Some Historical Remarks on the Positivity of Boundary Integral Operators

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Summary. Variational arguments go back a long time in the history of boundary integral equations. Energy methods have shown up very early, then virtually disappeared from the common knowledge and eventually resurfaced in the context of boundary element methods. We focus on some not so well known parts of classical works by well known classical authors and describe the relation of their ideas to modern variational principles in boundary element methods.

1 Introduction

The method of boundary integral equations has always had two important applications in the theory of boundary value problems for partial differential equations: As a theoretical tool for proving the existence of solutions and as a practical tool for the construction of solutions. This is one of the aspects that has remained constant since the times of Green and Gauss in the early 19th century until our times. Other ideas, in particular techniques of the analysis of integral equations, have of course greatly changed and evolved in the meantime, but it is curious to see how some of the very early questions and techniques are related to recent simple basic results about the structure of boundary integral equations.

This article has evolved from some observations made in the talk [6] about the scientific work of Wolfgang Wendland, connecting works by Carl Friedrich Gauss [11] and Carl Neumann [27] to the work by Wendland and his group on variational methods for boundary integral equations. In particular the curious case of "Gauss' missing theorem" on the positivity of the single layer potential operator – a proof of which could have been given by Gauss himself, but was in fact only given 135 years later by Nedelec and Planchard [26] – seemed to be sufficiently intriguing to merit a more detailed presentation. A secondary path concerning second kind boundary integral equations, leading from Neumann's observation of the contraction property of the double layer potential to the recent paper [31] by Steinbach and Wendland where energy methods

were used to prove the contraction property in very general cases, seemed to be less straightforward on the level of analytical tools and mathematical ideas. Following the early twists of this path, however, one comes across the monumental paper [29] by Henri Poincaré which uses, indeed, energy methods for proving the contraction property of Neumann's operator. The historical trail of Poincaré's paper which, after having been instantly famous initially, seems to have disappeared from the common knowledge of the boundary element community, is a second curiosity on which we will try to shed some light here. By taking this look, we will even find some "new" mathematical results.

This paper does not present a serious research into the history of mathematics, which would require much more space, time and knowledge than available to the author. It rather stays within the narrow viewing angle of the history of the analytical foundations of boundary element methods, but it tries to illustrate how a fresh look, however biased, can reveal new details of old monuments. We will consider a domain spanned by the four papers by Gauss [11], Poincaré [29], Nedelec and Planchard [26], and Steinbach and Wendland [31]. If one prefers a hexagonal constellation, one can add Neumann's book [27] and the paper [7] in which the generality of energy methods was emphasized. For a justification of this combination of papers, suffice it to say that in the sky, for giving the perception of a well-balanced constellation, the more distant objects have to be much bigger stars than the objects closer to the observer... Within this constellation, there exists a myriad of other points of light, only some of which will make a short appearance in the following. Other very bright stars in the vicinity of our constellation, from Lebesgue and Fredholm over Hilbert to Calderón–Zygmund and Mazya, will not be considered here.

The papers we are trying to connect belong, in fact, to three quite different galaxies: There is ours, extending over roughly the last 40 years, characterized by the availability of many simple but very powerful tools like the basic theory of Hilbert and Banach spaces, distributions, Fourier transforms and Sobolev spaces. At the distant end there is the early 19th century with Gauss, where the first general tools in potential theory and partial differential equations were being forged. In between there is the end of the 19th century, roughly from 1870 to 1910 with a condensation around 1895–1900, in the center of which we see Poincaré, where in close relation with the emergence of modern physics the first steps were done in directions that led to the subsequent explosion of functional analysis beginning quite soon afterwards.

What is common to all three periods is the strong primary motivation of the mathematical research by applications, which then led to the discovery of beautiful structures that were investigated for their own sake, the result being the creation not only of fine new mathematics, but also of more powerful tools for the applications. Let us quote from Gauss' introduction to [10] where he talks about some of the ambivalence in the relation between mathematics and its applications:

> Der rastlose Eifer, womit man in neuerer Zeit in allen Theilen der Erdoberfläche die Richtung und Stärke der magnetischen Kraft der Erde zu erforschen strebt, ist eine um so erfreulichere Erscheinung, je sichtbarer dabei das rein wissenschaftliche Interesse hervortritt. Denn in der That, wie wichtig auch für die Schifffahrt die möglichst vollständige Kenntniss der Abweichungslinie ist, so erstreckt sich doch ihr Bedürfniss eben nicht weiter, und was darüber hinausliegt, bleibt für jene beinahe gleichgültig. Aber die Wissenschaft, wenn gleich auch dem materiellen Interesse förderlich, lässt sich nicht auf dieses beschränken, sondern fordert für Alle Elemente ihrer Forschung gleiche Anstrengung.[1]

An earlier quote is the following quite modern-sounding grumble from 1825 [13]:

> Ihr gütiger Brief hat mir um so mehr Vergnügen gemacht, je seltener jetzt in Deutschland warmes Interesse an Mathematik ist. So erfreulich die gegenwärtige hohe Blüthe der Astronomischen Wissenschaften ist, so scheint doch die praktische Tendenz fast zu ausschliesslich vorherrschend, und die meisten sehen die abstracte Mathematik höchstens als Magd der Astronomie an, die nur deswegen zu toleriren ist.[2]

On a more technical level, all three periods have in common that *variational methods* play an important role. In Gauss' time, variational principles were commonly used for existence proofs, such as in Gauss' existence proof for the Dirichlet problem. In Poincaré's time, on the practical side their field of applications had been enlarged to cover the construction of eigenfunction systems via min-max principles, and on the theoretical side the problems caused by the perceived inadequacies of too naïve applications of variational principles (cf. Weierstrass' well-known criticism of Dirichlet's principle) were beginning to find solutions. Hilbert [14] is often credited with having given the first rigorous formulation and application of Dirichlet's principle. Here

[1] The restless eagerness with which in recent times one strives to investigate in all parts of the surface of the earth the direction and strength of the magnetic force of the earth, is a development which is all the more pleasing the clearer the purely scientific interest is standing out. For, in fact, how important the most complete knowledge of the deviation line may be for navigation, the need of the latter just does not extend further, and it remains almost indifferent to anything that lies beyond. But Science, albeit also beneficial to the material interest, cannot be restricted to this, but requires for All elements of its research the same effort.

[2] Your kind letter has given me all the more pleasure the rarer there is now warm interest in mathematics in Germany. As pleasant as the current high bloom of the astronomical sciences may be, the practical tendency seems to be almost too exclusively predominant, and most people consider abstract mathematics at most as a servant of astronomy which is only therefore to be tolerated.

is, however, a quote from a recent paper [1] by one of the specialists in the calculus of variations:

> In 1900, D. Hilbert, in a celebrated address, followed by a (slightly) more detailed paper in 1904 [14, 15, 16], announced that he had solved the Dirichlet problem [...] via the Dirichlet principle which had been discovered by G. Green in 1833, with later contributions by C. F. Gauss (1837), W. Thomson (=Lord Kelvin) (1847) and G. Riemann (1853). [...]
> The announcement of Hilbert turned out to be a little premature. Instead, it became a program which stimulated many people during the period 1900-1940: B. Levi, H. Lebesgue, L. Tonelli, R. Courant, S. L. Sobolev and many others. In 1940, H. Weyl [40] completed Hilbert's program. By 1940 the Calculus of Variations had been placed on firm grounds [...]

Thus a closer look at history tends to blur the boundaries of what constitutes a formal and complete proof. In any case, nowadays we have clearcut basic tools like Hilbert spaces, the Riesz representation theorem, the Lax-Milgram lemma, and Sobolev spaces, which allow us to teach Dirichlet's principle in a first course on finite element methods.

A final bridge between the present and the past should be mentioned that allows us to approach those distant galaxies far more easily than had been possible for a long time: The Internet. Almost all references in this article are readily and freely available online, thanks to enterprises like actamathematica, Gallica, GDZ, JSTOR, NUMDAM, SpringerLink.

In the following we will first make some remarks about Gauss and the first kind integral equation of the single layer potential. Then we describe some of Poincaré's ideas about the double layer potential. In the final part we list a few known and unknown results related to these old ideas.

