2

Complexity Theory of 3-Manifolds

Denote by $\mathcal{M}$ the set of all compact 3-manifolds. We wish to study it systematically and comprehensively. The crucial question is the choice of filtration in $\mathcal{M}$. It would be desirable to have a finite number of 3-manifolds in each term of the filtration, all of them being in some sense simpler than those in the subsequent terms. A useful tool here would be a measure of "complexity" of a 3-manifold. Given such a measure, we might hope to enumerate all "simple" manifolds before moving on to more complicated ones. There are several well-known candidates for such a complexity function. For example, take the Heegaard genus $g(M)$, defined to be the minimal genus over all Heegaard decompositions of M. Other examples include the minimal number of simplices in a triangulation of M and the minimal crossing number in a surgery presentation for M.

Each of these measures has its shortcomings. The Heegaard genus is additive with respect to connected sums of 3-manifolds, but for $g \geq 1$ there are infinitely many distinct manifolds of Heegaard genus g, and already for $g = 2$ one can hardly expect a simple classification. The surgery complexity has the same defect (because of framing). The minimal number of simplices in a triangulation is not a "natural" measure of complexity because the simplest possible closed manifold, S^3, already would have nonzero complexity, and we would have no chances to get the additivity.

In this chapter, an integral non-negative function $c\colon \mathcal{M} \to \mathcal{Z}$ is constructed, which has the following properties:

1. c is additive, that is, $c(M_1 \# M_2) = c(M_1) + c(M_2)$.
2. For any $k \in Z$, there are only finitely many closed irreducible manifolds $M \in \mathcal{M}$ with complexity $c(M) = k$.
3. $c(M)$ is relatively easy to estimate.

2.1 What is the Complexity of a 3-Manifold?

2.1.1 Almost Simple Polyhedra

As we know from Sect. 1.1.4, any homeomorphism between special spines can be extended to a homeomorphism between the corresponding manifolds (Theorem 1.1.17). This means that a special spine P of a 3-manifold M may serve as a presentation of M. Moreover, M can be reconstructed from a regular neighborhood $N(SP)$ in P of the singular graph SP of P: Starting from $N(SP)$, one can easily reconstruct P by attaching 2-cells to all the circles in $\partial N(SP)$, and then reconstruct M. If M is orientable, then $N(SP)$ can be embedded into R^3. This gives us a very convenient way for presenting 3-manifolds: we simply draw a picture, see Fig. 2.1.

Theorem 2.1.1. *For any integer k there exists only a finite number of special spines with k true vertices. All of them can be constructed algorithmically.*

Proof. We will construct a finite set of special polyhedra that a fortiori contains all special spines with k true vertices. First, one should enumerate all regular graphs of degree 4 with k true vertices. Clearly, there is only a finite number of them. Given a regular graph, we replace each true vertex v by a copy of the butterfly E that presents a typical neighborhood of a true vertex in a simple polyhedron, see Definition 1.1.8. Neighborhoods in ∂E of triple points of ∂E (we will call them *triodes*) correspond to edges having an endpoint at v. In Fig. 2.2 the triodes are shown by fat lines. For each edge e, we glue together the triodes that correspond to endpoints of e via a homeomorphism between them. It can be done in six different ways (up to isotopy). We get a simple polyhedron P with boundary. Attaching 2-discs to the circles in

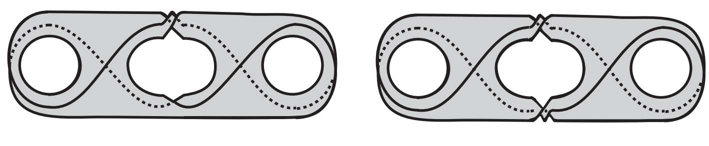

Fig. 2.1. Bing's House with two Rooms and its mutant (another special spine of the cube) presented as regular neighborhoods of their singular graphs

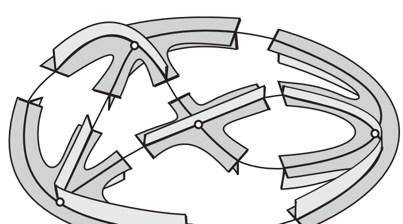

Fig. 2.2. A decomposition of $N(SP)$ into copies of E

∂P, we get a special polyhedron. Since at each step we have had only a finite number of choices, this method produces a finite set of special polyhedra. Not all of them are thickenable. Nevertheless, the set contains all special spines with k true vertices. □

It would be a natural idea to measure the complexity of a 3-manifold by the number of true vertices of its special spine. This characteristic is convenient in that there exists only a finite number of 3-manifolds having special spines with a given number of vertices. But it has two shortcomings. First, it is not additive with respect to connected sums. Second, restricting ourselves to special spines, we lose the possibility to consider very natural spines such as a point for the ball (and S^3), a circle for the solid torus, and a projective plane for the projective space RP^3. Also, working only with special spines, we are sometimes compelled to make artificial tricks to preserve the special polyhedra structure. For example, in the proof of Theorem 1.1.13 we used a delicate arch construction instead of simply making a hole in a 2-cell.

All these shortcomings have the same root: the property of being special is not hereditary. In other words, a subpolyhedron of a special polyhedron may not be special, even if it cannot be collapsed onto a smaller subpolyhedron. This is why we shall widen the class of special polyhedra by considering a class of what we call almost simple polyhedra. Roughly speaking, the class of almost simple polyhedra is the minimal class which contains special polyhedra and is closed with respect to the passage to subpolyhedra.

Definition 2.1.2. *A compact polyhedron P is said to be* almost simple *if the link of any of its points can be embedded into Γ_4, a complete graph with four vertices. A spine P of a 3-manifold M is* almost simple*, if it is an almost simple polyhedron.*

It is convenient to present Γ_4 as a circle with three radii or as the boundary of the standard butterfly. One usually considers only almost simple polyhedra that cannot be collapsed onto smaller subpolyhedra. It is easy to see that any proper subpolyhedron of the circle with three radii can be collapsed onto a polyhedron L having one of the following types:

(a) L is either empty or a finite set of $n \geq 2$ points.
(b) L is the union of a finite (possibly empty) set and a circle.
(c) L is the union of a finite (possibly empty) set and a circle with a diameter.
(d) L is a Γ_4.

The "cannot start" property assures us that an almost simple polyhedron P cannot be collapsed onto a smaller subpolyhedron if and only if the link L of any point of P is contained in the above list.

For example, a wedge of any simple polyhedron and any graph without free vertices satisfies this condition and hence cannot be collapsed onto a smaller subpolyhedron. This example is very typical, since any almost simple polyhedron P can be presented as the union of its 2-dimensional and its 1-dimensional parts. The 1-dimensional part (the closure of the set of points with 0-dimensional links) is a graph, the 2-dimensional part consists of points whose links contain an arc. If P cannot be collapsed onto a smaller subpolyhedron, then its 2-dimensional part is a simple polyhedron (maybe disconnected).

The notions of a true vertex, singular graph, 2-component of an almost simple polyhedron are introduced in the same way as for simple polyhedra, see Sect. 1.1.3. A *true vertex* of an almost simple polyhedron P is a point with the link $L = \Gamma_4$, the *singular graph* SP consists of points whose links contain a circle with a diameter, and *2-components* are the connected components of the set of all the points whose links contain a circle but do not contain a circle with a diameter. Note that the 1-dimensional part does not affect these notions. For instance, a 2-component may contain a point of the 1-dimensional part, and this point is not a true vertex of P.

Almost simple spines are easier to work with than special spines, since we may puncture cells and stay within the realm of almost simple spines. So, for example, the process we used to construct a special spine for a given manifold may be simplified to give an almost simple spine; there is no longer need for the arch construction, see Fig. 1.8.

2.1.2 Definition and Estimation of the Complexity

The complexity function adverted to in the introduction to this chapter can now be defined.

Definition 2.1.3. *The* complexity $c(P)$ *of a simple polyhedron P is equal to the number of its true vertices.*

Definition 2.1.4. *The* complexity $c(M)$ *of a compact 3-manifold M is equal to k if M possesses an almost simple spine with k true vertices and has no almost simple spines with a smaller number of true vertices. In other words, $c(M) = \min_P c(P)$, where the minimum is taken over all almost simple spines of M.*

Let us give some examples. The complexity of S^3, of the projective space RP^3, of the lens space $L_{3,1}$, and the manifold $S^2 \times S^1$ is equal to zero, since they possess almost simple spines without true vertices: the point, the projective plane, the triple hat, and the wedge of S^2 with S^1, respectively. Recall that by the triple hat we mean the quotient space of D^2 by a free action of the group Z_3 on ∂D^2. Among compact manifolds with boundary, zero complexity is possessed by all handlebodies, I-bundles over surfaces, as well as some other manifolds such as the complement of the trefoil knot. Indeed, any handlebody

collapses to a graph that (being considered as an almost simple polyhedron) has no true vertices. The I-bundles collapse to surfaces, and the complement of the trefoil collapses to the quotient space of the Möbius band by a free action of the group Z_3 on the boundary.

In general, the problem of calculating the complexity $c(M)$ is very difficult. Let us start with a simpler problem of estimating $c(M)$. To do that it suffices to construct an almost simple spine P of M. The number of true vertices of P will serve as an upper bound for the complexity. Since an almost simple spine can be easily constructed from practically any presentation of the manifold, the estimation problem does not give rise to any difficulties. Let us describe several estimates of the complexity based on different presentations of 3-manifolds. It is convenient to start with an observation that removing an open ball does not affect the complexity.

Proposition 2.1.5. *Suppose that B is a 3-ball in a 3-manifold M. Then $c(M) = c(M \setminus Int\ B)$.*

Proof. If M is closed, then $c(M) = c(M \setminus \mathrm{Int}\ B)$ since M and $M \setminus \mathrm{Int}\ B$ have the same spines by definition of the spine of a closed manifold. Let $\partial M \neq \emptyset$, and let P be an almost simple spine of $M \setminus \mathrm{Int}\ B$ possessing $c(M \setminus \mathrm{Int}\ B)$ true vertices. Denote by C the connected component of the space $M \setminus P$ containing B. Since M is not closed, there exists a 2-component α of P that separates C from another component of $M \setminus P$. Removing an open 2-disc from α and collapsing yields an almost simple spine $P_1 \subset P$ of M. The number of true vertices of P_1 is no greater than that of P, since puncturing α and collapsing results in no new true vertices. Therefore, $c(M) \leq c(M \setminus \mathrm{Int}\ B)$.

