Coupled Finite and Boundary Element Domain Decomposition Methods

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Summary. The finite element method and the boundary element method often have complementary properties in different situations. The domain decomposition technique allows to use the discretization method which is most appropriate for the subdomain under consideration. The coupling is based on the transmission conditions. The Dirichlet to Neumann (D2N) and Neumann to Dirichlet (N2D) maps are playing a crucial role in representing the transmission conditions. In this paper we study the D2N and N2D maps and their finite and boundary element approximations. Different formulations of the transmission conditions lead to different domain decomposition schemes with different properties. In any case we have to solve large scale systems of coupled finite and boundary element equations. The efficiency of iterative methods heavily depends on the availability of efficient preconditioners. We consider various solution strategies and provide appropriate preconditioners resulting in asymptotically almost optimal solvers.

1 Introduction

Domain Decomposition (DD) Methods provide not only the basic technology for parallelizing numerical algorithms for solving partial differential equations (PDEs) but also for coupling different physical fields and different discretization techniques. Beside the Finite Volume Method (FVM) and the Finite Element Method (FEM), the Boundary Element Method (BEM) is certainly one of the most popular discretization techniques for PDEs. If we compare the FEM with the BEM, then we observe that both methods have advantages and disadvantages in different situations. It is commonly known that the BEM can easily treat unbounded regions whereas the FEM requires special modifications for this case. On the other hand, the FEM is very flexible and can

be applied to very general problems including PDEs with varying coefficients and non–linear problems. However, beside unbounded computational regions, there are a lot of other problems where we can benefit from using boundary element discretization. Large air subdomains or rotating subdomains which are typical for electromagnetical problems, e.g., electrical machines, belong to this class of problems. Also the approximation of singularities can be handled much easier by a boundary mesh than by a volume mesh. Sometime only the complete Cauchy date are needed on the boundary of the computational domain or on the skeleton of a domain decomposition. In this situation, we would like to avoid the meshing of the domain or of the subdomains. A similar situation arises if we are only interested in the solution or in derivatives of the solution in some points or in some small subdomains. Therefore, it is certainly very attractive to develop coupling algorithms and software that can handle both the finite element and the boundary element technologies. There are many early contributions to the FEM–BEM coupling in the engineering literature, see, e.g., [6, 58, 59]. Most of them are using the collocation technique on the boundary element side that does not really fit to the finite element Galerkin technique. Moreover, there was some other drawback of the classical boundary element methods. They produce dense matrices. The breakthrough through this complexity barrier was achieved by developing data–sparse approximation techniques like the fast multipole method [9, 42], panel clustering [21], $\mathcal{H}$–matrix approaches [20], Adaptive Cross Approximation (ACA) methods [2, 3], and wavelet approximations [14, 43].

In the mathematical literature, there are also some early works on unsymmetric Galerkin BEM–FEM couplings by F. Brezzi and C. Johnson [8], C. Johnson and J. Nédélec [27] and others at the end of the 70ies and at the beginning of the 80ies. These results are based on the use of the first boundary integral equation using the single and double layer potentials only. In fact, the analysis requires the compactness of the double layer potential and therefore smooth boundaries have to be assumed. Since we are interested in domain decomposition techniques with Lipschitz subdomains, we heavily rely on the symmetric coupling that was first proposed by M. Costabel in [11]. This approach makes also use of the second boundary integral equation with the hypersingular boundary integral operator. The symmetric formulation can also be generalized to non–linear problems such as elastoplastic problems [13, 40]. G. C. Hsiao and W. L. Wendland first used the symmetric coupling technique for constructing symmetric boundary element domain decomposition equations [26]. The first fast solvers for coupled finite and boundary element domain decomposition equations were proposed and analyzed by U. Langer [32]. The classical Finite Element Tearing and Interconnecting (FETI) methods, which were introduced by C. Farhat and F.–X. Roux [17] in 1991 as a dual version of the classical iterative substructuring methods, and, in particular, the more recently developed dual–primal FETI (FETI–DP) and BDDC (Balanced Domain Decomposition by Constraints) methods are now well–established as efficient and robust parallel solvers for large–scale finite element

equations. We refer the reader to the recently published monograph [55] by A. Toselli and O. Widlund for more informations about the relevant references and for the analysis of FETI methods. U. Langer and O. Steinbach have recently introduced the Boundary Element Tearing and Interconnecting (BETI) methods [36] and the coupled BETI/FETI methods [37]. Inexact data–sparse BETI methods were discussed in [33]. The hybrid coupling of finite element methods and boundary element methods as a macro element was considered by G. C. Hsiao, E. Schnack and W. L. Wendland in [24] for general second order elliptic systems, and in [23] for applications in elasticity. Hybrid domain decomposition methods based on the approximation of the local Dirichlet to Neumann mappings by finite and boundary element methods and a related stability and error analysis were given by O. Steinbach in [48].

This paper provides a unified approach to the construction, analysis and solution of coupled finite and boundary domain decomposition equations. The potential equation with piecewise constant coefficients serves as a simple model problem. On an appropriate domain decomposition, such a special potential problem and similar elliptic boundary value problems in general can be reformulated as variational problems defined on the skeleton of the domain decomposition. These skeleton variational formulations reflect the transmission conditions which can be incorporated in different ways. The local Steklov–Poincaré (D2N) and Poincaré–Steklov (N2D) operators play an important role in these formulations. These operators can locally be approximated by finite and boundary element methods. We discuss and analyse these approximations. Finally we have to solve large scale coupled finite and boundary domain decomposition equations which are in general symmetric, but indefinite. Reductions to symmetric and positive definite Schur complement problems are always possible, but not always recommendable for efficiency reasons. Primal, primal–dual and dual iterative substructuring solvers require asymptotically almost optimal and robust preconditioners. Such preconditioners can be constructed by the use of boundary element technologies for both the boundary element and the finite element blocks.

The rest of the paper is organized as follows: In Section 2, we consider the Dirichlet boundary value problem for the potential equation with piecewise constant coefficients as a typical model problem and study the local Steklov–Poincaré and Poincaré–Steklov operators as well as their finite and boundary element approximations. Section 3 is devoted to different domain decomposition methods. We consider two types of symmetric coupling techniques. The Dirichlet domain decomposition methods presented in Subsection 3.1 require the strong continuity of the primal variable (the potentials) whereas the Neumann domain decomposition methods studied in Subsection 3.2 require the strong continuity of the dual variables (the fluxes). The tearing and interconnecting technology allows us to develop a unique approach to both domain decomposition techniques. In contrast to the primal–dual tearing and interconnecting methods, we prefer the all–floating technique that was introduced by G. Of [38]. In Section 4, we discuss the iterative solution of the linear

systems arising in Section 3 and provide preconditioners leading to asymptotically almost optimal and robust solvers. Finally, we draw some conclusions in Section 5.

2 Boundary Value Problems

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$. As a model problem, we consider the Dirichlet boundary value problem for the potential equation,

$$-\mathrm{div}[\alpha(x)\nabla u(x)] = f(x) \quad \text{for } x \in \Omega, \quad u(x) = g(x) \quad \text{for } x \in \Gamma \tag{1}$$

where $g \in H^{1/2}(\Gamma) \cap C(\Gamma)$ is a given continuous function. We assume that $\alpha(\cdot)$ is piecewise constant with $\alpha(x) = \alpha_i > 0$ for $x \in \Omega_i$ and for $i = 1, \ldots, p$, where we have given a non–overlapping domain decomposition

$$\overline{\Omega} = \bigcup_{i=1}^{p} \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset \quad \text{for } i \neq j, \quad \Gamma_i = \partial\Omega_i, \quad \overline{\Gamma}_{ij} = \overline{\Gamma}_i \cap \overline{\Gamma}_j$$

of the computational domain Ω into p Lipschitz subdomains Ω_i. Moreover,

$$\Gamma_S = \bigcup_{i=1}^{p} \Gamma_i = \Gamma \cup \Gamma_I \quad \text{and} \quad \overline{\Gamma}_I = \bigcup_{i<j} \overline{\Gamma}_{ij}$$

denote the skeleton and the interface of the domain decomposition, respectively. Instead of the global boundary value problem (1), we now consider the local boundary value problems

$$-\alpha_i \Delta u_i(x) = f_i(x) \quad \text{for } x \in \Omega_i, \quad u_i(x) = g(x) \quad \text{for } x \in \Gamma_i \cap \Gamma \tag{2}$$

together with the transmission conditions

$$u_i(x) = u_j(x), \quad \alpha_i \frac{\partial}{\partial n_i} u_i(x) + \alpha_j \frac{\partial}{\partial n_j} u_j(x) = 0 \quad \text{for } x \in \Gamma_{ij}, \tag{3}$$

where $f_i(x) = f(x)$ for $x \in \Omega_i$. In what follows, we will describe some variational formulations for domain decomposition methods which are based on the local solution of either Dirichlet or Neumann boundary value problems. The idea behind is that all solutions u_i of the local boundary value problems (2) are known as soon as the Cauchy data along the coupling boundaries Γ_{ij} satisfying the transmission conditions (3) are determined.

2.1 Dirichlet Boundary Value Problems

We start with the local Dirichlet boundary value problem for a given continuous function $g_i \in H^{1/2}(\Gamma_i) \cap C(\Gamma_i)$

$$-\alpha_i \Delta u_i(x) = f_i(x) \quad \text{for } x \in \Omega_i, \quad u_i(x) = g_i(x) \quad \text{for } x \in \Gamma_i \tag{4}$$

where the weak solution $u_i \in H^1(\Omega_i)$ is well defined. Moreover, the normal derivative $t_i = n_i \cdot \nabla u_i$ defines the associated Neumann datum. The solution of the local Dirichlet boundary value problem (4) therefore defines the local Dirichlet to Neumann map $g_i \longmapsto t_i$. Hence, we have to find the correct Dirichlet datum g_i such that the transmission boundary conditions (3), i.e.,

$$u_i(x) = u_j(x), \quad \alpha_i t_i(x) + \alpha_j t_j(x) = 0 \quad \text{for } x \in \Gamma_{ij},$$

are satisfied along the coupling interfaces Γ_{ij}. To describe the local Dirichlet to Neumann map we may consider either a domain variational formulation or boundary integral equations to obtain explicit representations of the local Steklov–Poincaré (Dirichlet to Neumann) operators involved.

Domain Variational Formulation

The associated variational formulation of the local Dirichlet boundary value problem (4) is to find $u_i \in H^1(\Omega_i)$, $u_i(x) = g_i(x)$ for $x \in \Gamma_i$, such that

$$\int_{\Omega_i} \alpha_i \nabla u_i(x) \cdot \nabla v_i(x) dx = \int_{\Omega_i} f_i(x) v_i(x) dx \tag{5}$$

is satisfied for all test functions $v_i \in H_0^1(\Omega_i)$. As usual, $H^1(\Omega_i)$ is the closure of $C^\infty(\Omega_i)$ with respect to the norm

$$\|v_i\|_{H^1(\Omega_i)} = \left[\|v_i\|^2_{L_2(\Omega_i)} + \|\nabla v_i\|^2_{L_2(\Omega)} \right]^{1/2}.$$

However, in what follows we will use an equivalent norm in $H^1(\Omega_i)$ which is given by

$$\|v_i\|_{H^1(\Omega_i),\Gamma_i} = \left[\left(\int_{\Gamma_i} v_i(x) ds_x \right)^2 + \|\nabla v_i\|^2_{L_2(\Omega_i)} \right]^{1/2}.$$

Moreover, $H_0^1(\Omega_i) = \{v_i \in H^1(\Omega_i), v_i(x) = 0 \text{ for } x \in \Gamma_i\}$. The bilinear form

$$a_{\Omega_i}(v_i, v_i) = \int_{\Omega_i} |\nabla v_i(x)|^2 dx = \|\nabla v_i\|^2_{L_2(\Omega_i)} = \|v_i\|^2_{H^1(\Omega_i),\Gamma_i} \quad \text{for } v_i \in H_0^1(\Omega_i)$$

defines an equivalent norm in $H_0^1(\Omega_i)$, i.e., $(a_{\Omega_i}(v_i, v_i))^{1/2}$ is the energy norm in $H_0^1(\Omega_i)$. By taking the trace of $H^1(\Omega_i)$ we may define the trace space $H^{1/2}(\Gamma_i)$ which is equipped with the norm

$$\|w_i\|_{H^{1/2}(\Gamma_i)} = \min_{v_i \in H^1(\Omega_i), v_{i|\Gamma_i} = w_i} \|v_i\|_{H^1(\Omega_i),\Gamma_i}.$$

For $g_i \in H^{1/2}(\Gamma_i)$, there exists a bounded extension $u_{g_i} = \mathcal{E}_i g_i \in H^1(\Omega_i)$ satisfying

$$\|u_{g_i}\|_{H^1(\Omega_i),\Gamma_i} = \|\mathcal{E}_i g_i\|_{H^1(\Omega_i),\Gamma_i} \leq c_{\mathcal{E}_i} \|g_i\|_{H^{1/2}(\Gamma_i)}.$$

A particular choice would be to consider the harmonic extension $u_{g_i} \in H^1(\Omega_i)$ as the unique solution of the variational problem

$$\int_{\Omega_i} \nabla u_{g_i}(x) \cdot \nabla v_i(x) dx = 0 \quad \text{for all } v_i \in H^1_0(\Omega_i).$$