2 Gauss and the Single Layer Potential

In 1838-40, Carl Friedrich Gauss published three famous works which stand at the beginning of our curious history of boundary integral equation methods: In two of them, [10] and [12], he introduced boundary integral equations (of the first kind!) as a tool in numerical computations and published extensive tables and graphs of numerical results obtained in part by employing this tool. It is truly amazing to see how much could be achieved with numerical calculations by hand when powerful analytical tools were used. In [10, §32] Gauss gives a simple description of the principle of boundary reduction, an idea from which another track leads to later successful methods for proving existence for the Dirichlet problem, namely Schwarz' alternating method and Poincaré's sweeping or "balayage" method.

> [32.] Die Art der wirklichen Vertheilung der magnetischen Flüssigkeiten in der Erde bleibt nothwendigerweise unbestimmt. In der That kann nach einem allgemeinen Theorem, welches bereits in der *Intensitas* Art. 2 erwähnt ist, und bei einer andern Gelegenheit ausführlich behandelt werden soll, anstatt jeder beliebigen Vertheilung der magnetischen Flüssigkeiten innerhalb eines körperlichen Raumes allemal substituirt werden eine Vertheilung auf der Oberfläche dieses Raumes, so dass die Wirkung in jedem Punkte des äusseren Raumes genau dieselbe bleibt, woraus man leicht schliesst, dass *einerlei* Wirkung im ganzen äussern Raume aus unendlich vielen *verschiedenen* Vertheilungen der magnetischen Flüssigkeiten im Innern abzuleiten ist.[3]

After this, he gives, for the case of a ball, an expansion in spherical harmonics of the unknown density on the surface. The "other occasion" where the mentioned Theorem was going to "be treated extensively", is the third paper [11, §36].

In this paper, Gauss not only lays down the foundations of potential theory, including the mean value property of harmonic functions (§20), the maximum principle and the principle of unique continuation (§21), but he also studies in detail the properties of single layer potentials. He presents the jump relations (§15) and the basic integration by parts formula (§21; now known as Green's first formula, because Green formulated this some years before Gauss, his works not yet being widely known at the time of Gauss' paper). We will quote these two results below in Gauss' own notation, as our pieces of evidence in the curious case of "Gauss' missing theorem." Let us first see, however, how Gauss considers the positivity of the single layer potential integral operator. In his own words:

> [30.] Es ist von selbst klar, dass, wie auch immer eine Masse M über eine Fläche *gleichartig* vertheilt sein möge, das daraus entspringende überall positive Potential V in jedem Punkte der Fläche grösser sein wird, als $\frac{M}{r}$, wenn r die grösste Entfernung zweier Punkte der Fläche voneinander bedeutet: diesen Werth selbst könnte das Potential nur in einem Endpunkt der Linie r haben, wenn die ganze Masse in dem andern Endpunkte concentrirt wäre, ein Fall, der hier gar nicht in Frage kommt, indem nur von stetiger Vertheilung die Rede sein soll, wo jedem Element der Fläche $\mathrm{d}s$ nur eine unendlich kleine Masse $m\,\mathrm{d}s$

[3] The specifics of the real distribution of the magnetic fluids in the earth remain necessarily undetermined. Indeed, according to a general theorem which has already been mentioned in the *Intensitas* Art. 2 and shall be treated extensively at another occasion, one can always substitute instead of any arbitrary distribution of the magnetic fluids inside a bodily space, a distribution on the surface of this space, so that the effect in every point of the exterior space remains exactly the same, from which one easily concludes that an *identical* effect in the entire exterior space is to be derived from infinitely many *different* distributions of the magnetic fluids in the interior.

> entspricht. Das Integral $\int Vm\,\mathrm{d}s$ über die ganze Fläche ausgedehnt, ist also jedenfalls grösser als $\int \frac{M}{r} m\,\mathrm{d}s$ oder $\frac{MM}{r}$, und so muss es nothwendig eine gleichartige Vertheilungsart geben, für welche jenes Integral einen Minimumwerth hat.[4]

The notion "gleichartig" (homogeneous) means not changing sign, in the case of a positive total mass M therefore non-negative.

In the paragraphs that follow, he considers a more general problem: Given a continuous function U on the surface, minimise the integral

$$\Omega = \int (V - 2U)m\,\mathrm{d}s\,.$$

This is then seen to be equivalent to the integral equation problem: Find a non-negative mass density m of total mass M and a constant C such that the single layer potential V satisfies $V + C = U$ on the surface. He also considers the case where C is given and M is not fixed, thus the basic first kind integral equation with the $1/r$ kernel.

For this problem he gives a detailed proof of existence and uniqueness. From this result he then deduces an existence proof for the Dirichlet problem in potential theory.

What jumps out at us when we read this argument is, *of course*, that Gauss commits the freshman error of confusing infimum and minimum and that as a consequence he has, in reality, no existence proof. This whole piece of analysis was, indeed, far ahead of its time, and we all know that the crucial question of completeness was only seriously studied after Weierstrass had criticized this naïve use of variational arguments. Weierstrass' main victim was the Dirichlet principle, that is, the variational method involving minimization of the Dirichlet integral over the domain. It is worth while noting, however, that although Dirichlet's principle was apparently formulated by Green before Gauss' work, the first serious mathematical existence proof for the Dirichlet problem was the one discussed here, which used *a boundary integral equation of the first kind*.

The second weak point of the above argument is one noticed by Gauss himself: His positivity argument is of a simple *geometric* nature: Since r is bounded by the diameter of the surface, the positive kernel $1/r$ is bounded

[4] It is self-evident that, however a mass M may be distributed *homogeneously* over a surface, the resulting everywhere positive potential V will be, in every point of the surface, greater than $\frac{M}{r}$ if r designates the greatest distance between two points of the surface: this value itself could be attained by the potential only in an endpoint of the line r if the entire mass was concentrated in the other endpoint, a case which cannot appear here, because we will only consider a continuous distribution, where every surface element $\mathrm{d}s$ corresponds only to an infinitely small mass $m\,\mathrm{d}s$. The integral $\int Vm\,\mathrm{d}s$, extended over the whole surface, is therefore in any case greater than $\int \frac{M}{r} m\,\mathrm{d}s$ or $\frac{MM}{r}$, and thus there must necessarily be a homogeneous kind of distribution for which that integral has a minimum value.

from below by a constant depending only on the domain. The quadratic form defined by the integral operator is therefore seen to be positive, but only for *non-negative* densities m. Having to respect this constraint makes the proof rather complicated: Only variations inside the positive cone are allowed, which means that in general, the solution of the minimisation problem solves only an integral *inequality*, turning into an *equation* only in those points where the solution is strictly positive. Gauss writes (Gauss' original emphasis as always):

> [33.] Der eigentliche Hauptnerv der im 32. Artikel entwickelten Beweisführung beruht auf der Evidenz, mit welcher die Existenz eines Minimumwerthes für Ω unmittelbar erkannt wird, solange man sich auf die gleichartigen Vertheilungen einer gegebenen Masse beschränkt. Fände die gleiche Evidenz auch ohne diese Beschränkung Statt, so würden die dortigen Schlüsse ohne weiteres zu dem Resultate führen, *dass es allemal, wenn nicht eine gleichartige, doch eine ungleichartige Vertheilung der gegebenen Masse gibt, für welche* $W = V - U$ *in allen Punkten der Fläche einen constanten Werth erhält*, indem dann die zweite Bedingung (Art. 31. II) wegfällt. Allein da jene Evidenz verloren geht, sobald wir die Beschränkung auf gleichartige Vertheilungen fallen lassen, so sind wir genöthigt, den strengen Beweis jenes wichtigsten Satzes unserer ganzen Untersuchung auf einem etwas künstlichern Wege zu suchen.[5]

Thus Gauss finds it desirable to prove the positivity of the quadratic form for not necessarily non-negative mass distributions. This would have given not only a much simpler proof, but even a much nicer theorem.