To prove the converse inequality, consider an almost simple spine P_1 of M with $c(M)$ true vertices. Let us take a 2-sphere S in M such that $S \cap P_1 = \emptyset$. Join S to P_1 by an arc ℓ that has no common points with $P_1 \cup S$ except the endpoints. Clearly, $P = P_1 \cup S \cup \ell$ is an almost simple spine of $M \setminus \mathrm{Int}\ B$. New true vertices do not arise. It follows that $c(M) \geq c(M \setminus \mathrm{Int}\ B)$. □

In Sect. 1.1.5 we described a relation between singular triangulations of closed 3-manifolds and special polyhedra. The same method works for estimating the complexity.

Proposition 2.1.6. *Suppose a 3-manifold M is obtained by pasting together n tetrahedra by affine identifications of their faces. Then $c(M) \leq n$.*

Proof. Recall that any tetrahedron Δ contains a canonical copy $P_\Delta = \cup|\ \mathrm{lk}_i(v_i, \Delta')|$ of the standard butterfly E, where $v_i, 0 \leq i \leq 3$, are the vertices of Δ. When pasting together the tetrahedra, these copies are glued together into a simple polyhedron $P \subset M$ that may have a boundary if M is not closed. P has n true vertices and is a spine of M with several balls removed from it. These balls are the neighborhoods of the points which are obtained by gluing the vertices of the tetrahedra and lie in the interior of M. It follows from Proposition 2.1.5 that $c(M) \leq n$. □

Remark 2.1.7. It follows from Corollary 1.1.27 that a closed 3-manifold M possesses a special spine with n true vertices if and only if it can be obtained by pasting together n tetrahedra. Further, we shall see that any minimal (in the sense of the number of true vertices) almost simple spine of a closed orientable irreducible 3-manifold M which differs from the "exceptional" manifolds $S^3, RP^3, L_{3,1}$, is special. Therefore, the complexity of such a manifold may be defined as the minimal number of tetrahedra that is sufficient to obtain M.

Proposition 2.1.8. *Suppose $M = H_1 \cup H_2$ is a Heegaard splitting of a closed 3-manifold M such that the meridians of the handlebody H_1 intersect the ones of H_2 transversally at n points. Suppose also that the closure of one of the components into which the meridians of H_1, H_2 decompose the Heegaard surface $\partial H_1 = \partial H_2$ contains m such points. Then $c(M) \leq n - m$.*

Proof. Denote by P the union of the Heegaard surface $F = \partial H_1 = \partial H_2$ with the meridional discs of the two handlebodies. Then P is a simple polyhedron whose true vertices are the crossing points of the meridians. Since the complement of P in M consists of two open 3-balls, P is a spine of M punctured twice. Removing from P the 2-component $\alpha \subset F$ whose closure contains m true vertices, we fuse together the balls and get an almost simple spine of M which has $n - m$ true vertices, since the vertices in the closure of α will cease to be true vertices, see Fig. 2.3. □

Proposition 2.1.9. *Suppose $\widetilde{M}$ is a k-fold covering space of a 3-manifold M. Then $c(\widetilde{M}) \leq kc(M)$.*

Proof. Let P be an almost simple spine of M having $c(M)$ true vertices. Consider the almost simple polyhedron $\widetilde{P} = p^{-1}(P)$, where $p: \widetilde{M} \to M$ is the covering map. Since the degree of the covering is k, the polyhedron $\widetilde{P}$ has $kc(M)$ true vertices. If $\partial M \neq \emptyset$, then $\widetilde{P}$ is an almost simple spine of $\widetilde{M}$, since the collapse of M onto P can be lifted to a collapse of $\widetilde{M}$ onto $\widetilde{P}$. Therefore, $c(\widetilde{M}) \leq kc(M)$.

If M is closed, $\widetilde{P}$ is a spine of the manifold $\widetilde{M} \setminus \pi^{-1}(V)$, where V is an open 3-ball in M. The inverse image $p^{-1}(V)$ consists of k open 3-balls, hence, by Proposition 2.1.5, we have $c(\widetilde{M}) = c(\widetilde{M} \setminus p^{-1}(V)) \leq kc(M)$. □

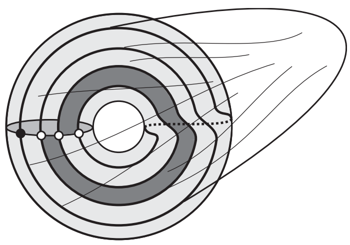

Fig. 2.3. Special spine of $L_{4,1}$ obtained from the standard Heegaard diagram of $L_{4,1}$

Remark 2.1.10. If M in the above proof is closed, then one can get an almost simple spine of $\widetilde{M}$ by puncturing those 2-components of $\widetilde{P}$ that separate different balls in $p^{-1}(V)$. To fuse k balls together, we must make $k-1$ punctures, and each of them decreases the total number of true vertices by the number of true vertices in the boundary of the 2-component we are piercing through. Thus, as a rule, $c(\widetilde{M})$ is significantly less than $kc(M)$.

Now we turn our attention to link complements and surgery presentations of 3-manifolds. Assume that a link L in the space $R^3 = S^3 \setminus \{*\}$ with coordinates x, y, z is in a general position with respect to the projection of R^3 onto the plane R^2 with the coordinates x, y. We will use the generally accepted way of presenting L by its projection $\bar{L}$, disconnecting it at the lower double points. The words *lower* and *higher* are understood in the sense of the value of the coordinate z. Connected components of the projection cut up in this way will be called *overpasses.* Each overpass is bounded by two lower points, and contains several upper crossing points. Their number will be called the *overpass degree.* We may look at the link from below and disconnect it at upper double points. Then we get *underpasses.* The number of lower points on an underpass is called the *underpass degree.* Let us call an overpass and an underpass *independent,* if the corresponding sets of double points (including the endpoints) are disjoint.

Often it is convenient to think of L as being contained in $S^3 = R^3 \cup \{*\}$ rather than in R^3. Then the projection $\bar{L}$ of L is in the sphere $S^2 = R^2 \cup \{*\}$. In this case the complement space $C(L) = S^3 \setminus \mathrm{Int}\, N(L)$, where $N(L)$ is an open tubular neighborhood of L in S^3, is a compact 3-manifold.

Proposition 2.1.11. *Suppose a link $L \subset S^3$ is given by a projection $\bar{L}$ with n crossing points so that there are an overpass of degree k and an independent underpass of degree m. Then the complexity of the complement space $C(L)$ of L is no greater than $4(n-m-k-2)$.*

Proof. Let us attach the annulus $S^1 \times I$ along the projection $\bar{L}$ to S^2 and to the other parts of the annulus previously pasted on. We get a “tunnel,” see Fig. 2.4, where the attaching procedure is shown in the neighborhood of a crossing point.

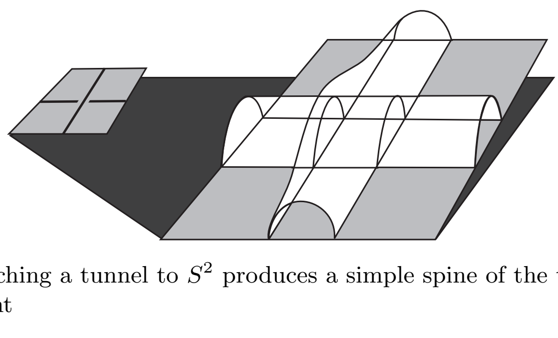

Fig. 2.4. Attaching a tunnel to S^2 produces a simple spine of the twice punctured link complement

The result will be a simple polyhedron P with $4n$ true vertices: each crossing point produces four of them. The complement to a regular neighborhood $N(P)$ of P in S^3 is the union of a tubular neighborhood $N(L)$ of L and two balls B_1, V_2 that lie inside and outside S^2, respectively. In order to get a spine of $C(L)$, one should fuse the balls with $N(L)$ by puncturing two 2-components of P that separate $N(L)$ from the balls. Choose for the puncture the 2-components $\alpha \subset S^1 \times I$ and $\beta \subset S^2$ that correspond to the overpass of degree k and the underpass of degree m, respectively. When we remove α, then $4k+4$ true vertices disappear (two pairs correspond to the endpoints of the overpass, and $4k$ are related with k crossing points). Removing β, we destroy $4m+4$ true vertices. It follows that after collapsing we get an almost simple spine of $C(L)$ with no more than $4(n-m-k-2)$ vertices. □

Remark 2.1.12. It can be shown that if the projection $\bar{L}$ has $n \geq 6$ crossings, then one can always find an overpass and independent underpass satisfying $k+m \geq 2$. The complexity of $C(L)$ can then be estimated by $4n-16$. If there are no independent overpasses and underpasses, then one can use dependent ones or, alternatively, puncture a 2-component that lies on S^2 and separates the balls B_1, B_2. The number of disappearing true vertices in this case may be smaller, since the same true vertex may be taken into account twice.

Consider now the surgery presentation of 3-manifolds [72]. For simplicity, we restrict ourselves to the case when M is presented by a framed knot K. Recall that the *writhe* $w(\bar{K})$ of a projection $\bar{K}$ may be defined as the framing number of the "vertical" framing of K by the vector field orthogonal to R^2. To get an arbitrary framing s, one should twist the vertical framing $|s-w(\bar{K})|$ times in the appropriate direction.