It remains to find $u_{i,0} \in H^1_0(\Omega_i)$ such that the homogenized variational problem

$$\int_{\Omega_i} \alpha_i \nabla u_{i,0}(x) \cdot \nabla v_i(x) dx = \int_{\Omega_i} f_i(x) v_i(x) dx - \int_{\Omega_i} \alpha_i \nabla u_{g_i}(x) \cdot \nabla v_i(x) dx \quad (6)$$

is satisfied for all $v_i \in H^1_0(\Omega_i)$. For $u_{i,0}, v_i \in H^1_0(\Omega_i)$, the bilinear form

$$a_{\Omega_i}(u_{i,0}, v_i) = \int_{\Omega_i} \nabla u_{i,0}(x) \cdot \nabla v_i(x) dx = \langle A_{\Omega_i,0} u_{i,0}, v_i \rangle_{\Omega_i},$$

induces, by the Riesz representation theorem, a bounded linear operator

$$A_{\Omega_i,0} : H^1_0(\Omega_i) \to H^{-1}(\Omega_i) = [H^1_0(\Omega_i)]'.$$

In addition, for $u_{g_i} \in H^1(\Omega_i)$ and $v_i \in H^1_0(\Omega_i)$ we define the bounded operator $A_{\Gamma_i} : H^1(\Omega_i) \to H^{-1}(\Omega_i)$ satisfying

$$\langle A_{\Gamma_i} u_{g_i}, v_i \rangle_{\Omega_i} = \int_{\Omega_i} \nabla u_{g_i}(x) \cdot \nabla v_i(x) dx.$$

Hence, we can write the variational problem (6) as an operator equation to find $u_{i,0} \in H^1_0(\Omega_i)$ such that

$$\alpha_i A_{\Omega_i,0} u_{i,0} = f_i - \alpha_i A_{\Gamma_i} \mathcal{E}_i g_i \in H^{-1}(\Omega_i). \quad (7)$$

The operator $A_{\Omega_i,0} : H^1_0(\Omega_i) \to H^{-1}(\Omega_i)$ is $H^1_0(\Omega_i)$–elliptic, i.e., for all $v_i \in H^1_0(\Omega_i)$, we have

$$\langle A_{\Omega_i,0} v_i, v_i \rangle_{\Omega_i} = \int_{\Omega_i} |\nabla v_i(x)|^2 dx = \|\nabla v_i\|^2_{L_2(\Omega_i)} = \|v_i\|^2_{H^1(\Omega_i),\Gamma_i}.$$

Hence, there exists the unique solution of the operator equation (7),

$$u_{i,0} = \frac{1}{\alpha_i} A^{-1}_{\Omega_i,0} f_i - A^{-1}_{\Omega_i,0} A_{\Gamma_i} \mathcal{E}_i g_i \in H^1_0(\Omega_i),$$

and, therefore, $u_i = u_{0,i} + u_{g_i} \in H^1(\Omega_i)$ is the weak solution of the Dirichlet boundary value problem (4). In particular, from

$$\begin{aligned}
\|f_i\|_{H^{-1}(\Omega_i)} &= \sup_{0\neq v_i\in H^1_0(\Omega_i)} \frac{\langle f_i, v_i\rangle_{\Omega_i}}{\|v_i\|_{H^1(\Omega_i),\Gamma_i}} \;\geq\; \frac{\langle f_i, u_{i,0}\rangle_{\Omega_i}}{\|u_{i,0}\|_{H^1(\Omega_i),\Gamma_i}} \\
&= \frac{\alpha_i}{\|u_{i,0}\|_{H^1(\Omega_i),\Gamma_i}} \int\limits_{\Omega_i} \nabla[u_{i,0}(x) + u_{g_i}(x)]\nabla u_{i,0}(x)\,dx \\
&\geq \frac{\alpha_i}{\|u_{i,0}\|_{H^1(\Omega_i),\Gamma_i}} \left[\|\nabla u_{i,0}\|^2_{L_2(\Omega_i)} - \|\nabla u_{i,0}\|_{L_2(\Omega_i)}\|\nabla u_{g_i}\|_{L_2(\Omega_i)}\right] \\
&= \alpha_i \left[\|\nabla u_{i,0}\|_{L_2(\Omega_i)} - \|\nabla u_{g_i}\|_{L_2(\Omega_i)}\right],
\end{aligned}$$

we find

$$\|\nabla u_{i,0}\|_{L_2(\Omega_i)} \;\leq\; \frac{1}{\alpha_i}\|f_i\|_{H^{-1}(\Omega_i)} + \|\nabla u_{g_i}\|_{L_2(\Omega_i)}.$$

In particular, for $f_i = 0$, we therefore have

$$\begin{aligned}
\|u_i\|^2_{H^1(\Omega_i),\Gamma_i} &= \left(\int\limits_{\Gamma_i} g_i(x)ds_x\right)^2 + \|\nabla u_{i,0}\|^2_{L_2(\Omega_i)} \\
&\leq \left(\int\limits_{\Gamma_i} g_i(x)ds_x\right)^2 + \|\nabla u_{g_i}\|^2_{L_2(\Omega_i)} \\
&= \|u_{g_i}\|^2_{H^1(\Omega_i),\Gamma_i} \;\leq\; c_{\mathcal{E}_i}\|g_i\|^2_{H^{1/2}(\Gamma_i)},
\end{aligned}$$

i.e.,

$$\|u_i\|_{H^1(\Omega_i),\Gamma_i} \;\leq\; c_{\mathcal{E}_i}\|g_i\|_{H^{1/2}(\Gamma_i)}. \tag{8}$$

It remains to find the associated Neumann datum $t_i = n_i \cdot \nabla u_i \in H^{-1/2}(\Gamma_i)$, where $H^{-1/2}(\Gamma_i) = [H^{1/2}(\Gamma_i)]'$ is the dual space which is equiped with the norm

$$\|\tau_i\|_{H^{-1/2}(\Gamma_i)} = \sup_{0\neq w_i\in H^{1/2}(\Gamma_i)} \frac{\langle \tau, w_i\rangle_{\Gamma_i}}{\|w_i\|_{H^{1/2}(\Gamma_i)}}.$$

Using Green's first formula, $t_i \in H^{-1/2}(\Gamma_i)$ solves the variational problem

$$\int\limits_{\Gamma_i} \alpha_i t_i(x)w_i(x)ds_x \;=\; \int\limits_{\Omega_i} \alpha_i \nabla u_i(x)\cdot\nabla\mathcal{E}_i w_i(x)dx - \int\limits_{\Omega_i} f_i(x)\mathcal{E}_i w_i(x)dx \tag{9}$$

for all test functions $w_i \in H^{1/2}(\Gamma_i)$. Using duality arguments, we then obtain

$$
\begin{aligned}
\|t_i\|_{H^{-1/2}(\Gamma_i)} &= \sup_{0\neq w_i\in H^{1/2}(\Gamma_i)} \frac{\langle t_i, w_i\rangle_{\Gamma_i}}{\|w_i\|_{H^{1/2}(\Gamma_i)}} \\
&= \sup_{0\neq w_i\in H^{1/2}(\Gamma_i)} \frac{1}{\|w_i\|_{H^{1/2}(\Gamma_i)}} \left[\langle \nabla u_i, \nabla \mathcal{E}_i w_i\rangle_{\Omega_i} - \frac{1}{\alpha_i}\langle f_i, \mathcal{E}_i w_i\rangle_{\Omega_i}\right] \\
&\leq \sup_{0\neq w_i\in H^{1/2}(\Gamma_i)} \frac{\|\mathcal{E}_i w_i\|_{H^1(\Omega_i)}}{\|w_i\|_{H^{1/2}(\Gamma_i)}} \left[\|u_i\|_{H^1(\Omega_i)} + \frac{1}{\alpha_i}\|f_i\|_{\widetilde{H}^{-1}(\Omega_i)}\right] \\
&\leq c_{\mathcal{E}_i}\left[\|u_i\|_{H^1(\Omega_i)} + \frac{1}{\alpha_i}\|f_i\|_{\widetilde{H}^{-1}(\Omega_i)}\right],
\end{aligned}
$$

where $\widetilde{H}^{-1}(\Omega_i) = [H^1(\Omega_i)]'$. The local Neumann datum t_i therefore depends only on the given right hand side f_i and on the prescribed Dirichlet datum g_i. Hence we have given a Dirichlet to Neumann map as

$$
\alpha_i t_i(x) \;=\; \alpha_i (S_i g_i)(x) - (N_i f_i)(x) \quad \text{for } x \in \Gamma_i,
$$

where we have used the local Steklov–Poincaré operator $S_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$, and the Newton potential $N_i : \widetilde{H}^{-1}(\Omega_i) \to H^{-1/2}(\Gamma_i)$ as given below. In particular, for $f_i = 0$, we therefore have $t_i = S_i g_i$ satisfying

$$
\|S_i g_i\|_{H^{-1/2}(\Gamma_i)} \;=\; \|t_i\|_{H^{-1/2}(\Gamma_i)} \;\leq\; c^2_{\mathcal{E}_i}\|g_i\|_{H^{1/2}(\Gamma_i)}. \tag{10}
$$

If we define the linear operator $A_{\Omega_i} : H^1(\Omega_i) \to \widetilde{H}^{-1}(\Omega_i)$ via the Riesz representation theorem as

$$
\langle A_{\Omega_i} u_i, v_i\rangle_{\Omega_i} = \int_{\Omega_i} \nabla u_i(x)\cdot \nabla v_i(x) dx \quad \text{for } u_i, v_i \in H^1(\Omega_i),
$$

we can rewrite the variational formulation (9) as

$$
\begin{aligned}
\alpha_i \langle t_i, w_i\rangle_{\Gamma_i} &= \alpha_i \int_{\Omega_i} \nabla u_i(x)\cdot \nabla \mathcal{E}_i w_i(x) dx - \int_{\Omega_i} f_i(x)\mathcal{E}_i w_i(x) dx \\
&= \alpha_i \langle A_{\Omega_i} u_i, \mathcal{E}_i w_i\rangle_{\Omega_i} - \langle f_i, \mathcal{E}_i w_i\rangle_{\Omega_i} \\
&= \alpha_i \langle A_{\Omega_i}[u_{i,0} + u_{g_i}], \mathcal{E}_i w_i\rangle_{\Omega_i} - \langle f_i, \mathcal{E}_i w_i\rangle_{\Omega_i} \\
&= \alpha_i \langle A'_{\Gamma_i} u_{i,0}, \mathcal{E}_i w_i\rangle_{\Omega_i} + \alpha_i \langle A_{\Omega_i} u_{g_i}, \mathcal{E}_i w_i\rangle_{\Omega_i} - \langle f_i, \mathcal{E}_i w_i\rangle_{\Omega_i} \\
&= \langle \alpha_i A'_{\Gamma_i} u_{i,0} + \alpha_i A_{\Omega_i} u_{g_i} - f_i, \mathcal{E}_i w_i\rangle_{\Omega_i},
\end{aligned}
$$

and, therefore, as the following operator equation,

$$
\begin{aligned}
\alpha_i t_i &= \mathcal{E}'_i\left[\alpha_i A_{\Omega_i} u_{g_i} + \alpha_i A'_{\Gamma_i} u_{i,0} - f_i\right] \\
&= \mathcal{E}'_i\left[\alpha_i A_{\Omega_i} u_{g_i} + \alpha_i A'_{\Gamma_i}\left(\frac{1}{\alpha_i} A^{-1}_{\Omega_i,0} f_i - A^{-1}_{\Omega_i,0} A_{\Gamma_i} u_{g_i}\right) - f_i\right] \\
&= \alpha_i \mathcal{E}'_i\left[A_{\Omega_i} - A'_{\Gamma_i} A^{-1}_{\Omega_i,0} A_{\Gamma_i}\right]\mathcal{E}_i g_i + \mathcal{E}'_i\left[A'_{\Gamma_i} A^{-1}_{\Omega_i,0} - I\right] f_i,
\end{aligned}
$$

where A'_{Γ_i} and $\mathcal{E}'_i$ are the corresponding adjoint operators. Hence we can represent the Steklov–Poincaré operator as

$$S_i = \mathcal{E}'_i \left[A_{\Omega_i} - A'_{\Gamma_i} A^{-1}_{\Omega_i,0} A_{\Gamma_i} \right] \mathcal{E}_i \, : \, H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i) \tag{11}$$

and the Newton potential as

$$N_i = \mathcal{E}'_i \left[A'_{\Gamma_i} A^{-1}_{\Omega_i,0} - I \right] : \widetilde{H}^{-1}(\Omega_i) \to H^{-1/2}(\Gamma_i). \tag{12}$$

Theorem 1. *The Steklov–Poincaré operator* $S_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ *as defined in* (11) *is bounded,*

$$\|S_i g_i\|_{H^{-1/2}(\Gamma_i)} \leq c_{\mathcal{E}_i} \|g_i\|_{H^{1/2}(\Gamma)} \quad \textit{for all } g_i \in H^{1/2}(\Gamma_i),$$

and $H^{1/2}(\Gamma_i)$*–semi–elliptic, i.e.,*

$$\langle S_i g_i, g_i \rangle_{\Gamma_i} \geq \|g_i\|^2_{H^{1/2}(\Gamma_i)} \quad \textit{for all } g_i \in H^{1/2}_0(\Gamma_i),$$

where

$$H^{1/2}_0(\Gamma_i) = \left\{ w_i \in H^{1/2}(\Gamma_i) \, : \int_{\Gamma_i} w_i(x) ds_x = 0 \right\}.$$

In particular, for $g_i \equiv 1$, *we have* $S_i g_i = 0$.