The truly odd observation is now that Gauss could easily have proved this general positivity himself by simply combining the jump relations and the integration by parts formula cited above. For completeness of this claim, here are Gauss' original formulations of these lemmas:

> [end of 15.] Man kann diesen wichtigen Satz auch so ausdrücken: der Grenzwerth von X, bei unendlich abnehmendem positiven x ist $X^0 - 2\pi k^0$, bei unendlich abnehmendem negativen x hingegen $X^0 + 2\pi k^0$, oder X ändert sich zweimal sprungsweise um $-2\pi k^0$, indem x aus einem negativen Werthe in einen positiven übergeht, das

[5] [33.] The actual main nerve of the line of proof developed in §32 rests on the self-evidence with which the existence of a minimum value for Ω is perceived immediately, as long as one restricts oneself to the homogeneous distributions of a given mass. If the same self-evidence held without this restriction, the above arguments would lead immediately to the result *that there is always, if not a homogeneous, then at least an inhomogeneous distribution of the given mass for which* $W = V - U$ *obtains in all points of the surface a constant value*, in that the second condition (§31. II) can then be omitted. However, since this self-evidence is lost as soon as we drop the restriction to homogeneous distributions, we are forced to search for the strict proof of this most important theorem of our whole investigation in a somewhat artificial way.

erstemal, indem x den Werth 0 erreicht, und das zweitemal, indem es ihn überschreitet.[6]

Here Gauss uses coordinates where the normal at a point on the surface coincides with the x axis and $X = \frac{\partial V}{dx}$ where V stands for the single layer potential with density k: $V = \int \frac{k\,\mathrm{d}s}{r}$ with the surface element $\mathrm{d}s$ and the distance r between observation point and point of integration.

[24.] Lehrsatz. Es ist

$$\int V \frac{dV}{dp} \mathrm{d}s = -\int qq\,\mathrm{d}T$$

wenn das erste Integral über die ganze Fläche, das zweite durch den ganzen Raum T ausgedehnt wird.[7]

Here Gauss denotes by q the gradient of the potential V, by T the interior domain, and $\frac{dV}{dp}$ is the interior normal derivative.

We see that he could have added the formulas from Lehrsatz 24 for the interior domain and the corresponding one for the exterior domain in order to get with Theorem 15 (in what would have been his formulation; he didn't write this, of course)

$$\int Vm\,\mathrm{d}s = \frac{1}{4\pi}\int qq\,\mathrm{d}T > 0$$

where the second integral is extended over the whole space. This gives positivity for any m, positive or negative. It is also physically intuitive (in electro- or magnetostatic terms that were familiar to Gauss), stating equality between the potential energy stored in the surface and the total energy of the field.

We can only speculate why Gauss didn't write this. It is also strange that this result about the positivity of the quadratic form defined by the $1/r$ kernel, which was, as we have seen, formulated as a useful and non-trivial open problem in one of the most famous and widely studied papers of its time, did apparently not become the object of serious study for a long time. The reason cannot be that the simple argument of adding the interior and exterior Green formulas did not occur to anyone. As an example, here is a quote from a paper [32, p.216] by W. Steklov, written 1900 in the wake of Poincaré's paper [29].

Posons

$$V = \frac{1}{4\pi}\int \frac{W}{r}\,\mathrm{d}s\ ,$$

[6][end of 15.] One can express this important theorem also as follows: The limit of X for infinitely decreasing positive x is $X^0 - 2\pi k^0$, whereas for infinitely decreasing negative x it is $X^0 + 2\pi k^0$, or X jumps twice by $-2\pi k^0$ when x passes from a negative value to a positive one, the first time when x reaches the value 0, and the second time when it goes beyond it.

[7][24.] Theorem. There holds [formula omitted] where the first integral is extended over the whole surface, the second one over the whole space T.

l'intégrale étant étendue à la surface (S) tout entière. Dans les suppositions faites par rapport à (S) nous pouvons employer le théorème connu de Green qui nous donne[8]

$$\int\sum(\frac{\partial V}{\partial x})^2\mathrm{d}\tau+\int\sum(\frac{\partial V}{\partial x})^2\mathrm{d}\tau' = \int V(\frac{\partial V_i}{\partial n}-\frac{\partial V_e}{\partial n})\mathrm{d}s = \int VW\,\mathrm{d}s > 0\,.$$

Steklov then uses this to prove that for a harmonic function the L^2 norm on the boundary is bounded by the diameter of the boundary times the H^1 seminorm on the domain. But he doesn't state this positivity as an interesting result in itself.

In 1935, Otto Frostman [8] finally formulated the positivity of this quadratic form as a theorem in order to complete Gauss' proof. But he considers the argument using Green's formula as easy to see, but too restrictive (p. 24: "Si le potentiel (newtonien) a des dérivées continues, cela résulte déjà des formules de Green et de Gauss; en effet on démontre facilement..."[9]). He then gives another proof using the composition property of Riesz potentials on the whole space which shows that the convolution with $1/r$ on $\mathbb{R}^3$ is a constant times the square of the convolution with $1/r^2$. This argument (which can easily be verified by taking Fourier transforms) is generalized by Frostman to other kernels of the form $1/r^\alpha$ with $\alpha > 0$. For these kernels, he then presents Gauss' complicated proof in the framework of positive measures using the maximum principle as a principal tool.

The, in our view, simpler and more general (because it applies to other equations of mathematical physics besides the Laplace equation) proof using the energy identity was not given before another 38 years, in 1973 by Nedelec and Planchard [26].

The difference between the two completions that have to be performed in order to complete Gauss' minimization argument is that on one hand, as Frostman showed, positive measures are complete in the energy norm. Thus in the well-understood framework of positive (Radon) measures, the infimum is indeed a minimum. One doesn't even have to know exactly what the finiteness of energy means for those measures (more about this point below); one can very well minimize a coercive lower semi-continuous functional that is not everywhere finite. On the other hand, as Nedelec and Planchard noticed, the space obtained by completion of a whole vector space (and not only the positive cone) in the energy norm is the Sobolev space $H^{-1/2}$ which is a space not of functions or measures, but of distributions.

Thus, whereas the efforts of Hilbert and others to complete the proof of Dirichlet's principle led to the introduction of the function spaces of Beppo

[8]Let [formula omitted], the integral being extended over the entire surface (S). With our assumptions on (S) we can apply the known theorem of Green which gives us [formula omitted].

[9]If the (Newtonian) potential has continuous derivatives, this follows already from the formulas of Green and Gauss; indeed one shows easily...

Levi and Sobolev already in the beginning of the 20th century (crudely stated: H^1 is a subspace of L^2, therefore a space of functions, once Lebesgue's notion of function is adopted), the energy space needed for Gauss' boundary integral form of the Dirichlet principle could only be constructed after the introduction of Schwartz' distributions and Sobolev spaces of fractional and negative index.

There is a glimpse of this difficulty in Henri Cartan's works in 1941 and 1945: In [2] he presents a proof of Frostman's theorem on the completeness of positive measures of finite energy (in fact a greatly generalized version thereof), but of the question of completeness of *all* signed measures of finite energy, he says (p.90) "C'est peu probable."[10]. In the paper [3] he gives a counterexample (p.87) showing that it is, indeed, not complete.

> On voit qu'en "complétant" l'espace $\mathcal{E}$ pour cette norme, on obtiendrait un espace de Hilbert. On vérifie sans peine que $\mathcal{E}$ lui-même *n'est pas complet ()*.[11]

But he does not want to quit the framework of measures (which he also calls "distributions") to investigate the nature of this Hilbert space.

Could it be that Gauss already had some intuition about the different nature of the minimizing objects that would appear when the condition of non-negativity was dropped? We can only speculate.

To finish this paragraph, here is another historic curiosity related to measures and their energy: As is well-known in the theory of the logarithmic single layer potential integral equation in two dimensions, the positivity is true there only under an additional condition on the boundary: Its capacity has to be less than one. It is also a classical result that the logarithmic capacity of a compact set in $\mathbb{R}^2$ is identical to its transfinite diameter and also to its exterior conformal radius (other names are Chebyshev constant or Robin constant). This was well known to Frostman in 1935, and the identity of transfinite diameter and exterior mapping radius for regular sets was already proved by Szegö in 1924 [39]. Now the standard reference (and the only available reference in book form, as far as I can tell) for a complete proof of this equivalence result is the book [17] by Einar Hille. Hille gives a detailed proof of all the equivalences, in particular (Theorem 16.4.4 p.284) a proof of the equality of transfinite diameter and logarithmic capacity by constructing a minimizing measure. He gets this measure as a limit of point measures supported by the Fekete points. This is also Corollary 1 (p. 285):

> Corollary 1. The equilibrium distribution $\nu(s)$ of E is the weak limit of the sequence of point distributions $\mu_n(S)$ associated with the zeros of the Fekete polynomials $F_n(z; E)$.

