boldsymbol{L}^2(\Omega_e)}} &= 0\,. \end{aligned} \tag{22}$$

Theorem 4. *The bilinear form associated with the variational problem* (22) *is* $\boldsymbol{H}(\mathbf{curl}; \Omega_c) \times \mathcal{V}$*-elliptic.*

Proof. Setting $\mathbf{W} := \mathbf{E}$, $\mathbf{V} := \mathbf{H}$, and subtracting both equations makes the "off-diagonal" terms cancel. □

Similarly as in the case of the **H**-based model, an equivalent transmission problem is also elusive for the variational problem (22).

In the sequel we are going to focus on the pure **E**/**H**-based formulations (18) and (19), respectively.

6 Boundary Integral Operators

The theory of boundary integral operators for strongly elliptic partial differential operators of second order is well developed [52, 28, 58]. Here, we summarize some of the results as a guidance for developing a similar theory for boundary integral operators for second-order partial differential equations involving the **curl**-operator. The relevance of this for the transmission problem (18) and the variational problem (19) is evident.

The starting point is a representation formula, the famous Green's representation formula for solutions of the homogeneous Helmholtz equation. It relies on the scalar single layer potential

$$\Psi_V^0(\varphi)(\mathbf{x}) := \int_\Gamma G_\kappa(\mathbf{x},\mathbf{y})\,\varphi(\mathbf{y})\,\mathrm{d}\mathbf{y} \quad \mathbf{x}\notin\Gamma\,, \quad \varphi\in H^{-\frac{1}{2}}(\Gamma)\,, \tag{23}$$

and the scalar double layer potential

$$\Psi_K^0(v)(\mathbf{x}) := \int_\Gamma \frac{\partial}{\partial\mathbf{n}(\mathbf{y})} G_\kappa(\mathbf{x},\mathbf{y})\,v(\mathbf{y})\,\mathrm{d}\mathbf{y} \quad \mathbf{x}\notin\Gamma\,, \quad v\in H^{\frac{1}{2}}(\Gamma)\,, \tag{24}$$

both based on the Helmholtz kernel [52, Ch. 9]

$$G_\kappa(\mathbf{x},\mathbf{y}) := \frac{\exp(-\kappa|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|} \quad , \quad \mathbf{x}\neq\mathbf{y}\,.$$

The potentials owe their significance to the following result [52, Thm. 6.10], [58, Thm. 3.1.6]:

Theorem 5. *Assume $\Re\kappa \geq 0$. Any distribution $U \in H^1_{\mathrm{loc}}(\Omega_c\cup\Omega_e)$ with $-\Delta U + \kappa^2 U = 0$ in $\Omega_c\cup\Omega_e$ and $|U(x)| = O(|x|^{-1})$ for $|x|\to\infty$ can be represented as*

$$U(\mathbf{x}) = -\Psi_V^\kappa([\partial_\mathbf{n} U]_\Gamma) + \Psi_K^\kappa([\gamma U]_\Gamma)\,, \quad \mathbf{x}\notin\Gamma\,.$$

It is well known [52, Thm. 6.11] that the potentials Ψ_V^κ and Ψ_K^κ provide continuous mappings

$$\Psi_V^\kappa : H^{-\frac{1}{2}}(\Gamma) \mapsto H^1_{\mathrm{loc}}(\mathbb{R}^3)\cap H(\Delta,\Omega_c\cup\Omega_e) \tag{25}$$

$$\Psi_K^\kappa : H^{\frac{1}{2}}(\Gamma) \mapsto H(\Delta,\Omega_c\cup\Omega_e)\,. \tag{26}$$

In fact, $(-\Delta+\kappa^2)\Psi_V^\kappa = (-\Delta+\kappa^2)\Psi_K^\kappa = 0$ away from Γ [58, Thm. 3.1.1]. We also recall the fundamental *jump relations* for the potentials

$$[\gamma\Psi_V^\kappa(\varphi)]_\Gamma = 0\,, \qquad [\partial_\mathbf{n}\Psi_V^\kappa(\varphi)]_\Gamma = -\varphi\,, \qquad \varphi\in H^{-\frac{1}{2}}(\Gamma)\,, \tag{27}$$

$$[\gamma\Psi_K^\kappa(v))]_\Gamma = v\,, \qquad [\partial_\mathbf{n}\Psi_K^\kappa(v)]_\Gamma = 0 \qquad v\in H^{\frac{1}{2}}(\Gamma)\,. \tag{28}$$

The mapping properties (25) and (26) of the potentials ensure that the *boundary integral operators*

$$\begin{aligned}
\mathsf{V}^\kappa &:= \gamma\Psi_V^\kappa && : H^{-\frac{1}{2}}(\Gamma)\mapsto H^{\frac{1}{2}}(\Gamma)\,,\\
\mathsf{K}^\kappa &:= \tfrac{1}{2}(\gamma^- + \gamma^+)\Psi_K^\kappa && : H^{\frac{1}{2}}(\Gamma)\mapsto H^{\frac{1}{2}}(\Gamma)\,,\\
\mathsf{K}^{\kappa,*} &:= \tfrac{1}{2}(\partial_\mathbf{n}^- + \partial_\mathbf{n}^+)\Psi_V^\kappa && : H^{-\frac{1}{2}}(\Gamma)\mapsto H^{-\frac{1}{2}}(\Gamma)\,,\\
\mathsf{D}^\kappa &:= -\partial_\mathbf{n}\Psi_K^\kappa && : H^{\frac{1}{2}}(\Gamma)\mapsto H^{-\frac{1}{2}}(\Gamma)\,.
\end{aligned} \tag{29}$$

are well defined and continous [58, Sect. 3.1.2]. Moreover, the single layer boundary integral operator V^κ and hypersingular boundary integral operator D^κ are elliptic in the following sense, see [52, Thms. 7.6,7.8]

$$|\left\langle\varphi,\mathsf{V}^0\varphi\right\rangle_{1/2,\Gamma}| \geq c\,\|\varphi\|^2_{H^{-\frac{1}{2}}(\Gamma)} \quad \forall\varphi\in H^{-\frac{1}{2}}(\Gamma)\,, \tag{30}$$

$$|\left\langle\mathsf{D}^0 v,v\right\rangle_{1/2,\Gamma}| \geq c\,\|v\|^2_{H^{\frac{1}{2}}(\Gamma)/\mathbb{R}} \quad \forall v\in H^{\frac{1}{2}}(\Gamma)\,, \tag{31}$$

with constants $c>0$ depending on Γ only.

Now we attempt to develop similar representation formulas and boundary integral operators related to the differential operator $\mathbf{curl}\,\mathbf{curl} + \kappa^2$. It is our first objective to derive a boundary integral representation formula for distributions satisfying the homogeneous equation $\mathbf{curl}\,\mathbf{curl}\,\mathbf{U} + \kappa^2\mathbf{U} = 0$ in $\Omega_c \cup \Omega_e$. In order to handle transmission conditions in the calculus of distributions, we introduce *currents*, that is, distributions supported on Γ. For a function $\varphi \in H^{-\frac{1}{2}}(\Gamma)$, a tangential vector-field $\xi \in \boldsymbol{H}_{\perp}^{-1}(\Gamma)$, and test functions $\Phi \in \mathcal{D}(\mathbb{R}^3)$, $\boldsymbol{\Phi} \in \mathcal{D}(\mathbb{R}^3) := (\mathcal{D}(\mathbb{R}^3))^3$, we define

$$(\varphi\delta_\Gamma)(\Phi) := \langle \varphi, \gamma\Phi \rangle_{1/2,\Gamma} \ , \quad (\xi\delta_\Gamma)(\boldsymbol{\Phi}) := \langle \xi, \gamma_{\mathbf{t}}\boldsymbol{\Phi} \rangle_\tau = \langle \xi, \gamma\boldsymbol{\Phi} \rangle_{-1,\Gamma} \ .$$

Now, in the sense of distributions, integration by parts yields, cf. [14, Sect. 2.3],

$$\begin{array}{lcr} \text{for } \mathbf{U} \in \boldsymbol{H}_{\mathrm{loc}}(\mathrm{div}; \Omega_c \cup \Omega_e) & : & \mathrm{div}\,\mathbf{U} = \mathrm{div}\,\mathbf{U}_{|\Omega_c\cup\Omega_e} + [\gamma_{\mathbf{n}}\mathbf{U}]_\Gamma\ \delta_\Gamma \ , \\ \text{for } \mathbf{U} \in \boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}; \Omega_c \cup \Omega_e) & : & \mathbf{curl}\,\mathbf{U} = \mathbf{curl}\,\mathbf{U}_{|\Omega_c\cup\Omega_e} - [\gamma_\times\mathbf{U}]_\Gamma\ \delta_\Gamma \ , \\ \text{for } \xi \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) & : & \mathrm{div}(\xi\,\delta_\Gamma) = (\mathrm{div}_\Gamma\,\xi)\,\delta_\Gamma \ . \end{array}$$

For notational simplicity, we introduce the average $\{\gamma\}_\Gamma = \frac{1}{2}(\gamma^+ + \gamma^-)$ for some trace operator γ. Remember that the superscripts $-$ and $+$ tag traces onto Γ from Ω_c and Ω_e respectively.

Now let $\mathbf{U}$ satisfy $\mathbf{curl}\,\mathbf{curl}\,\mathbf{U} + \kappa^2\mathbf{U} = 0$ along with $\mathrm{div}\,\mathbf{U} = 0$ in $\Omega_c \cup \Omega_e$. Then the following identity holds in the sense of distributions,

$$\begin{aligned} -\,\delta\mathbf{U} + \kappa^2\mathbf{U} &= \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - \mathbf{grad}\,\mathrm{div}\,\mathbf{U} + \kappa^2\mathbf{U} \\ &= \mathbf{curl}\left(\mathbf{curl}\,\mathbf{U}_{|\Omega_c\cup\Omega_e} - [\gamma_\times\mathbf{U}]_\Gamma\ \delta_\Gamma\right) - \mathbf{grad}\left([\gamma_{\mathbf{n}}\mathbf{U}]_\Gamma\ \delta_\Gamma\right) + \kappa^2\mathbf{U} \\ &= \mathbf{curl}\,\mathbf{curl}\,\mathbf{U}_{|\Omega_c\cup\Omega_e} - [\gamma_N\mathbf{U}]_\Gamma\ \delta_\Gamma - \mathbf{curl}([\gamma_\times\mathbf{U}]_\Gamma\ \delta_\Gamma) - \\ &\qquad\qquad - \mathbf{grad}([\gamma_{\mathbf{n}}\mathbf{U}]_\Gamma\ \delta_\Gamma) + \kappa^2\mathbf{U} \\ &= -\,[\gamma_N\mathbf{U}]_\Gamma\ \delta_\Gamma - \mathbf{curl}([\gamma_\times\mathbf{U}]_\Gamma\ \delta_\Gamma) - \mathbf{grad}([\gamma_{\mathbf{n}}\mathbf{U}]_\Gamma\ \delta_\Gamma) \ . \end{aligned}$$

