The hp–Version of the Boundary Element Method for the Lamé Equation in 3D

Matthias Maischak and Ernst P. Stephan

Institut für Angewandte Mathematik, Universität Hannover,
Welfengarten 1, 30167 Hannover, Germany
{maischak,stephan}@ifam.uni-hannover.de

Summary. We analyze the h-p version of the BEM for Dirichlet and Neumann problems of the Lamé equation on open surface pieces. With given regularity of the solution in countably normed spaces we show that the boundary element Galerkin solution of the h-p version converges exponentially fast on geometrically graded meshes. We describe in detail how to use an analytic integration for the computation of the entries of the Galerkin matrix. Numerical benchmarks correspond to our theoretical results.

1 Introduction

It is well-known that an appropriate combination of mesh refinement and polynomial degree distribution (the hp-version with geometrically refined graded meshes) may lead to an exponential rate of convergence, even in the presence of singularities (for the FEM see [6, 7], and for the BEM see [8, 10, 11, 17]). The approximation strategy for such hp-methods is to use polynomial degrees of lowest order where solutions behave singularly and to use high order polynomials where solutions are smooth. This strategy has the advantage that it completely avoids the approximation analysis of singular functions by high order polynomials. This differs from the situation for a pure p-version, see [3, 2].

In this paper we consider the hp-version of the boundary element method (BEM) for Dirichlet and Neumann problems of the Lamé equation in $\Omega_\Gamma := \mathbb{R}^3 \setminus \bar{\Gamma}$, where Γ is a smooth open surface piece with a piecewise smooth boundary curve. That is:

For given $\mathbf{u}_1, \mathbf{u}_2 \in (H^{1/2}(\Gamma))^3$ with $\mathbf{u}_1 - \mathbf{u}_2 \in (\tilde{H}^{1/2}(\Gamma))^3$ (Dirichlet) or for given $\mathbf{t}_1, \mathbf{t}_2 \in (H^{-1/2}(\Gamma))^3$ with $\mathbf{t}_1 - \mathbf{t}_2 \in (\tilde{H}^{-1/2}(\Gamma))^3$ (Neumann) find $\mathbf{u}$ satisfying

$$\Delta^* \mathbf{u} := \mu \boldsymbol{\Delta} \mathbf{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega_\Gamma, \tag{1}$$

$$\mathbf{u}|_{\Gamma_1} = \mathbf{u}_1, \mathbf{u}|_{\Gamma_2} = \mathbf{u}_2 \text{ (Dirichlet)} \tag{2}$$

$$\mathbf{T}(\mathbf{u})|_{\Gamma_1} = \mathbf{t}_1, \mathbf{T}(\mathbf{u})|_{\Gamma_2} = \mathbf{t}_2 \text{ (Neumann)} \tag{3}$$

$$\mathbf{u}(x) = o(1), \frac{\partial}{\partial x_j} \mathbf{u}(x) = o(|x|^{-1}), j = 1, 2, 3, \ |x| \to \infty. \tag{4}$$

Here, Γ_i, $i = 1, 2$, are the two sides of Γ and $\mu > 0$, $\lambda > -2/3\mu$ are the given Lamé constants.

The corresponding Neumann data of the linear elasticity problem are the tractions

$$\mathbf{T}(\mathbf{u}) = \lambda(\operatorname{div} \mathbf{u})\mathbf{n} + 2\mu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} + \mu \mathbf{n} \times \operatorname{curl} \mathbf{u} \text{ on } \Gamma_i, i = 1, 2, \tag{5}$$

where $\mathbf{n}$ is the normal vector exterior to a bounded domain Ω, such that $\Gamma \subset \partial\Omega$.

Let $G(x, y) \in \mathbb{R}^{3\times 3}$ denote the fundamental solution of the differential operator Δ^*, i.e.

$$G(x, y) \ = \ \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)} \left\{ \frac{1}{|x - y|} I_{3\times 3} + \frac{\lambda + \mu}{\lambda + 3\mu} \frac{(x - y)(x - y)^\top}{|x - y|^3} \right\}. \tag{6}$$

The problem (1)–(4) can be formulated as an integral equation of the first kind, see, e.g. [4, 5, 20, 21]:

Dirichlet:

$\mathbf{u} \in (H^1_{\mathrm{loc}}(\mathbb{R}^3 \backslash \bar{\Gamma}))^3$ is the solution of the Dirichlet problem (1), (2) and (4) if and only if the jump of the traction $\mathbf{t} := \mathbf{T}(u)|_{\Gamma_1} - \mathbf{T}(u)|_{\Gamma_2} \in (\tilde{H}^{-1/2}(\Gamma))^3$ solves the weakly singular integral equation

$$\mathbf{V}\mathbf{t}(x) := \int_\Gamma G(x, y)\mathbf{t}(y)\, ds_y = \mathbf{g}(x), \quad x \in \Gamma \tag{7}$$

where

$$\mathbf{g}(x) = \frac{1}{2}(\mathbf{u}_1 + \mathbf{u}_2)(x) + \int_\Gamma \mathbf{T}_y G(x, y)(\mathbf{u}_1 - \mathbf{u}_2)(y)\, ds_y. \tag{8}$$

The solution $\mathbf{t}$ of (7) yields the solution of the Dirichlet problem (1), (2) and (4) via the representation or Betti's formula

$$\mathbf{u}(x) = \int_\Gamma \left(G(x, y)\mathbf{t}(y) - (\mathbf{T}_y G(x, y))^t (\mathbf{u}_1(y) - \mathbf{u}_2(y)) \right)\, ds_y, \quad x \notin \Gamma.$$

The Galerkin scheme for (7) is given by: Find $\mathbf{t}_N \in S^{p,0}(\Gamma^n_\sigma) \subset (\tilde{H}^{-1/2}(\Gamma))^3$ such that for all $v \in S^{p,0}(\Gamma^n_\sigma)$

$$\langle \mathbf{V}\mathbf{t}, v \rangle \ = \ \langle \mathbf{g}, v \rangle \tag{9}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing of $(H^{1/2}(\Gamma))^3$ and $(\tilde{H}^{-1/2}(\Gamma))^3$. The symmetric bilinear form $\langle \mathbf{V}\cdot, \cdot \rangle$ is positive definite on $(\tilde{H}^{-1/2}(\Gamma))^3 \times (\tilde{H}^{-1/2}(\Gamma))^3$ giving the energy norm $\|t\|_V = \langle \mathbf{V}t, t \rangle^{1/2}$.