Unfortunately, in the proof it is used that the energy of μ_n is finite (and can even be given by a simple formula), which is not the case (Point measures are

[10] This is not very likely.

[11] One sees that by "completing" the space $\mathcal{E}$ in this norm, one would obtain a Hilbert space. One verifies with ease that $\mathcal{E}$ itself *is not complete*.

not in $H^{-1/2}$). Thus the standard reference for this basic (and well-known true) result has a hole that might still be open after more than 40 years!

3 Poincaré and the Double Layer Potential

After Gauss' work on the first kind integral equation of the single layer potential, the next major progress came with Carl Neumann's work on the double layer potential. Of his numerous publications on the subject of his "Methode des arithmetischen Mittels", we cite the book [27] from 1877 which is available online from the Gallica project of the BNF.

For convex domains, Neumann proves the convergence of the method of iterations which leads to the solution of the second kind integral equation by the Neumann series. The tool here is not positivity, but the contraction property of the integral operator in the maximum norm. Positivity comes in through the convexity of the domain which means that the measure defined by the double layer kernel

$$\mathrm{d}\theta_x(y) = \frac{1}{4\pi}\frac{n(y)\cdot(y-x)}{|x-y|^3}\mathrm{d}s(y) \tag{1}$$

is a positive measure of total mass 1. The idea that integration against such a measure should somehow level functions out and make iterations converge to a constant function seems to have been intuitive to physicists before Neumann. In a paper from 1856, quoted in its entirety by Neumann in his book (Chapter 6), the physicist Beer used an iterative method for the second kind integral equation of the normal derivative of the single layer potential (the adjoint equation to Neumann's). He formulates

> Dabei leuchtet ein, dass F' – welches innerhalb σ zwischen dem größten und kleinsten Werthe liegt, den die Funktion F auf der Fläche σ selbst annimmt – im Allgemeinen *gleichförmiger* als F verläuft.[12]

In a footnote, Neumann remarks that Beer does not offer any proof, and that the claim is not true, in general, unless the measure mentioned above is positive, that is, unless the domain is convex.

Neumann's proof of his result (and as a corollary also of Beer's result) uses highly non-trivial geometric and measure-theoretic arguments that constitute one of the early examples of "hard" analysis in potential theory. As a consequence, subsequent generalizations of his techniques were confined to hard harmonic analysis, too, see [24] and [25] for overviews.

Neumann's method of the arithmetic mean became famous, because it was at the time, besides Schwarz' alternating method and Poincaré's balayage method, the only rigorous way of proving existence for the Dirichlet

[12] Here it is clear that F' – which, inside σ, lies between the largest and smallest value that the function F takes on the surface σ itself – behaves in general *more uniformly* than F.

problem and for all the important theorems based on it like the Riemann mapping theorem. In addition, it looked like it was simpler to apply and more constructive than the other two methods. But the restrictive assumption of convexity of the domain was a mathematical challenge, and in 1895 Henri Poincaré published a paper [29] about Neumann's method which introduced a quite different argument for proving the contractivity that did not need convexity of the domain. The new method was based on positivity and energy identities.

In this paper, Poincaré presents an astonishing collection of techniques that were new at the time and that made the paper famous, at least for some years. Poincaré used this method only in one further work [30], a small paper on generalizations to elasticity theory which he himself characterizes as incomplete. But others continued and developed his ideas in various different directions, in particular Arthur Korn [19, 21, 22, 23], Vladimir Steklov [32, 33, 36, 37, 38] and Stanislaw Zaremba [41, 42, 43]. Korn and Steklov for some time engaged in a kind of race [34, 20, 35]. Here is a quote from [20] (our reference numbers):

> Dans une note [34] *sur la méthode de Neumann et le problème de Dirichlet*, M. W. Stekloff est arrivé à une démonstration de la méthode de la moyenne arithmétique de M. Neumann, qui est à peu près la même que celle que j'ai publiée il y a un an dans mon Cours sur la théorie du potentiel [19]. Ma démonstration, comme celle de M. Stekloff, a pour base le Mémoire ingénieux [29] de M. Poincaré, et nous avons éliminé tous les deux de la même manière la restriction de M. Poincaré, que l'existence d'une solution soit préalablement établie.[13]

The "fonctions fondamentales" mentioned in the titles of some of these papers, also called "universelle Funktionen" by Korn, are potentials generated by eigenfunctions of Neumann's integral operator or also by its adjoint, sometimes also the eigenfunctions of what is known as Steklov eigenvalue problem, or eigenfunctions of the Poincaré–Steklov operator.

These papers concentrated on eigenfunction expansions and eigenvalue estimates obtained by min-max principles as studied by Poincaré for the case of the eigenvalue problem for the Laplace operator with Dirichlet boundary conditions in his earlier important paper [28]. To prove existence of the eigenfunctions, regularity of the boundary had to be assumed, and after works by Hölder and Lyapunov, Hölder continuous functions on Lyapunov surfaces became the standard framework. During the same time, the new powerful Fredholm method for treating integral equations became widely accepted,

[13] In a note [34] *on Neumann's method and the Dirichlet problem*, Mr W. Stekloff arrived at a proof of Mr Neumann's method of the arithmetic mean which is more or less the same as the one that I have published a year ago in my Course on Potential Theory [19]. My proof, as the one of Mr Stekloff, is based on the ingenious paper [29] by Mr Poincaré, and we have both eliminated in the same manner the restriction of Mr Poincaré that the existence of a solution should be established beforehand.

and Hilbert published his book on integral equations which had the ambition to subsume all known results about integral equations. Hilbert and his group made big jumps forward by introducing the idea of function spaces and norms and developing the basics of modern functional analysis with the spectral theory of bounded and in particular compact selfadjoint operators in Hilbert spaces.

I mention all this well-known history as an explanation for the curious fact that the basic idea of Poincaré's paper on Neumann's method, namely to consider the convergence of the Neumann series in the energy norm, disappeared pretty much completely from the discussion. His estimates were only used for estimating the eigenvalues of the boundary integral operators considered as compact operators acting in spaces of continuous or Hölder continuous functions, and this remained the standard for a long time, see for example [25, Thm 12, p. 144]. One of the main advantages of Poincaré's method, namely its easy applicability to other elliptic problems having a positive energy, such as linear elasticity, remained present, but the other advantage, namely that it basically only uses Green's formula and is therefore valid for general Lipschitz domains, seems to have been forgotten.

Only very recently a similar point of view has been adopted in the paper by Steinbach and Wendland [31] where the contraction property of Neumann's operator in a norm related to the energy norm was proved for the first time for rather general positive second order elliptic systems on Lipschitz domains.

Poincaré's own estimates are being revisited and adapted to a modern standard in the very recent paper [18] which treats the same framework as Poincaré did, namely two- and three-dimensional potential theory on smooth domains. The full potential of Poincaré's main idea which easily generalizes to other positive elliptic operators and to domains with only Lipschitz continuous boundary, does not seem to have been exploited in a modern context yet. We will describe some of this in the next section.

Here is the basic estimate from [29, Chapter 2] in a notation similar to Poincaré's own notation: For a bounded domain Ω in $\mathbb{R}^3$ let W be a function harmonic in the domain and in the exterior domain $\Omega' = \mathbb{R}^3 \setminus \overline{\Omega}$, vanishing at infinity. Quantities related to the exterior domain are indicated by a prime. Let J and J' denote the interior and exterior Dirichlet integrals of W:

$$J = \int_\Omega |\nabla W|^2 \mathrm{d}x \; ; \quad J' = \int_{\Omega'} |\nabla W|^2 \mathrm{d}x \; .$$

Lemma 1. *There is a constant μ depending only on the domain such that (i) If W is a double layer potential, then*

$$J \le \mu J' \quad \text{and} \qquad J' \le \mu J \; . \tag{2}$$

(ii) If W is a single layer potential, then

$$J \le \mu J' \quad \text{and if} \int_{\partial\Omega} W \, \mathrm{d}s = 0, \text{ then} \qquad J' \le \mu J \; . \tag{3}$$

Here double and single layer potentials are defined by their jump properties: Single layer potentials are continuous across the surface $\partial\Omega$ and have a jump in their normal derivatives, whereas double layer potentials have a jump across the surface, but their normal derivatives from the interior and the exterior coincide. The difference between single and double layer potentials in the statement is caused by different behavior of potentials with vanishing Dirichlet integrals (constants): For double layer potentials, if W is constant in the interior domain, it is also constant (zero) in the exterior and vice versa, so that J and J' both vanish if one of them vanishes. For single layer potentials, W vanishing in the exterior implies W vanishing in the interior, so that $J' = 0$ implies $J = 0$, too, but there exists the non-trivial equilibrium density (Robin density) which has potential 1 in the interior and non-constant potential in the exterior, so that J' can be bounded by J only on a subspace of codimension one.