We know from [26, Theorem 6.7] that the Cartesian components of $\mathbf{U}$ will satisfy decay conditions and the scalar Helmholtz equation in $\Omega_c \cup \Omega_e$. Using the results from [52, Ch. 9], we can apply component-wise convolution with the outgoing fundamental solution G_κ for the operator $-\Delta + \kappa^2$. We find that almost everywhere in $\mathbb{R}^3$ the components of $\mathbf{U} = (u_1, u_2, u_3)^T$ satisfy

$$\begin{aligned} u_j(\mathbf{x}) = &-([\gamma_N\mathbf{U}]_\Gamma\ \delta_\Gamma)(G_\kappa(\mathbf{x}-\cdot)\mathbf{e}_j) - ([\gamma_\times\mathbf{U}]_\Gamma\ \delta_\Gamma)(\mathbf{curl}_{\mathbf{y}}(G_\kappa(\mathbf{x}-\cdot)\mathbf{e}_j)) + \\ &+ ([\gamma_{\mathbf{n}}\mathbf{U}]_\Gamma\ \delta_\Gamma)(\mathrm{div}(G_\kappa(\mathbf{x}-\cdot)\mathbf{e}_j)) \ , \quad j = 1, 2, 3 \ . \end{aligned}$$

Using $\mathbf{grad}_{\mathbf{x}}\,G_\kappa(\mathbf{x}-\mathbf{y}) = -\,\mathbf{grad}_{\mathbf{y}}\,G_\kappa(\mathbf{x}-\mathbf{y})$, we arrive at the famous Stratton–Chu representation formula for the electric field in $\Omega_c \cup \Omega_e$ [62], cf. [26, Sect. 6.2], [53, Sect. 5.5], [21, Ch. 3, Sect. 1.3.2], and [19, Sect. 4]

Theorem 6. *If, for* $\kappa \in \mathbb{C}$, $\Re\kappa \geq 0$, *a distribution* $\mathbf{U} \in \boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}; \Omega_c \cup \Omega_e)$ *satisfies* $\mathbf{curl}\,\mathbf{curl}\,\mathbf{U} + \kappa^2\mathbf{U} = 0$ *and* $\mathrm{div}\,\mathbf{U} = 0$ *in* $\Omega_c \cup \Omega_e$, *along with the decay condition* $|\mathbf{U}(\mathbf{x})| = O(|\mathbf{x}|^{-1})$ *for* $|\mathbf{x}| \to \infty$, *then it possess the representation*

$$\mathbf{U} = -\boldsymbol{\Psi}_A^\kappa([\gamma_N\mathbf{U}]_\Gamma) - \boldsymbol{\Psi}_M^\kappa([\gamma_{\mathbf{t}}\mathbf{U}]_\Gamma) - \mathbf{grad}\,\boldsymbol{\Psi}_V^\kappa([\gamma_{\mathbf{n}}\mathbf{U}]_\Gamma) \ .$$

Here, we used the notations $\boldsymbol{\Psi}^\kappa_A$ for the the vectorial single layer potential

$$\boldsymbol{\Psi}^\kappa_A(\lambda)(\mathbf{x}) := \int_\Gamma G_\kappa(\mathbf{x},\mathbf{y})\lambda(\mathbf{y})\,dS(\mathbf{y}) \quad \mathbf{x} \notin \Gamma ,$$

and $\boldsymbol{\Psi}^\kappa_M$ for the "Maxwell double layer potential"

$$\boldsymbol{\Psi}^\kappa_M(\mathbf{v}) := \mathbf{curl}\,\boldsymbol{\Psi}^\kappa_A(\mathsf{R}\mathbf{v}) .$$

From the representation formula it is clear that the potentials have the following mapping properties, see [39, Sect. 5]:

Theorem 7. *The potential mappings*

$$\begin{aligned}\boldsymbol{\Psi}^\kappa_A :& \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma) \mapsto \boldsymbol{W}^1(\mathbb{R}^3) \cap \boldsymbol{W}(\mathbf{curl}^2,\Omega_c \cup \Omega_e) ,\\ \boldsymbol{\Psi}^\kappa_M :& \boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma,\Gamma) \mapsto \boldsymbol{W}(\mathbf{curl}^2,\Omega_c \cup \Omega_e) ,\end{aligned}$$

are continuous.

We remark that any distribution complying with the assumptions of the theorem actually behaves like $|\mathbf{U}(\mathbf{x})| = O(|\mathbf{x}|^{-2})$ for $|\mathbf{x}| \to \infty$, see [3, Prop. 3.1].

In light of Thm. 7, the representation formula of Thm. 6 allows to deduce *jump relations*. For formal derivations please consult [39, Sect. 5], or [55], [53, Thm. 5.5.1], and [26, Thm. 6.11] for smooth boundaries.

Theorem 8. *The potentials satisfy the jump relations*

$$\begin{array}{lcl} [\gamma_{\mathbf{t}}\boldsymbol{\Psi}^\kappa_A]_\Gamma = 0 & , & [\gamma_N\boldsymbol{\Psi}^\kappa_A]_\Gamma = -Id ,\\ [\gamma_{\mathbf{t}}\boldsymbol{\Psi}^\kappa_M]_\Gamma = -Id & , & [\gamma_N\boldsymbol{\Psi}^\kappa_M]_\Gamma = 0 ,\\ [\gamma_{\mathbf{n}}\boldsymbol{\Psi}^\kappa_A]_\Gamma = 0 & , & [\gamma_{\mathbf{n}}\boldsymbol{\Psi}^\kappa_M]_\Gamma = 0 . \end{array}$$

If $\kappa \neq 0$, then, by virtue of (13),

$$\gamma^\pm_{\mathbf{n}}\mathbf{U} = -\frac{1}{\kappa^2}\gamma^\pm_{\mathbf{n}}\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{U} = -\frac{1}{\kappa^2}\,\mathrm{div}_\Gamma(\gamma^\pm_N\mathbf{U}) .$$

This permits us to rewrite the representation formula of Thm. 6 for the case $\kappa \neq 0$:

$$\mathbf{U} = -\boldsymbol{\Psi}^\kappa_A([\gamma_N\mathbf{U}]_\Gamma) - \boldsymbol{\Psi}^\kappa_M([\gamma_{\mathbf{t}}\mathbf{U}]_\Gamma) + \frac{1}{\kappa^2}\,\mathbf{grad}\,\boldsymbol{\Psi}^\kappa_V(\mathrm{div}_\Gamma([\gamma_N\mathbf{U}]_\Gamma)) . \quad (32)$$

After introducing the "Maxwell single layer potential"

$$\boldsymbol{\Psi}^\kappa_S(\mu) := \boldsymbol{\Psi}^\kappa_A(\mu) - \frac{1}{\kappa^2}\,\mathbf{grad}\,\boldsymbol{\Psi}^\kappa_V(\mathrm{div}_\Gamma\,\mu) , \quad \mu \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma) , \quad (33)$$

the formula (32) becomes a perfect analogue to the representation formula of Thm. 5:

$$\mathbf{U} = -\boldsymbol{\Psi}_S^\kappa([\gamma_N \mathbf{U}]_\Gamma) - \boldsymbol{\Psi}_M^\kappa([\gamma_\mathbf{t} \mathbf{U}]_\Gamma) . \tag{34}$$

Again, the analoguous roles of γ and $\gamma_\mathbf{t}$ as "Dirichlet traces" and $\partial_\mathbf{n}$ and γ_N as "Neumann traces" become apparent, cf. Sect. 3 and [19, Sect. 3].

For $\kappa = 0$, the jump of the normal trace cannot be elimintated from the Stratton-Chu representation formula. This stark difference between the situations $\kappa \neq 0$ and $\kappa = 0$ can be blamed on the divergence constraint, which is redundant for $\kappa \neq 0$, but becomes essential, if κ vanishes. This profoundly changes the characteristics of the differential operator and in the latter case we have to deal with $\gamma_N \mathbf{U}$ and $\gamma_\mathbf{n} \mathbf{U}$ as "Neumann data".

As above we introduce boundary integral operators by taking different traces of potentials. Their continuity properties can be directly inferred from those of the potentials, see Thm. 7, and those of the trace operators.

Theorem 9. *For $\kappa \neq 0$ the boundary integral operators*

$$\begin{aligned}
\mathsf{A}^\kappa &:= \gamma_\mathbf{t} \boldsymbol{\Psi}_S^\kappa &&: \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \mapsto \boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma) , \\
\mathsf{B}^\kappa &:= \tfrac{1}{2}(\gamma_N^- + \gamma_N^+) \boldsymbol{\Psi}_S^\kappa &&: \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \mapsto \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) , \\
\mathsf{C}^\kappa &:= \tfrac{1}{2}(\gamma_\mathbf{t}^- + \gamma_\mathbf{t}^+) \boldsymbol{\Psi}_M^\kappa &&: \boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma) \mapsto \boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma) , \\
\mathsf{N}^\kappa &:= \gamma_N \boldsymbol{\Psi}_M^\kappa &&: \boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma) \mapsto \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) ,
\end{aligned}$$

are well defined and continuous. The same holds for

$$\begin{aligned}
\mathsf{A}^0 &:= \gamma_\mathbf{t} \boldsymbol{\Psi}_A^0 &&: \boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma) \mapsto \boldsymbol{H}_{||}^{\frac{1}{2}}(\Gamma) , \\
\mathsf{B}^0 &:= \tfrac{1}{2}(\gamma_N^- + \gamma_N^+) \boldsymbol{\Psi}_A^0 &&: \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \mapsto \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) .
\end{aligned}$$

We know that the double layer boundary integral operators K^κ and $\mathsf{K}^{\kappa,*}$ are adjoints with respect to the sesquilinear duality pairing $\langle \cdot, \cdot \rangle_{1/2,\Gamma}$ [52, Thm. 6.17]. A similar property is enjoyed by their counterparts
$mathsfB^\kappa$ and C^κ:

Theorem 10. *If $\kappa \neq 0$, the boundary integral operators B^κ and C^κ satisfy*

$$\langle \mathsf{B}^\kappa \mu, \mathbf{v} \rangle_\tau = - \langle \mu, \mathsf{C}^\kappa \mathbf{v} \rangle_\tau \quad \forall \mu \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma), \, \mathbf{v} \in \boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma) .$$

The same relationship holds in the case $\kappa = 0$, if μ is restricted to

$$\boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma) := \{ \boldsymbol{\eta} \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) : \, \mathrm{div}_\Gamma \boldsymbol{\eta} = 0 \} .$$

Proof. We appeal to the relationship, see [39, Lemma 5.2] or [51, Lemma 2.3],

$$\mathrm{div}\, \boldsymbol{\Psi}_A^\kappa(\boldsymbol{\eta}) = \boldsymbol{\Psi}_V^\kappa(\mathrm{div}_\Gamma \boldsymbol{\eta}), \quad \boldsymbol{\eta} \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$$

to conclude

$$(\mathbf{curl}\,\mathbf{curl} + \kappa^2 Id)\boldsymbol{\Psi}_S^\kappa(\mu) = 0 \quad \text{for } \kappa \neq 0 \;,$$
$$(\mathbf{curl}\,\mathbf{curl} + \kappa^2 Id)\boldsymbol{\Psi}_A^\kappa(\mu) = 0 \quad \text{for } \mu \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma) \;.$$