Denote by ℓ the *preferred longitude* of K, i.e., the simple closed curve in $\partial N(K)$ that intersects a meridian m of $\partial N(K)$ at one point and is homologous to 0 in the complement to $N(K)$. Let K have the framing s. To convert S^3 to M, one should make two steps:

(1) Cut $N(K)$ out of S^3
(2) Glue in the solid torus $D^2 \times S^1$ so that the meridian $\partial D^2 \times \{*\}$ winds once around the longitude ℓ and s times around the meridian m

Proposition 2.1.13. *Suppose M is obtained by Dehn surgery along a knot K with framing s such that the projection $\bar{K}$ of K has $n \geq 1$ crossing points. Then $c(M) \leq 5n + |s - w(\bar{K})|$.*

Proof. First we assume that $s = w(\bar{K})$ or, equivalently, that the framing of K is vertical. Let P be a simple spine of the twice punctured complement $C(K)$ of K constructed in the proof of Proposition 2.1.11, i.e., the sphere S^2 with a tunnel attached along $\bar{K}$. Then one can get a simple spine P_1 of M punctured three times by attaching the disc D^2 along the top line of the tunnel. The disc plays the role of the meridional disc of the solid torus that is glued in instead of $N(K)$. Each time when the tunnel climbs onto itself, there appear two new

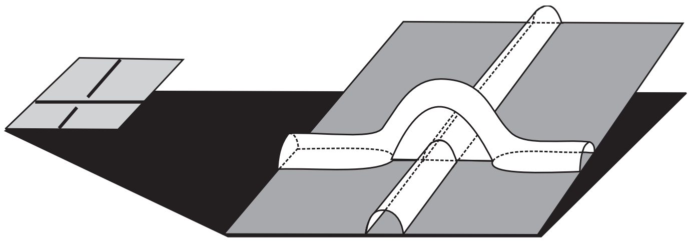

Fig. 2.5. An alternative construction of an almost simple spine of the link complement. The top line of the tunnel contains a smaller number of triple points, and each its winding around the meridian produces only one new true vertex

true vertices (where the base lines of the upper tunnel intersect the top line of the lower one). Thus P_1 possesses $6n$ true vertices (n is the number of crossing points of $\bar{K}$). To decrease the number of true vertices, we modify the construction of P as shown in Fig. 2.5. The new spine P of $C(K)$ punctured twice has the same number of true vertices, but the corresponding new spine P_1 of trice punctured M will have only $5n$ true vertices. The explanation is simple: if the tunnel climbs onto itself, then in the top line of the lower part of the tunnel there appears only one new true vertex.

If $s \neq w(\bar{K})$, one should force the top line of the tunnel to make $|s - w(\bar{K})|$ additional rotations. Each of them produces a new true vertex, so the total number of true vertices would be $5n + |s - w(\bar{K})|$. It remains to puncture two 2-components of P_1 that separate different balls and get an almost simple spine of M with a smaller number of true vertices. □

2.2 Properties of Complexity

2.2.1 Converting Almost Simple Spines into Special Ones

We have already stated the advantages of using almost simple spines, yet there are important downsides too. In general, almost simple spines determine 3-manifolds in a nonunique way, and cannot be represented by regular neighborhoods of their singular graphs alone. Since special spines, as has been mentioned before, are free from such liability, we would like to go from almost simple polyhedra to special ones whenever possible. So the question is: when is it possible? We shall study it in this section.

Let P be an almost simple spine of a 3-manifold M that is not a special one. Then P either possesses a 1-dimensional part or has 2-components not homeomorphic to a disc. Our aim is to transform P into a special spine of M without increasing the number of true vertices. In general this is impossible. For example, if M is reducible or has compressible boundary, any minimal almost simple spine of M must contain a 1-dimensional part. Nevertheless, in some cases it is possible. To give an exact formulation, we need to recall a few notions of 3-manifold topology.

Definition 2.2.1. *A 3-manifold M is called* irreducible *, if every 2-sphere in M bounds a 3-ball.*

If M is reducible, then either it can be decomposed into nontrivial connected sum, or is one of the manifolds $S^2 \times S^1$, $S^2 \tilde{\times} S^1$.

Recall that a compact surface F in a 3-manifold M is called *proper*, if $F \cap \partial M = \partial F$.

Definition 2.2.2. *A 3-manifold M is* boundary irreducible*, if for every proper disc $D \subset M$ the curve ∂D bounds a disc in ∂M.*

Definition 2.2.3. *Let M be an irreducible boundary irreducible 3-manifold. A proper annulus $A \subset M$ is called* inessential*, if either it is parallel rel ∂ to an annulus in ∂M, or the core circle of A is contractible in M (in the second case A can be viewed as a tube possessing a meridional disc). Otherwise A is called* essential*.*

Of course, these notions will be considered in more detail later.

Theorem 2.2.4. *Suppose M is a compact irreducible boundary irreducible 3-manifold such that $M \neq D^3, S^3, RP^3, L_{3,1}$ and all proper annuli in M are inessential. Then for any almost simple spine P of M there exists a special spine P_1 of M having the same or a fewer number of true vertices.*

Proof. Identify M (or M with a 3-ball removed, if M is closed) with a regular neighborhood of P. We will assume that P cannot be collapsed to a smaller subpolyhedron. We convert P into P_1 by a sequence of transformations (moves) of three types. To control the number of steps, we assign to any almost simple polyhedron P the following three numbers:

1. $c_2(P)$, the number of 2-components of P.
2. $-\chi_2(P) = -\sum_\alpha \chi(\alpha)$, where the sum is taken over all 2-components α of P and $\chi(\alpha)$ is the Euler characteristic.
3. $c_1(P) = \min e(X_P)$, where the 1-dimensional part X_P of P (i.e., the union of points having 0-dimensional links) is presented as a graph with $e(X_P)$ edges and the minimum is taken over all such presentations.

The triples $(c_2(P), -\chi_2(P), c_1(P))$ will be considered in the lexicographic order.

Move 1. Suppose that the 1-dimensional part X_P of P is nonempty. Consider an arc $\ell \subset X_P$ and a proper disc $D \subset M$ which intersects ℓ transversally at one point. Since M is irreducible and boundary irreducible, D cuts a 3-ball B out of M. Removing $B \cap P$ from P and collapsing the rest of P as long as possible, we get a new almost simple spine $P' \subset M$. If $B \cap P$ contains at least one 2-component of P, then $c_2(P') < c_2(P)$. If

$B \cap P$ is 1-dimensional, then the 2-dimensional parts of P, P' coincide and thus $c_2(P') = c_2(P), -\chi_2(P') = -\chi_2(P)$. Of course, $c_1(P') < c_1(P)$.

Assume that a 2-component α of P contains a nontrivial simple closed curve l so that the restriction to l of the normal bundle ν of α is trivial. If α is not D^2, S^2 or RP^2, then l always exists. It follows that one can find a proper annulus $A \subset M$ that intersects P transversally along l. Since all annuli are inessential, either A is parallel to the boundary or its core circle is contractible.

Move 2. Suppose that A is parallel to the boundary. Then it cuts off a solid torus V from M so that the remaining part of M is homeomorphic to M. Removing $V \cap P$ from P, we obtain (after collapsing) a new almost simple spine $P' \subset M$. This move annihilates α, so $c_2(P') < c_2(P)$.

Move 3. Suppose that the core circle of A is contractible. Then both circles of ∂A are also contractible. Choose one of them. By Dehn's Lemma [106], it bounds a disc in M and, since M is boundary irreducible, a disc D in ∂M. It follows that there is a disc $D \subset \text{Int } M$ such that $D \cap P = \partial D = l$. Since $M \setminus P$ is homeomorphic to $\partial M \times (0, 1]$, D cuts a proper open 3-ball B out of $M \setminus P$, see Definition 1.2.12. If we puncture D, collapse B and then collapse the rest of D, we return to P. However, if we get inside the ball B through another 2-component of the free boundary of B (see Fig. 2.6), we get after collapsing a new almost simple spine $P' \subset M$.

Let us analyze what happens to α under this move. If l does not separate α, then the collapse eliminates α completely together with D. In this case we have $c_2(P') < c_2(P)$.

Suppose that l separates α into two parts, α' and α'' (the notation is chosen so that the hole is in α''). Then the collapse destroys α'', and we are left with $\alpha' \cup D$. In this case either $c_2(P') < c_2(P)$ (if the collapse destroys some other 2-components of P), or $c_2(P') = c_2(P)$ and $-\chi_2(P') < \chi_2(P)$ since $-\chi(\alpha' \cup D) < -\chi(\alpha)$.

Now let us perform Steps 1, 2, 3 as long as possible. The procedure is finite, since each step strictly decreases the triple $(c_2(P), -\chi_2(P), c_1(P))$ and hence any monotonically decreasing sequence of triples is finite. Let P_1 be the resulting almost simple spine of M. By construction, P_1 has no 1-dimensional

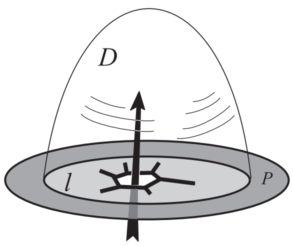

Fig. 2.6. Attaching D^2 along l and puncturing another 2-component produces a simpler spine

part and no 2-components different from D^2, S^2, and RP^2. The following cases are possible:

1. P_1 has no 2-components at all. Since it also has no 1-dimensional part, P_1 is a point and thus $M = S^3$ or $M = D^3$.
2. P_1 contains a 2-component which is not homeomorphic to the disc. In this case P_1 is either RP^2 or S^2. Suppose that P_1=RP^2. Then $M = RP^2 \times I$ or RP^3. We cannot have $M = RP^2 \tilde{\times} I$, since this manifold is a punctured projective space and hence is reducible. For the same reason we cannot have $P_1 = S^2$: the manifold $S^2 \times I$ is reducible.
3. All the 2-components of P_1 are discs and P_1 has no true vertices but contains triple points. Denote by k the number of 2-components of P_1. We cannot have $k = 3$, since the union of three discs with common boundary is a spine of S^3 with three punctures, which is a reducible manifold. The simple polyhedron obtained by attaching two discs to a circle is unthickenable, see Example 1.1.18. We may conclude that P_1 has only one 2-component, which is homeomorphic to the disc. In this case M is homeomorphic to $L_{3,1}$.
4. There remains only one possibility: P_1 has true vertices and all its 2-components are discs. In this case P_1 is special.

□

2.2.2 The Finiteness Property

Theorem 2.2.5. *For any integer k, there exists only a finite number of distinct compact irreducible boundary irreducible 3-manifolds that contain no essential annuli and have complexity k.*

Proof. Follows immediately from Theorems 2.2.4 and 2.1.1. □

Restricting ourselves to the most interesting case of closed orientable irreducible 3-manifolds, we immediately get Corollary 2.2.6.