Neumann:

$\mathbf{u} \in (H^1_{\text{loc}}(\mathbb{R}^3 \setminus \bar{\Gamma}))^3$ is the solution of the Neumann problem (1), (3) and (4) if and only if the jump of the displacement $\boldsymbol{\phi} := \mathbf{u}|_{\Gamma_1} - \mathbf{u}|_{\Gamma_2} \in (\tilde{H}^{1/2}(\Gamma))^3$ solves the hyper-singular integral equation

$$\mathbf{W}\boldsymbol{\phi}(x) := -\mathbf{T}_x \int_\Gamma (\mathbf{T}_y G(x,y))^t \, \boldsymbol{\phi}(y) \, ds_y = \mathbf{f}(x), \quad x \in \Gamma \tag{10}$$

where

$$\mathbf{f}(x) = \frac{1}{2}(\mathbf{t}_1 + \mathbf{t}_2)(x) - \mathbf{T}_x \int_\Gamma G(x,y)(\mathbf{t}_1 - \mathbf{t}_2)(y) \, ds_y. \tag{11}$$

The solution $\boldsymbol{\phi}$ of (10) yields the solution of the Neumann problem (1), (3) and (4) via the representation or Betti's formula

$$\mathbf{u}(x) = \int_\Gamma \big(G(x,y)(\mathbf{t}_1(y) - \mathbf{t}_2(y)) - (\mathbf{T}_y G(x,y))^t \boldsymbol{\phi}(y) \big) \, ds_y, \quad x \notin \Gamma.$$

The Galerkin scheme for (10) is given by: Find $\boldsymbol{\phi}_N \in S^{p,1}(\Gamma^n_\sigma) \subset (\tilde{H}^{1/2}(\Gamma))^3$ such that for all $\psi \in S^{p,1}(\Gamma^n_\sigma)$

$$\langle \mathbf{W}\boldsymbol{\phi}, \psi \rangle \ = \ \langle \mathbf{f}, \psi \rangle \tag{12}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing of $(H^{-1/2}(\Gamma))^3$ and $(\tilde{H}^{1/2}(\Gamma))^3$. The symmetric bilinear form $\langle \mathbf{W} \cdot, \cdot \rangle$ is positive definite on $(\tilde{H}^{1/2}(\Gamma))^3 \times (\tilde{H}^{1/2}(\Gamma))^3$ giving the energy norm $\|\phi\|_W = \langle \mathbf{W}\phi, \phi \rangle^{1/2}$.

Both Galerkin schemes (9) and (12) converge quasi-optimally in the energy norm with algebraic orders of convergence for the h- and p-versions, namely of order $\mathcal{O}(h^{1/2} p^{-1})$. This follows by extending corresponding results for the Laplacian [1, 3, 19, 20, 22, 26]. These low convergence rates result from the singular behavior of the solutions $\mathbf{t}$ of (7) and $\boldsymbol{\phi}$ of (10) near the boundary of Γ; this describes the well-known behavior of the displacement and traction near the edges of the crack [24, 26], cf. [25]. On the other hand, if we use an *hp*-version with a geometrically refined mesh towards the edges of the surface Γ we obtain even exponentionally fast convergence (cf. Fig. 3 and Fig. 4). Especially, as shown below, there hold the following error estimates for the exact solutions $\mathbf{t}$ of (7) and $\boldsymbol{\phi}$ of (10) and the Galerkin solutions $\mathbf{t}_N \in S^{p,0}(\Gamma^n_\sigma)$ of (9) and $\boldsymbol{\phi}_N \in S^{p,1}(\Gamma^n_\sigma)$ of (12), i.e.

$$\|\mathbf{t} - \mathbf{t}_N\|_V \leq C e^{-bN^{1/4}}, \qquad \|\boldsymbol{\phi} - \boldsymbol{\phi}_N\|_W \leq C e^{-bN^{1/4}} \tag{13}$$

with constants $C, b > 0$ independent of N (see Theorems 4 and 5 below, c.f. [10, 13, 18, 23]).

Another important issue is the implementation of the hp-version for the Galerkin equations itself. In this paper we explicitly describe how analytic integration can be used in the computation of the entries of the Galerkin matrices. The trick is to reduce the integrals for Lamé-case to simpler ones which already have been used for the computations of the integral operators belonging to the Laplacian [16]. Numerical benchmarks underline our theoretical results.

2 The hp-Version with Geometric Mesh

In this section we introduce the boundary element spaces for the hp-version together with countably normed spaces.

Now we define the geometric mesh on a triangle F. This is no loss of generality because every polygonal domain can be decomposed into triangles. We divide this triangle into three parallelograms and three triangles where each parallelogram lies in a corner of F and each triangle lies at an edge of F but does not touch the corners (see Fig. 1). By linear transformations φ_i we can map the parallelograms onto the reference square $Q = [0,1]^2$ such that the vertices of F are mapped to $(0,0)$. The triangles can be mapped by linear transformations $\tilde{\varphi}_i$ onto the reference triangle $\tilde{Q} = \{(x,y) \in Q \,|\, y \leq x\}$ such that the corner point of the triangle in the interior of F is mapped to $(1,1)$ of the reference triangle. By Definition 1 the geometric mesh and appropriate spline spaces are defined on the reference element Q. Analogously the geometric mesh can be defined on the reference triangle $\tilde{Q}$ (see Fig. 1).

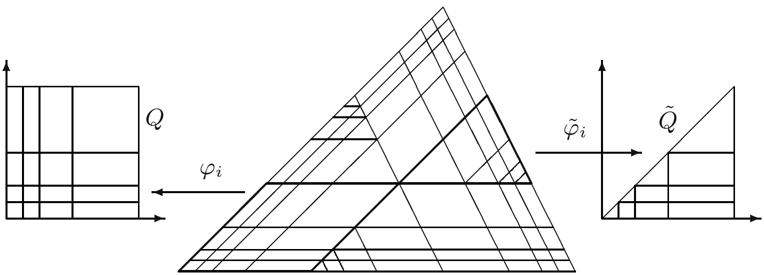

Fig. 1. Geometric mesh with $\sigma = 0.5$ on the triangle F

Via the transformations $\varphi_i^{-1}, \tilde{\varphi}_i^{-1}$ the geometric mesh Γ_σ^n can also be defined on the faces of a polyhedron. The approximation on the reference square is the more interesting case because it handles the corner-edge singularities. Therefore we deal in the following only with the approximation on the reference square.