We use these relationships together with the integration by parts formula (11): pick any $\mathbf{v} \in \boldsymbol{H}_\perp^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma)$, $\mu \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ ($\mu \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma)$, if $\kappa = 0$) and set $\mathbf{V} = \boldsymbol{\Psi}_M^\kappa(\mathbf{v})$ and $\mathbf{U} = \boldsymbol{\Psi}_S^\kappa(\mu)$ ($\mathbf{U} = \boldsymbol{\Psi}_A^0(\mu)$, if $\kappa = 0$). Then,

$$\begin{aligned}
\left\langle \gamma_N^+ \mathbf{U}, \gamma_\mathbf{t}^+ \mathbf{V} \right\rangle_\tau &= -\int_{\Omega_e} \mathbf{curl}\,\mathbf{U} \cdot \mathbf{curl}\,\overline{\mathbf{V}} - \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} \cdot \overline{\mathbf{V}}\,\mathrm{d}\mathbf{x} \\
&= -\int_{\Omega_e} \mathbf{curl}\,\mathbf{U} \cdot \mathbf{curl}\,\overline{\mathbf{V}} - \mathbf{U} \cdot \mathbf{curl}\,\mathbf{curl}\,\overline{\mathbf{V}}\,\mathrm{d}\mathbf{x} = \left\langle \gamma_N^+ \overline{\mathbf{V}}, \gamma_\mathbf{t}^+ \overline{\mathbf{U}} \right\rangle_\tau \\
&= \quad \left\langle \gamma_N^- \overline{\mathbf{V}}, \gamma_\mathbf{t}^- \overline{\mathbf{U}} \right\rangle_\tau \text{, by jump conditions of Thm. 8} \\
&= \int_{\Omega_c} \mathbf{curl}\,\mathbf{U} \cdot \mathbf{curl}\,\overline{\mathbf{V}} - \mathbf{U} \cdot \mathbf{curl}\,\mathbf{curl}\,\overline{\mathbf{V}}\,\mathrm{d}\mathbf{x} \\
&= \int_{\Omega_c} \mathbf{curl}\,\mathbf{U} \cdot \mathbf{curl}\,\overline{\mathbf{V}} - \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} \cdot \overline{\mathbf{V}}\,\mathrm{d}\mathbf{x} = \left\langle \gamma_N^- \mathbf{U}, \gamma_\mathbf{t}^- \mathbf{V} \right\rangle_\tau
\end{aligned}$$

We remark that "boundary terms at ∞" can be discarded due to the decay $O(|\mathbf{x}|^{-2})$ for $|\mathbf{x}| \to \infty$ of both fields. Thus, using the other set of jump relations from Thm. 8, we have obtained

$$\begin{aligned}
\left\langle \mathsf{B}^\kappa(\mu), \mathbf{v} \right\rangle_\tau &= -\frac{1}{2} \left\langle \gamma_N^+ \mathbf{U} + \gamma_N^- \mathbf{U}, \gamma_\mathbf{t}^+ \mathbf{V} - \gamma_\mathbf{t}^- \mathbf{V} \right\rangle_\tau \\
&= -\frac{1}{2} \left\langle \gamma_N^- \mathbf{U} - \gamma_N^+ \mathbf{U}, \gamma_\mathbf{t}^+ \mathbf{V} + \gamma_\mathbf{t}^- \mathbf{V} \right\rangle_\tau = -\left\langle \mu, \mathsf{C}^\kappa(\mathbf{v}) \right\rangle_\tau \;,
\end{aligned}$$

which finishes the proof. □

Ellipticity estimates corresponding to or extending (30) and (31) are available, too:

Theorem 11. *If* $\Re\kappa^2 \geq 0$ *and* $\Im\kappa^2 \geq 0$, *the following estimates hold true for all* $\forall \mu \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma)$ *and* $\mathbf{v} \in \boldsymbol{H}_\perp^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma)$

$$\Im\{\langle \mu, \mathsf{A}^\kappa \mu \rangle_\tau\} \geq 0 \;, \quad \Re\{\langle \mathsf{N}^\kappa \mathbf{v}, \mathbf{v} \rangle_\tau\} \geq 0 \;, \quad \Im\{\langle \mathsf{N}^\kappa \mathbf{v}, \mathbf{v} \rangle_\tau\} \geq 0 \;.$$

Moreover, with $c > 0$ *that may depend on* Γ *and* κ,

$$\Re\{\langle \mu, \mathsf{A}^\kappa \mu \rangle_\tau\} \geq c \,\|\mu\|^2_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)} \;, \; |\langle \mathsf{N}^\kappa \mathbf{v}, \mathbf{v} \rangle_\tau| \geq c \,\|\mathbf{v}\|^2_{\boldsymbol{H}_\perp^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)} \;.$$

Proof. As in the proof of Thm. 10, we rely on the integration by parts formula (11) and jump relations from Thm. 8 to get (for the case $\kappa \neq 0$)

$$\begin{aligned}\langle \mu, \mathsf{A}^\kappa \mu\rangle_\tau &= -\left\langle [\gamma_N \boldsymbol{\Psi}_S^\kappa(\mu)]_\Gamma , \gamma_{\mathbf{t}} \boldsymbol{\Psi}_S^\kappa(\mu)\right\rangle_\tau \\ &= \left\langle \gamma_N^- \boldsymbol{\Psi}_S^\kappa(\mu), \gamma_{\mathbf{t}}^- \boldsymbol{\Psi}_S^\kappa(\mu)\right\rangle_\tau - \left\langle \gamma_N^+ \boldsymbol{\Psi}_S^\kappa(\mu), \gamma_{\mathbf{t}}^+ \boldsymbol{\Psi}_S^\kappa(\mu)\right\rangle_\tau \\ &= \int_{\mathbb{R}^3\setminus\Gamma} |\,\mathbf{curl}\,\boldsymbol{\Psi}_S^\kappa(\mu)|^2 - \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\Psi}_S^\kappa(\mu)\cdot\overline{\boldsymbol{\Psi}_S^\kappa(\mu)}\,\mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^3\setminus\Gamma} |\,\mathbf{curl}\,\boldsymbol{\Psi}_S^\kappa(\mu)|^2 + \kappa^{-2}|\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\Psi}_S^\kappa(\mu)|^2\,\mathrm{d}\mathbf{x}\;.\end{aligned}$$

If $\kappa = 0$ we replace $\boldsymbol{\Psi}_S^\kappa$ with $\boldsymbol{\Psi}_A^0$, for which we know $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\Psi}_A^0(\mu) = 0$, if $\mathrm{div}_\Gamma\,\mu = 0$.

This identity can be combined with the continuity of the trace γ_N: with a constant $c = c(\Gamma) > 0$

$$\begin{aligned}\|\mu\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)} &= \|[\gamma_N \boldsymbol{\Psi}_S^\kappa(\mu)]_\Gamma\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)} \\ &\le c\left(\|\mathbf{curl}\,\boldsymbol{\Psi}_S^\kappa(\mu)\|_{\boldsymbol{L}^2(\Omega_c\cup\Omega_e)} + |\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\Psi}_S^\kappa(\mu)|_{\boldsymbol{L}^2(\Omega_c\cup\Omega_e)}\right)\;.\end{aligned}$$

Similar arguments show ellipticity for N^κ:

$$\begin{aligned}\langle \mathsf{N}^\kappa \mathbf{v}, \mathbf{v}\rangle_\tau &= -\left\langle \gamma_N \boldsymbol{\Psi}_M^\kappa(\mathbf{v}), [\gamma_{\mathbf{t}} \boldsymbol{\Psi}_M^\kappa(\mathbf{v})]_\Gamma\right\rangle_\tau \\ &= \int_{\mathbb{R}^3\setminus\Gamma} |\,\mathbf{curl}\,\boldsymbol{\Psi}_M^\kappa(\mathbf{v})|^2 - \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\Psi}_M^\kappa\cdot\overline{\boldsymbol{\Psi}_M^\kappa(\mathbf{v})}\,d\mathbf{x} \\ &= \int_{\mathbb{R}^3\setminus\Gamma} |\,\mathbf{curl}\,\boldsymbol{\Psi}_M^\kappa(\mathbf{v})|^2 + \kappa^2|\boldsymbol{\Psi}_M^\kappa(\mathbf{v})|^2\,d\mathbf{x} \\ &\ge c\,\|\boldsymbol{\Psi}_M^\kappa(\mathbf{v})\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega_c\cup\Omega_e)}\;.\end{aligned}$$

Now we have to make use of the continuity of the tangential trace $\gamma_{\mathbf{t}}$: for $c > 0$ independent of $\mathbf{v}$,

$$\|\mathbf{v}\|_{\boldsymbol{H}_\perp^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)} = \|[\gamma_{\mathbf{t}} \boldsymbol{\Psi}_M^\kappa(\mathbf{v})]_\Gamma\|_{\boldsymbol{H}_\perp^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)} \le c\,\|\boldsymbol{\Psi}_M^\kappa(\mathbf{v})\|_{\boldsymbol{H}(\mathbf{curl};\Omega_c\cup\Omega_e)}\;.$$

□

The same arguments confirm the following estimates for the scalar single layer potential boundary integral operator based on the Helmholtz kernel:

$$\Re\{\langle\varphi, \mathsf{V}^\kappa\varphi\rangle_{1/2,\Gamma}\} \ge c\,\|\varphi\|^2_{H^{-\frac{1}{2}}(\Gamma)}\;,\;\Im\{\langle\varphi, \mathsf{V}^\kappa\varphi\rangle_{1/2,\Gamma}\} \ge 0\;, \qquad (35)$$

for all $\varphi \in H^{-\frac{1}{2}}(\Gamma)$.

7 E-Based Model

Now we discuss the steps leading to a symmetrically coupled boundary element formulation for the transmission problem (18).

7.1 Coupled Problem

Now, let $(\mathbf{E}, \mathbf{E}_r)$ stand for the solution of the transmission problem (18) in Ω_c and Ω_e, respectively. Suitable trace operators can be applied to the representation formulas and this procedure yields the *Calderon identities*. From (34) we get

$$\begin{aligned} \gamma_{\mathbf{t}}^- \mathbf{E} &= \mathsf{A}^\kappa(\gamma_N^- \mathbf{E}) + (\tfrac{1}{2} Id + \mathsf{C}^\kappa)(\gamma_{\mathbf{t}}^- \mathbf{E}) \,, \\ \gamma_N^- \mathbf{E} &= (\tfrac{1}{2} Id + \\ maths f B^\kappa)&(\gamma_N^- \mathbf{E}) + \mathsf{N}^\kappa(\gamma_{\mathbf{t}}^- \mathbf{E}) \,, \end{aligned} \tag{36}$$

where $\kappa = \frac{1}{2}\sqrt{2}(1+i)\sqrt{\omega\sigma\mu_c}$. Thanks to Thm. 6 we have

$$\begin{aligned} \gamma_{\mathbf{t}}^+ \mathbf{E}_r &= -\mathsf{A}^0(\gamma_N^+ \mathbf{E}_r) + (\tfrac{1}{2} Id - \mathsf{C}^0)(\gamma_{\mathbf{t}}^+ \mathbf{E}_r) - \mathbf{grad}_\Gamma \mathsf{V}^0(\gamma_{\mathbf{n}}^+ \mathbf{E}_r) \,, \\ \gamma_N^+ \mathbf{E}_r &= (\tfrac{1}{2} Id - \mathsf{B}^0)(\gamma_N^+ \mathbf{E}_r) - \mathsf{N}^0(\gamma_{\mathbf{t}}^+ \mathbf{E}_r) \,, \\ \gamma_{\mathbf{n}}^+ \mathbf{E}_r &= -\gamma_{\mathbf{n}}^+ \boldsymbol{\Psi}_A^0(\gamma_N^+ \mathbf{E}_r) - \gamma_{\mathbf{n}}^+ \boldsymbol{\Psi}_M^0(\gamma_{\mathbf{t}}^+ \mathbf{E}_r) + (\tfrac{1}{2} Id - \mathsf{K}^0)(\gamma_{\mathbf{n}}^+ \mathbf{E}_r) \,. \end{aligned} \tag{37}$$

The boundary data for any solution of the interior/exterior $\mathbf{E}$-based eddy current equations will fulfill (36) and (37), respectively.