Proof. The boundedness of the Steklov–Poincaré operator S_i is just the estimate (10). Using (9) with $f_i = 0$, for $g_i \in H^{1/2}_0(\Gamma_i)$, we get

$$\begin{aligned} \langle S_i g_i, g_i \rangle_{\Gamma_i} = \langle t_i, g_i \rangle_{\Gamma_i} &= \int_{\Omega_i} \nabla u_i(x) \cdot \nabla u_{g_i}(x) dx \\ &= \int_{\Omega_i} \nabla u_i(x) \cdot \nabla [u_{g_i}(x) + u_{i,0}(x)] dx \, = \, |u_i|^2_{H^1(\Omega_i)} \\ &= \left(\int_{\Gamma_i} g_i(x) ds_x \right)^2 + |u_i|^2_{H^1(\Omega_i)} \, = \, \|u_i\|^2_{H^1(\Omega_i),\Gamma_i}. \end{aligned}$$

Now the $H^{1/2}(\Gamma_i)$–semi–ellipticity follows from the trace theorem. □

Finite Element Approximation

To define an approximate Dirichlet to Neumann map we first introduce the local finite element trial spaces

$$S^1_h(\Omega_i) \, = \, \text{span}\{\phi^1_{i,k}\}_{k=1}^{\widetilde{M}_i} \subset H^1(\Omega_i)$$

and

$$S^1_{h,0}(\Omega_i) \;=\; S^1_h(\Omega_i) \cap H^1_0(\Omega_i) \;=\; \mathrm{span}\{\phi^1_{i,k}\}_{k=M_i+1}^{\widetilde{M}_i}$$

of piecewise linear basis functions $\phi^1_{i,k}$ with respect to some regular finite element mesh $\Omega_{i,h}$ characterized by the local mesh–size parameter h_i. Note that the basis functions $\phi^1_{i,k}$, $k = M_i + 1, \ldots, \widetilde{M}_i$ correspond to the interior degrees of freedom, while the remaining basis functions $\phi^1_{i,k}$, $k = 1, \ldots, M_i$ are associated to degrees of freedom on the boundary.

Let

$$u_{g_i,h}(x) = \sum_{k=1}^{\widetilde{M}_i} u_{g_i}(x_{i,k})\phi^1_{i,k}(x)$$

be the piecewise linear interpolation of the continuous extension u_{g_i}. The finite element approximation of the local variational problem (6) is to find $u_{i,0,h} \in S^1_{h,0}(\Omega_i)$ such that

$$\int_{\Omega_i} \alpha_i \nabla u_{i,0,h} \cdot \nabla v_{i,h}(x)dx = \int_{\Omega_i} f_i(x)v_{i,h}(x)dx - \int_{\Omega_i} \alpha_i \nabla u_{g_i,h}(x) \cdot \nabla v_{i,h}(x)dx \tag{13}$$

is satisfied for all $v_{i,h} \in S^1_{h,0}(\Omega_i)$. This is equivalent to the Galerkin equations

$$\sum_{k=M_i+1}^{\widetilde{M}_i} [u_{i,0,k} + u_{g_i}(x_{i,k})] \int_{\Omega_i} \alpha_i \nabla\phi^1_{i,k}(x) \cdot \nabla\phi^1_{i,\ell}(x)dx$$
$$= \int_{\Omega_i} f_i(x)\phi^1_{i,\ell}(x)dx - \sum_{k=1}^{M_i} g_i(x_{i,k}) \int_{\Omega_i} \alpha_i \nabla\phi^1_{i,k}(x) \cdot \nabla\phi^1_{i,\ell}(x)dx$$

for all $\ell = M_i + 1, \ldots, \widetilde{M}_i$. Introducing the nodal values

$$u_{I,i,k} = u_{i,0,k} + u_{g_i}(x_{i,k}) \quad \text{for } k = M_i + 1, \ldots, \widetilde{M}_i,$$
$$u_{C,i,k} = g_i(x_{i,k}) \quad \text{for } k = 1, \ldots, M_i$$

as new unknowns, this is equivalent to a linear system

$$\alpha_i K_{II,i}\underline{u}_{I,i} \;=\; \underline{f}_{I,i} - \alpha_i K_{CI,i}\underline{u}_{C,i},$$

where the local stiffness matrix is given by

$$K_{II,i}[\ell, k] = \int_{\Omega_i} \nabla\phi^1_{i,k}(x) \cdot \nabla\phi^1_{i,\ell}(x)dx$$

for $k, \ell = M_i + 1, \ldots, \widetilde{M}_i$, while the vector of the right hand side is determined by

$$f_{I,i,\ell} = \int_{\Omega_i} f_i(x)\phi^1_{i,\ell}(x)dx.$$

In addition,

$$K_{CI,i}[\ell,k] = \int_{\Omega_i} \nabla\phi^1_{i,k}(x)\cdot\nabla\phi^1_{i,\ell}(x)dx$$

for $k = 1,\ldots,M_i$, $\ell = M_i+1,\ldots,\widetilde{M}_i$. The solution vector

$$\underline{u}_{I,i} = \frac{1}{\alpha_i}K^{-1}_{II,i}\underline{f}_{I,i} - K^{-1}_{II,i}K_{CI,i}\underline{u}_{C,i}$$

defines an approximate solution $u_{i,h} = u_{i,0,h} + u_{g_i,h}$ for which the error estimate

$$\|u_i - u_{i,h}\|_{H^1(\Omega_i),\Gamma_i} \le c\,h_i\,|u_i|_{H^2(\Omega_i)}$$

provided that the regularity assumption $u_i \in H^2(\Omega_i)$ holds.

Now, instead of the variational problem (9), we have to consider a perturbed formulation to find $\widetilde{t}_i \in H^{-1/2}(\Gamma_i)$ such that

$$\int_{\Gamma_i} \alpha_i\widetilde{t}_i(x)w_i(x)ds_x = \int_{\Omega_i} \alpha_i\nabla u_{i,h}(x)\cdot\nabla\mathcal{E}_i w_i(x)dx - \int_{\Omega_i} f_i(x)\mathcal{E}_i w_i(x)dx \quad (14)$$

is satisfied for all test functions $w_i \in H^{1/2}(\Gamma_i)$. This implies an approximated Dirichlet to Neumann map

$$\alpha_i\widetilde{t}_i(x) = \alpha_i(\widetilde{S}_i g_i)(x) - (\widetilde{N}_i f_i)(x) \quad \text{for } x \in \Gamma_i,$$

where $\widetilde{S}_i$ is an approximate Steklov–Poincaré operator which is defined via the solution of the Galerkin variational formulation (13).

Theorem 2. [48] *The approximate Steklov–Poincaré operator $\widetilde{S}_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ as defined above is bounded and $H^{1/2}_0(\Gamma_i)$–elliptic. Moreover, there holds the a priori error estimate*

$$\|(S_i - \widetilde{S}_i)g_i\|_{H^{-1/2}(\Gamma_i)} \le c\,h_i\,|u_i|_{H^2(\Omega_i)}$$

when assuming $u_i \in H^2(\Omega_i)$.

When choosing in (14) $\phi^1_{i,\ell}$, $\ell = 1,\ldots,M_i$ as a test function, this gives

$$\begin{aligned}
\alpha_i \int_{\Gamma_i} \widetilde{t}_i(x)\phi^1_{i,\ell}(x)ds_x &= \alpha_i \int_{\Omega_i} \nabla u_{i,h}(x)\cdot\nabla\phi^1_{i,\ell}(x)dx - \int_{\Omega_i} f_i(x)\phi^1_{i,\ell}(x)dx \\
&= \sum_{k=1}^{\widetilde{M}_i} \alpha_i u_{I,i,k} \int_{\Omega_i} \nabla\phi^1_{i,k}(x)\cdot\nabla\phi^1_{i,\ell}(x)dx \\
&\quad + \sum_{k=\widetilde{M}_i+1}^{M_i} u_{C,i,k} \int_{\Omega_i} \nabla\phi^1_{i,k}(x)\cdot\nabla\phi^1_{i,\ell}(x)dx - \int_{\Omega_i} f_i(x)\phi^1_{i,\ell}(x)dx
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\widetilde{M}_i} \alpha_i K_{IC,i}[\ell,k] u_{I,i,k} + \sum_{k=\widetilde{M}_i+1}^{M_i} \alpha_i K_{CC,i}[\ell,k] u_{C,i,k} - f_{C,i,\ell} \\
&= \alpha_i \left(K_{CC,i} \underline{u}_{C,i} + K_{IC,i} \underline{u}_{I,i} \right)_\ell - f_{C,i,\ell} \\
&= \alpha_i \left(\left[K_{CC,i} - K_{IC,i} K_{II,i}^{-1} K_{CI,i} \right] \underline{u}_{C,i} \right)_\ell + \left(K_{CI,i} K_{II,i}^{-1} \underline{f}_{I,i} \right)_\ell - f_{C,i,\ell}.
\end{aligned}
$$

Hence we obtain the discrete Dirichlet to Neumann map

$$
\alpha_i \widetilde{\underline{t}}_i \;=\; \alpha_i \widetilde{S}_{i,h}^{\mathrm{FEM}} \underline{u}_C + K_{CI,i} K_{II,i}^{-1} \underline{f}_{I,i} - \underline{f}_{C,i} \tag{15}
$$

with the finite element approximation of the Steklov–Poincaré operator

$$
\widetilde{S}_{i,h}^{\mathrm{FEM}} \;=\; K_{CC,i} - K_{IC,i} K_{II,i}^{-1} K_{CI,i}. \tag{16}
$$

Boundary Integral Equations

Instead of using finite element discretizations of domain variational formulations for the numerical solution of the local Dirichlet boundary value problem (4), we now consider boundary integral formulations and their boundary element discretization. The starting point is the representation formula

$$
u_i(x) = \int_{\Gamma_i} U^*(x,y) t_i(y) ds_y - \int_{\Gamma_i} \frac{\partial}{\partial n_y} U^*(x,y) g_i(y) ds_y + \frac{1}{\alpha_i} \int_{\Omega_i} U^*(x,y) f_i(y) dy,
$$

that holds for $x \in \Omega_i$, where

$$
U^*(x,y) \;=\; \frac{1}{4\pi} \frac{1}{|x-y|}
$$

is the fundamental solution of the Laplace operator. To find the yet unknown Neumann datum $t_i \in H^{-1/2}(\Gamma_i)$, we first consider the boundary integral equation which results from the representation formula for $x \to \Gamma_i$,

$$
\int_{\Gamma_i} U^*(x,y) t_i(y) ds_y = \frac{1}{2} g_i(x) + \int_{\Gamma_i} \frac{\partial}{\partial n_y} U^*(x,y) g_i(y) ds_y - \frac{1}{\alpha_i} \int_{\Omega_i} U^*(x,y) f(y) dy,
$$

or,

$$
(V_i t_i)(x) \;=\; (\frac{1}{2} I + K_i) g_i(x) - \frac{1}{\alpha_i} (\widetilde{N}_{i,0} f_i)(x) \quad \text{for } x \in \Gamma_i. \tag{17}
$$

Here, $x \in \Gamma_i$ is assumed to be on a smooth part of the boundary Γ_i. Since we are using a Galerkin approach, such an assumption is sufficient. Moreover, $V_i : H^{-1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i)$ is the single layer potential, $K_i : H^{1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i)$ is the double layer potential, and $\widetilde{N}_{i,0} : \widetilde{H}^{-1}(\Omega_i) \to H^{1/2}(\Gamma_i)$ is

the Newton potential. Since the single layer potential operator is $H^{-1/2}(\Gamma_i)$–elliptic and therefore invertible, we find the Dirichlet to Neumann map

$$\begin{aligned}\alpha_i t_i(x) &= \alpha_i V_i^{-1}(\frac{1}{2}I + K_i)g_i(x) - V_i^{-1}\widetilde{N}_{0,i}f(x) \\ &= \alpha_i(S_i g_i)(x) - (N_i f)(x), \quad x \in \Gamma_i\end{aligned}$$

with the boundary integral operator representation of the Steklov–Poincaré operator

$$S_i \;=\; V_i^{-1}(\frac{1}{2}I + K_i) \;:\; H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i), \tag{18}$$

and with the operator

$$N_i \;=\; V_i^{-1}\widetilde{N}_{0,i} \;:\; \widetilde{H}^{-1}(\Omega_i) \to H^{-1/2}(\Gamma_i).$$

Although the Steklov–Poincaré operator (18) is self–adjoint in the continuous case, an approximation of this composed operator results in a non–symmetric stiffness matrix in general. Hence we are interested in alternative representations which result in symmetric boundary element approximations.