In 1900, Steklov [32, p.224], after stating the above estimate for single layer potentials, gets quite enthusiastic and writes (his emphasis):

> Nous appellerons ce théorème *théorème fondamental*.
> ...
> Nous verrons dans ce qui va suivre, que *la solution de tous les problèmes fondamentaux de la Physique mathématique se ramène à la démonstration complète du théorème fondamental*.[14]

Writing this in a year when Planck introduced his quantum constant and Poincaré was already working on the theory of relativity seems, in hindsight, slightly exaggerated, but it underlines the importance of these estimates for potential theory and for related models of classical mathematical physics like elasticity, heat conduction, acoustics, electrostatics and electrodynamics, fluid dynamics and so on. Such applications were studied by Steklov, Korn, Zaremba and others, who also worked on removing some of the hypotheses Poincaré had to make in order to prove Lemma 1. They proved the lemma essentially for arbitrary connected Lyapunov (i.e. $C^{1,\alpha}$) surfaces.

Poincaré proved the lemma under the condition that the domain is diffeomorphic to a ball (actually for a simply connected smooth boundary; the question of the existence of a diffeomorphism to the ball is a first simple case of the famous Poincaré conjecture), and he used the diffeomorphism to reduce the estimates to the case of a ball where he could show them explicitly by expansion in spherical harmonics.

Nowadays, the lemma is easy to prove even for Lipschitz surfaces by noticing that the H^1 seminorm of a harmonic function on the interior or exterior domain is equivalent to both the $H^{1/2}$ seminorm of its trace and the $H^{-1/2}$ norm of its normal derivative on the boundary. This equivalence is seen immediately in one direction from the standard trace theorem (sometimes called

[14] We shall call this theorem the *fundamental theorem*.... We shall see in the following that *the solution of all the fundamental problems of Mathematical Physics can be reduced to a complete proof of the fundamental theorem*.

Gagliardo's trace theorem in the case of a merely Lipschitz continuous boundary) plus Poincaré's inequality (the one estimating the L^2 norm modulo constants by the H^1 seminorm) and the weak definition of the normal derivative, and in the other direction from the variational solution of the Dirichlet and the Neumann problems. But one should keep in mind that without the introduction of the fractional Sobolev space $H^{1/2}$ on the surface, which at first seems like overkill for proving a statement mentioning only Dirichlet integrals, one has no way of stating or proving that the trace spaces from the exterior and from the interior are the same, which is one of the crucial points in this argument. In fact, one can consider Poincaré's procedure of using a diffeomorphism to the sphere and estimating the coefficients of the expansion in spherical harmonics as an early definition of the space $H^{1/2}$, although the idea of function spaces and norms was not expressed in that paper.

Poincaré uses the estimates in Lemma 1 to prove the contraction property of Neumann's operator in the energy norm, and with this the convergence of Neumann's series solution for the Dirichlet problem in the same norm. He then shows trace estimates, first for the boundary L^2 norm modulo constants (Chapter 4), and then (Chapter 5) for the L^∞ norm of the double layer operator applied to the trace on the boundary. The latter estimate uses difficult geometric constructions, is not yet optimal, and is subsequently generalized by the above-mentioned authors and others like Lebesgue, Plemelj and Radon, one famous later observation being that whereas Neumann's operator is not a contraction in the L^∞ norm when the domain is not convex, the square of the operator is a contraction, at least when the domain is smooth. In any case, Poincaré completes the proof of the uniform convergence in the whole space of Neumann's series for general smooth domains.

Neumann's operator, as defined by Neumann himself and in the same way by Poincaré, is the mapping from the difference of the boundary traces of a double layer potential to the sum of the traces. If we denote the interior and exterior traces of the double layer potential W by V and V', respectively, then Neumann's operator N maps $V - V'$ to $V + V'$, which corresponds in our notation of the next section below to

$$N = -2K .$$

The problem studied by Poincaré (his equation (1)) is written not as an integral equation, but as a transmission problem with a parameter λ:

$$V - V' = \lambda(V + V') + 2\Phi . \tag{4}$$

The choice of $\lambda = 1$ corresponds to the exterior Dirichlet problem, and $\lambda = -1$ to the interior Dirichlet problem. Poincaré proves convergence (modulo constant functions) of the Neumann series solution of (4) for $|\lambda| < \frac{\mu+1}{\mu-1}$, where μ is the constant from Lemma 1.

4 Positive Boundary Integral Operators and the Convergence of Neumann's Series

In this section we will give a modern expression of Poincaré's idea that the estimate (2) of Lemma 1 implies that Neumann's operator is a contraction. We start by an abstract observation whose simple proof we leave to the reader. No tools more advanced than the Cauchy-Schwarz inequality are required for the proof.

Lemma 2. *Let A and B be bounded selfadjoint operators on a Hilbert space X satisfying $A + B = I$, where I is the identity operator.*
(i) *If $B - A$ is a contraction, then A and B are contractions with norms bounded by $(1 + \|B - A\|)/2$. The inverse A^{-1} can be represented in two different ways by convergent Neumann series*

$$A^{-1} \;=\; \sum_{\ell=0}^{\infty} B^{\ell} \;=\; 2\sum_{\ell=0}^{\infty} (B-A)^{\ell}\;. \tag{5}$$

(ii) *If A is positive definite and B positive semidefinite:*

$$\exists \alpha > 0\,, \exists \beta \geq 0 \;:\; \forall u \in X \;:\quad (Au,u) \geq \alpha \|u\|^2\,; \quad (Bu,u) \geq \beta \|u\|^2\,,$$

then B is a contraction with norm $\|B\| \leq 1 - \alpha$. If in addition $\beta > 0$, then $B - A$ is a contraction with norm $\|B - A\| \leq \max\{1 - 2\alpha, 1 - 2\beta\}$.

A situation where this lemma can easily be applied is the following:

Lemma 3. *Let a and b be symmetric bilinear forms on a vector space X_0. We assume that a and b are positive semidefinite and that a is non-degenerate:*

$$\forall u \in X_0 \;:\quad a(u,u) > 0 \;\text{if}\; u \neq 0\,;\; b(u,u) \geq 0\,.$$

Let X be the Hilbert space completion of X_0 with respect to the inner product

$$(u,v) \;=\; a(u,v) + b(u,v)$$

and let A and B be the operators on X defined by the bilinear forms a and b. If there exists $\mu > 0$ such that

$$\forall u \in X_0 \;:\quad b(u,u) \;\leq\; \mu\, a(u,u)\,,$$

then A and B satisfy the hypothesis of Lemma 2 (ii) *with $\alpha = \frac{1}{\mu+1}$.*
In particular, B is a contraction with norm $\|B\| \leq \frac{\mu}{\mu+1}$. If, in addition,

$$\forall u \in X_0 \;:\quad a(u,u) \;\leq\; \mu\, b(u,u)\,,$$

then $B - A$ is a contraction with norm $\|B - A\| \leq \frac{\mu-1}{\mu+1}$.

Note that the Riesz representation theorem implies that the existence of an estimate $b \leq \mu a$ is equivalent to the positive definiteness of a on the Hilbert space X.

Another remark which is easy to verify is that the non-degeneracy of a alone is sufficient to show that all *eigenvalues* of $B-A$ and of B are of absolute value strictly less than 1. One does not need the estimate $b \leq \mu a$ for this, but one also does not get the contractivity from it. If, however, $B - A$ has a pure point spectrum, for example if it is compact, then the contractivity follows. This may provide a partial explanation why Poincaré's mutual estimates of the interior and exterior energies were later forgotten: If the Fredholm-Riesz theory can be applied as is the case for Neumann's operator on a smooth surface, then they are not needed. They are then, in fact, a consequence of the Fredholm alternative: If a is positive semidefinite and non-degenerate and the corresponding operator A is Fredholm, then a is positive definite.