Corollary 2.2.6. *For any integer k, there exists only a finite number of distinct closed orientable irreducible 3-manifolds of complexity k.*

Recall that a compact 3-manifold M is *hyperbolic* if Int M admits a complete hyperbolic metrics of a finite volume. It is known (see [136]) that any hyperbolic 3-manifold is irreducible, has incompressible boundary, and contains no essential annuli.

Corollary 2.2.7. *For any integer k, there exists only a finite number of distinct orientable hyperbolic 3-manifolds of complexity k.*

Both corollaries follow immediately from Theorem 2.2.5. Let $n_c(k)$ and $n_h(k)$ be the numbers of all closed orientable irreducible 3-manifolds of complexity k and all orientable hyperbolic 3-manifolds of complexity k, respectively. Then for small k the exact values of these numbers are listed in the table below.

k	0 1 2 3
$n_c(k)$	3 2 4 7
$n_h(k)$	0 0 2 9

Remark 2.2.8. To show that the assumptions of Theorem 2.2.5 are essential, let us describe three infinite sets of distinct 3-manifolds of complexity 0. The sets consist of manifolds that are either reducible (1), or boundary reducible (2), or contain essential annuli (3).

(1) For any integer n the connected sum M_n of n copies of the projective space RP^3 is a closed manifold of complexity 0. To construct an almost simple spine of M_n without true vertices, one may take n exemplars of the projective plane RP^2 and join them by arcs. Alternatively, one can start with $L_{3,1}$ and the triple hat instead of RP^3 and RP^2.
(2) The genus n handlebody H_n is irreducible, but boundary reducible. Since it can be collapsed onto a 1-dimensional spine, $c(H_n) = 0$.
(3) Manifolds $\partial H_n \times I$ are irreducible and boundary irreducible, but contain essential annuli. They have complexity 0 since can be collapsed onto the corresponding surfaces.

2.2.3 The Additivity Property

Recall that the *connected sum* $M_1 \# M_2$ of two compact 3-manifolds M_1, M_2 is defined as the manifold $(M_1 \setminus \text{Int } B_1) \cup_h (M_2 \setminus \text{Int } B_2)$, where $B_1 \subset \text{Int } M_1$, $B_2 \subset \text{Int } M_2$ are 3-balls, and h is a homeomorphism between their boundaries. If the manifolds are orientable, their connected sum may depend on the choice of h. In this case $M_1 \# M_2$ will denote any of the two possible connected sums. Alternatively, one can use signs and write $M_1 \# (\pm M_2)$

To define the boundary connected sum, consider two discs $D_1 \subset \partial M_1$, $D_2 \subset \partial M_2$ in the boundaries of two 3-manifolds. Glue M_1 and M_2 together by identifying the discs along a homeomorphism $h \colon D_1 \to D_2$. Equivalently, one can attach an index 1 handle to $M_1 \cup M_2$ such that the base of the handle coincides with $D_1 \cup D_2$. The manifold M thus obtained is called the *boundary connected sum* of M_1, M_2 and is denoted by $M_1 \perp\!\!\!\perp M_2$. Of course, M depends on the choice of the discs (if at least one of the manifolds has disconnected boundary), and on the choice of h (homeomorphisms that differ by a reflection may produce different results). Thus the notation $M_1 \perp\!\!\!\perp M_2$ is slightly ambiguous, like the notation for the connected sum. When shall use it to mean that $M_1 \perp\!\!\!\perp M_2$ is one of the manifolds that can be obtained by the above gluing.

Theorem 2.2.9. *For any 3-manifolds M_1, M_2 we have:*

1. $c(M_1 \# M_2) = c(M_1) + c(M_2)$
2. $c(M_1 \perp\!\!\!\perp M_2) = c(M_1) + c(M_2)$

Proof. We begin by noticing that the first conclusion of the theorem follows from the second one. To see that, we choose 3-balls $V_1 \subset \mathrm{Int}\ M_1, V_2 \subset \mathrm{Int}\ M_2$, and $V_3 \subset \mathrm{Int}\ (M_1 \# M_2)$. It is easy to see that $(M_1 \setminus \mathrm{Int}\ V_1) \perp\!\!\!\perp (M_2 \setminus \mathrm{Int}\ V_2)$ and $(M_1 \# M_2) \setminus V_3$ are homeomorphic, where the index 1 handle realizing the boundary connected sum is chosen so that it joins ∂V_1 and ∂V_2. Assuming (2) and using Proposition 2.1.5, we have: $c(M_1 \# M_2) = c((M_1 \# M_2) \setminus V_3) = c(M_1 \setminus \mathrm{Int}\ V_1) + c(M_2 \setminus \mathrm{Int}\ V_2) = c(M_1) + c(M_2)$.

Let us prove the second conclusion. The inequality $c(M_1 \perp\!\!\!\perp M_2) \leq c(M_1) + c(M_2)$ is obvious, since if we join minimal almost simple spines of M_1, M_2 by an arc, we get an almost simple spine of $M_1 \perp\!\!\!\perp M_2$ having $c(M_1) + c(M_2)$ true vertices.

The proof of the inverse inequality is based on Haken's theory of normal surfaces (see Chap. 3). So we restrict ourselves to a reference to Corollary 4.2.10, which states that attaching an index 1 handle preserves complexity. □

2.3 Closed Manifolds of Small Complexity

2.3.1 Enumeration Procedure

It follows from the finiteness property that for any k there exist finitely many closed orientable irreducible 3-manifolds of complexity k. The question is: *how many?* The constructive proof of Theorem 2.1.1 allows us to organize a computer enumeration of special spines with k true vertices. Of course, the list of corresponding 3-manifolds can contain duplicates as well as nonorientable, nonclosed, or reducible manifolds. All such manifolds must be removed.

Let us briefly describe the enumeration results in historical order. First, Matveev and Savvateev tabulated closed irreducible orientable manifolds up to complexity 5, see [91]. The manifolds were listed with the help of a computer and recognized manually. This was the first paper on computer tabulation of 3-manifolds. It contained all basic elements of the corresponding theory, which much later have been rediscovered by various mathematicians. This table was extended to the level of complexity 6 in [80,83]. The same approach was used by Ovchinnikov [102,103] in composing the table of complexity 7. The manifolds were still recognized manually, although by an improved method (by distinguishing and using elementary blocks). Later Martelli wrote a computer program which is based on the same principle, but tabulates 3-manifolds in two steps. First, it enumerates some special building blocks (bricks), and only then assembles bricks into 3-manifolds. An interesting relative version of the complexity theory (see [74]) serves as a theoretical background for the program. We describe it in Sect. 7.7.

Let us present the results of these enumeration processes for $k \leq 7$ (see Sect. 7.5 for the similar results for $k \leq 12$).

Theorem 2.3.1. *The number $n_c(k)$ of closed orientable irreducible 3-manifolds of complexity k for $k \leq 7$ is given by the following table:*

k	0	1	2	3	4	5	6	7
$n_c(k)$	3	2	4	7	14	31	74	175

Closed orientable irreducible 3-manifolds of complexity 0 are the following ones: the sphere S^3, the projective space RP^3, and the lens space $L_{3,1}$. Their almost simple spines without true vertices were described in Sect. 2.1.2. The complexity of $S^2 \times S^1$ is also equal to 0, but this manifold is reducible. Closed orientable irreducible 3-manifolds of complexity 1 are lens spaces $L_{4,1}$ and $L_{5,2}$. There are four 3-manifolds of complexity 2. They are the lens spaces $L_{5,1}$, $L_{7,2}$, $L_{8,3}$, and the manifold S^3/Q_8, where $Q_8 = \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$ is the quaternion unit group (the action of Q_8 on S^3 is linear). See Sect. 2.3.3 and the Appendix for the description and the complete table of manifolds of complexity $k \leq 6$.

Let us give a nonformal description of the computer program that was used for creating the table up to complexity 7. The computer enumerates all the regular graphs of degree 4 with a given number of vertices. The graphs may be considered as work-pieces for singular graphs of special spines. For each graph, the computer lists all possible gluings together of butterflies that are taken instead of true vertices (see the proof of Theorem 2.1.1). Note that if the graph has k vertices, then there are $2k$ edges, and thus potentially 6^{2k} different gluings of the triodes. Not all of them produce spines of orientable manifolds: it may happen that we get a special polyhedron which is not a spine or is a spine of a nonorientable manifold. To avoid this, we supply each copy of E with an orientation (in an appropriate sense), and use orientation reversing identifications of the triodes. This leaves us with no more than $2^{k-1}3^k$ spines of orientable manifolds. One may decrease this number by selecting spines of closed manifolds, but it still remains too large. The problem is that we get a list of spines, while it is a list of manifolds we are interested in (as we know, any 3-manifold has many different special spines). Also, some manifolds from the list thus created would be reducible. A natural idea to obtain a list of manifolds that does not contain duplicates and reducible manifolds consists in considering minimal spines, i.e., spines of minimal complexity. Unfortunately, there are no general criteria of minimality. The good news here is that there are a lot of partial criteria of nonminimality. In Sect. 2.3.2 we present two of them that appeared to be sufficient for casting out all reducible manifolds and almost all duplicates up to $k = 6$.

The completion of the table of closed orientable irreducible 3-manifolds up to complexity 6 was made by hand. It was a big job indeed: for each pair of spines that had passed the minimality tests one must decide whether or not they determine homeomorphic manifolds. In practice we calculated their invariants: homology groups and, in worst cases, fundamental groups [83, 91]. Later, after Turaev–Viro invariants had been discovered, we used them to verify the table. If the invariants did not help to distinguish the manifolds, we tried to transform one spine into the other by different moves that preserve the manifold. In all cases a definitive answer was obtained.

We point out that the Turaev–Viro invariants are extremely powerful for distinguishing 3-manifolds. In particular, invariants of order ≤ 7 distinguish all orientable closed irreducible 3-manifolds up to complexity 6 having the same homology groups. The only exception are lens spaces, for which there is no need to apply these invariants.