Since the solution of the local Dirichlet boundary value problem (4) is given by the representation formula, the application of the normal derivative to the representation formula gives a second, the so–called hypersingular boundary integral equation,

$$\begin{aligned}t_i(x) = \frac{1}{2}t_i(x) + \int_{\Gamma_i} \frac{\partial}{\partial n_x}U^*(x,y)t_i(y)ds_y - \frac{\partial}{\partial n_x}\int_{\Gamma_i}\frac{\partial}{\partial n_y}U^*(x,y)g_i(y)ds_y \\ -\frac{1}{\alpha_i}\frac{\partial}{\partial n_x}\int_{\Omega_i} U^*(x,y)f_i(y)dy,\end{aligned}$$

or,

$$t_i(x) \;=\; \frac{1}{2}t_i(x) + (K_i' t_i)(x) + (D_i g_i)(x) - \frac{1}{\alpha_i}(\widetilde{N}_{i,1}f_i)(x) \quad \text{for } x \in \Gamma_i. \tag{19}$$

Here, $K_i' : H^{-1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ is the adjoint double layer potential, $D_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ is the hypersingular boundary integral operator, and $\widetilde{N}_{i,1} : \widetilde{H}^{-1}(\Omega_i) \to H^{-1/2}(\Gamma_i)$ is the normal derivative of the Newton potential. Inserting the first boundary integral representation of the Dirichlet to Neumann map into (19) gives the relations

$$\begin{aligned}\alpha_i t_i(x) &= \alpha_i(D_i g_i)(x) + (\frac{1}{2}I + K_i')(\alpha_i t_i)(x) - (\widetilde{N}_{i,1}f_i)(x) \\ &= \alpha_i(D_i g_i)(x) + (\frac{1}{2}I + K_i')\left[\alpha_i V_i^{-1}(\frac{1}{2}I + K_i)g_i(x) - V_i^{-1}\widetilde{N}_{i,0}f_i(x)\right] \\ &\qquad -(\widetilde{N}_{1,i}f)(x)\end{aligned}$$

$$= \alpha_i \left[D_i + (\frac{1}{2} I + K_i') V_i^{-1} (\frac{1}{2} I + K_i) \right] g_i(x)$$
$$- (\widetilde{N}_{i,1} f_i)(x) - (\frac{1}{2} I + K_i') V_i^{-1} \widetilde{N}_{i,0} f_i(x)$$
$$= \alpha_i (S_i g_i)(x) - (N_i f)(x) \quad \text{for } x \in \Gamma_i$$

with the so–called symmetric boundary integral operator representation of the Steklov–Poincaré operator,

$$S_i = D_i + (\frac{1}{2} I + K_i') V_i^{-1} (\frac{1}{2} I + K_i) : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i), \tag{20}$$

and with an alternative representation of

$$N_i = \widetilde{N}_{i,1} + (\frac{1}{2} I + K_i') V_i^{-1} \widetilde{N}_{i,0} : \widetilde{H}^{-1}(\Omega_i) \to H^{-1/2}(\Gamma_i).$$

While the Steklov–Poincaré operator $S_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ is bounded, it is not obvious which Sobolev norms in $H^{1/2}(\Gamma_i)$ and $H^{-1/2}(\Gamma_i)$ have to be used, respectively. When using appropriate norms, explicit estimates can be derived as in Theorem 1, where we used a trace norm to characterize $H^{1/2}(\Gamma_i)$.

In the case of boundary integral operators a natural choice is to use norms which are induced by the single layer potential and its inverse. In particular,

$$\|w_i\|_{V_i} = \sqrt{\langle V_i w_i, w_i \rangle_{\Gamma_i}} \quad \text{and} \quad \|v_i\|_{V_i^{-1}} = \sqrt{\langle V_i^{-1} v_i, v_i \rangle_{\Gamma_i}}$$

define equivalent norms of the Sobolev spaces $H^{-1/2}(\Gamma_i)$ and $H^{1/2}(\Gamma_i)$, respectively. Using both boundary integral representations (18) and (20) of the Steklov–Poincaré operator S_i, we obtain the estimate [52]

$$\|(\frac{1}{2} I + K_i) v_i\|_{V_i^{-1}} \leq c_K(\Gamma_i) \, \|v_i\|_{V_i^{-1}} \quad \text{for all } v_i \in H^{1/2}(\Gamma_i), \tag{21}$$

where

$$c_K(\Gamma_i) = \frac{1}{2} + \sqrt{\frac{1}{4} - c_1^{V_i} c_1^{D_i}} < 1$$

is the contraction constant of the double layer potential $\frac{1}{2} I + K_i$ defined by the ellipticity constants $c_1^{V_i}$ and $c_1^{D_i}$ of the single layer potential V_i and of the hypersingular boundary integral operator D_i, respectively.

Using (21) we find the boundedness estimate [52]

$$\|S_i g_i\|_{V_i} \leq c_K(\Gamma_i) \, \|g_i\|_{V_i^{-1}} \quad \text{for all } g_i \in H^{1/2}(\Gamma_i) \tag{22}$$

as well as the ellipticity estimate

$$\langle S_i g_i, g_i \rangle_{\Gamma_i} \geq [1 - c_K(\Gamma_i)] \, \|g_i\|^2_{V_i^{-1}} \quad \text{for all } g_i \in H_0^{1/2}(\Gamma_i). \tag{23}$$

Since all representations of the local Steklov–Poincaré operators S_i coincide, the boundedness estimate (22) and the ellipticity estimate (23) are also true for the definition (11) based on a domain variational formulation. Note that the contraction rate $c_K(\Gamma_i)$ only reflects the shape of the subdomain, but does not reflect the size or the diameter of the subdomain Ω_i.

Boundary Element Methods

For $g_i \in H^{1/2}(\Gamma_i)$, the application of the Steklov–Poincaré operator S_i in its symmetric representation (20) can be rewritten as

$$S_i g_i \;=\; D_i g_i + (\frac{1}{2} I + K_i')V_i^{-1}(\frac{1}{2} I + K_i) g_i \;=\; D_i g_i + (\frac{1}{2} I + K_i')w_i,$$

where $w_i = V_i^{-1}(\frac{1}{2} I + K_i) g_i \in H^{-1/2}(\Gamma_i)$ is the unique solution of the local variational problem

$$\langle V_i w_i, \tau_i \rangle_{\Gamma_i} \;=\; \langle (\frac{1}{2} I + K_i) g_i, \tau_i \rangle_{\Gamma_i} \quad \text{for all } \tau_i \in H^{-1/2}(\Gamma_i).$$

Let

$$S_h^0(\Gamma_i) \;=\; \mathrm{span}\{\psi_{i,n}^0\}_{n=1}^{N_i} \subset H^{-1/2}(\Gamma_i)$$

be the boundary element space of piecewise constant basis functions $\psi_{i,n}^0$. Using the Galerkin solution $w_{i,h} \in S_h^0(\Gamma_i)$ satisfying

$$\langle V_i w_{i,h}, \tau_{i,h} \rangle_{\Gamma_i} \;=\; \langle (\frac{1}{2} I + K_i) g_i, \tau_{i,h} \rangle_{\Gamma_i} \quad \text{for all } \tau_{i,h} \in S_h^0(\Gamma_i), \tag{24}$$

we may define an approximate Steklov–Poincaré operator by the relation

$$\widetilde{S}_i g_i \;=\; D_i g_i + (\frac{1}{2} I + K_i')w_{i,h}. \tag{25}$$

Theorem 3. [48] *The approximate Steklov–Poincaré operator* $\widetilde{S}_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ *as defined in* (25) *is bounded and* $H_0^{1/2}(\Gamma_i)$*–elliptic, i.e.,*

$$\langle \widetilde{S}_i g_i, g_i \rangle_{\Gamma_i} \;\geq\; \langle D_i g_i, g_i \rangle_{\Gamma_i} \;\geq\; c_1^{D_i} \|g_i\|^2_{H^{1/2}(\Gamma_i)} \quad \textit{for all } g_i \in H_0^{1/2}(\Gamma_i).$$

Moreover, there holds the a priori error estimate

$$\|(S_i - \widetilde{S}_i) g_i\|_{H^{-1/2}(\Gamma_i)} \;\leq\; c\, h_i^{3/2} \|S_i g_i\|_{H^1_{pw}(\Gamma_i)}$$

when assuming $S_i g_i \in H^1_{\mathrm{pw}}(\Gamma_i)$*, i.e.,* $u_i \in H^{5/2}(\Omega_i)$*. Note that* $H^1_{\mathrm{pw}}(\Gamma_i)$ *is the Sobolev space which is defined piecewise.*

In the same way as above we may also introduce some approximation of the volume potential $N_i f = V_i^{-1} \widetilde{N}_{0,i} f$. In particular, $N_{i,h} f \in S_h^0(\Gamma_i)$ is defined as the unique solution of the Galerkin variational problem

$$\langle V_i N_{i,h} f, \tau_{i,h} \rangle_{\Gamma_i} = \langle \widetilde{N}_{0,i} f, \tau_{i,h} \rangle_{\Gamma_i} \quad \text{for all } \tau_{i,h} \in S_h^0(\Gamma_i).$$

Let

$$g_{i,h} = \sum_{k=1}^{M_i} u_{C,i} \varphi_{i,k}^1 \in S_h^1(\Gamma_i)$$

be some piecewise linear approximation of the given Dirichlet datum g_i. For the approximate Dirichlet to Neumann map we then find

$$\begin{aligned}
&\int_{\Gamma_i} \alpha_i \widetilde{t}_i(x) \varphi_\ell^1(x)(x) ds_x \\
&\quad = \int_{\Gamma_i} \left[\alpha_i (D_i g_{i,h})(x) + \alpha_i (\frac{1}{2} I + K_i') w_{i,h}(x) - N_{i,h} f \right] \varphi_{i,\ell}^1(x) ds_x \\
&\quad = \sum_{k=1}^{M_i} u_{C,i,k} \alpha_i \langle D_i \varphi_{i,k}^1, \varphi_{i,\ell}^1 \rangle_{\Gamma_i} + \sum_{n=1}^{N_i} w_{i,n} \alpha_i \langle (\frac{1}{2} I + K_i') \psi_{i,n}^0, \varphi_{i,\ell}^1 \rangle_{\Gamma_i} \\
&\qquad\qquad - \sum_{n=1}^{N_i} N_{i,h,n} \langle \psi_{i,n}^0, \varphi_{i,\ell}^1 \rangle_{\Gamma_i}
\end{aligned}$$

for $\ell = 1, \ldots, M_i$ where

$$\sum_{n=1}^{N_i} w_{i,n} \langle V_i \psi_{i,n}^0, \psi_{i,m}^0 \rangle_{\Gamma_i} = \sum_{k=1}^{M_i} u_{C,i,k} \langle (\frac{1}{2} I + K_i) \varphi_{i,k}^1, \psi_{i,m}^0 \rangle_{\Gamma_i}$$

and

$$\sum_{n=1}^{N_i} N_{i,h,n} \langle V_i \psi_{i,n}^0, \psi_{i,m}^0 \rangle_{\Gamma_i} = \langle \widetilde{N}_{0,i} f, \psi_{i,m}^0 \rangle_{\Gamma_i} = f_{N,i,m}$$

for $m = 1, \ldots, N_i$. Hence we obtain the discrete Dirichlet to Neumann map

$$\alpha_i \underline{\widetilde{t}}_i = \alpha_i \widetilde{S}_{i,h}^{\mathrm{BEM}} \underline{u}_{C,i} - M_{i,h}^\top V_{i,h}^{-1} \underline{f}_{N,i} \tag{26}$$

with the boundary element approximation of the Steklov–Poincaré operator

$$\widetilde{S}_{i,h}^{\mathrm{BEM}} = D_{i,h} + \widetilde{K}_{i,h}^\top V_{i,h}^{-1} \widetilde{K}_{i,h} \tag{27}$$

and

$$\begin{aligned}
V_{i,h}[m,n] &= \langle V_i \psi_{i,n}^0, \psi_{i,m}^0 \rangle_{\Gamma_i}, \\
D_{i,h}[\ell,k] &= \langle D_i \varphi_{i,k}^1, \varphi_{i,\ell}^1 \rangle_{\Gamma_i}, \\
\widetilde{K}_{i,h}[m,k] &= \langle (\frac{1}{2} I + K_i) \varphi_{i,k}^1, \psi_{i,m}^0 \rangle_{\Gamma_i}, \\
M_{i,h}[m,k] &= \langle \varphi_{i,k}^1, \psi_{i,m}^0 \rangle_{\Gamma_i}
\end{aligned}$$

for $m, n = 1, \ldots, N_i$, $k, \ell = 1, \ldots, M_i$.

Instead of using the symmetric representation (20) we may use also the first representation (18) to define an approximate Steklov–Poincaré operator as

$$\widetilde{S}_i g_i \;=\; w_{i,h} \tag{28}$$

where $w_{i,h} \in S_h^0(\Gamma_i)$ is the unique solution of the variational problem (24). Although this approximated Steklov–Poincaré operator $\widetilde{S}_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ is bounded and satisfies an approximation property as in Theorem 3, it is in general not stable. Let $S_h^1(\Gamma_i) \subset H^{1/2}(\Gamma_i)$ be some boundary element space of piecewise linear basis functions. To ensure the $S_h^1(\Gamma_i)$–semi–ellipticity of the approximate Steklov–Poincaré operator $\widetilde{S}_i$ as defined in (28), we need to assume the discrete stability condition

$$c_S \, \|g_{i,h}\|_{H^{1/2}(\Gamma_i)} \;\leq\; \sup_{0 \neq \tau_{i,h} \in S_h^0(\Gamma_i)} \frac{\langle g_{i,h}, \tau_{i,h} \rangle_{\Gamma_i}}{\|\tau_{i,h}\|_{H^{-1/2}(\Gamma_i)}} \quad \text{for all } g_{i,h} \in S_h^1(\Gamma_i). \tag{29}$$

Note that the discrete stability condition (29) is satisfied, for example, when using a sufficiently small mesh size to define the trial space $S_h^0(\Gamma_i)$ compared to the mesh size or $S_h^1(\Gamma_i)$ [56], or when using piecewise linear basis functions to define both trial spaces [47].

2.2 Neumann Boundary Value Problems

We now consider the local Neumann boundary value problem

$$-\alpha_i \Delta u_i(x) = f_i(x) \quad \text{for } x \in \Omega_i, \quad \alpha_i \frac{\partial}{\partial n_i} u_i(x) = \lambda_i(x) \quad \text{for } x \in \Gamma_i, \tag{30}$$

where we have to assume the solvability condition

$$\int_{\Omega_i} f_i(x) dx + \int_{\Gamma_i} \lambda_i(x) ds_x \;=\; 0\,. \tag{31}$$

Moreover, the solution of the local Neumann boundary value problem (30) is only unique up to an additive constant, i.e., if u_i is a solution of (30), then $u_i + \gamma_i$ is also a solution of (30) for any constant $\gamma_i \in \mathbb{R}$.