In the following, we present some applications of these simple estimates. In all cases, the quadratic forms a and b will correspond to the energy integrals in the exterior and interior domains, respectively, so that the Hilbert space X will be endowed with the norm of the total energy. Which concrete boundary integral operators correspond to the abstract operators A and B can vary, however, according to how the abstract vector space X_0 is represented by a concrete function space.

We choose the same general situation as considered in the paper [31] by Steinbach and Wendland. This covers some of the most important applications such as potential theory and elasticity theory (basically "every fundamental problem of mathematical physics" in the sense of Steklov quoted above).

The same ideas for proving the contraction property of second kind boundary integral operators could be applied to higher order strongly elliptic partial differential operators that have a positive energy form in the framework studied in [7], or to other situations where positivity of first kind integral operators has been shown by using Green's formulas like for parabolic problems in [5]. In this paper we will stay within the framework of positive second order systems as in [31]. This will allow an easy comparison in order to see similarities and differences with the arguments of [31]. Note, however, that while we consider the same objects as in [31], we will not always use the same letters to denote them.

Let then L be a second order selfadjoint elliptic partial differential operator on $\mathbb{R}^n$ with smooth, not necessarily constant coefficients about which we will make a certain number of further hypotheses. First we assume that L has a real-valued fundamental solution $U^*(x, y)$. Given a density ψ on the boundary Γ of the bounded Lipschitz domain Ω, the single layer potential $\mathcal{S}$ is defined in the interior domain Ω and in the exterior domain $\Omega^c = \mathbb{R}^n \setminus \overline{\Omega}$ by

$$\mathcal{S}\psi(x) = \int_\Gamma U^*(x, y)\psi(y)\,\mathrm{d}y\,. \tag{6}$$

Before defining the double layer potential, we need to assume that there exists a first Green formula

$$\int_\Omega (Lu(x))^\top v(x)\,\mathrm{d}x = \Phi(u,v) - \int_\Gamma (Tu(x))^\top v(x)\,\mathrm{d}s(x)\,. \tag{7}$$

Here T is the conormal derivative, defined by this formula. The energy bilinear form Φ is a first order symmetric integro-differential form which we assume to be positive in the sense that it is non-negative and elliptic: There are constants α, c, k with $\alpha > 0$ such that $|\Phi(u,v)| \le c\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$ and

$$\Phi(u,u) \ge 0 \quad \text{and} \quad \Phi(u,u) \ge \alpha\|u\|^2_{H^1(\Omega)} - k\|u\|^2_{L^2(\Omega)}\,. \tag{8}$$

As a consequence of the Gårding inequality (8) and the compact embedding of the Sobolev space $H^1(\Omega)$ into $L^2(\Omega)$, the space of functions of vanishing energy

$$\mathcal{R} = \{u \mid \Phi(u,u) = 0\} \tag{9}$$

is finite-dimensional. For $u \in \mathcal{R}$ one has also $\Phi(u,v) = 0$ for all v, which according to (7) is the weak formulation of the homogeneous Neumann problem $Lu = 0$ in Ω, $Tu = 0$ on Γ, so that $\mathcal{R}$ can also be defined as solution space of the homogeneous Neumann problem.

For the exterior domain, we also assume the first Green formula

$$\int_{\Omega^c} (Lu(x))^\top v(x)\,\mathrm{d}x = \Phi^c(u,v) + \int_\Gamma (Tu(x))^\top v(x)\,\mathrm{d}s(x)\,. \tag{10}$$

Whereas the previous equations (6)–(9) were assumed to be valid for all smooth functions – with the idea of extending the domain of validity by continuity to some larger Hilbert spaces of functions afterwards – in the Green formula (10) for the exterior domain we have to assume that u and v are, in addition, of compact support. For such functions, we assume then positivity of the exterior energy form:

$$\forall u \in C_0^\infty(\mathbb{R}^n)\ :\quad \Phi^c(u,u) > 0 \quad \text{unless } u \equiv 0\,. \tag{11}$$

The final assumption we have to make is that *potentials have finite energy*. This is an assumption on the behavior of the fundamental solution at infinity which can be phrased as follows: If γ and δ are multi-indices and $\chi \in C^\infty(\mathbb{R}^n)$ is a cut-off function which is zero on a large enough ball and equal to one on a neighborhood of infinity, then the function u defined by

$$u(x) = \chi(x)\partial_x^\gamma \partial_y^\delta U^*(x,y)$$

satisfies $\Phi^c(u,u) < \infty$.

The assumptions made so far cover some important standard examples:
- The Laplace equation in dimension $n \ge 3$ with its standard fundamental solution. Here the conormal derivative T is the exterior normal derivative. The

space $\mathcal{R}$ consists of the constant functions on Ω. The condition that potentials have finite energy excludes the logarithmic potentials in the plane.
- The equations of linear elasticity in dimension $n \geq 3$. The conormal derivative T corresponds to the normal traction on the boundary, and the space $\mathcal{R}$ consists of the rigid motions.
- The mathematically simplest case is a strictly positive operator such as $-\Delta + \lambda I$ with $\lambda > 0$ in any dimension, or similarly any strongly elliptic constant coefficient operator plus λI with a sufficiently large λ. In this case, the energy form in the interior is positive definite, too, the space $\mathcal{R}$ is reduced to $\{0\}$, and the energy forms in both the interior and the exterior domain are equivalent to the square of the H^1 norm.

The double layer potential $\mathcal{D}$ with density φ is given for $x \notin \Gamma$ by

$$\mathcal{D}\varphi(x) = \int_\Gamma \left(T_y U^*(x,y)\right)^\top \varphi(y)\, \mathrm{d}s(y)\,. \tag{12}$$

It is well known [4] that the definitions (6) and (12) of the single and double layer potentials can be extended by continuity to densities $\psi \in H^{-1/2}(\Gamma)$ and $\varphi \in H^{1/2}(\Gamma)$, respectively, and that the potentials $v = \mathcal{S}\psi$ and $w = \mathcal{D}\varphi$ then satisfy

$$Lv = 0\,,\ Lw = 0 \text{ in } \Omega \cup \Omega^c\ ;\ v \in H^1_{\mathrm{loc}}(\mathbb{R}^n)\ ;\ w \in H^1(\Omega) \text{ and } w \in H^1_{\mathrm{loc}}(\overline{\Omega^c})\,.$$

If we denote the interior and exterior traces by γ and γ^c and the interior and exterior conormal derivatives (both taken with respect to the exterior normal) by γ_1 and γ_1^c, then these can also be extended by continuity to the potentials with this weak regularity, and there hold the jump relations

$$\begin{aligned} &(\gamma^c - \gamma)\mathcal{S}\psi = 0\ ;\ (\gamma_1^c - \gamma_1)\mathcal{S}\psi = -\psi\ ; \\ &(\gamma^c - \gamma)\mathcal{D}\varphi = \varphi\ ;\ (\gamma_1^c - \gamma_1)\mathcal{D}\varphi = 0\,. \end{aligned} \tag{13}$$

The four classical boundary integral operators are then defined as the operators of
- the single layer potential: $V = \gamma\mathcal{S} = \gamma^c\mathcal{S}$
- the normal derivative of the single layer potential: $K' = \frac{1}{2}(\gamma_1 + \gamma_1^c)\mathcal{S}$
- the double layer potential: $K = \frac{1}{2}(\gamma + \gamma^c)\mathcal{D}$
- the normal derivative of the double layer potential: $W = -\gamma_1\mathcal{D} = -\gamma_1^c\mathcal{D}$.