2.3.2 Simplification Moves

We describe here only two types of moves. The moves have the following advantage: It is extremely easy to determine whether or not one can apply them to a given special spine.

Definition 2.3.2. *Let P be a special polyhedron and c a 2-component of P. Then we say that the boundary curve of c has a* counterpass*, if it passes along one of the edges of P twice in opposite directions. We say that the boundary curve is* short *, if it passes through no more than 3 true vertices of P and through each of them no more than once.*

For instance, Bing's House contains two 2-components with boundary curves of length 1 while the boundary curve of the third 2-component has a few counterpasses (see Fig. 1.6).

Proposition 2.3.3. *Suppose that P is a special spine of a 3-manifold M such that either:*

1. *P has a 2-component with a short boundary curve.*
2. *M is closed, orientable, and the boundary curve of one of the 2-components of P has a counterpass.*

Then M possesses an almost simple spine with a smaller number of true vertices.

Proof. Assume that P has a 2-component c with a short boundary curve. A regular neighborhood of $\mathrm{Cl}(c)$ in P can be presented as a lateral surface of a cylinder with $k \leq 3$ wings and the 2-component c as a middle disc. Attach to P a disc parallel to c and drill a hole in a lateral face of the cylinder thus obtained, see Fig. 2.7. Collapsing the resulting polyhedron, we get a new

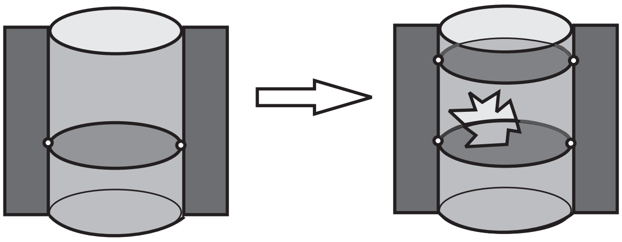

Fig. 2.7. Attaching a new 2-cell and making a hole decreases the number of true vertices

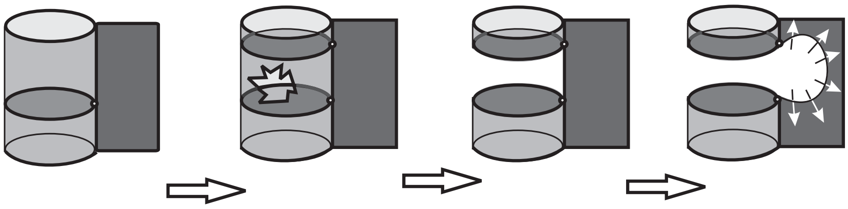

Fig. 2.8. Collapsing the unique wing

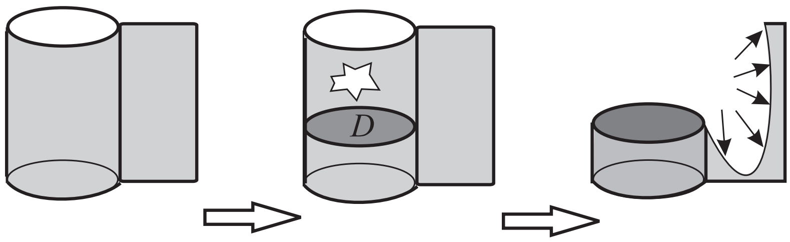

Fig. 2.9. Simplification by a counterpass

almost simple spine of M. It has a smaller number of vertices, since attaching the disc creates k new true vertices, and piercing the lateral face and collapsing destroys at least four of them if $k > 1$, and at least two if $k = 1$. It may be illuminating to note that the above transformation of P coincides with the move L^{-1} if $k = 2$, and with the move T^{-1} if $k = 3$. For $k = 1$ the result is drastic: We collapse not only the pierced 2-component, but also the unique wing of the cylinder. See Fig. 2.8.

Assume now that M is closed and orientable, and the boundary curve of a 2-component c of P has a counterpass on an edge e. Then there exists a simple closed curve $l \subset \mathrm{Cl}(c)$ that intersects e transversally at exactly one point. It decomposes c into two 2-cells c', c''. Since M is closed and orientable, one can easily find a disc $D \subset M$ such that $D \cap P = \partial D = l$. To construct D, one may push l by an isotopy to the boundary of a regular neighborhood of P and span it by a disc in the complementary ball. The polyhedron $P \cup D$ is a special spine of the twice punctured M, that is, of M with two balls B_1, B_2 cut out of it. To get a spine of M, we make a hole in c' or c'' depending on which of them is a common face of these balls. After collapsing we get an almost simple spine of M having a smaller number of true vertices, see Fig. 2.9. □

Remark 2.3.4. Suppose P has a 2-component such that its boundary curve visits four true vertices, and each of them exactly once. If we apply the same trick (glue in a parallel 2-cell and puncture a lateral one), we get another spine of M having the same number of true vertices. Sometimes this transformation is useful for recognition of duplicates.

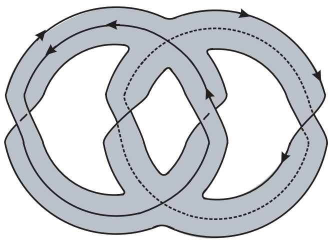

Fig. 2.10. The minimal spine of the complement of the figure eight knot has counterpasses

Remark 2.3.5. The assumption that M is closed in item 2 of Proposition 2.3.3 can be replaced by the requirement that ∂M consists of spheres. If ∂M contains tori or surfaces of higher genus, in general the counterpass simplification does not work. The reason is that the curve l in the proof may not bound a disc in the complement to the spine. For example, the special spine of the complement to the figure eight knot shown in Fig. 2.10 has counterpasses but cannot be simplified since it is minimal.

2.3.3 Manifolds of Complexity ≤ 6

The list of all closed orientable irreducible 3-manifolds up to complexity 6 contains 135-manifolds, see Sect. A.2 and its description in the Appendix. Each manifold is presented by a regular neighborhood of the singular graph of its minimal special spine. If the manifold has several minimal spines, all of them are included in the table. Let us comment on which kinds of 3-manifolds can be found in the table.

A. All closed orientable irreducible 3-manifolds up to complexity 6 are Seifert manifolds. All the manifolds of complexity ≤ 5 and many manifolds of complexity 6 have finite fundamental groups. They are elliptic, that is, can be presented as quotient spaces of S^3 by free linear actions of finite groups. Groups which can linearly act on S^3 without fixed points are well known (see [94]). They are:

1. The finite cyclic groups
2. The groups $Q_{4n}, n \geq 2$
3. The groups $D_{2^k(2n+1)}, k \geq 3, n \geq 1$
4. The groups P_{24}, P_{48}, P_{120}, and $P'_{8(3^k)}, k \geq 2$
5. The direct product of any of these groups with a cyclic group of coprime order

Lower indices show the orders of the groups. Presentations by generators and relations are given in the preliminary to Sect. A.2, see the Appendix.

B. The list contains representatives of all the five series of elliptic manifolds. In particular, the manifolds $S^3/P_{24}, S^3/P_{48}$, and the Poincaré homology sphere S^3/P_{120} have complexities 4,5, and 5, respectively. The first manifold with a nonabelian fundamental group is S^3/Q_8, where Q_8 is the quaternion unit group. It has complexity 2. More generally, for $2 \leq n \leq 6$ the manifolds S^3/Q_{4n} have complexity n. The simplest manifold of the type $S^3/D_{2^k(2n+1)}$, that is, S^3/D_{24}, has complexity 4 while the simplest manifold of the type $S^3/P'_{8(3^k)}$, the manifold S^3/P'_{72}, has complexity 5. There also occur quotient spaces of S^3 by actions of direct products of the above-mentioned groups with cyclic groups of relatively prime orders. The simplest of these (the manifold $S^3/Q_8 \times Z_3$) has complexity 4.

C. All six flat closed orientable 3-manifolds have complexity 6, among them the torus $S^1 \times S^1 \times S^1$ and the Whitehead manifold obtained from S^3 by Dehn surgery on the Whitehead link with trivially framed components. The last two are the only closed orientable irreducible manifolds of complexity ≤ 6 having the first homology group of rank ≥ 2. Recall that the Whitehead manifold coincides with the mapping torus of a homeomorphism $S^1 \times S^1 \to S^1 \times S^1$ which is the Dehn twist along nontrivial simple closed curve.

D. Among the manifolds of complexity ≤ 6 there is just one nontrivial homology sphere S^3/P_{120}. It has a unique minimal special spine with five true vertices. The singular graph of the spine is the complete graph on five vertices.

E. If the complexity of the lens space $L_{p,q}$ with $p > 2$ does not exceed 6, then it can be computed by the formula $c(L_{p,q}) = S(p,q) - 3$, where $S(p,q)$ is the sum of all partial quotients in the expansion of p/q as a regular continued fraction. Most probably, the formula $c(L_{p,q}) = S(p,q) - 3$ holds for all lens spaces, but we know only how to prove the inequality $c(L_{p,q}) \leq S(p,q) - 3$: it follows from Remark 2.3.8.

In practice, it is more convenient to calculate $c(L_{p,q})$ by the following empirical rule: if $p > 2q$, then $c(L_{p,q}) = c(L_{p-q,q}) + 1$. For example, $c(L_{33,10}) = c(L_{23,10}) + 1 = c(L_{13,10}) + 2 = c(L_{13,3}) + 2 = c(L_{10,3}) + 3 = c(L_{7,3}) + 4 = c(L_{4,3}) + 5 = c(L_{4,1}) + 5 = c(L_{3,1}) + 6 = 6$ since $c(L_{3,1}) = 0$ (we have used twice that lens spaces $L_{p,q}$ and $L_{p,p-q}$ are homeomorphic). This shows once again how natural the notion of complexity is.