Domain Variational Formulation

The associated variational formulation of the Neumann boundary value problem (30) is to find $u_i \in H_*^1(\Omega_i)$ such that

$$\alpha_i \int_{\Omega_i} \nabla u_i(x) \nabla v_i(x) dx \;=\; \int_{\Omega_i} f_i(x) v_i(x) dx + \int_{\Gamma_i} \lambda_i(x) v_i(x) ds_x \tag{32}$$

is satisfied for all $v_i \in H_*^1(\Omega_i)$, where

$$H^1_*(\Omega_i) = \left\{ v_i \in H^1(\Omega_i) \, : \, \int\limits_{\Omega_i} v_i(x)dx = 0 \right\}$$

is a suitable choosen subspace of $H^1(\Omega_i)$. Since

$$\|v_i\|^2_{H^1(\Omega_i),\Omega_i} \, = \, \left[\int\limits_{\Omega_i} v_i(x)dx \right]^2 + \int\limits_{\Omega_i} |\nabla v_i(x)|^2 dx$$

defines an equivalent norm in $H^1(\Omega)$, the operator $A_{\Omega_i} : H^1(\Omega_i) \to \widetilde{H}^{-1}(\Omega_i)$ defined via the Riesz representation theorem, i.e.,

$$\langle A_{\Omega_i} u_i, v_i \rangle_{\Omega_i} \, = \int\limits_{\Omega_i} \nabla u_i(x) \nabla v_i(x) dx \quad \text{for all } u_i, v_i \in H^1(\Omega),$$

is $H^1_*(\Omega_i)$–elliptic. Hence there exists a unique solution $u_i \in H^1_*(\Omega_i)$ of the variational problem (32). Using the trace operator $B_i : H^1(\Omega_i) \to H^{1/2}(\Gamma_i)$ and its adjoint, $B_i' : H^{-1/2}(\Gamma_i) \to \widetilde{H}^{-1}(\Omega)$, we can write the general solution of the Neumann boundary value problem (30) as

$$u_i \, = \, A^+_{\Omega_i}[f_i + B_i' \lambda_i] + \gamma_i, \quad \gamma_i \in \mathbb{R}$$

where $A^+_{\Omega_i}$ is the associated pseudoinverse. From this we obtain the Neumann to Dirichlet map as

$$g_i \, = \, B_i A^+_{\Omega_i}[B_i' \lambda_i + f_i] + \gamma_i, \quad \gamma_i \in \mathbb{R}, \quad \langle f_i, 1 \rangle_{\Omega_i} + \langle \lambda_i, 1 \rangle_{\Gamma_i} = 0. \tag{33}$$

Instead of the variational formulation (32), where the side condition $\langle v_i, 1 \rangle_{\Omega_i} = 0$ was included in the definition of the function space $H^1_*(\Omega)$, we now consider an extended variational problem to find $u_i \in H^1(\Omega_i)$ satisfying

$$\begin{aligned} \alpha_i \int\limits_{\Omega_i} u_i(x)dx \int\limits_{\Omega_i} v_i(x)dx + \alpha_i \int\limits_{\Omega_i} \nabla u_i(x) \nabla v_i(x) dx \\ = \int\limits_{\Omega_i} f_i(x) v_i(x) dx + \int\limits_{\Gamma_i} \lambda_i(x) v_i(x) ds_x \end{aligned} \tag{34}$$

for all $v_i \in H^1(\Omega_i)$. Since the operator $\bar{A}_{\Omega_i} : H^1(\Omega_i) \to \widetilde{H}^{-1}(\Omega)$ defined by

$$\langle \bar{A}_{\Omega_i} u_i, v_i \rangle_{\Omega_i} \, = \int\limits_{\Omega_i} u_i(x)dx \int\limits_{\Omega_i} v_i(x)dx + \int\limits_{\Omega_i} \nabla u_i(x) \nabla v_i(x) dx$$

for all $u_i, v_i \in H^1(\Omega_i)$ is $H^1(\Omega_i)$–elliptic, we find

$$u_i \, = \, \frac{1}{\alpha_i} \bar{A}^{-1}_{\Omega_i}[f_i + B_i' \lambda_i]$$

as the unique solution of the variational problem (34) for any given data $f_i \in \widetilde{H}^{-1}(\Omega_i)$ and $\lambda_i \in H^{-1/2}(\Gamma_i)$. Moreover, when assuming the solvability condition (31), we obtain $u_i \in H^1_*(\Omega)$, and therefore, the general Neumann to Dirichlet map

$$g_i = \frac{1}{\alpha_i} B_i \bar{A}^{-1}_{\Omega_i} [B_i' \lambda_i + f_i] + \gamma_i, \quad \gamma_i \in \mathbb{R}, \quad \langle f_i, 1\rangle_{\Omega_i} + \langle \lambda_i, 1\rangle_{\Gamma_i} = 0. \tag{35}$$

The involved Poincaré–Steklov operator

$$T_i = B_i \bar{A}^{-1}_{\Omega_i} B_i' : H^{-1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i)$$

is bounded and $H^{-1/2}(\Gamma_i)$–elliptic.

Finite Element Approximation

Let

$$S^1_h(\Omega_i) = \operatorname{span}\{\phi^1_{i,k}\}_{k=1}^{\widetilde{M}_i} \subset H^1(\Omega_i)$$

be the local finite element space of piecewise linear basis functions $\phi^1_{i,k}$ which are again defined with respect to some regular finite element mesh $\Omega_{i,h}$ with the mesh–size parameter h_i. In addition, let

$$\lambda_{i,h} \in S^0_h(\Gamma_i) = \operatorname{span}\{\psi^0_{i,n}\}_{n=1}^{N_i}$$

be some approximation of the given Neumann data by using piecewise constant basis functions $\psi^0_{i,n}$. The Galerkin formulation of the extended variational problem (34) is to find $u_{i,h} \in S^1_h(\Omega_i)$ such that

$$\begin{aligned} \alpha_i \int_{\Omega_i} u_{i,h}(x)dx \int_{\Omega_i} v_{i,h}(x)dx + \alpha_i \int_{\Omega_i} \nabla u_{i,h}(x) \nabla v_{i,h}(x)dx \\ = \int_{\Omega_i} f_i(x) v_{i,h}(x)dx + \int_{\Gamma_i} \lambda_{i,h}(x) v_{i,h}(x) ds_x \end{aligned} \tag{36}$$

is satisfied for all $v_{i,h} \in S^1_h(\Omega_i)$. This is equivalent to a linear system of algebraic equations,

$$\alpha_i \bar{A}_{\Omega_i,h} \underline{u}_i = \underline{f}_i + B^\top_{i,h} \underline{\lambda}_i,$$

with

$$\begin{aligned} \bar{A}_{\Omega_i,h}[\ell,k] &= \int_{\Omega_i} \phi^1_{i,k}(x)dx \int_{\Omega_i} \phi^1_{i,\ell}(x)dx + \int_{\Omega_i} \nabla \phi^1_{i,k}(x) \nabla \phi^1_{i,\ell}(x)dx, \\ f_{i,\ell} &= \int_{\Omega_i} f_i(x) \phi^1_{i,\ell}(x)dx, \\ B_{i,h}[n,k] &= \int_{\Gamma_i} \phi^1_{i,k}(x) \psi^0_{i,n}(x) ds_x \end{aligned}$$

for $k, \ell = 1, \ldots, \widetilde{M}_i$, $n = 1, \ldots, N_i$. Hence we find

$$\underline{u}_i = \bar{A}^{-1}_{\Omega_i,h}[\underline{f}_i + B^\top_{i,h}\underline{\lambda}_i]$$

yielding the approximate solution $u_{i,h} \in S^1_h(\Omega_i)$. Taking the trace $u_{i,h|\Gamma_i}$, this defines an approximation of the Neumann to Dirichlet map (35), i.e., an approximate Poincaré–Steklov operator $\widetilde{T}_i$.

Theorem 4. *The approximate Poincaré–Steklov operator $\widetilde{T}_i : H^{-1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i)$ as introduced above is bounded and $H^{-1/2}(\Gamma_i)$–elliptic. Moreover, there holds the a priori error estimate*

$$\|(T_i - \widetilde{T}_i)\lambda_i\|_{H^{1/2}(\Gamma_i)} \leq c\, h_i\, \|u_i\|_{H^2(\Omega_i)}$$

when assuming $u_i \in H^2(\Omega_i)$.

Boundary Integral Equations

Using the hypersingular boundary integral equation (19) the unknown Dirichlet datum $g_i \in H^{1/2}(\Gamma_i)$ is a solution of

$$\alpha_i(D_i g_i)(x) = \frac{1}{2}\lambda_i(x) - (K'_i\lambda_i)(x) + (\widetilde{N}_{i,1}f_i)(x) \quad \text{for } x \in \Gamma_i.$$

The local hypersingular boundary integral operator $D_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ is only $H^{1/2}(\Gamma_i)$–semi–elliptic, i.e.

$$\langle D_i v_i, v_i\rangle_{\Gamma_i} \geq c_1^{D_i}\, \|v_i\|^2_{H^{1/2}(\Gamma_i)} \quad \text{for all } v_i \in H^{1/2}_0(\Gamma_i).$$

As for the extended variational formulation (34) we may consider an extended variational problem [39] to find $g_i \in H^{1/2}(\Gamma_i)$ such that

$$\alpha_i\,[\langle D_i u_i, v_i\rangle_{\Gamma_i} + \langle u_i, 1\rangle_{\Gamma_i}\langle v_i, 1\rangle_{\Gamma_i}] = \langle(\frac{1}{2}I - K'_i)\lambda_i, v_i\rangle_{\Gamma_i} + \langle\widetilde{N}_{i,1}f_i, v_i\rangle_{\Gamma_i} \quad (37)$$

is satisfied for all $v_i \in H^{1/2}(\Gamma_i)$. Since the modified hypersingular boundary integral operator $\widetilde{D}_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ which is defined via the bilinear form

$$\langle\widetilde{D}_i u_i, v_i\rangle_{\Gamma_i} = \langle D_i u_i, v_i\rangle_{\Gamma_i} + \langle u_i, 1\rangle_{\Gamma_i}\langle v_i, 1\rangle_{\Gamma_i},$$

is $H^{1/2}(\Gamma_i)$–elliptic, the extended variational problem (37) has a unique solution $u_i \in H^{1/2}(\Gamma_i)$ for any given data $f_i \in \widetilde{H}^{-1}(\Omega_i)$ and $\lambda_i \in H^{-1/2}(\Gamma_i)$. If the solvability condition (31) is satisfied, then we have $u_i \in H^{1/2}_0(\Gamma_i)$ and the general solution of the local Neumann boundary value problem is given by

$$u_i = \frac{1}{\alpha_i}\widetilde{D}_i^{-1}(\frac{1}{2}I - K'_i)\lambda_i + \frac{1}{\alpha_i}\widetilde{D}_i^{-1}\widetilde{N}_{i,1}f_i + \gamma_i, \quad \gamma_i \in \mathbb{R}.$$

Inserting this into the first boundary integral equation (17), we obtain

$$\begin{aligned}\alpha_i u_i &= V_i\lambda_i + \alpha_i(\frac{1}{2}I - K_i)u_i + \widetilde{N}_{i,0}f_i \\ &= V_i\lambda_i + \alpha_i(\frac{1}{2}I - K_i)\left[\frac{1}{\alpha_i}\widetilde{D}_i^{-1}(\frac{1}{2}I - K_i')\lambda_i + \frac{1}{\alpha_i}\widetilde{D}_i^{-1}\widetilde{N}_{i,1}f_i\right] + \widetilde{N}_{i,0}f_i \\ &= \left[V_i + (\frac{1}{2}I - K_i)\widetilde{D}_i^{-1}(\frac{1}{2}I - K_i')\right]\lambda_i + (\frac{1}{2}I - K_i)\widetilde{D}_i^{-1}\widetilde{N}_{i,1}f_i + \widetilde{N}_{i,0}f_i\end{aligned}$$

and therefore the Neumann to Dirichlet map

$$u_i(x) \;=\; \frac{1}{\alpha_i}\,(T_i\lambda_i)(x) + \frac{1}{\alpha_i}(\overline{N}_i f_i)(x) + \gamma_i \quad \text{for } x \in \Gamma_i, \gamma_i \in \mathbb{R},$$

where

$$T_i \;=\; V_i + (\frac{1}{2}I - K_i)\widetilde{D}_i^{-1}(\frac{1}{2}I - K_i') \;:\; H^{-1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i)$$

is again the Poincaré–Steklov operator.