With these definitions, the traces of the single layer and double layer potentials take the form

$$\begin{aligned} \gamma\mathcal{S} &= \gamma^c\mathcal{S} = V \quad ;\ \gamma_1\mathcal{S} = \tfrac{1}{2}I + K'\ ;\ \gamma_1^c\mathcal{S} = -\tfrac{1}{2}I + K'\ ; \\ \gamma_1\mathcal{D} &= \gamma_1^c\mathcal{D} = -W\ ;\ \gamma\mathcal{D} = -\tfrac{1}{2}I + K\ ;\ \gamma^c\mathcal{D} = \tfrac{1}{2}I + K\,. \end{aligned} \tag{14}$$

As mentioned above, this way of defining the boundary integral operator K of the double layer potential corresponds to Neumann's and Poincaré's definitions for the case of potential theory. If one defines $K_0\varphi$ as the double layer potential of density φ *evaluated on the surface* Γ in the sense of a Cauchy

principal value integral (which in potential theory is the same as integrating with respect to the solid angle measure (1)), then it is well known that $K\varphi(x) = K_0\varphi(x)$ for smooth boundary points x, but for corner points the two definitions differ. The operator whose contraction property is studied by Neumann is $N = -2K$. If N has a norm less than one in some function space, then the four operators $\frac{1}{2}I \pm K$ and $\frac{1}{2}I \pm K'$ will also have norms less than one.

We can now begin to apply Lemma 3 to various incarnations of vector space X_0 and bilinear forms a and b. We will always represent a by the energy integral Φ^c and b by Φ. According to the Green formulas (7) and (10), we have for a function u satisfying $Lu = 0$ in Ω and in Ω^c and any v:

$$\Phi(u,v) = \langle \gamma_1 u, \gamma v \rangle \; ; \qquad \Phi^c(u,v) = -\langle \gamma_1^c u, \gamma^c v \rangle \; . \tag{15}$$

Here we write $\langle \cdot, \cdot \rangle$ for the L^2 inner product (integral) on Γ, extended to the duality product between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

4.1 Single Layer Potentials

The first possibility is to take for the space X_0 some space of integrable functions on Γ, for example the continuous functions, or $L^2(\Gamma)$. For $\varphi, \psi \in X_0$, we define the bilinear forms a and b as energy forms of the corresponding single layer potentials:

$$a(\varphi,\psi) = \Phi^c(\mathcal{S}\varphi, \mathcal{S}\psi) \; ; \qquad b(\varphi,\psi) = \Phi(\mathcal{S}\varphi, \mathcal{S}\psi) \; . \tag{16}$$

With the boundary reduction by Green's formula (15) and the expressions (14) for the traces of the single layer potential, we find the boundary integral forms

$$a(\varphi,\psi) = \langle (\frac{1}{2}I - K')\varphi, V\psi \rangle \; ; \qquad b(\varphi,\psi) = \langle (\frac{1}{2}I + K')\varphi, V\psi \rangle \; . \tag{17}$$

For the total energy $a + b$ we find the bilinear form defined by the single layer potential integral operator which is therefore positive definite (Gauss' missing theorem); and the Hilbert space X is the completion of our space X_0 in this energy norm which we know from Nedelec and Planchard [26] to be the Sobolev space $H^{-1/2}(\Gamma)$:

$$a(\varphi,\psi) + b(\varphi,\psi) = \langle \varphi, V\psi \rangle \; ; \; X = H^{-1/2}(\Gamma) \text{ with norm } \|\varphi\|_V^2 = \langle \varphi, V\varphi \rangle \; . \tag{18}$$

The operators A and B are defined by $(A\varphi,\psi)_V = a(\varphi,\psi)$ and $(B\varphi,\psi)_V = b(\varphi,\psi)$, hence

$$A = \frac{1}{2}I - K' \; ; \qquad B = \frac{1}{2}I + K' \; . \tag{19}$$

We conclude from our construction that the hypotheses of Lemma 2 are satisfied. In particular, $\frac{1}{2}I \pm K'$ are bounded operators in $H^{-1/2}(\Gamma)$, selfadjoint

and positive semidefinite with respect to the inner product $(\varphi,\psi)_V = \langle\varphi, V\psi\rangle$. As we explained after Lemma 1, the positive definiteness of A or, equivalently, the Poincaré estimate $b \le \mu a$ is a simple consequence of the identity between X and $H^{-1/2}(\Gamma)$: $b(\varphi,\varphi)$ is the energy integral $\Phi(\mathcal{S}\varphi,\mathcal{S}\varphi)$, and $u = \mathcal{S}\varphi$ is the solution of the Dirichlet problem $Lu = 0$ in Ω, $u = V\varphi$ on Γ, hence $\Phi(\mathcal{S}\varphi,\mathcal{S}\varphi)$ is bounded by $\|V\varphi\|^2_{H^{1/2}(\Gamma)}$. Now V is continuous from $H^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$, so we get an estimate by $\|\varphi\|^2_{H^{-1/2}(\Gamma)}$. That this in turn can be estimated by $a(\varphi,\varphi)$ is an a-priori estimate for the solution of the exterior Neumann problem which follows from its variational formulation.

In this way we obtain that B is a contraction. If we want to show that A is a contraction, too, or even stronger that $B - A$ is a contraction, we need the positive definiteness of B, and this is not satisfied, in general, if the space $\mathcal{R}$ of functions of vanishing energy in Ω is non-trivial. The nullspace of the form b consists of densities whose single layer potential has vanishing energy on Ω:

$$b(\psi,\psi) = 0 \Longleftrightarrow \mathcal{S}\psi \in \mathcal{R} \Longleftrightarrow V\psi \in \gamma\mathcal{R}\ .$$

To make B positive definite, we have to factor this kernel out, which is done by the definition [31]

$$\begin{aligned} H_0^{-1/2}(\Gamma) &= \{\varphi \in H^{-1/2}(\Gamma) \mid \forall\psi \in \ker B : (\varphi,\psi)_V = 0\} \\ &= \{\varphi \in H^{-1/2}(\Gamma) \mid \forall u \in \mathcal{R} : \langle\varphi,\gamma u\rangle = 0\} \end{aligned} \tag{20}$$

Equivalently, we could have passed to the quotient space $H^{-1/2}(\Gamma)/\gamma\mathcal{R}$. In any case, we then find that B is positive definite, which by Lemma 2 implies that both $B - A$ and A are contractions. We also note that since A and B commute, $\ker B$ and its orthogonal complement are invariant subspaces of A. We summarize these results:

Theorem 1. *The operators $A = \frac{1}{2}I - K'$ and $B = \frac{1}{2}I + K'$ are positive semidefinite bounded selfadjoint operators on the Hilbert space $H^{-1/2}(\Gamma)$ equipped with the inner product $(\cdot,\cdot)_V$. The operator $\frac{1}{2}I - K'$ is positive definite, and the operator $\frac{1}{2}I + K'$ is a contraction. The Neumann series*

$$(\frac{1}{2}I - K')^{-1} = \sum_{\ell=0}^{\infty}(\frac{1}{2}I + K')^{\ell}$$

converges in $H^{-1/2}(\Gamma)$ in the operator norm associated with the norm $\|\cdot\|_V$. On the subspace $H_0^{-1/2}(\Gamma)$, the operator $\frac{1}{2}I + K'$ is positive definite, and the operators $\frac{1}{2}I - K'$ and $B - A = 2K'$ are contractions. On this subspace, there are the convergent Neumann series:

$$(\frac{1}{2}I - K')^{-1} = 2\sum_{\ell=0}^{\infty}(2K')^{\ell}$$
$$(\frac{1}{2}I + K')^{-1} = \sum_{\ell=0}^{\infty}(\frac{1}{2}I - K')^{\ell}$$
$$(\frac{1}{2}I + K')^{-1} = 2\sum_{\ell=0}^{\infty}(-2K')^{\ell}$$

4.2 Double Layer Potentials

As a second possibility, we now look at double layer potentials. In order to have finite energy, we have to take a space of more regular functions for our classical departure space X_0, Hölder continuous functions for example. For $\varphi, \psi \in X_0$, we now define the bilinear forms a and b as energy forms of the corresponding double layer potentials:

$$a(\varphi,\psi) = \Phi^c(\mathcal{D}\varphi, \mathcal{D}\psi)\ ; \qquad b(\varphi,\psi) = \Phi(\mathcal{D}\varphi, \mathcal{D}\psi)\ . \tag{21}$$

With the boundary reduction by Green's formula (15) and the expressions (14) for the traces of the double layer potential, we find the boundary integral forms

$$a(\varphi,\psi) = \langle W\varphi, (\frac{1}{2}I + K)\psi\rangle\ ; \qquad b(\varphi,\psi) = \langle W\varphi, (\frac{1}{2}I - K)\psi\rangle\ . \tag{22}$$

The total energy $a+b$ is now given by the bilinear form defined by the operator W of the conormal derivative of the double layer potential. It is easy to see that the nullspace of W is given by the traces of the zero-energy fields $\mathcal{R}$. Densities in $\gamma\mathcal{R}$ generate double layer potentials that are identically zero in the exterior domain Ω^c and belong to $\mathcal{R}$ in Ω. In order to be able to apply our program, to get a positive definite bilinear form a and hence Hilbert space X, we have to factor these densities out from the beginning. Our Hilbert space is therefore a quotient space

$$X = H^{1/2}(\Gamma)/\gamma\mathcal{R} \text{ with norm } \|\varphi\|_W^2 = \langle W\varphi, \varphi\rangle\ . \tag{23}$$

This is the natural dual space of $H_0^{-1/2}(\Gamma)$ with respect to $L^2(\Gamma)$ duality. We know from the variational solution of the Dirichlet problem $Lu = 0$ in Ω or Ω^c, $\gamma u = \varphi$ or $\gamma^c u = \varphi$, that on this space the square of the (quotient) norm is equivalent to each one of the energy forms $\Phi(u,u)$ and $\Phi^c(u,u)$. Thus both quadratic forms a and b can be mutually estimated, and we get the full result of Lemmas 3 and 2.