It should be noted that the number of true vertices of a special spine as a measure of complexity of 3-manifolds was implicitly used by numerous authors. Ikeda proved that any simply-connected manifold having a simple spine with ≤ 4 vertices is homeomorphic to S^3 [49]. Together with Yoshinobu [50] he listed all closed 3-manifolds which in our terminology possess complexity ≤ 2. A complete list of all closed orientable irreducible 3-manifolds of complexity ≤ 5 was obtained by means of a computer as early as 1973 by Matveev and Savvateev [91]. Gillman and Laszlo, who were interested only in homology spheres [35], with the help of a computer proved that among manifolds of complexity ≤ 5 only S^3/P_{120} and S^3 have trivial homology. Actually, this fact can be extracted easily from the Matveev and Savvateev list. A list of closed orientable irreducible 3-manifolds of complexity 7 was obtained by

Ovchinnikov [102, 103]. It consists of 175-manifolds and is too large to be presented in full. Fortunately, a major part of the manifolds can be divided into four series admitting clear descriptions. In Appendix we present these descriptions and list the remaining exceptional manifolds.

It is interesting to note that not all regular graphs can be realized as singular graphs of minimal special spines of 3-manifolds. Let us try to single out several types of graphs that produce the majority of 3-manifolds up to complexity 6.

Definition 2.3.6. *A regular graph G of degree 4 is called a* nonclosed chain *if it contains two loops, and all the other edges are double. G is a* closed chain*, if it has only double edges. Finally, G is called a* triangle with a tail*, if it is homeomorphic to a wedge of a closed chain with three vertices and a nonclosed chain such that the base point of the tail* (*i.e., the common point of these two chains*) *lies on a loop of the nonclosed chain. See Fig. 2.11.*

We will say that a special spine of a closed orientable 3-manifold is *pseudominimal* if it has no counterpasses and short boundary curves. In particular, any minimal special spine is pseudominimal. For brevity we will say that a special spine P is *modeled* on a graph G if G is homeomorphic to the singular graph of P.

Proposition 2.3.7. *A closed orientable 3-manifold M has a pseudominimal special spine modeled on a nonclosed chain if and only if M is a lens space $L_{p,q}$ with $p > 3$.*

Proof. Let P be a pseudominimal special spine of M modeled on a closed chain G with n vertices. Denote by i an involution on G having $n+2$ fixed points: n vertices and one additional point on each loop. The involution permutes edges having common endpoints. Since the boundary curves have no counterpasses, they are symmetric with respect to i. Moreover, there is a boundary curve that passes a loop of G twice. Remove from P the corresponding 2-component, and denote by P_1 the resulting polyhedron. Note that P_1 is a spine of $M \setminus \mathrm{Int} H_1$, where H_1 is a solid torus in M.

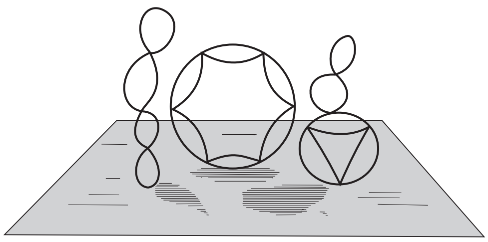

Fig. 2.11. Three useful types of singular graphs: a *nonclosed chain*, *closed chain*, and a *triangle with a tail*

Let us collapse P_1 for as long as possible by removing other 2-components together with their free edges, and free edges together with their free vertices. Using the above-mentioned symmetry of the boundary curves, one can easily show that we get a circle (actually, the second loop of G). To visualize this, one may take the spine of any lens space from Sect. A.2 and carry out the collapsing by hand. It follows that $M \setminus \mathrm{Int} H_1$ is a regular neighborhood of a circle, that is, a solid torus. Thus M is a lens space. □

Remark 2.3.8. Let us describe a simple method for calculating parameters of the lens space presented by a picture that shows a regular neighborhood of the singular graph of its pseudominimal special spine. The correctness of this method can be easily proved by induction on the number of true vertices of the spine. Assign to each double edge and to each loop of the singular graph a letter ℓ or r as shown in Fig. 2.12. We get a string w of letters that we will consider as a composition of operators $r, \ell\colon Z \oplus Z \to Z \oplus Z$ given by $r(a, b) = (a, a+b)$ and $\ell(a, b) = (a+b, b)$. Then the lens space has parameters $p = m + n, q = m$, where $(m, n) = w(1, 1)$. For example, for the lens space shown in Fig. 2.12 we have $w = rrrr\ell\ell\ell$, $(m, n) = (4, 17)$, and $(p, q) = (21, 4)$, since by our interpretation of r, ℓ we have

$$(1,1) \xrightarrow{\ell} (2,1) \xrightarrow{\ell} (3,1) \xrightarrow{\ell} (4,1) \xrightarrow{r} (4,5) \xrightarrow{r} (4,9) \xrightarrow{r} (4,13) \xrightarrow{r} (4,17).$$

The same method can be used for constructing a pseudominimal special spine of a given lens space $L_{p,q}$: One should apply to the pair $(p-q, q)$ operators r^{-1}, ℓ^{-1} until we get $(1,1)$, and then use the string of letters r, ℓ thus obtained for constructing the spine.

Proposition 2.3.9. *A closed orientable 3-manifold M has a pseudominimal special spine modeled on a triangle with a tail if and only if M is an orientable Seifert fibered manifold of the type $(S^2, (2,1), (2,-1), (n, \beta))$, where $\beta, n > 0$, and $(n, \beta) \neq (1,1)$.*

Proof. Let P be a pseudominimal special spine of M modeled on a triangle with a tail. Since the boundary curves have no counterpasses, they pass over

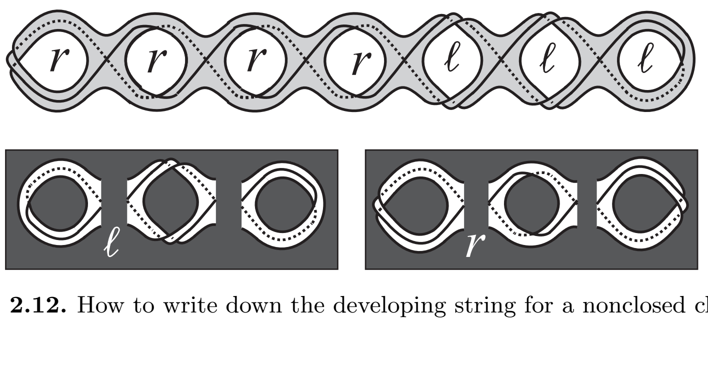

Fig. 2.12. How to write down the developing string for a nonclosed chain

the tail in a symmetric way with respect to the involution of the tail that permutes the double edges and reverses the orientation of the loop. For the same reason one of the boundary curves passes the loop twice. Remove from P the corresponding 2-component and denote by P_1 the resulting polyhedron. Of course, P_1 is a spine of $M_1 = M \setminus \mathrm{Int}H$, where H is a solid torus in M. Let us collapse P_1 as long as possible. Using the above-mentioned symmetry of the boundary curves, one can easily show that the tail disappears completely, together with all 2-cells that have common points with it, including all the 2-components whose boundary curves pass through the base point of the tail. It follows that we get a simple spine without singular points, that is, a closed surface K. Since the boundary of M_1 is a torus, K is the Klein bottle, and $M_1 = K\tilde{\times}I$. It is well known that $K\tilde{\times}I$ can be presented as the Seifert fibered manifold over the disc with two exceptional fibers of types $(2,1)$ and $(2,-1)$, see Example 6.4.14. Thus attaching the solid torus H converts M_1 to a Seifert fibered manifold $(S^2,(2,1),(2,-1),(n,\beta))$. The parameters (n,β) show how H is glued to $K\tilde{\times}I$. We can always get $n,\beta > 0$ by reversing the orientation of the manifold.

Just as in the proof of Proposition 2.3.7, let us describe a simple method for calculating the parameters (n,β) starting from a pseudominimal special spine modeled on a triangle with a tail. The correctness of the method can be easily proved by induction on the number of vertices of the tail. Assign to the loop and the double edges of the tail and to the pair of edges adjacent to it letters ℓ and r as shown in Fig. 2.13. We get a string w of letters which, as above, can be considered as a composition of operators $r,\ell\colon Z\oplus Z \to Z\oplus Z$ given by $r(a,b) = (a,a+b)$ and $\ell(a,b) = (a+b,b)$. Then $(n,\beta) = w(1,1)$. For example, for the spine shown in Fig. 2.13 we have $w = \ell r\ell\ell$ and $w(1,1) = (n,\beta) = (7,4)$.

The same method can be used for constructing a pseudominimal special spine of a given Seifert fibered manifold $(S^2,(2,1),(2,-1),(n,\beta))$: one should recover the string of r,ℓ by transforming (n,β) into $(1,1)$, and then use it for choosing the correct tail. Since $n \neq 0$ and $(n,\beta) \neq (1,1)$, the string and the tail exist. □

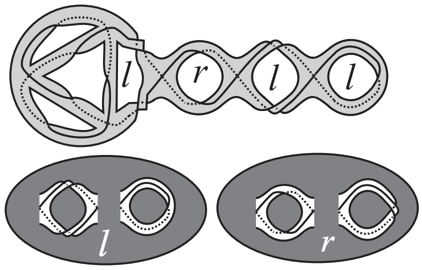

Fig. 2.13. How to write down the developing string for a tail

Note that for any pair of coprime positive integers (n, β) with $n \geq 1$ the fundamental group of the manifold $M_{n,\beta} = (S^3, (2,1), (2,-1), (n,\beta))$ is finite and has the presentation

$$\langle c_1, c_2, c_3, t | c_1^2 = t, c_2^2 = t^{-1}, c_3^n = t^\beta, c_1 c_2 c_3 = 1 \rangle$$

The order of the homology group $H_1(M_{n,\beta}; Z)$ is 4β. Using this, it is not hard to present $M_{n,\beta}$ as the quotient space of S^3 by a linear action of a group from the Milnor list [94] presented above. It turns out that the following is true:

(1) If $n > 1$ and β is odd, then $M_{n,\beta} = S^3/Q_{4n} \times Z_\beta$
(2) If $n > 1$ and β is even, then $M_{n,\beta} = S^3/D_{2^{k+2}n} \times Z_{2m+1}$, where k and m can be found from the equality $\beta = 2^k(2m+1)$
(3) If $n = 1$ and $\beta \neq 0, 1$, then $M_{n,\beta} = L_{4\beta, 2\beta+1}$

If $n = 1$ or $\beta = 1$, the pseudominimal special spine of $M_{n,\beta}$ modeled on the triangle with a tail is not minimal. An easy way to see that is to apply the transformation described in Remark 2.3.4. This is possible since the spine possesses a boundary curve that passes through four true vertices, and visits each of them exactly once. After the transformation we get a spine that has the same number of true vertices but possesses a boundary curve of length 3. Therefore, one can simplify the spine. In the case $n = 1$ we get a spine of a lens space modeled on a nonclosed chain with smaller number of true vertices. If $\beta = 1$, we get a simple spine of the manifold S^3/Q_{4n}.