Definition 1 (geometric mesh). *Let $I = [0,1]$. For $0 < \sigma < 1$ we use the partition I_σ^n of I into n subintervals $[x_{k-1}, x_k]$, $k = 1, \ldots, n$, where*

$$x_0 = 0, \qquad x_k = \sigma^{n-k}, \quad k = 1, \ldots, n. \tag{14}$$

With I_σ^n we associate a degree-vector $p = (p_1, \ldots, p_n)$ and define $S^{p,r}(I_\sigma^n) \subset H^r(I)$ as the vector space of all piecewise polynomials w on I having degree p_j on (x_{j-1}, x_j), $j = 1, \ldots, n$, i.e. $w|_{(x_{j-1},x_j)} \in P_{p_j}((x_{j-1}, x_j))$.

Let $Q = [0,1] \times [0,1]$. For $0 < \sigma < 1$ we use the partition Q^n_σ of Q into n^2 subsquares R_{kl}

$$R_{kl} = [x_{k-1}, x_k] \times [x_{l-1}, x_l], \quad (k,l = 1,\ldots,n), \qquad Q = \bigcup_{k,l=1}^{n} R_{kl}. \tag{15}$$

With Q^n_σ we associate a degree vector $p = (p_1,\ldots,p_n)$ and define $S^{p,r}(Q^n_\sigma) \subset H^r(Q)$ as the vector space of all piecewise polynomials $v(x,y)$ on Q having degree p_k in x and p_l in y on R_{kl}, $k,l = 1,\ldots,n$, i.e. $v|_{R_{kl}} \in P_{p_k,p_l}(R_{kl})$. The index $r \in \{0,1\}$ in $S^{p,r}(I^n_\sigma)$ and $S^{p,r}(Q^n_\sigma)$ determines the regularity of the piecewise polynomials, i.e. discontinuity in case of $r = 0$ and continuity in case of $r = 1$. For the differences $h_k = x_k - x_{k-1}$ we have with $\lambda = (1-\sigma)/\sigma$

$$h_k = x_k - x_{k-1} = x_{k-1}(\frac{1}{\sigma} - 1) \le x(\frac{1}{\sigma} - 1) = x\lambda, \quad \forall x \in [x_{k-1}, x_k] \ (2 \le k \le n) \tag{16}$$

Then we have by construction:

$$S^{p,r}(I^n_\sigma) \times S^{p,r}(I^n_\sigma) \subset S^{p,r}(Q^n_\sigma) \tag{17}$$

Fig. 2 shows the geometric meshes for $\sigma = 1/2$ and $n = 4$.

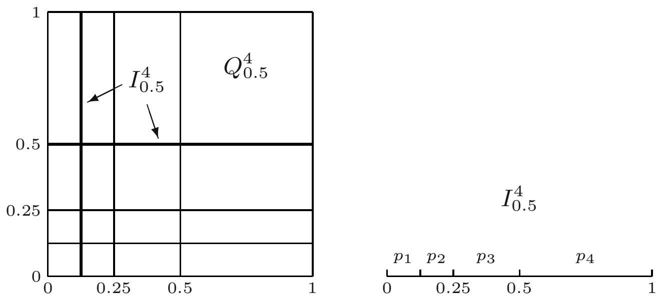

Fig. 2. Geometric mesh on the square plate ($\sigma = 0.5$, $n = 4$).

Now we define countably normed spaces on the reference element Q using Cartesian coordinates.

Definition 2 (countably normed spaces $B^l_\beta(Q)$). *Let β be a real number with $0 < \beta < 1$. The weight function $\Phi_{\beta,\alpha,l} = \Phi_{\beta,\alpha,l}(x,y)$ is for $\alpha = (\alpha_1, \alpha_2)$ and an integer $l \ge 1$ defined by*

$$\Phi_{\beta,\alpha,l} = x^\beta \sum_{\gamma_1=\max(\alpha_1-l,0)}^{\min(\alpha_1-1,\alpha_1+\alpha_2-l)} x^{\gamma_1} y^{\alpha_1+\alpha_2-l-\gamma_1} + y^\beta \sum_{\gamma_2=\max(\alpha_2-l,0)}^{\min(\alpha_2-1,\alpha_1+\alpha_2-l)} x^{\alpha_1+\alpha_2-l-\gamma_2} y^{\gamma_2}. \tag{18}$$

Let

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} = \partial_x^{\alpha_1} \partial_y^{\alpha_2}.$$

The weighted Sobolev spaces for integers m, l with $m \geq l \geq 1$ are defined by

$$H_\beta^{m,l}(Q) = \Big\{ u : u \in H^{l-1}(Q) \text{ for } l > 0, \quad \|\Phi_{\beta,\alpha,l} D^\alpha u\|_{L^2(Q)} < \infty \text{ for } l \leq |\alpha| \leq m \Big\}, \tag{19}$$

with the norm

$$\|u\|^2_{H_\beta^{m,l}(Q)} = \|u\|^2_{H^{l-1}(Q)} + \sum_{k=l}^{m} \sum_{|\alpha|=k} \int_Q |D^\alpha u(x,y)|^2 \Phi^2_{\beta,\alpha,l}(x,y)\, dy\, dx \tag{20}$$

and the semi norm

$$|u|^2_{H_\beta^{m,l}(Q)} = \sum_{k=l}^{m} \sum_{|\alpha|=k} \int_Q |D^\alpha u(x,y)|^2 \Phi^2_{\beta,\alpha,l}(x,y)\, dy\, dx. \tag{21}$$

The countably normed spaces for $l \geq 1$ are defined by

$$B_\beta^l(Q) = \Big\{ u : u \in H^{l-1}(Q), \|\Phi_{\beta,\alpha,l} D^\alpha u\|_{L^2(Q)} \leq C\, d^{k-l}(k-l)! \text{ for } |\alpha| = k = l, l+1, \ldots, ; \quad C \geq 1, d \geq 1 \text{ independent of } k \Big\}. \tag{22}$$

If we would like to emphasize the dependence on the constants C, d we will write $B_\beta^l(Q) = B_{\beta,C,d}^l(Q)$, etc.

Theorem 1. [12] *Let Q be the reference element and let φ be the linear transformation from a parallelogram, lying in a corner of the triangle F, to the reference element Q. Then, for $l = 1, 2$, $u \in \mathcal{B}^l_{\beta,\tilde{C},\tilde{d}}(\varphi(Q))$ implies $u \circ \varphi^{-1} \in B^l_{\beta,C,d}(Q)$ where C, d (resp. $\tilde{C}, \tilde{d}$) are the constants in the definition of $B_\beta^l(Q)$ (resp. $\mathcal{B}_\beta^l(\varphi(Q))$). For the case $l = 1$ the reverse implication holds as well.*

The exponentially good approximation properties of splines on our geometric meshes for general functions $u \in B_\beta^l(Q)$ $(l = 1, 2)$ are given by the following theorem (see also [12, 15, 17, 18]).

Theorem 2.