Boundary Element Methods

For $\lambda_i \in H^{-1/2}(\Gamma_i)$, the application of the Poincaré–Steklov operator T_i in its symmetric representation reads as

$$T_i\lambda_i \;=\; V_i\lambda_i + (\frac{1}{2}I - K_i)\widetilde{D}_i^{-1}(\frac{1}{2}I - K_i')\lambda_i \;=\; V_i\lambda_i + (\frac{1}{2}I - K_i)z_i,$$

where $z_i = \widetilde{D}_i^{-1}(\frac{1}{2}I - K_i')\lambda_i \in H^{1/2}(\Gamma_i)$ is the unique solution of the local variational problem

$$\langle\widetilde{D}_i z_i, v_i\rangle_{\Gamma_i} \;=\; \langle(\frac{1}{2}I - K_i')\lambda_i, v_i\rangle_{\Gamma_i} \quad \text{for all } v_i \in H^{1/2}(\Gamma_i).$$

Let

$$S_h^1(\Gamma_i) = \operatorname{span}\{\varphi_{i,k}^1\}_{k=1}^{M_i} \subset H^{1/2}(\Gamma_i)$$

be some boundary element space of piecewise linear basis functions $\varphi_{i,k}^1$. Using the Galerkin solution $z_{i,h} \in S_h^1(\Gamma_i)$ satisfying

$$\langle\widetilde{D}_i z_{i,h}, v_{i,h}\rangle_{\Gamma_i} \;=\; \langle(\frac{1}{2}I - K_i')\lambda_i, v_{i,h}\rangle_{\Gamma_i} \quad \text{for all } v_{i,h} \in S_h^1(\Gamma_i),$$

we may define an approximate Poincaré–Steklov operator as

$$\widetilde{T}_i\lambda_i \;=\; V_i\lambda_i + (\frac{1}{2}I - K_i)z_{i,h}. \tag{38}$$

Theorem 5. *The approximate Poincaré–Steklov operator $\widetilde{T}_i : H^{-1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i)$ as defined in* (38) *is bounded and $H^{-1/2}(\Gamma_i)$–elliptic. Moreover, there holds the a priori error estimate*

$$\|(T_i - \widetilde{T}_i)\lambda_i\|_{H^{1/2}(\Gamma_i)} \;\le\; c\,h_i^{3/2}\,\|T_i\lambda_i\|_{H^2(\Gamma_i)}$$

when assuming $T_i\lambda_i \in H^2(\Gamma_i)$, i.e. $u_i \in H^{5/2}(\Omega_i)$.

3 Domain Decomposition Methods

Using the local Dirichlet to Neuman map

$$\alpha_i t_i(x) \;=\; \alpha_i (S_i u_i)(x) - (N_i f_i)(x) \quad \text{for } x \in \Gamma_i$$

with the Steklov–Poincaré operator S_i as defined in (11), (18) or in (20), we can reformulate the coupled domain decomposition formulation (2) and (3) as

$$\begin{aligned} \alpha_i t_i(x) = \alpha_i (S_i u_i)(x) - (N_i f)(x) \quad & \text{for } x \in \Gamma_i, \\ u_i(x) = g(x) \quad & \text{for } x \in \Gamma_i \cap \Gamma, \\ u_i(x) = u_j(x) \quad & \text{for } x \in \Gamma_{ij}, \\ \alpha_i t_i(x) + \alpha_j t_j(x) = 0 \quad & \text{for } x \in \Gamma_{ij}. \end{aligned} \tag{39}$$

3.1 Dirichlet Domain Decomposition Methods

Eliminating the local Neumann data t_i in (39) gives the transmission conditions

$$u_i(x) = u_j(x), \quad \alpha_i (S_i u_i)(x) + \alpha_j (S_j u_j)(x) = (N_i f_i)(x) + (N_j f_j)(x)$$

for $x \in \Gamma_{ij}$. Let $H^{1/2}(\Gamma_S)$ be the skeleton trace space of $H^1(\Omega)$. To ensure the Dirichlet transmission condition $u_i(x) = u_j(x)$ for $x \in \Gamma_{ij}$ we may define $u_i(x) = u(x)$, $x \in \Gamma_i$, as the restriction of a globally defined funtion $u \in H^{1/2}(\Gamma_S)$ with $u(x) = g(x)$ for $x \in \Gamma$. Hence we have to find $u \in H^{1/2}(\Gamma_S)$, $u(x) = g(x)$ for $x \in \Gamma$, such that

$$\alpha_i (S_i u_{|\Gamma_i})(x) + \alpha_j (S_j u_{|\Gamma_j})(x) \;=\; (N_i f_i)(x) + (N_j f_j)(x) \quad \text{for } x \in \Gamma_{ij}.$$

The associated variational problem is to find $u \in H^{1/2}(\Gamma_S)$ such that $u = g$ on Γ and

$$\sum_{i=1}^{p} \int_{\Gamma_i} \alpha_i (S_i u_{|\Gamma_i})(x) v_{|\Gamma_i}(x) ds_x = \sum_{i=1}^{p} \int_{\Gamma_i} (N_i f_i)(x) v_{|\Gamma_i}(x) ds_x \tag{40}$$

is satisfied for all $v \in H^{1/2}(\Gamma_S)$ vanishing on Γ.

Let

$$S_h^1(\Gamma_S) \;=\; \mathrm{span}\{\varphi_k^1\}_{k=1}^M \subset H^{1/2}(\Gamma_S)$$

be some global finite element space of piecewise linear basis functions φ_k^1 which are defined with respect to some regular finite element mesh $\Gamma_{S,h}$ of the skeleton Γ_S. By $S_h^1(\Gamma_i)$ we denote the restriction of $S_h^1(\Gamma_S)$ onto the local subdomain boundary Γ_i. In particular, for any $v_h \in S_h^1(\Gamma_S)$ we find the local restriction $v_{i,h} \in S_h^1(\Gamma_i)$ via a transformation of the associated coefficients, $\underline{v}_i = A_i \underline{v}$, where $A_i : \mathbb{R}^M \to \mathbb{R}^{M_i}$ is the connectivity matrix. Moreover, let

$S_h^1(\Gamma)$ be the restriction of $S_h^1(\Gamma_S)$ onto the Dirichlet boundary $\Gamma = \partial\Omega$, where the associated connectivity matrix is $A_0 \in \mathbb{R}^{M_0 \times M}$. Let $\underline{g} \in \mathbb{R}^{M_0}$ result from some piecewise linear approximation $g_h \in S_h^1(\Gamma)$ of the given Dirichlet datum g.

Using one of the previous introduced approximate Dirichlet to Neumann maps, the Galerkin variational formulation of (40) is to find $u_h \in S_h^1(\Gamma_S)$ satisfying the Dirichlet boundary condition $u_h(x_k) = g(x_k)$ for $x_k \in \Gamma$ such that

$$\sum_{i=1}^{p} \int_{\Gamma_i} \alpha_i (\widetilde{S}_i u_{h|\Gamma_i})(x) v_{h|\Gamma_i}(x) ds_x = \sum_{i=1}^{p} \int_{\Gamma_i} (\widetilde{N}_i f_i)(x) v_{h|\Gamma_i}(x) ds_x \tag{41}$$

is satisfied for all $v_h \in S_h^1(\Gamma_S)$ with $v_h(x) = 0$, $x \in \Gamma$. This is equivalent to a linear system of algebraic equations to find $\underline{u} \in \mathbb{R}^M$ such that

$$\sum_{i=1}^{p} \alpha_i A_i^\top \widetilde{S}_{i,h} A_i \underline{u} = \sum_{i=1}^{p} A_i^\top \underline{f}_i, \quad A_0 \underline{u} = \underline{g}. \tag{42}$$

In (42), the approximate stiffness matrices $\widetilde{S}_{i,h}$ and the local vectors $\underline{f}_i$ of the right hand side correspond to the discretization of the locally defined approximate Steklov–Poincaré operators $\widetilde{S}_i$. In particular, when using the finite element approximation (15) this gives

$$\widetilde{S}_{i,h}^{\mathrm{FEM}} = K_{CC,i} - K_{IC,i} K_{II,i}^{-1} K_{CI,i}, \quad \underline{f}_i^{\mathrm{FEM}} = \underline{f}_{C,i} - K_{CI,i} K_{II,i}^{-1} \underline{f}_{I,i}.$$

When using the symmetric boundary element approximation (27) of the Steklov–Poincaré operator this gives

$$\widetilde{S}_{i,h}^{\mathrm{BEM}} = D_{i,h} + \widetilde{K}_{i,h}^\top V_{i,h}^{-1} \widetilde{K}_{i,h} \quad \underline{f}_i^{\mathrm{BEM}} = M_{i,h}^\top V_{i,h}^{-1} \underline{f}_{N,i}.$$

When using a boundary element approximation in the first $q \leq p$ subdomains Ω_i, and a finite element approximation in the remaining subdomains, the linear system (42) can be written as

$$\sum_{i=1}^{q} \alpha_i A_i^\top \widetilde{S}_{i,h}^{\mathrm{BEM}} A_i \underline{u} + \sum_{i=q+1}^{p} \alpha_i A_i^\top \widetilde{S}_{i,h}^{\mathrm{FEM}} A_i \underline{u} = \sum_{i=1}^{q} A_i^\top \underline{f}_i^{\mathrm{BEM}} + \sum_{i=q+1}^{p} A_i^\top \underline{f}_i^{\mathrm{FEM}} \tag{43}$$

together with the side condition $A_0 \underline{u} = \underline{g}$.

The solution $\underline{u} \in \mathbb{R}^M$ of the assembled linear system (43) is also characterized as the unique solution of the constrained minimization problem

$$F(\underline{u}) = \min_{\underline{v} \in \mathbb{R}^M, A_0 \underline{v} = \underline{g}} F(\underline{v}), \tag{44}$$

where

$$F(v \qquad \underline{\ }) \;=\; \sum_{i=1}^{p} \left\{ \frac{\alpha_i}{2} (\widetilde{S}_{i,h}^{\mathrm{BEM/FEM}} A_i \underline{v}, A_i \underline{v}) - (\underline{f}_i^{\mathrm{BEM/FEM}}, A_i \underline{v}) \right\}.$$

By introducing the local vectors $\underline{v}_i = A_i \underline{v} \in \mathbb{R}^{M_i}$ we have to minimize

$$\widetilde{F}(\underline{v}_1, \ldots, \underline{v}_p) \;=\; \sum_{i=1}^{p} \left\{ \frac{\alpha_i}{2} (\widetilde{S}_{i,h}^{\mathrm{BEM/FEM}} \underline{v}_i, \underline{v}_i) - (\underline{f}_i^{\mathrm{BEM/FEM}}, \underline{v}_i) \right\}$$

where we have to add the constraints $A_0 \underline{v} = \underline{g}$ due to the Dirichlet boundary condition and $\underline{v}_i = A_i \underline{v}$ which ensures the global continuity $v_{i,i_k} = v_{j,j_k}$ along Γ_{ij}. Here, v_{i,i_k} is the local degree of freedom which belongs to a global node $x_k \in \Gamma_S$, i.e. $A_i[i_k, k] = 1$. Now we can formulate all above constraints as

$$\sum_{i=1}^{p} B_i \underline{v}_i = A_0^\top \underline{g} = \widetilde{\underline{g}} \in \mathbb{R}^{\overline{M}}$$

where the nonzero elements of the matrices $B_i \in \mathbb{R}^{\overline{M} \times M_i}$ are defined as follows:

- $x_k \in \Gamma_i \cap \Gamma$ is on the Dirichlet boundary:

$$B_i[k, i_k] = 1;$$

- $x_k \in \Gamma_{ij} = \Gamma_i \cap \Gamma_j$ is on the interface:

$$B_i[\ell_k, i_k] = 1, \quad B_j[\ell_k, j_k] = -1, \quad i < j.$$

Note that the above constraints are defined in a redundant manner, i.e. ℓ_k corresponds to the multiplicity of constraints which are associated to the node x_k. Now, instead of the minimization problem (44), we have to solve a modified constrained minimization problem, i.e.,

$$\widetilde{F}(\underline{u}_1, \ldots, \underline{u}_p) = \inf_{\sum_{i=1}^{p} B_i \underline{v}_i = \widetilde{\underline{g}}} \widetilde{F}(\underline{v}_1, \ldots, \underline{v}_p). \tag{45}$$

By introducing the Lagrange multiplier $\underline{\lambda} \in \mathbb{R}^{\overline{M}}$, we have to minimize the extended functional

$$\widetilde{F}_\lambda(\underline{v}_1, \ldots, \underline{v}_p) \;=\; \widetilde{F}(\underline{v}_1, \ldots, \underline{v}_p) - (\underline{\lambda}, \sum_{i=1}^{p} B_i \underline{v}_i - \widetilde{\underline{g}}).$$

The necessary conditions give the equations

$$\alpha_i \widetilde{S}_{i,h}^{\mathrm{BEM/FEM}} \underline{u}_i - \underline{f}_i^{\mathrm{BEM/FEM}} - B_i^\top \underline{\lambda} = \underline{0} \tag{46}$$

by taking the derivative with respect to $\underline{v}_i$ for $i = 1, \ldots, p$, and,

$$\sum_{i=1}^{p} B_i \underline{v}_i = \widetilde{\underline{g}}$$

by taking the derivative with respect to $\underline{\lambda}$. Hence, we have to solve the linear system

$$\begin{pmatrix} S_{\mathrm{BEM}} & & -B_{\mathrm{BEM}}^{\top} \\ & S_{\mathrm{FEM}} & -B_{\mathrm{FEM}}^{\top} \\ B_{\mathrm{BEM}} & B_{\mathrm{FEM}} & \end{pmatrix} \begin{pmatrix} \underline{u}_{\mathrm{BEM}} \\ \underline{u}_{\mathrm{FEM}} \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f}_{\mathrm{BEM}} \\ \underline{f}_{\mathrm{FEM}} \\ \widetilde{\underline{g}} \end{pmatrix}, \tag{47}$$

where

$$S_{\mathrm{BEM}} = \mathrm{diag}\left(\alpha_i [D_{i,h} + \widetilde{K}_{i,h}^{\top} V_{i,h}^{-1} \widetilde{K}_{i,h}]\right)_{i=1,\ldots,q},$$
$$S_{\mathrm{FEM}} = \mathrm{diag}\left(\alpha_i [K_{CC,i} - K_{IC,i} K_{II,i}^{-1} K_{CI,i}]\right)_{i=q+1,\ldots,p}.$$