The gist of the symmetric coupling approach according to Costabel [27] is to use all of the equations of the Calderon identities in conjunction with the transmission conditions. However, here we have to grapple with a mismatch of interior and exterior boundary data due to the presence of $\gamma_{\mathbf{n}}^+ \mathbf{E}_r$ in (37). A remedy is motivated by the observation

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}_r = 0 \quad \text{in } \Omega_e \quad \Rightarrow \quad \mathrm{div}_\Gamma(\gamma_N^+ \mathbf{E}_r) = 0 \,,$$

which is an immediate consequence of the identity (13). We observe that $\gamma_N^+ \mathbf{E}_r$ has to be sought in the space $\boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma)$!

By the transmission condition for γ_N and the fact that $\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}_s = 0$ in a neighborhood of Γ, $\gamma_N^- \mathbf{E}$ has to be div_Γ-free, as well. Hence, we can restrict our attention to boundary data $\gamma_N^- \mathbf{E}, \gamma_N^+ \mathbf{E}_r$ in $\boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma)$ throughout. Recalling the dualities, this is a proper test space for those equations of the Calderon identities that are set in $\boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma)$. Since div_Γ is the $\boldsymbol{L}^2(\Gamma)$-adjoint of $\mathbf{grad}_\Gamma$, we find

$$\mu \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma) \quad \Rightarrow \quad \langle \mu, \mathbf{grad}_\Gamma \varphi \rangle_\tau = 0 \quad \forall \varphi \in H^{\frac{1}{2}}(\Gamma) \,.$$

This makes the undesirable terms disappear, when switching to a weak form of the top equations in the Calderon identities (36) and (37)! For all $\mu \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma)\boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma)$ we obtain

$$\begin{aligned} \left\langle \mu, \gamma_{\mathbf{t}}^- \mathbf{E} \right\rangle_\tau &= \left\langle \mu, \mathsf{A}^\kappa(\gamma_N^- \mathbf{E}) \right\rangle_\tau + \left\langle \mu, (\tfrac{1}{2} Id + \mathsf{C}^\kappa)\gamma_{\mathbf{t}}^- \mathbf{E} \right\rangle_\tau \,, \\ \left\langle \mu, \gamma_{\mathbf{t}}^+ \mathbf{E}_r \right\rangle_\tau &= \left\langle \mu, -\mathsf{A}^0(\gamma_N^+ \mathbf{E}_r) \right\rangle_\tau + \left\langle \mu, (\tfrac{1}{2} Id - \mathsf{C}^0)\gamma_{\mathbf{t}}^+ \mathbf{E}_r \right\rangle_\tau \,. \end{aligned}$$

From the transmission conditions we know $\gamma_{\mathbf{t}}^{+}\mathbf{E}_r - \gamma_{\mathbf{t}}^{-}\mathbf{E} = -\gamma_{\mathbf{t}}^{+}\mathbf{E}_s$. Thus, subtracting the above equations leads to

$$-\left\langle \mu, \mathsf{A}^0(\gamma_N^+\mathbf{E}_r) + \mathsf{A}^\kappa(\gamma_N^-\mathbf{E})\right\rangle_\tau - \left\langle \mu, \mathsf{C}^0(\gamma_{\mathbf{t}}^+\mathbf{E}_r) + \mathsf{C}^\kappa(\gamma_{\mathbf{t}}^-\mathbf{E}_r)\right\rangle_\tau = -\tfrac{1}{2}\left\langle \mu, \gamma_{\mathbf{t}}^+\mathbf{E}_s\right\rangle_\tau$$

for all $\mu \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma)$. From the transmission condition $\frac{1}{\mu_0}\gamma_N^+\mathbf{E}_r - \frac{1}{\mu_r}\gamma_N^-\mathbf{E} = -\frac{1}{\mu_0}\gamma_N^+\mathbf{E}_s$ and the second equations of the Calderon identities we directly conclude

$$\frac{1}{\mu_0}(\tfrac{1}{2}Id - \mathsf{B}^0)\gamma_N^+\mathbf{E}_r - \frac{1}{\mu_0}\mathsf{N}^0(\gamma_{\mathbf{t}}^+\mathbf{E}_r) - \tfrac{1}{\mu_c}(\tfrac{1}{2}Id + \mathsf{B}^\kappa)\gamma_N^-\mathbf{E} - \tfrac{1}{\mu_c}\mathsf{N}^\kappa(\gamma_{\mathbf{t}}^-\mathbf{E}) = -\frac{1}{2\mu_0}\gamma_N^+\mathbf{E}_s\ .$$

As final unknown quantities we introduce the tangential trace of the electric field $\mathbf{u} := \gamma_{\mathbf{t}}^-\mathbf{E} \in \boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma)$ and the tangential trace of the magnetic field $\lambda := \frac{1}{\mu_c}\gamma_N^-\mathbf{E} \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma)$. The latter is also known as *equivalent surface current*. The transmission conditions enable us to express the exterior traces in these unknowns. We end up with the coupled variational problem: Seek $\mathbf{u} \in \boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma)$, $\lambda \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma)$ such that

$$\boxed{\begin{aligned} \left\langle (\tfrac{1}{\mu_0}\mathsf{N}^0 + \tfrac{1}{\mu_c}\mathsf{N}^\kappa)\mathbf{u}, \mathbf{v}\right\rangle_\tau + \left\langle (\mathsf{B}^0 + \mathsf{B}^\kappa)\lambda, \mathbf{v}\right\rangle_\tau &= f(\mathbf{v})\ , \\ \left\langle \mu, (\mathsf{C}^0 + \mathsf{C}^\kappa)\mathbf{u}\right\rangle_\tau + \left\langle \mu, (\mu_0\mathsf{A}^0 + \mu_c\mathsf{A}^\kappa)\lambda\right\rangle_\tau &= g(\mu) \end{aligned}} \tag{38}$$

for all $\mathbf{v} \in \boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma)$, $\mu \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma)$. The right hand side is given by

$$\begin{aligned} f(\mathbf{v}) &:= \frac{1}{\mu_0}\left\langle (\tfrac{1}{2}Id + \mathsf{B}^0)\gamma_N\mathbf{E}_s, \mathbf{v}\right\rangle_\tau + \frac{1}{\mu_0}\left\langle \mathsf{N}^0(\gamma_{\mathbf{t}}\mathbf{E}_s), \mathbf{v}\right\rangle_\tau\ , \\ g(\mu) &:= \left\langle \mu, (\tfrac{1}{2}Id + \mathsf{C}^0)\gamma_{\mathbf{t}}\mathbf{E}_s\right\rangle_\tau + \left\langle \mu, \mathsf{A}^0(\gamma_N\mathbf{E}_s)\right\rangle_\tau\ . \end{aligned} \tag{39}$$

Theorem 12. *The bilinear form* $\mathbf{d}$ *associated with the variational problem* (38) *is* $\boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma) \times \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma)$*-elliptic in the sense that there is* $c = c(\Gamma, \kappa, \mu_0, \mu_c) > 0$ *such that*

$$\left|\mathbf{d}\left(\begin{pmatrix}\mathbf{v}\\ \mu\end{pmatrix}, \begin{pmatrix}\mathbf{v}\\ \mu\end{pmatrix}\right)\right| \geq c\left(\|\mathbf{v}\|^2_{\boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)} + \|\mu\|^2_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)}\right)$$

for all $\mathbf{v} \in \boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma)$ *and* $\mu \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma)$.

Proof. As a simple consequence of the block skew-symmetric structure of the variational problem (cf. Thm 10) we find for $\mathbf{v} \in \boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\mathbf{curl}_{\Gamma}, \Gamma)$, $\mu \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma}\, 0, \Gamma)$ that

$$\mathbf{d}((\mathbf{v}, \mu), (\mathbf{v}, \mu)) = \left\langle (\mathsf{N}^0 + \tfrac{1}{\mu_r}\mathsf{N}^{\kappa})\mathbf{v}, \mathbf{v} \right\rangle_{\tau} + \left\langle \mu, (\mathsf{A}^0 + \mu_r \mathsf{A}^{\kappa})\mu \right\rangle_{\tau} .$$

Subsequently, the estimates of Thm. 11 permit us to conclude ellipticity of the whole bilinear form from separate estimates for the individual terms. □

Corollary 1. *The variational problem* (38) *has a unique solution* $(\mathbf{u}, \lambda) \in \boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\mathbf{curl}_{\Gamma}, \Gamma) \times \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma}\, 0, \Gamma)$.

By the derivation of the boundary integral equations we can be certain that traces $\gamma_{\mathbf{t}}^{-}\mathbf{E}$ and $\gamma_{\times}^{-}\mathbf{H}$ will always give rise to solutions of (38). Their uniqueness then confirms that we get the traces of solutions of the $\mathbf{E}$-based eddy current model (17). These traces are fixed regardless of the gauging of $\mathbf{E}$ employed in Ω_e.

7.2 Galerkin Discretization

We select a conforming Galerkin boundary element discretization of (38) and (39) that relies on finite dimensional subspaces $\mathcal{W}_h \subset \boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\mathbf{curl}_{\Gamma}, \Gamma)$ and $\mathcal{V}_h \subset \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma}\, 0, \Gamma)$. These should be boundary element spaces in the sense that

1. the functions in both $\mathcal{W}_h$ and $\mathcal{V}_h$ are piecewise polynomial tangential vector fields with respect to a mesh Γ_h of Γ consisting of flat triangles.
2. there are bases of $\mathcal{W}_h$ and $\mathcal{V}_h$ that only comprise locally supported functions.