(i) Let $u \in B_\beta^1(Q)$ with $0 < \beta < 1$. Let Q_σ^n be a geometric mesh and assume $p = (p_1, \ldots, p_n)$, $p_k = [\mu(k-1)]$ for some $\mu > 0$. Set $N = \dim S^{p,0}(Q_\sigma^n)$. Then there exist constants $C_1, b_1 > 0$ independent of N, but depending on σ, μ, β, such that the L^2-projection $u_N \in S^{p,0}(Q_\sigma^n)$ of u satisfies

$$\|u - u_N\|_{L^2(Q)} \leq C_1\, e^{-b_1 \sqrt[4]{N}}. \tag{23}$$

(ii) Let $v \in B^2_\beta(Q)$ *with* $0 < \beta < 1$*. Let* Q^n_σ *be a geometric mesh and assume* $p = (p_1, \dots, p_n)$*,* $p_1 = 1$*,* $p_k = \max(2, [\mu(k-1)] + 1)$ $(k > 1)$ *for some* $\mu > 0$*. Set* $N = \dim S^{p,1}(Q^n_\sigma)$*. Then there is a spline function* $v_N \in S^{p,1}(Q^n_\sigma)$ *and constants* $C_2, b_2 > 0$ *independent of* N*, but dependent on* σ, μ, β*, such that*

$$\|v - v_N\|_{H^1(Q)} \le C_2\, e^{-b_2 \sqrt[4]{N}}. \tag{24}$$

(iii) Let $v \in B^1_\beta(Q) \cap C^0(\bar{Q})$*,* $v|_{\partial Q} = 0$ *with* $0 < \beta < 1/2$*. Let* Q^n_σ *be a geometric mesh and assume* $p = (p_1, \dots, p_n)$*,* $p_1 = 1$*,* $p_k = \max(2, [\mu(k-1)] + 1)$ $(k > 1)$ *for some* $\mu > 0$*. Set* $N = \dim S^{p,1}(Q^n_\sigma)$*. Then there is a spline function* $v_N \in S^{p,1}(Q^n_\sigma)$ *and constants* $C_3, b_3 > 0$ *independent of* N*, but dependent on* σ, μ, β*, such that*

$$\|v - v_N\|_{\tilde{H}^{1/2}(Q)} \le C_3\, e^{-b_3 \sqrt[4]{N}}. \tag{25}$$

Now, we want to recall the typical structure of the solutions of our problems for sufficiently smooth right-hand side functions g and f.

Theorem 3. [24, Theorem 2.3, 2.4 and 2.5] *Let* V *and* E *denote the sets of vertices and edges of* Γ*, respectively. For* $v \in V$*, let* $E(v)$ *denote the set of edges with* v *as an end point. Then, the solution* $\mathbf{t}$ *of* (7) *has the form*

$$\mathbf{t} = \mathbf{t}_{\text{reg}} + \sum_{e \in E} \mathbf{t}^e + \sum_{v \in V} \mathbf{t}^v + \sum_{v \in V} \sum_{e \in E(v)} \mathbf{t}^{ev}, \tag{26}$$

with a regular part $\mathbf{t}_{\text{reg}}$*, edge singularities* $\mathbf{t}^e$*, vertex singularities* $\mathbf{t}^v$ *and edge-vertex singularities* $\mathbf{t}^{ev}$*. These terms result from applying boundary traction to the corresponding decomposition of the solution.*

Accordingly, the solution $\boldsymbol{\phi}$ *of* (10) *has the form*

$$\boldsymbol{\phi} = \boldsymbol{\phi}_{\text{reg}} + \sum_{e \in E} \boldsymbol{\phi}^e + \sum_{v \in V} \boldsymbol{\phi}^v + \sum_{v \in V} \sum_{e \in E(v)} \boldsymbol{\phi}^{ev}. \tag{27}$$

Checking the specific terms (26) and (27) , which are given in [24], one realizes that these terms $\mathbf{t}^e, \mathbf{t}^v, \mathbf{t}^{ev}$ and $\boldsymbol{\phi}^e, \boldsymbol{\phi}^v, \boldsymbol{\phi}^{ev}$ belong to countably normed spaces. Therefore we can argue as done in [10] and obtain the following convergence results.

Theorem 4. *Let the right hand side* g *in equation* (7) *be piecewise analytic, let* $\mathbf{t}$ *be the solution of* (7) *and let* $\mathbf{t}_N \in S^{p,0}(\Gamma^n_\sigma)$ *be its Galerkin approximation defined by* (9)*. Then, with* $N = \dim S^{p,0}(\Gamma^n_\sigma)$*, there holds for any* $\alpha > 0$

$$\|\mathbf{t} - \mathbf{t}_N\|_{(\tilde{H}^{-1/2}(\Gamma))^3} \le C\, e^{-b \sqrt[4]{N}} + \mathcal{O}(N^{-\alpha}) \tag{28}$$

for constants $C, b > 0$*, depending on* σ*,* μ *and* α*, but independent of* N*.*

Theorem 5. *Let the right hand side f in equation* (10) *be piecewise analytic and let $\boldsymbol{\phi}$ be the solution of* (10) *and let $\boldsymbol{\phi}_N \in S^{p,1}(\Gamma_\sigma^n)$ be its Galerkin approximation defined by* (12)*. Then there holds for all $\alpha > 0$*

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_N\|_{(\tilde{H}^{1/2}(\Gamma))^3} \leq C\, e^{-b\sqrt[4]{N}} + \mathcal{O}(N^{-\alpha}) \tag{29}$$

for constants $C, b > 0$ depending on σ, μ and α, but not depending on $N = \dim S^{p,1}(\Gamma_\sigma^n)$.

Remark 1. Due to the splittings (26) and (27) into finitely many singularity terms the regular remainder terms $\mathbf{t}_{\text{reg}}$ and $\boldsymbol{\phi}_{\text{reg}}$ have only restricted regularity, even for given smooth right hand sides. On the other hand, even taking infinitely many singularity terms, would not automatically guarantee that the solutions $\mathbf{t}$ and $\boldsymbol{\phi}$ themselves belong to countably normed spaces. To our knowledge this is an open problem. Therefore we get the additional $\mathcal{O}(N^{-\alpha})$-terms in the estimates (28) and (29).