Proposition 2.3.10. *A closed orientable 3-manifold M has a pseudominimal special spine modeled on a closed chain with $n \geq 2$ vertices if and only if M is S^3/Q_{4n}.*

Proof. One can easily show that any pseudominimal spine of a closed manifold modeled on a closed chain contains a boundary curve that goes twice around the chain and passes through all the edges. For any $n \geq 2$ there is only one such spine (see Fig. 2.14), and its fundamental group is Q_{4n}. Removing the

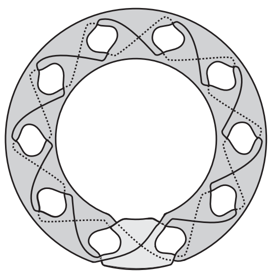

Fig. 2.14. The unique pseudominimal special spine modeled on a closed chain with n vertices is a spine of S^3/Q_{4n}

corresponding 2-component and collapsing, we get the Klein bottle. Therefore, M is a Seifert fibered manifold over S^2 with three exceptional fibers of degree $2, 2$, and n. Among such manifolds only S^3/Q_{4n} has the fundamental group Q_{4n}. □

The following conjectures are motivated by Propositions 2.3.7–2.3.10 and the results of the computer enumeration.

Conjecture 2.3.11. Any lens space $L_{p,q}$ with $p \geq 3$ has a unique minimal special spine. This spine is modeled on a nonclosed chain.

Conjecture 2.3.12. For any $n \geq 2$ the manifold S^3/Q_{4n} has a unique minimal special spine. This spine is modeled on a closed chain with n links.

Conjecture 2.3.13. Manifolds of the type $S^3/Q_{4n} \times Z_\beta, n > 1, \beta \neq \pm 1$ and $S^3/D_{2^{k+2}n} \times Z_{2m+1}$ have minimal special spine modeled on triangles with a tail.

Section A.2 shows that the conjectures are true for manifolds of complexity ≤ 7.

Remark 2.3.14. One can prove that any pseudominimal special spine modeled on a triangle with three tails is a spine of a Seifert fibered manifold M over S^2 with three exceptional fibers. Let $w_i, 1 \leq i \leq 3$ be the developing rl-strings of the tails. Then $M = (S^2, (n_1, \beta_1), (n_2, \beta_2), (n_3, \beta_3), (1, -1))$, where $(n_i, \beta_i) = w_i(1, 1)$ for $1 \leq i \leq 3$. We have inserted the regular fiber of the type (1,-1) to preserve the symmetry of the expression. Certainly, one may write $M = (S^2, (n_1, \beta_1), (n_2, \beta_2), (n_3, \beta_3 - n_3))$. The formula works also for triangles with < 3 tails, if we adopt the convention that the developing string for the empty tail is ℓ and produces the exceptional fiber of type (2,1). See Fig. 2.15.

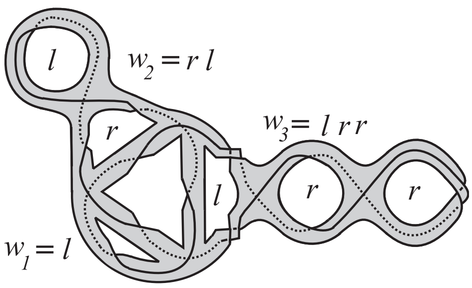

Fig. 2.15. The developing strings are ℓ (for the empty tail), $r\ell$, and ℓrr. Thus $M = (S^2, (2, 1), (2, 3), (4, 3), (1, -1))$

2.4 Graph Manifolds of Waldhausen

Our discussion in Sect. 2.3.3 shows that all closed orientable irreducible 3-manifolds of complexity ≤ 6 belong to the class $\mathcal{G}$ of graph manifolds of Waldhausen. $\mathcal{G}$ contains all Seifert manifolds and all Stallings and quasi-Stallings 3-manifolds with fiber $S^1 \times S^1$. Its advantage is that it is closed with respect to connected sums. It follows that all closed orientable (not necessarily irreducible) 3-manifolds of complexity ≤ 6 are also graph manifolds. In Sect. 2.4.2 we will show that the same is true for 3-manifolds of complexity ≤ 8, but first we should study $\mathcal{G}$ in more detail.

2.4.1 Properties of Graph Manifolds

Graph manifolds have been introduced and classified by Waldhausen in two consecutive papers [129]. They turned out to be very important for understanding the structure of 3-manifolds. Indeed, it follows from JSJ-decomposition theorem (see [55, 57] and Sect. 6.4) that for any orientable closed irreducible 3-manifold M there exists a finite system $\mathcal{T}$ of disjoint incompressible tori $T_1, T_2, \ldots, T_n$ in M (unique up to isotopy) such that the following holds:

(1) $\mathcal{T}$ decomposes M into Seifert manifolds and manifolds which are not Seifert and contain no essential tori. We will call these submanifolds *JSJ-chambers*.
(2) $\mathcal{T}$ has the minimal number of tori among all systems possessing (1).

If $\mathcal{T} \neq \emptyset$, then all the JSJ-chambers are sufficiently large (see Sect. 4.1.6). Therefore, one can apply Thurston's results [62, 111, 122, 123] and prove that every non-Seifert JSJ-chamber is a hyperbolic manifold. The union of all Seifert JSJ-chambers is not necessarily a Seifert manifold, but it is composed of Seifert manifolds. In particular, it can have Stallings and quasi-Stallings components (see Definitions 6.4.16 and 6.5.12) with fiber $S^1 \times S^1$. Manifolds which can be obtained from Seifert manifolds by gluing their boundary tori are known as *graph manifolds of Waldhausen*.

Roughly speaking, the role of graph manifolds in 3-manifold topology may be expressed by the informal relation

$$\mathcal{M} = (\mathcal{G} + \mathcal{H}) \cup (?).$$

Here $\mathcal{M}, \mathcal{G}$, and $\mathcal{H}$ are the classes of all closed irreducible 3-manifolds, graph manifolds, and hyperbolic manifolds, respectively. The class $\mathcal{G} + \mathcal{H}$ consists of manifolds that can be decomposed by incompressible tori into graph and hyperbolic manifolds. The additional term (?) stands for the class of closed irreducible manifolds which contain no essential tori and are neither hyperbolic nor graph manifolds. If Thurston's Geometrization Conjecture [111] is true, then (?) is empty.

The class of graph manifolds was rediscovered by Fomenko [30]. It turned out that there is a close relationship between the integrability of Hamiltonian mechanical systems on symplectic 4-manifolds and the topological structure of level surfaces of the Hamiltonian: If the system is integrable, then each nonsingular level surface is a graph manifold. See [12] for further development of the theory.

To give a rigorous formal definition of graph manifolds, we prefer to compose them from Seifert manifolds of two very simple types. Denote by N^2 the disc D^2 with two holes. Then the manifold $N^2 \times S^1$ can be presented as the solid torus $D^2 \times S^1$ with two drilled out solid tori H_1, H_2 that are parallel to the core circle $\{*\} \times S^1$ of $D^2 \times S^1$. A more general way to view $N^2 \times S^1$ is to cut out a regular neighborhood of $c \cup \ell$ from $D^2 \times S^1$, where c is a core circle of $D^2 \times S^1$ and ℓ is any simple closed curve in Int $(D^2 \times S^1)$ that is parallel to a nontrivial curve in $\partial D^2 \times S^1$. The result does not depend on the choice of ℓ since all nontrivial simple closed curves in the boundary of $S^1 \times S^1 \times I$ are equivalent up to homeomorphisms of $S^1 \times S^1 \times I$. We will call the manifolds $D^2 \times S^1$ and $N^2 \times S^1$ *elementary blocks*.

Definition 2.4.1. *A compact 3-manifold M is called a* graph manifold *if it can be obtained by pasting together several elementary blocks $D^2 \times S^1$ and $N^2 \times S^1$ along some homeomorphisms of their boundary tori.*

It is often convenient to present the gluing schema by a graph having vertices of valence 1 and 3. The vertices of valence 3 correspond to blocks $N^2 \times S^1$. Vertices of valence 1 correspond either to blocks $D^2 \times S^1$ or to free boundary component of M. In Fig. 2.16 we represent them by black and white fat dots, respectively.

We next recall some well-known properties of graph manifolds, accompanying them with short explanations or informal proofs.

Proposition 2.4.2. *The class $\mathcal{G}$ contains all orientable Seifert manifolds.*

Proof. Suppose M is a Seifert manifold fibered over a surface F. Note that any surface can be decomposed by disjoint circles into the following elementary

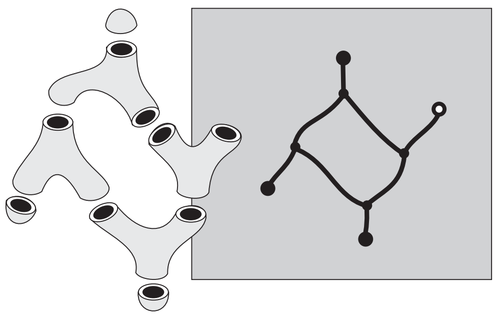

Fig. 2.16. A graph structure of a graph manifold

pieces: discs, copies of N^2, and Möbius bands. One may assume that all exceptional fibers correspond to the centers of the discs. Then the decomposition of F induces a decomposition of M into inverse images of elementary pieces. It remains to note that the inverse image of each piece P is either an elementary block (if $P = D^2, N^2$), or can be decomposed into three elementary blocks (if P is a Möbius band). The latter is true since the twisted product of a Möbius band and S^1 admits an alternative Seifert structure: it fibers over D^2 with two exceptional fibers of types (2,1), (2,-1), see Example 6.4.14. □

Proposition 2.4.3. *The class $\mathcal{G}$ is closed with respect to connected sums, that is, $M_1 \# M_2 \in \mathcal{G} \iff M_1, M_2 \in \mathcal{G}$.*

Proof. To prove the implication $\Leftarrow$, it suffices to find a graph presentation of $D^2 \times S^1 \# D^2 \times S^1$. Let c be the core circle of $D^2 \times S^1$ and m a circle obtained from a meridian of $D^2 \times S^1$ by pushing it inward $D^2 \times S^1$. Denote by $N(c), N(m)$ regular neighborhoods of the circles. Then the manifold $D^2 \times S^1 \setminus \mathrm{Int} N(m)$ is homeomorphic to the connected sum of two solid tori. On the other hand, it can be obtained from the manifold $D^2 \times S^1 \setminus (\mathrm{Int}\ N(c) \cup \mathrm{Int} N(m))$ homeomorphic to $N^2 \times S^1$ by pasting back the torus $N(c)$.