For the construction of $\mathcal{W}_h$ we start from $\boldsymbol{H}(\mathbf{curl}; \Omega_c)$-conforming finite element schemes for the approximation of vector potentials. The simplest is provided by the so-called edge elements [38]. Keeping in mind that

$$\boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\mathbf{curl}_{\Gamma}, \Gamma) := \gamma_{\mathbf{t}}(\boldsymbol{H}(\mathbf{curl}; \Omega_c)),$$

we simply take the tangential projections of edge element functions on a mesh Ω_h with $\Omega_{h|\Gamma} = \Gamma_h$ as space $\mathcal{W}_h$. This will give a space of piecewise linear vector fields on Γ, whose tangential components are continuous across edges of triangles. This is a well-known sufficient condition for $\mathcal{W}_h \subset \boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\mathbf{curl}_{\Gamma}, \Gamma)$. The local shape functions on a triangle T are given by the formula

$$\mathbf{b}_{i,j}^{T} := \lambda_i \, \mathbf{grad}_{\Gamma} \lambda_j - \lambda_j \, \mathbf{grad}_{\Gamma} \lambda_i \;, \quad 1 \leq i < j \leq 3 \;, \tag{40}$$

where λ_i, $i = 1, 2, 3$, are the local linear barycentric coordinate functions in T. These basis functions are sketched in Fig. 4. They are associated with the edges of Γ_h so that $\dim \mathcal{W}_h$ will agree with the total number of edges of Γ_h. Note that $\mathcal{W}_h$ can also be obtained by 90°-rotation of the lowest order div-conforming Raviart-Thomas elements in 2D, *cf.* [11, Ch. 3]. More details can be found in [6, Sect. 2.2].

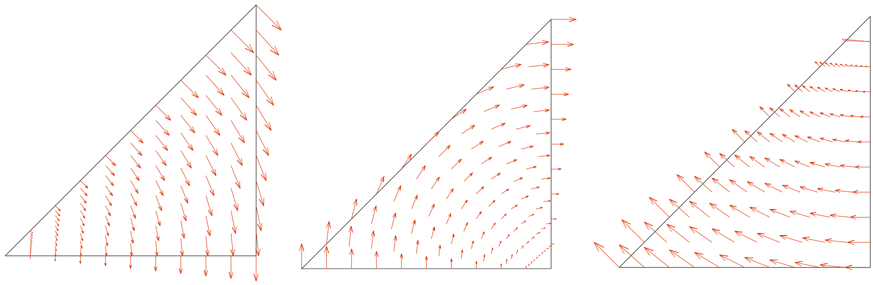

Fig. 4. Local shape functions of $\mathcal{W}_h$.

In order to find $\mathcal{V}_h$ we remember that λ is the rotated tangential trace of the magnetic field $\mathbf{H}$. As $\boldsymbol{H}(\mathbf{curl};\Omega)$ is the right function space for $\mathbf{H}$, too, we get the right boundary element space for magnetic traces by rotating functions in $\mathcal{W}_h$ by 90°. This will give surface vector fields with continuous fluxes across edges of triangles, which is a very desirable property for discrete equivalent surface currents. However, ellipticity of (38) only holds provided that $\operatorname{div}_\Gamma \lambda = 0$. Therefore, this property has to be enforced on $\mathcal{V}_h$. Formally, we may choose

$$\mathcal{V}_h := \{\mu_h \in \mathcal{W}_h \times \mathbf{n},\ \operatorname{div}_\Gamma \mu = 0\}\ . \tag{41}$$

Using the formula (40), we readily see that $\mathcal{V}_h$ only contains piecewise constant vector fields.

By Thm. 12 and Cea's lemma [24, Thm. 2.4.1] conformity of the Galerkin method directly translates into the quasi-optimal error estimate in energy norm

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_{\boldsymbol{H}_\perp^{-\frac{1}{2}}(\operatorname{curl}_\Gamma,\Gamma)} + \|\lambda-\lambda_h\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\operatorname{div}_\Gamma,\Gamma)} \le \\ \le C\left(\inf_{\mathbf{v}_h\in\mathcal{W}_h}\|\mathbf{u}-\mathbf{v}_h\|_{\boldsymbol{H}_\perp^{-\frac{1}{2}}(\operatorname{curl}_\Gamma,\Gamma)} + \inf_{\boldsymbol{\zeta}_h\in\mathcal{V}_h}\|\lambda-\boldsymbol{\zeta}_h\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\operatorname{div}_\Gamma,\Gamma)}\right)\ ,\end{aligned} \tag{42}$$

where $\mathbf{u}_h$ and λ_h stand for the boundary element solutions, and $C > 0$ depends on the ellipticity and continuity constants of the continuous variational problem (38). Hence, approximation error estimates for the finite element spaces will directly provide us with rates of convergence. Let us assume quasi-uniform and shape regular families of surface meshes Γ_h, where h denotes the meshwidth. Provided that the continuous solutions $\mathbf{u}$ and λ are sufficiently smooth, we arrive at

$$\|\mathbf{u}-\mathbf{u}_h\|_{\boldsymbol{H}_\perp^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)} + \|\lambda-\lambda_h\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\div_\Gamma,\Gamma)} \leq$$
$$\leq C\left(h^{\min\{\frac{3}{2},\eta+\frac{1}{2}\}}\|\mathbf{u}\|_{\boldsymbol{H}^\eta(\mathbf{curl}_\Gamma,\Gamma)} + h^{\min\{\frac{3}{2},\rho+\frac{1}{2}\}}\|\lambda\|_{\boldsymbol{H}^\rho(\Gamma)}\right) , \quad (43)$$

for some $\eta,\rho>0$. The constant $C>0$ now depends on the shape-regularity of the meshes, too. Details about approximation by functions in $\mathcal{W}_h$ can be found in [14, Sect. 4.2.2]. The possible ranges of η and ρ depend on the geometry of Γ: the presence of edges and corners will impose limits on η,ρ. At worst, these may only be slightly larger than zero.

The divergence constraint is essential in the definition (41) of the boundary element trial space for the surface currents. We cannot simply use rotated shape functions from Fig. 4 to get a locally supported basis, because the constraint has to be enforced. Two options are available:

Lagrangian Multipliers

We may take the cue from mixed finite element schemes for second order elliptic boundary value problems[11, Ch. 4] and use Lagrangian multipliers to impose the linear constraints $\mathrm{div}_\Gamma\,\lambda_h=0$. The natural discrete Lagrangian multiplier space is

$$\begin{aligned}\mathcal{M}_h &:= \mathrm{div}_\Gamma(\mathcal{W}_h\times\mathbf{n}) \qquad (44)\\ &= \{\nu\in L^2(\Gamma):\ \mu_{h|K}\equiv\mathrm{const}\ \forall K\in\Gamma_h,\ \int_\Gamma \mu_h\,\mathrm{d}S=0\} .\end{aligned}$$

Care must be taken when selecting the sesqui-linear form $m(\cdot,\cdot)$ that brings the Lagrangian multiplier to bear on λ_h in the sense that

$$\mathcal{V}_h=\{\mu_h\in\mathcal{W}_h\times\mathbf{n}:\ m(\mu_h,\nu_h)=0\quad\forall\nu_h\in\mathcal{M}_h\} .$$

For the sake of asymptotic stability of the discrete problem, the form m must be both h-uniformly continuous and satisfy inf-condition [11, Ch. 3]

$$\sup_{\mu_h\in\mathcal{W}_h\times\mathbf{n}}\frac{|m(\mu_h,\nu_h)|}{\|\mu_h\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)}}\geq c\,\|\nu_h\|_M\quad\forall\nu_h\in\mathcal{M}_h , \qquad (45)$$

where $c>0$ should not depend on the meshwidth h. The norm $\|\cdot\|_M$ with which $\mathcal{M}$ is endowed is still at our disposal.

Next, note that the tempting choice $\|\cdot\|_M=\|\cdot\|_{L^2(\Gamma)}$ and

$$m(\mu_h,\nu_h):=(\mathrm{div}\,\mu_h,\nu_h)_{L^2(\Gamma)}$$

must be ruled out, though (45) is easily seen to hold, because this m will fail to be continuous on $\boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)\times L^2(\Omega)$.

A viable option is

$$\|\cdot\|_M = \|\cdot\|_{H^{-\frac{1}{2}}(\Gamma)} \quad \text{and} \quad m(\mu_h, \nu_h) := \left\langle \operatorname{div}_\Gamma \mu_h, \mathsf{V}^0 \nu_h \right\rangle_{1/2,\Gamma} .$$

Here, continuity is immediate from (29) and the h-uniform inf-sup condition (45) has been shown in [22], see also [10, Sect. 5.3].

Surface Stream Functions

Another way to deal with the divergence constraint resorts to scalar *surface stream functions*. Let $\mathcal{S}_h$ stand for the space of Γ_h-piecewise linear and continuous functions on Γ. Then, if Γ is simply connected, we know from deRham's theorem [38, Cor. 3.3] that $\mathcal{V}_h = \mathbf{curl}_\Gamma \mathcal{S}_h$. Hence, we may simply use the surface rotation of the "hat basis functions" of $\mathcal{S}_h$ as a basis for $\mathcal{V}_h$, see Fig. 5 (left).

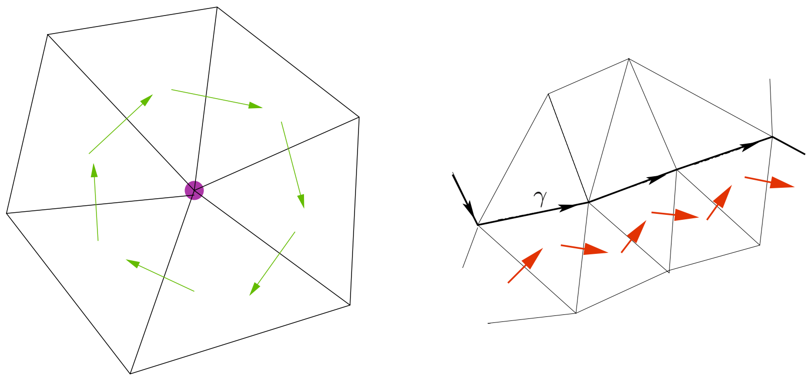

Fig. 5. Basis function of $\mathcal{V}_h$ associated with a vertex (left). Current sheet along a section of a path γ (right).

Because we have not ruled out more general topologies of Γ, surface cohomology vector fields can also contribute to the kernel of $\operatorname{div}_\Gamma$:

$$\mathcal{V}_h = \mathbf{curl}_\Gamma \mathcal{S}_h \oplus \mathcal{H}_h \quad , \quad \dim \mathcal{H}_h = \beta_1(\Gamma) , \tag{46}$$

where $\beta_1(\Gamma)$ is the first Betti number of Γ, which is twice the number of holes drilled through Ω_c. This means that $\dim \mathcal{V}_h$ will be equal to the number of vertices of Γ_h plus $\beta_1(\Gamma)$.

To find a basis of $\mathcal{H}_h$ we need representatives γ_k, $k = 1 \ldots, \beta_1(\Gamma)$, of a basis of the cohomology group $H_1(\Gamma_h, \mathbb{Z})$ in the form of oriented closed edge paths (cycles). In other words, we need a maximal set of closed curves on the surface that do not cut the surface into two separate parts, and cannot be deformed into each other by sweeping them over parts of Γ. Typical choices for the torus are depicted in Fig. 3. We can always find such curves that run along edges of Γ_h and this can be done with a computational effort proportional to the number of edges in Γ_h [41]. To each such path γ a "current sheet" $\boldsymbol{\eta}_\gamma$ can be associated, a circular current traveling along the path, see Fig. 5 (right).