3 Implementation of Galerkin Scheme

Assume that the surface piece $\Gamma \subset \mathbb{R}^3$ can be decomposed into triangles and parallelograms, i.e. $\Gamma = \bigcup_{i=1}^N \Gamma_i$, with Γ_i pairwise disjoint and Γ_i is the affine image of the reference square $\square = [-1,1]^2$ or the reference triangle $\Delta = \{(t_1,t_2) : 0 \leq t_1 \leq 1 - t_2 \leq 1\}$. That means

$$\Gamma_i = \{a_i t_1 + b_i t_2 + x_i \,:\, (t_1,t_2) \in Q\}, \quad Q \in \{\Delta, \square\} \tag{30}$$

depending on whether Γ_i is a triangle or a parallelogram, with $a_i, b_i, x_i \in \mathbb{R}^3$, $i = 1, \ldots, N$. Here we investigate only basis functions whose restriction to Γ_i are polynomials. Effectively, we compute the integrals only for monomials as test- and trial-functions, from which all other basis functions can be constructed.

For $Q \in \{\Delta, \square\}$ let

$$F_i \,:\, \begin{cases} Q \Rightarrow \Gamma_i \\ t = (t_1,t_2) \to x = a_i t_1 + b_i t_2 + x_i \end{cases} \tag{31}$$

be the affine transformation from the reference element Δ or $\square$ to Γ_i with $|\frac{\partial F_i}{\partial t}| = |a_i \times b_i|$. We will write Q for Δ or $\square$, respectively, if the expressions hold for both cases. Then the basis functions on Γ_i are defined by

$$\varphi_{kl}^i(x) = \tilde{\varphi}_{kl}(F_i^{-1}(x)) = \tilde{\varphi}_{kl} \circ F_i^{-1}(x) \tag{32}$$

with $\tilde{\varphi}_{kl}(t_1,t_2) = t_1^k t_2^l$ for $x \in \Gamma_i$ and $\varphi_{kl}^i(x) = 0$ otherwise. The vector valued test and trial functions $\boldsymbol{\phi}$ restricted to an element Γ_i can be represented as linear combination of this monomial basis functions $\varphi_{kl}^i(x)$, i.e. we have

$$\boldsymbol{\phi}(x)|_{\Gamma_i} = \sum_{r=1}^3 \mathbf{e}_r \phi_r(x)|_{\Gamma_i}, \quad \phi_r(x)|_{\Gamma_i} = \sum_{kl} c_{kl}^{i,r} \varphi_{kl}^i(x)$$

with $\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0), \mathbf{e}_3 = (0,0,1)$.

Single layer potential

Using (6) the single layer potential is then given by

$$\mathbf{V}\boldsymbol{\phi}(x) := \int_\Gamma G(x,y)\boldsymbol{\phi}(y)\,ds_y = \sum_{i=1}^N \sum_{r=1}^3 \mathbf{e}_r \sum_{s=1}^3 \sum_{kl} c_{kl}^s \int_{\Gamma_i} G_{rs}(x,y)\varphi_{kl}^i(y)\,ds_y \tag{33}$$

and the corresponding bilinear form reads

$$\langle \mathbf{V}\boldsymbol{\phi}, \boldsymbol{\psi}\rangle = \int_\Gamma \int_\Gamma \psi_r(x) G_{rs}(x,y)\phi_s(y)\,ds_y\,ds_x. \tag{34}$$

In the following we are interested in the computation of the term

$$V_{kl}^{i,rs}(x) := \int_{\Gamma_i} G_{rs}(x,y)\varphi_{kl}^i(y)\,ds_y. \tag{35}$$

We will use the following form of the fundamental solution (6)

$$\begin{aligned} G_{rs}(x,y) &= \frac{\lambda+3\mu}{4\pi\mu(\lambda+2\mu)} \left\{ \frac{1}{|x-y|}\,\delta_{rs} + \frac{\lambda+\mu}{\lambda+3\mu}\,\frac{(x_r-y_r)(x_s-y_s)}{|x-y|^3} \right\} \\ &= \frac{1}{4\pi\mu}\frac{1}{|x-y|}\delta_{rs} - \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\frac{\partial}{\partial y_r}\frac{y_s-x_s}{|y-x|}. \end{aligned} \tag{36}$$

By extending the affine transformation F_i to

$$F_i(t_1,t_2,t_3) = a_i t_1 + b_i t_2 + n_i t_3 + x_i,$$

where n_i is the normal direction on the patch Γ_i, we obtain the following integral

$$\begin{aligned} V_{kl}^{i,rs}(x) &= \frac{1}{4\pi\mu}\int_{\Gamma_i} \frac{1}{|x-y|}\delta_{rs}\varphi_{kl}^i(y)\,ds_y \\ &\qquad - \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\int_{\Gamma_i} \frac{\partial}{\partial y_r}\frac{y_s-x_s}{|y-x|}\varphi_{kl}^i(y)\,ds_y \\ &= \frac{1}{4\pi\mu}\left|\frac{\partial F_i}{\partial t}\right|\delta_{rs}\int_Q \frac{1}{|F_i(t)-x|}\tilde{\varphi}_{kl}(t)\,dt \\ &\qquad - \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\left|\frac{\partial F_i}{\partial t}\right| \int_Q \sum_{p=1}^3 \frac{\partial t_p}{\partial y_r}\frac{\partial}{\partial t_p}\frac{(F_i(t)-x)_s}{|F_i(t)-x|}\tilde{\varphi}_{kl}(t)\,dt \\ &=: \frac{1}{4\pi\mu}\left|\frac{\partial F_i}{\partial t}\right|\delta_{rs} A_{kl}^i(x) - \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}\left|\frac{\partial F_i}{\partial t}\right| B_{kl}^{i,rs}(x) \end{aligned}$$

Defining the following elementary integrals, analyzed in [16]

$$I_{kl}^{Q,p}(a,b,c) := \int_Q t_1^k t_2^l |at_1+bt_2+c|^{2p}\,dt_2\,dt_1, \quad Q \in \{\Delta, \Box\} \tag{37}$$

we can identify

$$A^i_{kl}(x) = \int_Q \frac{\tilde{\varphi}_{kl}(t)}{|F_i(t) - x|}\,dt = \int_Q \frac{t_1^k t_2^l}{|a_i t_1 + b_i t_2 + x_i - x|}\,dt_2\,dt_1$$
$$= I^{Q,-\frac{1}{2}}_{kl}(a_i, b_i, x_i - x). \tag{38}$$

It remains to reduce the integral $B^{i,rs}_{kl}(x)$ to a linear combination of elementary integrals. We can compute

$$\left(\frac{\partial t_p}{\partial y_r}\right) = \left(\frac{\partial y_r}{\partial t_p}\right)^{-1} = \left(\, a_i \big| b_i \big| n_i \,\right)^{-1} = \frac{1}{a_i(b_i \times n_i)} \begin{pmatrix} b_i \times n_i \\ \hline n_i \times a_i \\ \hline a_i \times b_i \end{pmatrix}. \tag{39}$$