In what follows we proceed as for the solution of a local Neumann boundary value problem. Due to

$$S_{i,h}^{\mathrm{BEM}} \underline{1}_i = S_{i,h}^{\mathrm{FEM}} \underline{1}_i = \underline{0},$$

we can write the local variables $\underline{u}_i \in \mathbb{R}^{M_i}$ as

$$\underline{u}_i = \underline{u}_{i,0} + \gamma_i \underline{1}_i, \quad (\underline{u}_{i,0}, \underline{1}_i) = 0 \tag{48}$$

and therefore we have to solve

$$\alpha_i S_{i,h}^{\mathrm{BEM/FEM}} \underline{u}_{i,0} - B_i^{\top} \underline{\lambda} = \underline{f}_i^{\mathrm{BEM/FEM}} \quad \text{for } i = 1, \ldots, p$$

as well as

$$\sum_{i=1}^{p} B_i \underline{u}_{i,0} + \sum_{i=1}^{p} \gamma_i B_i \underline{1}_i = \widetilde{\underline{g}}_i.$$

On the other hand, for $i = 1, \ldots, p$, we find

$$(B_{i,h}^{\top} \underline{\lambda} + \underline{f}_i^{\mathrm{BEM/FEM}}, \underline{1}_i) = \alpha_i (S_{i,h}^{\mathrm{BEM/FEM}} \underline{u}_i, \underline{1}_i) = \alpha_i (\underline{u}_i, S_{i,h}^{\mathrm{BEM/FEM}} \underline{1}_i) = 0$$

and, therefore, the additional constraints

$$(\underline{\lambda}, B_{i,h} \underline{1}_i) = -(\underline{f}_i^{\mathrm{BEM/FEM}}, \underline{1}_i) \quad \text{for } i = 1, \ldots, p.$$

Hence, we obtain $\underline{u}_{i,0} \in \mathbb{R}^{M_i}$ as the unique solution of

$$\alpha_i [S_{i,h}^{\mathrm{BEM/FEM}} + \underline{1}_i \underline{1}_i^{\top}] \underline{u}_{i,0} - B_i^{\top} \underline{\lambda} = \underline{f}_i^{\mathrm{BEM/FEM}}$$

for $i = 1, \ldots, p$. Now, instead of (47), we may solve the extended system

$$\begin{pmatrix} \overline{S}_{\mathrm{BEM}} & & -B_{\mathrm{BEM}}^{\top} & \\ & \overline{S}_{\mathrm{FEM}} & -B_{\mathrm{FEM}}^{\top} & \\ B_{\mathrm{BEM}} & B_{\mathrm{FEM}} & & G \\ & & G^{\top} & \end{pmatrix} \begin{pmatrix} \underline{u}_{\mathrm{BEM},0} \\ \underline{u}_{\mathrm{FEM},0} \\ \underline{\lambda} \\ \underline{\gamma} \end{pmatrix} = \begin{pmatrix} \underline{f}_{\mathrm{BEM}} \\ \underline{f}_{\mathrm{FEM}} \\ \widetilde{\underline{g}} \\ \underline{e} \end{pmatrix}, \tag{49}$$

where

$$\overline{S}_{\rm BEM} = {\rm diag}\left(\alpha_i[D_{i,h} + \widetilde{K}_{i,h}^\top V_{i,h}^{-1}\widetilde{K}_{i,h} + \underline{1}_i\underline{1}_i^\top]\right)_{i=1,\dots,q},$$
$$\overline{S}_{\rm FEM} = {\rm diag}\left(\alpha_i[K_{CC,i} - K_{IC,i}K_{II,i}^{-1}K_{CI,i} + \underline{1}_i\underline{1}_i^\top]\right)_{i=q+1,\dots,p}$$

and

$$G = (B_1\underline{1}_1, \dots, B_p\underline{1}_p), \quad e_i = -(\underline{f}_i^{\rm BEM/FEM}, \underline{1}_i) \quad \text{for } i = 1, \dots, p.$$

The local boundary element equations in (49) can be written as

$$\alpha_i[D_{i,h} + \underline{1}_i\underline{1}_i^\top]\underline{u}_{i,0} + \alpha_i\widetilde{K}_{i,h}^\top\underline{w}_i - B_i^\top\underline{\lambda} = \underline{f}_i^{\rm BEM},$$
$$\alpha_i V_{i,h}\underline{w}_i - \alpha_i\widetilde{K}_{i,h}\underline{u}_{i,0} = \underline{0},$$

while the finite element equations are equivalent to

$$\alpha_i[K_{CC,i} + \underline{1}_i\underline{1}_i^\top]\underline{u}_{i,0} + \alpha_i K_{IC,i}\underline{u}_{I,i} - B_i^\top\underline{\lambda} = \underline{f}_{C,i},$$
$$\alpha_i K_{II,i}\underline{u}_{I,i} + \alpha_i K_{CI,i}\underline{u}_{i,0} = \underline{f}_{I,i}.$$

Hence, we have to solve the linear system

$$\begin{pmatrix} V_h & -\widetilde{K}_h & & & & \\ \widetilde{K}_h^\top & \widetilde{D}_h & & & -B_{\rm BEM}^\top & \\ & & K_{II} & K_{CI} & & \\ & & K_{CI} & \widetilde{K}_{CC} & -B_{\rm FEM}^\top & \\ & B_{\rm BEM} & & B_{\rm FEM} & & G \\ & & & & G^\top & \end{pmatrix} \begin{pmatrix} \underline{w} \\ \underline{u}_{\rm BEM,0} \\ \underline{u}_I \\ \underline{u}_{\rm FEM,0} \\ \underline{\lambda} \\ \underline{\gamma} \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \underline{f}_{\rm BEM} \\ \underline{f}_I \\ \underline{f}_C \\ \widetilde{\underline{g}} \\ \underline{e} \end{pmatrix}, \qquad (50)$$

where

$$V_h = {\rm diag}\,(\alpha_i V_{i,h})_{i=1}^q, \qquad \widetilde{D}_h = {\rm diag}\left(\alpha_i[D_{i,h} + \underline{1}_i\underline{1}_i^\top]\right)_{i=1}^q,$$
$$K_{II} = {\rm diag}\,(\alpha_i K_{II,i})_{i=q+1}^p, \qquad \widetilde{K}_{CC} = {\rm diag}\left(\alpha_i[K_{CC,i} + \underline{1}_i\underline{1}_i^\top]\right)_{i=q+1}^p,$$
$$\widetilde{K}_h = \begin{pmatrix} \alpha_1\widetilde{K}_{1,h} \\ \vdots \\ \alpha_q\widetilde{K}_{q,h} \end{pmatrix}, \qquad K_{CI} = \begin{pmatrix} \alpha_{q+1}\widetilde{K}_{CI,1} \\ \vdots \\ \alpha_p\widetilde{K}_{CI,p} \end{pmatrix}.$$

Next we use a subspace projection in order to separate the determination of $\underline{\gamma}$ from the determination of the rest of the unknowns in (50). Thus, we introduce the orthogonal projection

$$P = I - QG(G^\top QG)^{-1}G^\top \qquad (51)$$

where Q is some suitable diagonal scaling matrix [7, 30]. Since $P^\top G = 0$, the application of $P^\top$ to the fifth equation of (50) gives $P^\top G\underline{\gamma} = 0$ that excludes $\underline{\gamma}$ from the first five equations of (50). Let us represent $\underline{\lambda}$ in the form

$$\underline{\lambda} = T_0\underline{\lambda}_0 + \underline{\lambda}_e, \quad \underline{\lambda}_e = QG(G^\top QG)^{-1}\underline{e}. \tag{52}$$

Hence we have to find $T_0\underline{\lambda}_0 \in \ker G^\top$, i.e. $G^\top T_0\underline{\lambda}_0 = 0$. In particular, the columns of T_0 span a basis of $\ker G^\top = (\operatorname{range} G)^\perp$. Hence we also conclude $T_0\underline{\lambda}_0 = PT_0\underline{\lambda}_0$. Therefore we have to solve the reduced linear system

$$\begin{pmatrix} V_h & -\widetilde{K}_h & & & \\ \widetilde{K}_h^\top & \widetilde{D}_h & & & -B_{\text{BEM}}^\top PT_0 \\ & & K_{II} & K_{CI} & \\ & & K_{CI} & \widetilde{K}_{CC} & -B_{\text{FEM}}^\top PT_0 \\ & T_0^\top P^\top B_{\text{BEM}} & & T_0^\top P^\top B_{\text{FEM}} & \end{pmatrix} \begin{pmatrix} \underline{w} \\ \underline{u}_{\text{BEM},0} \\ \underline{u}_I \\ \underline{u}_{\text{FEM},0} \\ \underline{\lambda}_0 \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \underline{f}_{\text{BEM}} + B_{\text{BEM}}^\top \underline{\lambda}_e \\ \underline{f}_I \\ \underline{f}_C + B_{\text{FEM}}^\top \underline{\lambda}_e \\ T_0^\top P^\top \underline{\widetilde{g}} \end{pmatrix}. \tag{53}$$

Once the vectors $\underline{w}$, $\underline{u}_{\text{BEM},0}$, $\underline{u}_I$, $\underline{u}_{\text{FEM},0}$ and $\underline{\lambda}_0$ are defined from (53), we get $\underline{\lambda}$ from (52), $\underline{\gamma}$ from the fifth equation in (50), i.e.,

$$\underline{\gamma} = (G^\top QG)^{-1} G^\top Q \left[\underline{\widetilde{g}} - B_{\text{BEM}}\underline{u}_{\text{BEM},0} - B_{\text{FEM}}\underline{u}_{\text{FEM},0}\right],$$

and, finally, $\underline{u}$ from (48).

3.2 Neumann Domain Decomposition Methods

Instead of eliminating the Neumann data in (39) we are now going to eliminate the Dirichlet data. For this, we introduce global Neumann data as follows: For any interface $\Gamma_{ij} = \Gamma_i \cap \Gamma_i$, we introduce $t_{ij} \in \widetilde{H}^{-1/2}(\Gamma_{ij})$ and set

$$t_i(x) = \frac{1}{\alpha_i} t_{ij}(x), t_j(x) = -\frac{1}{\alpha_j} t_{ij}(x) \quad \text{for } x \in \Gamma_{ij}, i < j,$$

and for the Dirichlet boundary we introduce $t_0 \in H^{-1/2}(\Gamma)$ and set $t_i = t_{0|\Gamma_i}/\alpha_i$. Hence, we have satisfied the Neumann transmission condition in (39) in a strong sense. Therefore, we have to impose the Dirichlet transmission conditions and the Dirichlet boundary conditions in some weak sense, i.e.,

$$\int_{\Gamma_{ij}} [u_i(x) - u_j(x)]\tau_{ij}(x) ds_x = 0 \quad \text{for all } \tau_{ij} \in \widetilde{H}^{-1/2}(\Gamma_{ij}),$$

and

$$\int_{\Gamma_i \cap \Gamma} [u_i(x) - g(x)] \tau_{0|\Gamma_i} ds_x = 0 \quad \text{for all } \tau_0 \in H^{-1/2}(\Gamma),$$

where $\widetilde{H}^{-1/2}(\Gamma_{ij}) = (\widetilde{H}^{1/2}(\Gamma_{ij}))'$. For the interfaces Γ_{ij}, we find from the weak formulations of the Dirichlet to Neumann map

$$\langle t_{ij}, v_{i|\Gamma_{ij}} \rangle_{\Gamma_{ij}} = \langle \alpha_i t_i, v_{i|\Gamma_{ij}} \rangle_{\Gamma_{ij}} = \langle \alpha_i S_i u_i - N_i f, v_{i|\Gamma_{ij}} \rangle_{\Gamma_{ij}}, \quad v_i \in H^{1/2}(\Gamma_i),$$

and

$$-\langle t_{ij}, v_{j|\Gamma_{ij}} \rangle_{\Gamma_{ij}} = \langle \alpha_j t_j, v_{j|\Gamma_{ij}} \rangle_{\Gamma_{ij}} = \langle \alpha_j S_j u_j - N_j f, v_{j|\Gamma_{ij}} \rangle_{\Gamma_{ij}}, \ v_j \in H^{1/2}(\Gamma_j).$$

Hence,

$$\begin{aligned} \langle \alpha_i S_i u_i, v_{i|\Gamma_{ij}} \rangle_{\Gamma_{ij}} + \langle \alpha_j S_j u_j, v_{j|\Gamma_{ij}} \rangle_{\Gamma_{ij}} - \langle t_{ij}, v_{i|\Gamma_{ij}} - v_{j|\Gamma_{ij}} \rangle_{\Gamma_{ij}} \\ = \langle N_i f, v_{i|\Gamma_{ij}} \rangle_{\Gamma_{ij}} + \langle N_j f, v_{j|\Gamma_{ij}} \rangle_{\Gamma_{ij}}. \end{aligned}$$

Moreover, on the local Dirichlet boundaries $\Gamma_i \cap \Gamma$, we have

$$\langle \alpha_i S_i u_i, v_{i|\Gamma_i \cap \Gamma} \rangle_{\Gamma_i \cap \Gamma} - \langle t_0, v_{i|\Gamma_i \cap \Gamma} \rangle_{\Gamma_i \cap \Gamma} = \langle N_i f, v_{i|\Gamma_i \cap \Gamma} \rangle_{\Gamma_i \cap \Gamma}.$$

The associated variational formulation is to find $u_i \in H^{1/2}(\Gamma_i)$ for $i = 1, \ldots, p$, $t_{ij} \in \widetilde{H}^{-1/2}(\Gamma_{ij})$ for all $i < j$ and $t_0 \in H^{-1/2}(\Gamma)$ such that

$$\begin{aligned} &\sum_{i=1}^{p} \langle \alpha_i S_i u_i, v_i \rangle_{\Gamma_i} - \sum_{i<j} \langle t_{ij}, v_{i|\Gamma_{ij}} - v_{j|\Gamma_{ij}} \rangle_{\Gamma_{ij}} - \sum_{i=1}^{p} \langle t_0, v_i \rangle_{\Gamma_i \cap \Gamma} = \sum_{i=1}^{p} \langle N_i f, v_i \rangle_{\Gamma_i} \\ &\langle u_{i|\Gamma_{ij}} - u_{j|\Gamma_{ij}}, \tau_{ij} \rangle_{\Gamma_{ij}} = 0 \\ &\langle u_i, \tau_0 \rangle_{\Gamma_i \cap \Gamma} = \langle g, \tau_0 \rangle_{\Gamma_i \cap \Gamma} \end{aligned} \tag{54}$$

is satisfied for all $v_i \in H^{1/2}(\Gamma_i)$, $\tau_{ij} \in \widetilde{H}^{-1/2}(\Gamma_{ij})$, and $\tau_0 \in H^{-1/2}(\Gamma)$.