Consider a non-bounding surface edge cycle γ that is bounding with respect to Ω_e, that is, there is an oriented surface $\Sigma \subset \Omega_e$ such that $\gamma = \partial\Sigma$. Then we get from Stokes theorem

$$\int_\gamma (\gamma_N \mathbf{E}_r \times \mathbf{n}) \cdot d\mathbf{s} = \int_\Sigma \mathbf{curl}\,\mathbf{curl}\,\mathbf{E}_r \cdot \mathbf{n}\,\mathrm{d}S = 0\ .$$

As $\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}_r = 0$ in Ω_e, this means that, in the discrete variational problem (47), we can confine ourselves to those $\lambda_h \in \mathcal{V}_h$ that satisfy $\int_\gamma (\lambda_h \times \mathbf{n}) \cdot d\mathbf{s} = 0$ for all cycles γ bounding relative to Ω_e. This means that we only have to take into account current sheets along cycles bounding relative to the exterior. An algorithm for the construction of these cycles has been developed in [41]. The resulting basis of the relevant subspace of $\mathcal{H}_h$ will be denoted by $\iota_1, \ldots, \iota_L$, $L =: \frac{1}{2}\beta_1(\Gamma)$. Then the discrete linear variational problem arising from (38) read search for $\mathbf{u}_h \in \mathcal{W}_h$, $\varphi_h \in \mathcal{S}_h/\mathbb{R}$, $(\alpha_1, \ldots, \alpha_L)^T \in \mathbb{C}^L$ such that

$$\begin{aligned}
&-\left\langle \widetilde{\mathsf{N}^0}\mathbf{u}_h, \mathbf{v}_h \right\rangle_\tau - \left\langle \widetilde{\mathsf{B}^0}\,\mathbf{curl}_\Gamma\,\varphi_h, \mathbf{v}_h \right\rangle_\tau - \sum_{k=1}^{L} \alpha_k \left\langle \widetilde{\mathsf{B}^0}\iota^k, \mathbf{v}_h \right\rangle_\tau \\
&\qquad\qquad = f(\mathbf{v}_h)\ , \\
&\left\langle \widetilde{\mathsf{B}^0}\,\mathbf{curl}_\Gamma\,\psi_h, \mathbf{u}_h \right\rangle_\tau + \left\langle \mathbf{curl}_\Gamma\,\psi_h, \widetilde{\mathsf{A}^0}\,\mathbf{curl}_\Gamma\,\varphi_h \right\rangle_\tau + \sum_{k=1}^{L} \alpha_k \left\langle \mathbf{curl}_\Gamma\,\psi_h, \widetilde{\mathsf{A}^0}\iota_k \right\rangle_\tau \\
&\qquad\qquad = g(\mathbf{curl}_\Gamma\,\psi_h)\ , \\
&\left\langle \widetilde{\mathsf{B}^0}\iota^j, \mathbf{u}_h \right\rangle_\tau + \left\langle \iota^j, \widetilde{\mathsf{A}^0}\,\mathbf{curl}_\Gamma\,\varphi_h \right\rangle_\tau + \sum_{k=1}^{L} \alpha_k \left\langle \iota^j, \widetilde{\mathsf{A}^0}\iota_k \right\rangle_\tau \\
&\qquad\qquad = g(\iota^j)\ ,
\end{aligned} \tag{47}$$

for all $\mathbf{v}_h \in \mathcal{W}_h$, $\psi_h \in \mathcal{S}_h/\mathbb{R}$, $j = 1, \ldots, L$. We abbreviated $\widetilde{\mathsf{A}^0} := \mu_0\mathsf{A}^0 + \mu_c\mathsf{A}^\kappa$, $\widetilde{\mathsf{B}^0} = \mathsf{B}^0 + \mathsf{B}^\kappa$, $\widetilde{\mathsf{N}^0} := \frac{1}{\mu_0}\mathsf{N}^0 + \frac{1}{\mu_c}\mathsf{N}^\kappa$. From (47) we can retrieve $\lambda_h = \mathbf{curl}_\Gamma\,\varphi_h + \sum_{k=1}^{L} \alpha_k \iota^k$.

Remark 2. If surface stream functions are used, Non-local inductive excitation can taken into account in an amazingly simple fashion: for each loop of the conductor there is basis cycle of $H_1(\Gamma, \mathbb{Z})$ that "winds around it", see Fig. 3 for an example. We realize that the circulation of the magnetic field along that fundamental cycle, which is equal to the flux of λ through it, agrees with the total current in the loop. Hence, inductive excitation amounts to fixing some of the α_k in the variational formulation (47). More details are given in [39, Sect. 8].

8 H-Based Model

For want of a transmission problem, the derivation of symmetrically coupled boundary integral equations starts from the variational problem (19).

8.1 Boundary Reduction

In order to be able to perform a reduction to the boundary through integration by parts we have to resort to scalar potentials. Therefore we use (15) to replace $\mathcal{V}$ by

$$\mathcal{V}[\mathbf{H}_s] = \{(\mathbf{V}, \Phi) \in \boldsymbol{H}(\mathbf{curl}; \Omega_c) \times H^1_\Sigma(\Omega_e),\ \gamma_\mathbf{t}^- \mathbf{V} - \gamma_\mathbf{t}^+ \widetilde{\mathbf{grad}\Phi} = \gamma_\mathbf{t}^+ \mathbf{H}_s \text{ on } \Gamma\} .$$

For the notations we refer to Sect. 4. Thus, (42) is converted into: Seek $(\mathbf{H}, \Psi) \in \mathcal{V}[\mathbf{H}_s]$ such that

$$\left(\sigma^{-1} \mathbf{curl}\,\mathbf{H}, \mathbf{curl}\,\mathbf{V}\right)_{\boldsymbol{L}^2(\Omega_c)} + i\omega\mu_c \left(\mathbf{H}, \mathbf{V}\right)_{\boldsymbol{L}^2(\Omega_c)} + \\ + i\omega\mu_0 \left(\mathbf{H}_s + \widetilde{\mathbf{grad}\Psi}, \widetilde{\mathbf{grad}\Phi}\right)_{\boldsymbol{L}^2(\Omega_e)} = 0 , \quad (48)$$

for all $(\mathbf{V}, \Phi) \in \mathcal{V}[0]$. As $\operatorname{div}\mathbf{H}_s = 0$ in Ω_e, testing with functions compactly supported either in Ω_c or Ω_e shows that for $k = 1, \dots, N$

$$\mathbf{curl}\,\sigma^{-1}\,\mathbf{curl}\,\mathbf{H} + i\omega\mu_c \mathbf{H} = 0 \quad \text{in } \Omega_c , \quad (49)$$

$$-\Delta\Psi = 0 \text{ in } \Omega' \quad , \quad [\partial_\mathbf{n}\,\mathbf{grad}\,\Psi]_{\Sigma_k} = 0,\ [\gamma\Psi]_{\Sigma_k} = \text{const.} . \quad (50)$$

Integration by parts can be carried out on both Ω_c and Ω'. Thus, setting $\tau = (i\omega\sigma\mu_0)^{-1}$, (48) becomes

$$\tau \left\langle \sigma^{-1}\gamma_N^- \mathbf{H}, \gamma_\mathbf{t}^- \mathbf{V} \right\rangle_\tau - \langle \partial'_\mathbf{n}\Psi, \gamma\Phi \rangle_{1/2,\partial\Omega'} = \langle \gamma_\mathbf{n}\mathbf{H}_s, \gamma'\Phi \rangle_{1/2,\partial\Omega'} . \quad (51)$$

Here, γ' and $\partial'_\mathbf{n}$ are the standard trace and conormal derivative onto $\partial\Omega'$. The definition of $\partial'_\mathbf{n}$ relies on the interior unit normal vectorfield on $\partial\Omega'$.

Remark 3. Splitting the duality pairing $\langle \gamma_\mathbf{n}\mathbf{H}_s, \gamma'\Phi \rangle_{1/2,\partial\Omega'}$ into contributions of Γ and of the cuts cannot be done immediately, because the individual integrals are no continuous functionals on the space $H^{\frac{1}{2}}(\partial\Omega')$. This procedure must be postponed until after discretization.

8.2 Coupled Problem

For both (49) and (50) we need a realization of the Dirichlet-to-Neumann operator by boundary integral operators. For (50) we can rely on the exterior Calderon projector for the Laplacian on Ω' [58, Sect. 3.6], which gives the identities

$$\begin{aligned}\gamma'\Psi &= (\tfrac{1}{2}Id + \mathsf{K}')(\gamma'\Psi) - \mathsf{V}'(\partial'_{\mathbf{n}}\Psi)\,,\\ \partial'_{\mathbf{n}}\Psi &= -\mathsf{D}'(\gamma'\Psi) + (\tfrac{1}{2}Id - (\mathsf{K}')^*)(\partial'_{\mathbf{n}}\Psi)\,.\end{aligned} \tag{52}$$

The integral operators match those introduced in the beginning of Sect. 6, but this time they are defined on $\partial\Omega'$ and based on a unit normal vectorfield pointing into the interior of Ω': K' is the double layer potential integral operator for Δ, $(\mathsf{K}')^*$ its $\boldsymbol{L}^2(\partial\Omega')$-adjoint, and D' stands for the hypersingular operator, see (29).

What is not reflected in the statement of the Calderon identities is the special nature of the "Dirichlet trace" $\gamma'\Psi$ and "Neumann trace" $\partial'_{\mathbf{n}}\Psi$ entailed by the transmission conditions of (50). They imply that

$$\gamma'\Psi \in H^{\frac{1}{2}}_{\Sigma}(\partial\Omega') := \{v \in H^{\frac{1}{2}}(\partial\Omega'),\, [v]_{\Sigma_j} = \text{const.},\, j = 1, \dots, N\}\,,$$

$$\partial'_{\mathbf{n}}\Psi \in H^{-\frac{1}{2}}_{\Sigma}(\partial\Omega') := \{\phi \in H^{-\frac{1}{2}}(\partial\Omega'),\, \phi^+ + \phi^- = 0 \text{ on } \Sigma_j,\, j = 1, \dots, N\}\,.$$

For the interior problem (49) we can reuse the Calderon identities (36) with $\kappa = \frac{1}{\sqrt{2}}(1+i)\sqrt{\omega\sigma\mu_c}$ and $\mathbf{H}$ instead of $\mathbf{E}$:

$$\begin{aligned}\gamma_{\mathbf{t}}^-\mathbf{H} &= \mathsf{A}^\kappa(\gamma_N^-\mathbf{H}) + (\tfrac{1}{2}Id + \mathsf{C}^\kappa)(\gamma_{\mathbf{t}}^-\mathbf{H})\,,\\ \gamma_N^-\mathbf{H} &= (\tfrac{1}{2}Id +\\ &maths f B^\kappa)(\gamma_N^-\mathbf{H}) + \mathsf{N}^\kappa(\gamma_{\mathbf{t}}^-\mathbf{H})\,.\end{aligned} \tag{53}$$