Therefore we obtain

$$\sum_{p=1}^{3} \frac{\partial t_p}{\partial y_r} \frac{\partial}{\partial t_p} = \frac{1}{a_i(b_i \times n_i)} \left((b_i \times n_i)_r \partial_{t_1} + (n_i \times a_i)_r \partial_{t_2} + (a_i \times b_i)_r \partial_{t_3}\right) \tag{40}$$

and consequently

$$B^{i,rs}_{kl}(x) = \int_Q \sum_{p=1}^{3} \frac{\partial t_p}{\partial y_r} \frac{\partial}{\partial t_p} \frac{(F_i(t) - x)_s}{|F_i(t) - x|} \tilde{\varphi}_{kl}(t)\,dt$$
$$= \int_Q \frac{\left((b_i \times n_i)_r \partial_{t_1} + (n_i \times a_i)_r \partial_{t_2} + (a_i \times b_i)_r \partial_{t_3}\right)}{a_i(b_i \times n_i)} \frac{(F_i(t) - x)_s}{|F_i(t) - x|} t_1^k t_2^l\,dt_2\,dt_1$$
$$=: \frac{(b_i \times n_i)_r}{a_i(b_i \times n_i)} C^{i,s}_{kl}(x) + \frac{(n_i \times a_i)_r}{a_i(b_i \times n_i)} D^{i,s}_{kl}(x) + \frac{(a_i \times b_i)_r}{a_i(b_i \times n_i)} E^{i,s}_{kl}(x).$$

For the last integral we obtain

$$E^{i,s}_{kl}(x) = \int_Q \partial_{t_3} \frac{(a_i t_1 + b_i t_2 + n_i t_3 + x_i - x)_s}{|a_i t_1 + b_i t_2 + n_i t_3 + x_i - x|} t_1^k t_2^l\,dt_2\,dt_1$$
$$= \int_Q \frac{(n_i)_s}{|a_i t_1 + b_i t_2 + x_i - x|} t_1^k t_2^l\,dt_2\,dt_1$$
$$- \int_Q \frac{(a_i t_1 + b_i t_2 + x_i - x)_s (n_i(x_i - x))}{|a_i t_1 + b_i t_2 + x_i - x|^3} t_1^k t_2^l\,dt_2\,dt_1$$
$$= (n_i)_s I^{Q,-\frac{1}{2}}_{kl}(a_i, b_i, x_i - x) - (a_i)_s (n_i(x_i - x)) I^{Q,-\frac{3}{2}}_{k+1,l}(a_i, b_i, x_i - x)$$
$$- (b_i)_s (n_i(x_i - x)) I^{Q,-\frac{3}{2}}_{k,l+1}(a_i, b_i, x_i - x)$$
$$- (x_i - x)_s (n_i(x_i - x)) I^{Q,-\frac{3}{2}}_{k,l}(a_i, b_i, x_i - x).$$

The integrals $C^{i,s}_{kl}(x)$, $D^{i,s}_{kl}(x)$ can be treated by partial integration, but we have to distinguish between triangles and parallelograms. On parallelograms we simply obtain

$$C_{kl}^{i,s}(x) = \int_{\Box} \partial_{t_1} \frac{(a_i t_1 + b_i t_2 + x_i - x)_s}{|a_i t_1 + b_i t_2 + x_i - x|} t_1^k t_2^l \, dt_2 \, dt_1$$
$$= \int_{-1}^{1} \frac{(a_i t_1 + b_i t_2 + x_i - x)_s}{|a_i t_1 + b_i t_2 + x_i - x|} t_1^k t_2^l \, dt_2 \Big|_{t_1=-1}^{t_1=1}$$
$$-k \int_{\Box} \frac{(a_i t_1 + b_i t_2 + x_i - x)_s}{|a_i t_1 + b_i t_2 + x_i - x|} t_1^{k-1} t_2^l \, dt_2 \, dt_1$$

and on triangles there holds

$$C_{kl}^{i,s}(x) = \int_{\Delta} \partial_{t_1} \frac{(a_i t_1 + b_i t_2 + x_i - x)_s}{|a_i t_1 + b_i t_2 + x_i - x|} t_1^k t_2^l \, dt_2 \, dt_1$$
$$= \int_0^1 t_1^k (1-t_1)^l \frac{(a_i t_1 + b_i(1-t_1) + x_i - x)_s}{|a_i t_1 + b_i(1-t_1) + x_i - x|} \, dt_1$$
$$- \delta_{k,0} \int_0^1 t_2^l \frac{(b_i t_2 + x_i - x)_s}{|b_i t_2 + x_i - x|} \, dt_2 - k \int_{\Delta} \frac{(a_i t_1 + b_i t_2 + x_i - x)_s}{|a_i t_1 + b_i t_2 + x_i - x|} t_1^{k-1} t_2^l \, dt_2 \, dt_1.$$

Double layer potential

Using the traction operator

$$(T\boldsymbol{\phi}(y))_r = \lambda n_r \frac{\partial}{\partial y_t} \phi_t(y) + \mu n_t \frac{\partial}{\partial y_t} \phi_r(y) + \mu n_t \frac{\partial}{\partial y_r} \phi_t(y),$$

we can define the double layer potential operator by

$$\mathbf{K}\boldsymbol{\phi}(x) := \int_{\Gamma} (T_y G(x,y))^t \boldsymbol{\phi}(y) \, ds_y = \sum_{i=1}^{N} \sum_{r=1}^{3} \mathbf{e}_r \sum_{s=1}^{3} \sum_{kl} c_{kl}^s K_{kl}^{i,rs}(x) \tag{41}$$

with

$$K_{kl}^{i,rs}(x) := \sum_{t=1}^{3} \Big(\lambda n_{i,s} F_{kl}^{i,rtt}(x) + \mu n_{i,t} F_{kl}^{i,rst}(x) - \mu n_{i,t} F_{kl}^{i,rts}(x) \Big) \tag{42}$$

and

$$F_{kl}^{i,rst}(x) := \int_{\Gamma_i} \frac{\partial}{\partial y_t} G_{rs}(x,y) \varphi_{kl}^i(y) \, ds_y. \tag{43}$$