The saddle point formulation (54) describes a hybrid domain decomposition method [1] which is also known as a mortar domain decomposition method to couple locally different trial spaces [4].

For a Galerkin discretization of (54), we introduce local boundary element spaces

$$S_h^1(\Gamma_i) = \operatorname{span}\{\varphi_{i,k}^1\}_{k=1}^{M_i} \subset H^{1/2}(\Gamma_i)$$

of, e.g., piecewise linear basis functions $\varphi_{i,k}^1$. Moreover, for each coupling boundary Γ_{ij}, we consider a trial space to discretize the local Neumann datum t_{ij},

$$S_h(\Gamma_{ij}) = \operatorname{span}\{\psi_{ij,n}\}_{n=1}^{N_{ij}} \subset \widetilde{H}^{-1/2}(\Gamma_{ij})$$

where $\psi_{ij,n}$ are some basis functions to be defined in an appropriate way. In the same manner we introduce

$$S_h(\Gamma) = \operatorname{span}\{\psi_{0,n}\}_{n=1}^{N_0} \subset H^{-1/2}(\Gamma)$$

to discretize the unknown Neumann datum on Γ. The choice of the basis functions $\psi_{ij,n}$ and $\psi_{0,n}$ is very sensitive, since we have to ensure local inf–sup conditions which are related to the saddle point formulation (54), i.e.,

$$c_S \, \|\tau_{ij,h}\|_{\widetilde{H}^{-1/2}(\Gamma_{ij})} \leq \sup_{(v_{i,h},v_{j,h}) \in S_h^1(\Gamma_i) \times S_h^1(\Gamma_j)} \frac{\langle \tau_{ij}, v_{i|\Gamma_{ij}} - v_{j|\Gamma_{ij}} \rangle_{\Gamma_{ij}}}{\sqrt{\|v_{i,h|\Gamma_{ij}}\|^2_{H^{1/2}(\Gamma_{ij})} + \|v_{j,h|\Gamma_{ij}}\|^2_{H^{1/2}(\Gamma_{ij})}}}.$$

For appropriate choices of the trial spaces $S_h(\Gamma_{ij})$ and $S_h(\Gamma)$ see, for example, [57] and the references given therein.

The Galerkin discretization of the variational problem (54) is equivalent to a set of linear equations which can be written as

$$\begin{aligned}
&\sum_{i=1}^{p} \widetilde{S}_{i,h}^{\mathrm{FEM/BEM}} \underline{u}_i - \sum_{i<j} (M_{ij,h}^\top - M_{ji,h}^\top) \underline{t}_{ij} - \sum_{i=1}^{p} M_{0i,h}^\top \underline{t}_0 = \underline{f}_0, \\
&M_{ij,h} \underline{u}_i - M_{ji,h} \underline{u}_j = \underline{0}, \\
&M_{0i,h} \underline{u}_i = \underline{g}
\end{aligned}$$

with the discrete Steklov–Poincaré operator as defined in (16) for a finite element approximation, and as given in (27) for a boundary element discretization. Moreover,

$$\begin{aligned}
M_{ij,h}[m,k] &= \langle \varphi_{i,k}^1, \psi_{ij,m} \rangle_{\Gamma_{ij}}, \\
M_{ji,h}[m,k] &= \langle \varphi_{j,k}^1, \psi_{ij,m} \rangle_{\Gamma_{ij}}, \\
M_{0i,h}[m,k] &= \langle \varphi_{i,k}^1, \psi_{0,m} \rangle_{\Gamma_{ij}}.
\end{aligned}$$

By reordering all degrees of freedom we then obtain the coupled linear system

$$\begin{pmatrix} S_{\mathrm{BEM}} & & -M_{\mathrm{BEM}}^\top \\ & S_{\mathrm{FEM}} & -M_{\mathrm{FEM}}^\top \\ M_{\mathrm{BEM}} & M_{\mathrm{FEM}} & \end{pmatrix} \begin{pmatrix} \underline{u}_{\mathrm{BEM}} \\ \underline{u}_{\mathrm{FEM}} \\ \underline{t} \end{pmatrix} = \begin{pmatrix} \underline{f}_{\mathrm{BEM}} \\ \underline{f}_{\mathrm{FEM}} \\ \underline{g} \end{pmatrix} \tag{55}$$

which is of the same structure as the linear system (47). In fact, when considering conforming local trial spaces $S_h^1(\Gamma_i)$ and choosing $S_h(\Gamma_{ij})$ and $S_h(\Gamma)$ to be spanned by biorthogonal basis functions $\psi_{ij,n}$ and $\psi_{0,n}$, respectively, both linear systems (47) and (55) will coincide. In general, we may apply all the transformations which are used to reformulate the linear system (47) to solve the linear system (55) in a similar way, we skip the details.

4 Preconditioned Iterative Solution Techniques

In this section we describe some preconditioned CG–like iterative methods for solving the linear system (53),

$$\begin{pmatrix} V_h & -\widetilde{K}_h & & & \\ \widetilde{K}_h^\top & \widetilde{D}_h & & & -B_{\rm BEM}^\top P T_0 \\ & & K_{II} & K_{CI} & \\ & & K_{CI} & \widetilde{K}_{CC} & -B_{\rm FEM}^\top P T_0 \\ & T_0^\top P^\top B_{\rm BEM} & & T_0^\top P^\top B_{\rm FEM} & \end{pmatrix} \begin{pmatrix} \underline{w} \\ \underline{u}_{{\rm BEM},0} \\ \underline{u}_I \\ \underline{u}_{{\rm FEM},0} \\ \underline{\lambda}_0 \end{pmatrix}$$
$$= \begin{pmatrix} \underline{0} \\ \underline{f}_{\rm BEM} + B_{\rm BEM}^\top \underline{\lambda}_e \\ \underline{f}_I \\ \underline{f}_C + B_{\rm FEM}^\top \underline{\lambda}_e \\ T_0^\top P^\top \underline{\widetilde{g}} \end{pmatrix}.$$

Since $V_h = {\rm diag}(\alpha_i V_{i,h})$ and $K_{II} = {\rm diag}(\alpha_i K_{II,i})$ are block diagonal and therefore easily invertible we may first eliminate the vectors $\underline{w}$ and $\underline{u}_I$ to obtain

$$\begin{pmatrix} \overline{S}_{\rm BEM} & & -B_{\rm BEM}^\top P T_0 \\ & \overline{S}_{\rm FEM} & -B_{\rm FEM}^\top P T_0 \\ T_0^\top P^\top B_{\rm BEM} & T_0^\top P^\top B_{\rm FEM} & \end{pmatrix} \begin{pmatrix} \underline{u}_{{\rm BEM},0} \\ \underline{u}_{{\rm FEM},0} \\ \underline{\lambda}_0 \end{pmatrix} \tag{56}$$
$$= \begin{pmatrix} \underline{f}_{\rm BEM} + B_{\rm BEM}^\top \underline{\lambda}_e \\ \underline{f}_C - K_{CI} K_{II}^{-1} \underline{f}_I + B_{\rm FEM}^\top \underline{\lambda}_e \\ T_0^\top P^\top \underline{\widetilde{g}} \end{pmatrix}.$$

Eliminating $\underline{u}_{{\rm BEM},0}$ and $\underline{u}_{{\rm FEM},0}$ we have to solve the Schur complement system of (56),

$$F \underline{\lambda}_0 = T_0^\top P^\top B \overline{S} B^\top P T_0 \underline{\lambda}_0 = \underline{\widetilde{f}}. \tag{57}$$

Since the system matrix in (57) is symmetric and positive definite one may use a preconditioned conjugate gradient scheme to solve (57). For this, an appropriate preconditioner C_F is needed, which is spectrally equivalent to the Schur complement matrix F. Another choice is the application of a Bramble–Pasciak conjugate gradient scheme [5] to the one–fold saddle point problem (56). For this, besides C_F also preconditioners $C_S = {\rm diag}(C_{S_i})$ for the local discrete Steklov–Poincaré operators $\overline{S}_{i,h}^{\rm BEM/FEM}$ are needed. A third possibility is to use a CG–like iterative method to solve the two–fold saddle point problem (53), see [33, 60]. Then, also preconditioners C_{V_i} and C_{K_i} for the local matrices $V_{i,h}$ and $K_{II,i}$ are needed, respectively.

Following [36] we can define the scaled hypersingular BETI preconditioner

$$C_F^{-1} = (B C_\alpha^{-1} B^\top)^{-1} B C_\alpha^{-1} \overline{D}_h C_\alpha^{-1} B^\top (B C_\alpha^{-1} B^\top)^{-1} \tag{58}$$

where C_α is some diagonal scaling. Note that there hold the spectral equivalence inequalities [36, Theorem 3.2]

$$c_1^F\,(C_F\underline{\mu},\underline{\mu}) \;\le\; (F\underline{\mu},\underline{\mu}) \;\le\; c_2^F(1+\log(H/h))^2(C_F\underline{\mu},\underline{\mu})$$

for all $\underline{\mu} \in \ker G^\top$ where the positive constants c_1^F and c_2^F are independent of the local mesh size h, the subdomain diameter H, the number p of subdomains, and of the coefficient jumps. The preconditioner (58) is based on local realizations of the discrete stabilized hypersingular boundary integral operator with respect to all subdomains, independent of whether a finite or boundary element discretization is used locally.

To construct preconditioning matrices C_{S_i} for the local discrete Schur complement matrices $\widetilde{S}_{i,h}^{\mathrm{BEM/FEM}}$ we will apply the concept of boundary integral operators of the opposite order [51]. Based on the local trial space $S_h^1(\Gamma_i)$ of piecewise linear basis functions $\varphi_{i,k}^1$ as used for the Galerkin discretization of the local hypersingular boundary integral operators D_i we define the Galerkin matrices

$$\bar{V}_{i,h}[\ell,k] \;=\; \langle V_i\varphi_{i,k}^1,\varphi_{i,\ell}^1\rangle_{\Gamma_i}, \quad \bar{M}_{i,h}[\ell,k] \;=\; \langle\varphi_{i,k}^1,\varphi_{i,\ell}^1\rangle_{\Gamma_i}$$

for $k,\ell = 1,\ldots,M_i$ and the application of the resulting preconditioning matrix is given by

$$C_{S_i}^{-1} \;=\; \bar{M}_{i,h}^{-1}\bar{V}_{i,h}\bar{M}_{i,h}^{-1} \quad \text{for } i = 1,\ldots,p. \tag{59}$$

Moreover, there hold the spectral equivalence inequalities

$$c_1^{S_i}\,(C_{S_i}\underline{v}_i,\underline{v}_i) \;\le\; (\overline{S}_i^{\mathrm{BEM/FEM}}\underline{v}_i,\underline{v}_i) \;\le\; c_2^{S_i}\,(C_{S_i}\underline{v}_i,\underline{v}_i)$$

for all $\underline{v}_i \in \mathbb{R}^{M_i}$.

For the definition of preconditioners C_{V_i} for the local discrete single layer potentials $V_{i,h}$, there exists a wide variety of different possible choices. Here, we only mention multilevel methods [18, 53] which are based on a given mesh hierarchy or algebraic multilevel techniques [35, 38, 50].

For finite element subdomains one may also use geometric or algebraic multigrid methods to construct preconditioners C_{K_i} for the local finite element stiffness matrices $K_{II,i}$, see, for example, [15] and the references given therein.

5 Conclusions

In this paper we have provided a unique approach to both the Dirichlet and the Neumann domain decomposition techniques. The all–floating tearing and interconnecting technology is a very general and powerful technique. Eliminating more or less variables results in symmetric and positive definite Schur complement problems, one–fold or two–fold saddle point problems which can be solved by preconditioned conjugate gradient methods. We have used boundary element technologies for constructing the required block preconditioners for both the boundary element and the finite element blocks. There are many papers showing the efficiency of FETI methods including the efficiency in

large–scale parallel computations, see, e.g., [16, 29, 44]. Numerical results for BETI and coupled BETI–FETI methods can be found in [33, 34, 38].

The methods and techniques discussed in this paper are not restricted to the potential problem. They can be extended to linear elasticity problems as well [38]. The generalization to three–dimensional electromagnetic problems usually considered in $H(\mathrm{curl})$ is certainly more challenging, see [22] for the symmetric coupling and [54] for FETI–DP methods. Coupled finite and boundary element tearing and interconnecting solvers for nonlinear potential problems were discussed in [34].

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