Now we can merge (51), (52), and (53), making use of $\gamma_{\mathbf{t}}^-\mathbf{V} = \mathbf{grad}_\Gamma\,\gamma^+\Phi$ and $\gamma_{\mathbf{t}}^-\mathbf{H} = \mathbf{grad}_\Gamma\,\gamma^+\Psi + \gamma_{\mathbf{t}}\mathbf{H}_s$. This results in: Seek $u \in H^{\frac{1}{2}}_{\Sigma}(\partial\Omega')/\mathbb{R}$, $\psi \in H^{-\frac{1}{2}}_{\Sigma}(\partial\Omega')$, $\boldsymbol{\eta} \in \boldsymbol{H}^{-\frac{1}{2}}_{||}(\operatorname{div}_\Gamma, \Gamma)$ such that

$$\begin{aligned} n'(u,v) + b(\boldsymbol{\eta}, v) - k'(\psi, v) &= f(v)\,,\\ -b(\mu, u) + a(\boldsymbol{\eta}, \mu) &= g(\mu)\,,\\ k'(\phi, u) + d'(\psi, \phi) &= 0\,.\end{aligned} \tag{54}$$

for all $v \in H^{\frac{1}{2}}_{\Sigma}(\partial\Omega')/\mathbb{R}$, $\phi \in H^{-\frac{1}{2}}_{\Sigma}(\partial\Omega')$, $\mu \in \boldsymbol{H}^{-\frac{1}{2}}_{||}(\operatorname{div}_\Gamma, \Gamma)$, where

$$\begin{aligned} n'(u,v) &:= \tau\,\langle \mathsf{N}^\kappa(\mathbf{grad}_\Gamma\, u), \mathbf{grad}_\Gamma\, v\rangle_\tau + \langle \mathsf{D}'u, v\rangle_{1/2,\partial\Omega'}\,,\\ b(\boldsymbol{\eta}, v) &:= \tau\,\left\langle (\tfrac{1}{2}Id + \mathsf{B}^\kappa)\boldsymbol{\eta}, \mathbf{grad}_\Gamma\, v\right\rangle_\tau\,,\\ k'(\psi, v) &:= \left\langle \psi, (\tfrac{1}{2}Id - \mathsf{K}')v\right\rangle_{1/2,\partial\Omega'}\,,\\ a(\boldsymbol{\eta}, \mu) &:= \tau\,\langle \mu, \mathsf{A}^\kappa\boldsymbol{\eta}\rangle_\tau\,,\\ d'(\psi, \phi) &:= \langle \phi, \mathsf{V}'\psi\rangle_{1/2,\partial\Omega'}\,,\\ f(v) &:= \langle \gamma'_{\mathbf{n}}\mathbf{H}_s, v\rangle_{1/2,\partial\Omega'} - \langle \mathsf{N}^\kappa(\gamma_{\mathbf{t}}\mathbf{H}_s), \mathbf{grad}_\Gamma\, v\rangle_\tau\,,\\ g(\mu) &:= \tau\,\left\langle \mu, (\tfrac{1}{2}Id - \mathsf{C}^\kappa)\gamma_{\mathbf{t}}\mathbf{H}_s\right\rangle_\tau\,.\end{aligned}$$

It is worth noting that (11) yields the identity (cf. [25, Formula (2.86)])

$$\langle \mathsf{N}^\kappa\mathbf{u}, \mathbf{v}\rangle_\tau = \kappa^2\,\langle \gamma_{\mathbf{t}}\boldsymbol{\Psi}^\kappa_A(\mathsf{R}\mathbf{u}), \mathsf{R}\mathbf{v}\rangle_\tau + \langle \mathsf{V}^\kappa(\operatorname{curl}_\Gamma \mathbf{u}), \operatorname{curl}_\Gamma \mathbf{v}\rangle_{1/2,\Gamma}\,. \tag{55}$$

This leads to an alternative expression for the first contribution to $n'(u,v)$:

$$\langle \mathsf{N}^{\kappa}(\mathbf{grad}_{\Gamma}\, u), \mathbf{grad}_{\Gamma}\, v\rangle_{\tau} = \kappa^2 \langle \gamma_{\mathbf{t}} \boldsymbol{\Psi}_A^{\kappa}(\mathbf{curl}_{\Gamma}\, u), \mathbf{curl}_{\Gamma}\, v\rangle_{\tau} \ .$$

The surface gradient of the u-component of the solution of (54) provides the tangential trace of $\mathbf{H}$, whereas $\psi := \partial'_{\mathbf{n}}\Psi$ can be viewed as the (scaled) magnetic flux through $\partial\Omega'$. The meaning of $\boldsymbol{\eta} := \gamma_N^- \mathbf{H}$ is that of a (scaled) twisted tangential trace of the electric field.

Theorem 13. *The bilinear form associated with the variational problem* (54) *is* $H_{\Sigma}^{\frac{1}{2}}(\partial\Omega')/\mathbb{R} \times H_{\Sigma}^{-\frac{1}{2}}(\partial\Omega') \times \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma}, \Gamma)$*-elliptic*

Proof. As in the proof of Thm. 12 we can exploit the block skew symmetric structure, because the bilinear forms on the diagonal are elliptic on their respective spaces, see Thm. 11 and (35). □

Remark 4. Actually, the coupled variational problem for the $\mathbf{H}$-based model fails the condition that only equations on Γ may be involved, because some integral operators rely on cutting surfaces, too. This is an enormous practical obstacle to the use of the $\mathbf{H}$-based model, because the construction of cutting surfaces requires a triangulation of some part of the air region and can be prohibitively expensive [46, 37].

One might wonder why this drawback is inevitable with the $\mathbf{H}$-based model but not encountered in the case of the $\mathbf{E}$-based model. We owe this to the second nature of $\mathbf{E}$ as a vector potential. For this reason we do not have to introduce another potential to carry out boundary reduction. On top of that a vector potential always exists and is not tied to any topological constraints.

8.3 Galerkin Discretization

Assume that a combined triangulation Γ'_h of Γ and the cuts Σ_k, $k = 1, \ldots, N$, is supplied. As before, we write Γ_h for its restriction to Γ. Thanks to Thm. 13 a conforming Galerkin discretization will yield quasi-optimal approximations of solutions u, ψ, and $\boldsymbol{\eta}$ of (54).

In particular, the space $\mathcal{F}_h(\Gamma_h) \subset \boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma}, \Gamma)$ can be reused as trial space for $\boldsymbol{\eta}$. To approximate u and ψ we can employ the usual conforming boundary element spaces for $H^{\frac{1}{2}}(\partial\Omega')$ and $H^{-\frac{1}{2}}(\partial\Omega')$. Let $\mathcal{S}_h(\Gamma'_h)$ and $\mathcal{Q}_h(\Gamma'_h)$ stand for these.

A common trait of the boundary element spaces is that they offer far more regularity than required by mere conformity. For instance, all boundary element functions will belong to $L^{\infty}(\partial\Omega')$. Then the constraints inherent in the spaces $H_{\Sigma}^{\frac{1}{2}}(\partial\Omega')$ and $H_{\Sigma}^{-\frac{1}{2}}(\partial\Omega')$ permit us to restrict the operators V', K', and, D' to Γ: Straightforward manipulations using the integral operator representations of V', D', and K' show that for $u, v \in H_{\Sigma}^{\frac{1}{2}}(\partial\Omega') \cap L^{\infty}(\partial\Omega')$ and $\phi, \psi \in H_{\Sigma}^{-\frac{1}{2}}(\partial\Omega') \cap L^{\infty}(\partial\Omega')$

$$
\begin{aligned}
\langle \psi, \mathsf{V}'\phi \rangle_{1/2,\partial\Omega'} &= \left\langle \psi, \mathsf{V}^0\phi \right\rangle_{1/2,\Gamma} \quad , \quad \langle \mathsf{D}'u, v \rangle_{1/2,\partial\Omega'} = \left\langle \mathsf{D}^0 u, v \right\rangle_{1/2,\Gamma} \ , \\
\langle \phi, \mathsf{K}'v \rangle_{1/2,\partial\Omega'} &= \left\langle \phi, \mathsf{K}^0 v \right\rangle_{1/2,\Gamma} + \sum_{k=1}^{N} [v]_{\Sigma_k} \int_{\Gamma} \int_{\Sigma_k} \frac{\partial G_0(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} \phi(\mathbf{x}) \, dS(\mathbf{y}) dS(\mathbf{x}) \ , \\
\langle \phi, v \rangle_{1/2,\partial\Omega'} &= \langle \phi, v \rangle_{1/2,\Gamma} + \sum_{k=1}^{N} [v]_{\Sigma_k} \cdot \int_{\Sigma_k} \phi(\mathbf{x}) \, dS(\mathbf{x}) \ .
\end{aligned}
\tag{56}
$$

We observe that the cuts will enter the discrete variational problem only through some global integral quantities that are not sensitive to the choice of boundary elements on the cuts. Sloppily speaking, this permits us to cover each cut by only a single surface element. More precisely, we may choose

$$
\begin{aligned}
\mathcal{S}_h(\Gamma'_h) &= \mathcal{S}_h(\Gamma_h) + \mathrm{Span}\left\{c_h^1, \dots, c_h^N\right\} \ , \\
\mathcal{Q}_h(\Gamma'_h) &= \mathcal{Q}_h(\Gamma_h) + \mathrm{Span}\left\{\chi_1, \dots, \chi_N\right\} \ .
\end{aligned}
$$

Here, c_h^k is a Γ'_h-piecewise linear function $\in C^0(\partial\Omega') \cap H_\Sigma^{\frac{1}{2}}(\partial\Omega')$, whose restriction to Γ has a jump of height 1 across the edge cycle σ_k and is continuous across any other σ_j, $j \neq k$. The function $\chi_k \in L^\infty(\partial\Omega') \cap H_\Sigma^{-\frac{1}{2}}(\partial\Omega')$ assumes the values $+1$ and -1 on Σ_k^+ and Σ_k^-, respectively, and vanishes on $\partial\Omega' \setminus \Sigma_k$.