We can decompose $F_{kl}^{i,rst}(x)$ as follows

$$F_{kl}^{i,rst}(x) := \frac{\lambda + 3\mu}{4\pi\mu(\lambda+2\mu)} \delta_{rs} \int_{\Gamma_i} \frac{\partial}{\partial y_t} \frac{1}{|x-y|} \varphi_{kl}^i(y) \, ds_y$$
$$+ \frac{\lambda + \mu}{4\pi\mu(\lambda+2\mu)} \int_{\Gamma_i} \frac{\partial}{\partial y_t} \frac{(x_r - y_r)(x_s - y_s)}{|x-y|^3} \varphi_{kl}^i(y) \, ds_y$$
$$= \frac{\lambda + 3\mu}{4\pi\mu(\lambda+2\mu)} \Big|\frac{\partial F_i}{\partial t}\Big| \, \delta_{rs} \int_Q \sum_{p=1}^{3} \frac{\partial t_p}{\partial y_t} \frac{\partial}{\partial t_p} \frac{1}{|F_i(t) - x|} \tilde{\varphi}_{kl}(t) \, dt$$
$$+ \frac{\lambda + \mu}{4\pi\mu(\lambda+2\mu)} \Big|\frac{\partial F_i}{\partial t}\Big| \int_Q \sum_{p=1}^{3} \frac{\partial t_p}{\partial y_t} \frac{\partial}{\partial t_p} \frac{(F_i(t) - x)_r (F_i(t) - x)_s}{|F_i(t) - x|^3} \tilde{\varphi}_{kl}(t) \, dt$$

$$=: \frac{\lambda+3\mu}{4\pi\mu(\lambda+2\mu)}|\frac{\partial F_i}{\partial t}|\,\delta_{rs}H_{kl}^{i,t}(x)+\frac{\lambda+\mu}{4\pi\mu(\lambda+2\mu)}|\frac{\partial F_i}{\partial t}|J_{kl}^{i,rst}(x).$$

As before, we can represent the integrals $H_{kl}^{i,t}(x)$ and $J_{kl}^{i,rst}(x)$ in terms of the elementary integrals $I_{kl}^{Q,p}(a_i,b_i,x_i-x)$. We have

$$\begin{aligned} H_{kl}^{i,t}(x) &= \int_Q \sum_{p=1}^3 \frac{\partial t_p}{\partial y_t}\frac{\partial}{\partial t_p}\frac{1}{|F_i(t)-x|}\tilde{\varphi}_{kl}(t)\,dt \\ &=: \frac{(b_i\times n_i)_t}{a_i(b_i\times n_i)}L_{kl}^i(x)+\frac{(n_i\times a_i)_t}{a_i(b_i\times n_i)}M_{kl}^i(x)+\frac{(a_i\times b_i)_t}{a_i(b_i\times n_i)}N_{kl}^i(x). \end{aligned}$$

The last integral becomes

$$\begin{aligned} N_{kl}^i(x) &= \int_Q \partial_{t_3}\frac{1}{|a_it_1+b_it_2+n_it_3+x_i-x|}t_1^kt_2^l\,dt_2\,dt_1 \\ &= -\int_Q \frac{n_i(x_i-x)}{|a_it_1+b_it_2+x_i-x|^3}t_1^kt_2^l\,dt_2\,dt_1 = -n_i(x_i-x)I_{k,l}^{Q,-\frac{3}{2}}(a_i,b_i,x_i-x). \end{aligned}$$

The integrals $L_{kl}^i(x)$, $M_{kl}^i(x)$ can be treated like $C_{kl}^{i,s}(x)$, $D_{kl}^{i,s}(x)$ by partial integration and $J_{kl}^{i,rst}(x)$ is analyzed analogously.

Hypersingular integral operator

We implement the Galerkin matrix of the hypersingular integral operator via integration by parts which yields [9, 16]

$$\begin{aligned} \langle \mathbf{W}\boldsymbol{\phi},\boldsymbol{\psi}\rangle &= \int_\Gamma\int_\Gamma \frac{\mu}{2\pi}\frac{1}{|x-y|}\sum_{r,s=1}^3(\operatorname{curl}_\Gamma \phi_r(x))_s(\operatorname{curl}_\Gamma \psi_r(y))_s\,ds_y\,ds_x \\ &+ \frac{\mu}{2\pi}\int_\Gamma\int_\Gamma \sum_{r,s,k,l,m,n=1}^3 \varepsilon_{rsl}(\operatorname{curl}_\Gamma \phi_l(x))_s\frac{\delta_{rn}}{|x-y|}\varepsilon_{nkm}(\operatorname{curl}_\Gamma \psi_m)_k\,ds_y\,ds_x \\ &- 4\mu^2\int_\Gamma\int_\Gamma \sum_{r,s,k,l,m,n=1}^3 \varepsilon_{rsl}(\operatorname{curl}_\Gamma \phi_l(x))_s G_{rn}(x,y)\varepsilon_{nkm}(\operatorname{curl}_\Gamma \psi_m)_k\,ds_y\,ds_x \\ &- \int_\Gamma\int_\Gamma \frac{\mu}{4\pi}\frac{1}{|x-y|}\sum_{r,s=1}^3(\operatorname{curl}_\Gamma \phi_r(x))_r(\operatorname{curl}_\Gamma \psi_s(y))_s\,ds_y\,ds_x \qquad (44) \end{aligned}$$

where $\operatorname{curl}_\Gamma u(x) = \mathbf{n}(x)\times \operatorname{grad}_\Gamma u(x)$, and ε_{ijk} is the total antisymmetric tensor ($\varepsilon_{123}=1$). Using (44) the entries of the Galerkin matrix are computed analytically with the software package *maiprogs* [14].

4 Numerical Results

In this section we present numerical results of the above described Galerkin scheme for various examples. We perform *h*-, *p*- and *hp*-versions. Young's modulus (E-modulus) is $E = 2000$ and the Poisson number is $\nu = 0.3$.

For the computation of the error we use $\|\boldsymbol{\phi} - \boldsymbol{\phi}_N\|_W^2 = \|\boldsymbol{\phi}\|_W^2 - \|\boldsymbol{\phi}_N\|_W^2$ and $\|\mathbf{t} - \mathbf{t}_N\|_V^2 = \|\mathbf{t}\|_V^2 - \|\mathbf{t}_N\|_V^2$.

Example 1. For the Dirichlet problem of the Lamé equation with boundary data $\mathbf{g}(x_1, x_2, x_3) = (-x_2, x_1, 0)$ in (7) on the square $\Gamma = [-1, 1]^2$ we know the energy norm of the exact solution by extrapolation

$$\|\mathbf{t}\|_V = 115.0355908.$$

In Fig. 3 we present the numerical results for the Dirichlet problem. The convergence rates which are given in Table 1, clearly confirm the exponentially fast convergence of the hp-version with geometric mesh, which is expected due to Theorem 4.