It remains to identify the operators A and B. We have for all $\varphi, \psi \in X$:

$$\langle W\varphi, (\frac{1}{2}I + K)\psi\rangle = (\varphi, A\psi)_W = \langle W\varphi, A\psi\rangle$$

and similarly for B. This shows that if $\pi_{\mathcal{R}} : H^{1/2}(\Gamma) \to X$ is the canonical projection on the quotient space, we have

$$A = \pi_{\mathcal{R}}(\frac{1}{2}I + K) ; \qquad B = \pi_{\mathcal{R}}(\frac{1}{2}I - K) .$$

In the case of the operator A, we can omit the extra factor $\pi_{\mathcal{R}}$, because $\ker(\frac{1}{2}I+K) = \gamma\mathcal{R}$, and therefore $\frac{1}{2}I+K$ is well-defined on the quotient space and commutes with the projector. This remark does not apply in the same way to the operator B, but since $\frac{1}{2}I - K$ commutes with $\frac{1}{2}I + K$, the kernel $\gamma\mathcal{R}$ of the latter is an invariant subspace of the former, so that $\frac{1}{2}I - K$ is also defined in a natural way on the quotient space. The operator $\frac{1}{2}I - K$ actually acts as the identity on the subspace $\gamma\mathcal{R}$, so that its inverse on the whole space $H^{1/2}(\Gamma)$ can be obtained from the inverse on the quotient space. Altogether, we can simply write without ambiguity

$$A = \frac{1}{2}I + K ; \qquad B = \frac{1}{2}I - K . \tag{24}$$

We can now summarize the conclusion of Lemma 2 in this case:

Theorem 2. *The operators $A = \frac{1}{2}I+K$ and $B = \frac{1}{2}I-K$ are positive definite bounded selfadjoint operators on the quotient space $H^{1/2}(\Gamma)/\gamma\mathcal{R}$ equipped with the inner product $(\cdot,\cdot)_W$. Both operators, as well as the operator $B-A = -2K$ (Neumann's operator) are contractions in the corresponding operator norm. The Neumann series*

$$(\frac{1}{2}I - K)^{-1} = \sum_{\ell=0}^{\infty}(\frac{1}{2}I + K)^{\ell}$$

$$(\frac{1}{2}I - K)^{-1} = 2\sum_{\ell=0}^{\infty}(2K)^{\ell}$$

$$(\frac{1}{2}I + K)^{-1} = \sum_{\ell=0}^{\infty}(\frac{1}{2}I - K)^{\ell}$$

$$(\frac{1}{2}I + K)^{-1} = 2\sum_{\ell=0}^{\infty}(-2K)^{\ell}$$

all converge in the operator norm in the quotient space, which corresponds to convergence in $H^{1/2}(\Gamma)$ modulo the traces $\gamma\mathcal{R}$ of the zero-energy fields in Ω. The first Neumann series for the operator $(\frac{1}{2}I - K)^{-1}$ converges in the whole Sobolev space $H^{1/2}(\Gamma)$.

4.3 Single Layer Potentials via Dirichlet Data

The bijectivity of the single layer integral operator V offers another possible interpretation of the results of Section 4.1: Instead of representing a single

layer potential $v = \mathcal{S}\psi$ by its density ψ, one can represent it by its Dirichlet trace $\gamma v = V\psi$. Since $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is bijective, it can be used to transport the Hilbert space structure on $H^{-1/2}(\Gamma)$ which we considered before to $H^{1/2}(\Gamma)$. From the relation

$$(\varphi, \psi)_V = \langle \varphi, V\psi \rangle = \langle V^{-1}V\varphi, V\psi \rangle$$

we see that if we define the inner product on $H^{1/2}(\Gamma)$ by

$$(u, v)_{V^{-1}} = \langle V^{-1}u, v \rangle \ ,$$

then $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ becomes an isometry. Instead of writing our whole program once again with a new space X, we can simply transport all the results of Section 4.1 via this Hilbert space isomorphism. Positivity, operator norms and convergence of Neumann series are conserved, the only question that has to be settled is the form of the operators A and B in this new representation.

The answer to this question is provided by the well-known relation

$$KV \ = \ VK'$$

which is one of the relations that give the projection property of the Calderón projector, obtained from the representation of a single layer potential as a sum of a single layer potential and a double layer potential of its own Cauchy data.

The operator $A = \frac{1}{2}I - K'$ on $H^{-1/2}(\Gamma)$ is therefore transported to the operator $VAV^{-1} = V(\frac{1}{2}I - K')V^{-1} = \frac{1}{2}I - K$, and $B = \frac{1}{2}I + K'$ is transported to the operator $\frac{1}{2}I + K$. In this way, we can transport all of Theorem 1. In particular, $\frac{1}{2}I + K$ is a contraction on $H^{1/2}(\Gamma)$ equipped with the norm $\|\cdot\|_{V^{-1}}$. For the other results we have to transport the subspace $H_0^{-1/2}(\Gamma)$. We find

$$VH_0^{-1/2}(\Gamma) = \{\varphi \in H^{1/2}(\Gamma) \mid \forall u \in \mathcal{R} : (\varphi, \gamma u)_{V^{-1}} = 0\}$$

On this space, the operator $\frac{1}{2}I - K$ and Neumann's operator $-2K$ are contractions.

Thus we get similar results as in Section 4.2, with a different norm on $H^{1/2}(\Gamma)$. The results in this form (except for the operator $-2K$) were first proved by Steinbach and Wendland in [31].

4.4 Final Remarks

Although our results obtained here from Poincaré's estimates are largely similar to the results of Steinbach and Wendland in [31], their method for proving the contraction property of $\frac{1}{2}I \pm K$ and $\frac{1}{2}I \pm K'$ is different:

The simple idea here was that if two positive numbers add up to 1, then both of them must be smaller than 1; with Lemma 2 as a transposition of this idea to the class of selfadjoint operators on a Hilbert space.

The corresponding simple idea in [31] is that if a number is bigger than its square, then it must lie between 0 and 1. For operators, this idea can be stated as follows:

Let A and B be bounded selfadjoint operators on a Hilbert space. If

$$B = B^2 + A \text{ and } A \text{ is positive definite, } A \geq \alpha I,$$

then B is a contraction with norm $\|B\| \leq \frac{1}{2} + \sqrt{\frac{1}{4} - \alpha}$.

This lemma can be applied to the well-known relations

$$(\frac{1}{2}I + K)(\frac{1}{2}I - K) = VW\ ; \qquad (\frac{1}{2}I + K')(\frac{1}{2}I - K') = WV$$

which are a consequence of the symmetry of the energy form $\Phi(u, v)$ between a double layer potential u and a single layer potential v, or also of the projection property of the Calderón projector. Since WV is positive semi-definite in the inner product $(\cdot, \cdot)_V$ and VW is positive definite in the inner product $(\cdot, \cdot)_W$ and positive semi-definite in the inner product $(\cdot, \cdot)_{V^{-1}}$, the respective contraction properties for $\frac{1}{2}I \pm K$ and $\frac{1}{2}I \pm K'$ follow.

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