Using the identities (56), the discrete variational problem can be rephrased as: Seek $\tilde{u}_h \in \mathcal{S}_h(\Gamma_h)$, $\tilde{\psi}_h \in \mathcal{Q}_h(\Gamma_h)$, $\boldsymbol{\eta}_h \in \mathcal{F}_h(\Gamma_h)$, $\alpha_1, \dots, \alpha_N \in \mathbb{C}$, $\beta_1, \dots, \beta_N \in \mathbb{C}$ such that

$$
\begin{array}{llllll}
n(\tilde{u}_h, v_h) & + \, b(\boldsymbol{\eta}_h, v_h) & - \, k(\tilde{\psi}_h, v_h) & + \sum_k \alpha_k n(c_k, v_h) & & = f(v_h) \ , \\
-b(\mu_h, \tilde{u}_h) & + \, a(\boldsymbol{\eta}_h, \mu_h) & & - \sum_k \alpha_k b(\mu_h, c_k) & & = g(\mu_h) \ , \\
k(\phi_h, \tilde{u}_h) & & + \, d(\psi_h, \phi_h) & + \sum_k \alpha_k k'(\phi_h, c_k) & & = 0 \ , \\
n(\tilde{u}_h, c_j) & + \, b(\boldsymbol{\eta}_h, c_j) & - \, k'(\tilde{\psi}_h, c_j) & + \sum_k \alpha_k n(c_k, c_j) & - \sum_k \beta_k k'(\chi_k, c_j) & = f(c_j) \ , \\
 & & & \sum_k \alpha_k k'(\chi_l, c_k) & & = 0 \ .
\end{array}
$$

for all $v_h \in \mathcal{S}_h(\Gamma_h)$, $\mu_h \in \mathcal{F}_h(\Gamma_h)$, $\phi_h \in \mathcal{Q}_h(\Gamma_h)$, $j = 1, \dots, N$, $l = 1, \dots, N$. Here we set, using [32, Thm. 7, Ch. XI],

$$
\begin{aligned}
n(u, v) &:= \tfrac{1}{\tau^2} \left(\mathsf{N}^\kappa(\mathbf{grad}_\Gamma u), \mathbf{grad}_\Gamma v\right)_{\boldsymbol{L}^2(\Gamma)} - \left(\mathsf{A}^0 \widetilde{\mathbf{curl}_\Gamma} u, \widetilde{\mathbf{curl}_\Gamma} v\right)_{\boldsymbol{L}^2(\Gamma)} \ , \\
d(\psi, \phi) &:= \left(\phi, \mathsf{V}^0 \psi\right)_{\boldsymbol{L}^2(\Gamma)} \ , \quad k(\psi, v) := \left(\psi, (\tfrac{1}{2} Id - \mathsf{K}^0) v\right)_{\boldsymbol{L}^2(\Gamma)} \ ,
\end{aligned}
$$

for bilinear forms induced by integral operators on Γ alone. The discrete solution can be obtained as $u_h = \tilde{u}_h + \sum_k \alpha_k c_k$ and $\psi_h = \tilde{\psi}_h + \sum_k \beta_k \chi_k$. A closer study of the boundary integral operators shows that the cuts only come into play through integrals of the form

$$\int_{\Gamma}\int_{\Sigma_k} \frac{\partial G_0(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} \phi_h(\mathbf{x})\, dS(\mathbf{y},\mathbf{x}) \quad , \quad \int_{\Sigma_k} \mathbf{H}_s \cdot \mathbf{n}\, dS \; ,$$

for $\phi_h \in \mathcal{Q}_h(\Gamma_h)$. Obviously, by Gauß' divergence theorem, Σ_k can be replaced by any other surface homologous in $H_2(\Omega_e, \mathbb{Z})$ without changing the values of the integrals. Paradoxically, information about the concrete geometry of the Σ_k seems to be indispensable for the evaluation of the integrals.

The case of lumped parameter excitation is treated in a similar fashion as in the case of the **E**-based model. First, note that α_k measures the jump of the magnetic scalar potential across Σ_k. According to Ampere's law the height of this jump agrees with the total current in the loop of the conductor corresponding to Σ_k. Hence, a prescribed total current in a loop of the conductor can be taken into account by fixing the value of α_k for the related cut.

Remark 5. The intrinsic use of a (multivalued) magentic scalar potential in Ω_e paves the way for accommodating non-local inductive current excitation: by Ampere's law, we only need to fix the jump of ψ across a cut associated with a current carrying loop of the conductor. In the above variational formulation, this boils down to fixing some of the α_k.

Remark 6. The values of the β_k agree with the total magnetic flux through the cut Σ_k. By Faraday's law it is proportional to the electromotive force along σ_k. Hence, if the voltage around a loop of the conductor is to be imposed, we can do so by fixing the value of the associated β_k. The possibility to take into account lumped parameter voltage excitation is only available with the **H**-based model.

9 Postprocessing

As we have remarked in the introduction, getting approximate Cauchy data $(\gamma_{\mathbf{t}}\mathbf{E}, \gamma_\times \mathbf{H})$ on Γ might not be the eventual goal of the computation. Thus, we have to figure out how to get (i) the total Ohmic losses in Ω_C, and (ii) the total force acting on Ω_C. Here, we focus on the **E**-based formulation of Sect. 7 and assume that by solving (47) we have obtained approximate Cauchy data $(\mathbf{u}_h, \lambda_h)$.

Ohmic losses are the only mechanism for the dissipation of field energy in the eddy current model. Moreover, since all fields are harmonic in time, the total field energy inside Ω_C will not change over one period. Therefore, we get the averaged Ohmic losses by appealing to Poynting's theorem

$$\bar{P}_{\mathrm{Ohm}} = -\frac{1}{2}\Re\left\{\int_{\Gamma} (\mathbf{E} \times \overline{\mathbf{H}}) \cdot \mathbf{n}\, \mathrm{d}S\right\} = \frac{1}{2}\Re\left\{\langle \mathbf{u}, \lambda\rangle_\tau\right\} \; .$$

A natural approximation is

$$\bar{P}_{\mathrm{Ohm}} \approx \bar{P}^h_{\mathrm{Ohm}} := \tfrac{1}{2}\Re\,\langle \mathbf{u}_h, \lambda_h\rangle_\tau \ .$$

The error can be estimated by

$$\begin{aligned}\bar{P}_{\mathrm{Ohm}} - \bar{P}^h_{\mathrm{Ohm}} &= \tfrac{1}{2}\Re\,\{\langle \mathbf{u}, \lambda - \lambda_h\rangle_\tau + \langle \mathbf{u}-\mathbf{u}_h, \lambda_h\rangle_\tau\} \\ &\leq \tfrac{1}{2}\Big(\|\mathbf{u}\|_{H_\perp^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)} \|\lambda-\lambda_h\|_{H_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)} \\ &\qquad + \|\mathbf{u}-\mathbf{u}_h\|_{H_\perp^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)} \|\lambda_h\|_{H_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)}\Big) \ ,\end{aligned}$$

where we have exploited the continuity of the pairing $\langle\cdot,\cdot\rangle_\tau$. This shows that $\bar{P}^h_{\mathrm{Ohm}}$ will converge with the same rate as observed for the Cauchy data.

To compute the total force on the conductor we can resort to the magnetic Maxwell stress tensor for linear materials [43, Sect. 6.7]

$$\mathbb{T} := \mathcal{B}\cdot\mathcal{H}^T - \tfrac{1}{2}(\mathcal{B}\cdot\mathcal{H})\,\mathbb{I}\ , \tag{57}$$

where, $\mathcal{B}$ and $\mathcal{H}$ denote the real, time dependent fields. Ignoring the electric forces is consistent with the eddy current model, which rests on the assumption of negligible electric field energy. Next, we consider $\mathbb{T}$ on Γ and split both the magnetic induction $\mathcal{B}$ and $\mathcal{H}$ into tangential and normal components, *cf.* [49, Sect. 6].

$$\mathcal{B}(\mathbf{x}) = \mathcal{B}_n(\mathbf{x})\mathbf{n}(\mathbf{x}) + \mathcal{B}_\mathbf{t}(\mathbf{x}) \quad , \quad \mathcal{H}(\mathbf{x}) = \mathcal{H}_n(\mathbf{x})\mathbf{n}(\mathbf{x}) + \mathcal{H}_\mathbf{t}(\mathbf{x})\ , \quad \mathbf{x}\in\Gamma\ .$$

Using the constitutive equation $\mathcal{B} = \mu_0\mathcal{H}$, that is valid in Ω_e, we express

$$\mathcal{H}_n(\mathbf{x}) = \frac{1}{\mu_0}\mathcal{B}_n \quad , \quad \mathcal{B}_\mathbf{t}(\mathbf{x}) = \mu_0\mathcal{H}_\mathbf{t}\ .$$

and get on Γ

$$\mathbb{T}(\mathbf{x})\mathbf{n}(\mathbf{x}) = \frac{1}{2}(\frac{1}{\mu_0}\mathcal{B}_n^2(\mathbf{x}) - \mu_0|\mathcal{H}_\mathbf{t}|^2)\mathbf{n}(\mathbf{x}) + \mathcal{B}_n(\mathbf{x})\,\mathcal{H}_\mathbf{t}(\mathbf{x})\ , \quad \mathbf{x}\in\Gamma\ .$$

Hence, the total force on the conductor at a particular time is given by

$$\begin{aligned}F_{\mathrm{tot}} &= \int_\gamma \mathbb{T}(\mathbf{y})\mathbf{n}(\mathbf{y})\,\mathrm{d}S(\mathbf{y}) \\ &= \int_\gamma \frac{1}{2}(\frac{1}{\mu_0}\mathcal{B}_n^2(\mathbf{y}) - \mu_0|\mathcal{H}_\mathbf{t}(\mathbf{y})|^2)\mathbf{n}(\mathbf{y}) + \mathcal{B}_n(\mathbf{y})\,\mathcal{H}_\mathbf{t}(\mathbf{y})\,\mathrm{d}S(\mathbf{y})\ .\end{aligned}$$

Let us revert to complex amplitudes $\mathbf{B}$ and $\mathbf{H}$, for which the averaged force over one period is given by

$$\bar{F}_{\mathrm{tot}} = \int\limits_{\gamma} \frac{1}{4}(\frac{1}{\mu_0}|B_n(\mathbf{y})|^2 - \mu_0|\mathbf{H_t}(\mathbf{y})|^2)\mathbf{n}(\mathbf{y}) + \frac{1}{2}\Re\{B_n(\mathbf{y})\,\mathbf{H_t}(\mathbf{y})\}\;\mathrm{d}S(\mathbf{y})\;.$$

From $\mathbf{B} = (i\omega)^{-1}\,\mathbf{curl}\,\mathbf{E}$ we infer $B_n = (i\omega)^{-1}\,\mathrm{curl}_\Gamma\,\mathbf{u}$, where curl_Γ stands for the scalar surface rotation (div_Γ applied to the rotated field). On the other hand, it is straightforward that $\mathbf{H_t} = -\lambda \times \mathbf{n}$. Thus, we can rewrite

$$\bar{F}_{\mathrm{tot}} = \int\limits_{\gamma} (\frac{1}{4\mu_0\omega^2}|\,\mathrm{curl}_\Gamma\,\mathbf{u}(\mathbf{y})|^2 - \frac{\mu_0}{4}|\lambda(\mathbf{y})|^2)\mathbf{n}(\mathbf{y}) - \frac{1}{2\omega}\Re\{\mathrm{curl}_\Gamma\,\mathbf{u}(\mathbf{y})\,(\lambda(\mathbf{y}) \times \mathbf{n}(\mathbf{y}))\}\;\mathrm{d}S(\mathbf{y})\;. \tag{58}$$

Finally, we have expressed the total force in terms of quantities that occur as unknowns in the variational problem (38). Now, it is straightforward how to compute an approximation of $\bar{F}_{\mathrm{tot}}$ from the boundary element solution $(\mathbf{u}_h, \lambda_h)$. As far as the approximation error is concerned, the same considerations apply as for the energy flux.

It is important to be aware that the force as given by (58) is by no means a continuous functional in the natural trace norms, because the inclusions $\boldsymbol{H}_\perp^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma) \subset \boldsymbol{L}^2(\Gamma)$ and $\boldsymbol{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \subset \boldsymbol{L}^2(\Gamma)$ do **not** hold (Compare the case to the Neumann trace space $H^{-\frac{1}{2}}(\Gamma)$ for second order elliptic problems). Of course, (58) can easily be evaluated for the boundary element functions, but unlike in the case of the total energy flux, rates of convergence for $\bar{F}_{\mathrm{tot}}$ cannot be inferred from (42).

Remark 7. We emphasize that approximations for the traces of the fields onto Γ are directly available, because we have relied on a *direct boundary element method*. If an indirect method had been used, it would have taken expensive post-processing, in order to get the same information.

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