Fig. 3 shows clearly the exponentially fast convergence of the *hp*-version on the geometric mesh with mesh grading parameter $\sigma = 0.17$. The parameter $\mu = 0.5$ describes the increase of the polynomial degree, namely $(q, p), (q, p), (q, p+1), (q, p+1), (q, p+2), (q, p+2), \ldots$ in the x_2 direction and correspondingly in the x_1 direction, for a geometric mesh consisting of rectangles only and refined towards the edges. Very good results are also obtained for the h-version on an algebraically graded mesh towards the edges with mesh grading parameter $\beta = 4.0$; this is in agreement with the theoretical results in [26]. Also Fig. 3 and Table 1 show that the uniform p-version converges twice as fast as the uniform h-version [3].

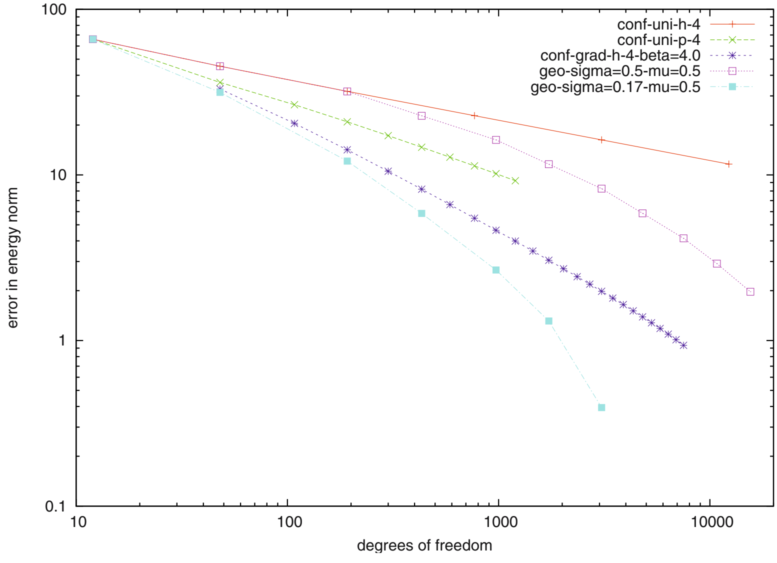

Fig. 3. Weakly singular integral equation (Lamé), Example 1.

Table 1. Convergence rates for the weakly singular integral equation on the Square.

N	$\|\mathbf{t}-\mathbf{t}_N\|_V$	α	p	N	$\|\mathbf{t}-\mathbf{t}_N\|_V$	α	N	$\|\mathbf{t}-\mathbf{t}_N\|_V$	α
h-Version, p=1			p-Version, 4 elements				hp-Version, $\sigma=0.17, \mu=0.5$		
12	65.977067		0	12	65.977067		12	65.977067	
48	45.338115	0.271	1	48	36.205111	0.433	48	31.511011	0.533
192	31.978059	0.252	2	108	26.548835	0.382	192	12.121016	0.689
768	22.804025	0.244	3	192	20.914871	0.415	432	5.8540817	0.897
3072	16.289194	0.243	4	300	17.265718	0.430	972	2.6642368	0.971
12228	11.618080	0.245	5	432	14.701526	0.441	1728	1.3123139	1.231
			6	588	12.801060	0.449	3072	0.3934324	2.094
			7	768	11.335587	0.455			
			8	972	10.170859	0.460			
			9	1200	9.2227497	0.464			
theoretically: 0.250			theoretically: 0.500						

Example 2. For the Neumann problem of the Lamé equation we consider the square $\Gamma=[-1,1]^2$ and choose $\mathbf{f}=(-x_2,x_1,0)$ in (10). Via extrapolation we get $\|\phi\|_W=0.04005011548$.

In Fig. 4 we present the numerical results for the Neumann problem. The convergence rates which are given in Table 2, clearly confirm the exponentially fast convergence of the hp-version with geometric mesh, which is expected due to Theorem 5.

Table 2. Convergence rates for the hypersingular integral equation on the square.

N	$\|\phi-\phi_N\|_W$	α	p	N	$\|\phi-\phi_N\|_W$	α	N	$\|\phi-\phi_N\|_W$	α
h-Version, p=1			p-Version, 4 elements				hp-Version, $\sigma=0.17, \mu=0.5$		
27	0.0258942		1	27	0.0258942		3	0.0400501	
147	0.0170821	0.245	2	147	0.0139794	0.364	27	0.0153835	0.435
675	0.0114749	0.261	3	363	0.0094512	0.433	147	0.0061827	0.538
2883	0.0078521	0.261	4	675	0.0071976	0.439	363	0.0035278	0.621
			5	1083	0.0058224	0.448	867	0.0012488	1.193
			6	1587	0.0048894	0.457	1587	0.0004945	1.532
			7	2187	0.0042117	0.465			
			8	2883	0.0037193	0.450			
theoretically: 0.250			theoretically: 0.500						

Fig. 4 shows clearly the exponentially fast convergence of the hp-version on the geometric mesh with $\sigma=0.17$ and $\mu=0.5$. Again we obtain very good results for the h-version on an algebraically graded mesh towards the edges with mesh grading parameter $\beta=4.0$; which agrees with [26]. Also

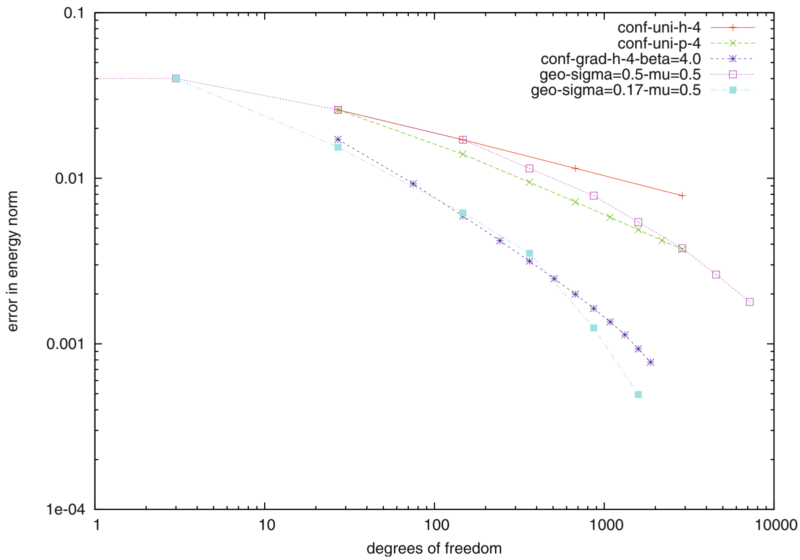

Fig. 4. Hypersingular integral equation (Lamé), Example 2.

Fig. 4 shows that the uniform p-version converges twice as fast as the uniform h-version [3].

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