To prove the inverse implication, assume that a graph manifold M contains a nontrivial 2-sphere S. Consider a decomposition of M into *extended* elementary blocks, where each extended block is the union of an elementary block $N^2 \times S^1$ and all the solid tori adjacent to it. Applying the innermost circle argument to the intersection of S with the boundaries of extended blocks, we locate an extended block B with compressible boundary. Recall that the boundary of any Seifert manifold is incompressible unless it is the solid torus. It follows that for B we have the following possibilities:

1. *B is a solid torus (presented as a union of smaller elementary blocks).* We consolidate the initial structure of a graph manifold by considering B as a new block.
2. *B is not a Seifert manifold.* This can happen only in the case when B is composed of $N^2 \times S^1$ and solid tori so that the meridian of one of the solid tori is isotopic to a fiber $\{*\} \times S^1$ of $N^2 \times S^1$. Then B can be presented as $B_1 \# B_2$, where each B_i is either a solid torus or a lens space. Thus we can decompose M into a connected sum of either simpler graph manifolds or a simpler graph manifold and $S^2 \times S^1$.

Continuing this process for as long as possible, we get a decomposition of M into a connected sum of prime graph manifolds. Since the topological types of the summands are determined by M, the prime decomposition summands for M_1, M_2 have the same types. It follows that M_1 and M_2 are graph manifolds. □

Now we investigate the behavior of $\mathcal{G}$ with respect to boundary connected sums and, more generally, to cutting along discs. Of course, the statement $M_1 \perp\!\!\!\perp M_2 \in \mathcal{G} \iff M_1, M_2 \in \mathcal{G}$ is not true anymore. For example, let

V be a solid torus and B a 3-ball. Then $V \amalg V \notin \mathcal{G}$ although $V \in \mathcal{G}$ and $V \amalg B \in \mathcal{G}$ although $B \notin \mathcal{G}$. To formulate the correct corresponding statement, it is convenient to introduce the following notation: If M is a 3-manifold, then $\hat{M}$ denotes the 3-manifolds obtained from M by attaching 3-balls to all the spherical components of ∂M. In particular, if ∂M contains no spherical components, then $\hat{M} = M$. Recall also that if D is a proper disc in M, then M_D denotes the 3-manifold obtained from M by cutting along D.

Corollary 2.4.4. *Let D be a proper disc in a connected 3-manifold M such that ∂M consists of tori. Then M is a graph manifold if and only if so is $\hat{M}_D$.*

Proof. Denote by T the torus component of ∂M containing ∂D and by $N = N(D \cup T)$ a regular neighborhood of $D \cup T$ in M. Then M_D is homeomorphic to the manifold $M_1 = \mathrm{Cl}(M \setminus N)$. It is easy to see that $M_1 \cap N$ is either a sphere S (if ∂D does not decompose T) or the union of a sphere S and a torus T_1 (if ∂D decomposes T). There are three different cases, see Fig. 2.17. Let us list all of them together with the corresponding relation between $\hat{M}$ and $\hat{M}_D$ we wish to prove.

Case 1. *D decomposes M into two components M', M''.* Then $M = \hat{M}' \# \hat{M}''$.

Case 2. *∂D decomposes T, but D does not decompose M.* Then $M = \hat{M}_D \# (S^2 \times S^1)$.

Case 3. *∂D does not decompose T.* Then $M = \hat{M}_D \# (D^2 \times S^1)$.

Let us prove that. Suppose $M_1 \cap N = S \cup T_1$ is as in Cases 1, 2. Denote by M_S the 3-manifold obtained from $M = M_1 \cup N$ by cutting along S. If we attach N to M_1 along T_1, we get M_S. It follows that $\hat{M}_S$ can be obtained by attaching $\hat{N}$ to $\hat{M}_1$ along T_1. On the other hand, $\hat{N}$ is homeomorphic to $T_1 \times I$, so $\hat{M}_S$ and $\hat{M}_1$ (hence $\hat{M}_S$ and $\hat{M}_D$) are homeomorphic. We can conclude that M can be obtained from $\hat{M}_D$ by cutting out two 3-balls and identifying the two boundary spheres of the manifold thus obtained. This operation is equivalent to taking the connected sum with $S^2 \times S^1$ (if $\hat{M}_D$ is connected) or to taking the connected sum of its components (if not).

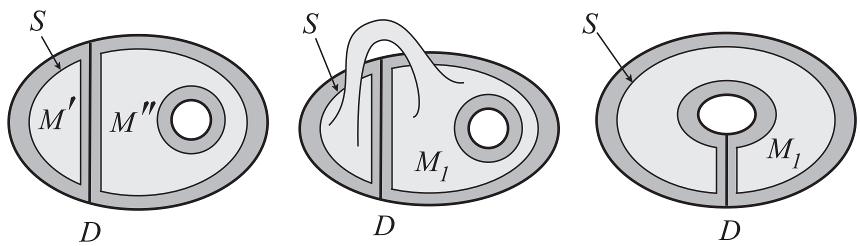

Fig. 2.17. M is either $\hat{M}' \# \hat{M}''$ (*on the left*), or $\hat{M}_D \# (S^2 \times S^1)$ (*in the middle*), or $\hat{M}_D \# (D^2 \times S^1)$ (*on the right*)

If $M_1 \cap N = S$ as in Case 3, then M_S is the disjoint union of M_1 (which is homeomorphic to M_D) and N (which is a punctured solid torus). It follows that $M = \hat{M}_D \#(D^2 \times S^1)$.

To conclude the proof of the corollary, it remains to recall that $S^2 \times S^1$ and $D^2 \times S^1$ are graph manifolds and apply Proposition 2.4.3. □

Remark 2.4.5. One can easily generalize Corollary 2.4.4 as follows. Let D be a proper disc in a connected 3-manifold M so that ∂M consists of spheres and tori. Then $\hat{M}$ is a graph manifold if and only if so is $\hat{M}_D$. Indeed, if ∂D lies on a torus of ∂M, then the same proof works. Let ∂D lies on a sphere $S \subset \partial M$. Then either $\hat{M} = \hat{M}_D \#(S^1 \times S^2)$ (if D does not separate M), or $\hat{M} = \hat{M}' \# \hat{M}''$ (if it decomposes M into two components M', M'').

By Definition 2.4.1, any graph manifold can be decomposed onto elementary blocks by a finite system of disjoint tori. Our next goal is to decrease the number of tori by amalgamating the elementary blocks into Seifert manifolds called *Seifert blocks*. We will restrict ourselves to considering graph manifolds which are irreducible and boundary irreducible. This restriction is not very important. Indeed, the behavior of graph manifolds with respect to connected sums is already known (Proposition 2.4.3), and the only connected graph manifold which is irreducible but boundary reducible is the solid torus.

Definition 2.4.6. *A system $\mathcal{T} = \{T_1, T_2, \dots, T_n\}$ of disjoint incompressible tori embedded into an irreducible boundary irreducible graph manifold M is called* canonical *if:*

(1) *$\mathcal{T}$ decomposes M into a collection of Seifert blocks (that is, Seifert manifolds).*
(2) *For any torus $T_i \subset \mathcal{T}$ and for any choice of Seifert fibrations on the adjacent blocks, the two S^1-fibrations on T induced from the both sides are not isotopic.*

Proposition 2.4.7. *Let a system $\mathcal{T} = \{T_1, T_2, \dots, T_n\}$ of disjoint incompressible tori in an irreducible boundary irreducible graph manifold M decompose it into Seifert blocks. Then $\mathcal{T}$ contains a canonical subsystem.*

Proof. We introduce two moves that decrease the number of blocks for $\mathcal{T}$.

(1) If the two S^1-fibration on $T \subset \mathcal{T}$ induced by some Seifert fibrations of the adjacent blocks are isotopic, we remove the torus T from $\mathcal{T}$. The new block arising in this way is a Seifert manifold, the Seifert structure being composed from the Seifert structures of the old blocks.
(2) Suppose a torus $T \subset \mathcal{T}$ is compressible but is not the boundary of a block $D^2 \times S^1$. By irreducibility of M, it bounds a solid torus B in M. We amalgamate all the blocks lying in B into the new block B by erasing all the tori of $\mathcal{T}$ contained in B.

Let us now apply the moves to $\mathcal{T}$ as long as possible. Evidently, the resulting system (still denoted by $\mathcal{T}$) is canonical. Indeed, since the first move is impossible, any torus $T \in \mathcal{T}$ can inherit only distinct S^1-fibrations from the neighboring blocks. Also, all the tori in $\mathcal{T}$ are incompressible, because all moves of the second type are performed. $\square$

At first glance a graph manifold M can contain many canonical systems. Indeed, the initial decomposition into, say, elementary blocks is not unique, and the blocks can be amalgamated into larger blocks in many different ways. Nevertheless, if M is irreducible and boundary irreducible, then the canonical system is unique up to isotopy. This result follows from the JSJ-decomposition theorem (see Corollary 6.4.30 and Theorem 6.4.31) and, having been obtained 10 years earlier, can be considered as its infant stage. As a matter of fact, the Waldhausen classification of graph manifolds is nothing more than the JSJ-decomposition theorem for them. To supply a graph manifold M with a unique "name" which distinguishes it from all other graph manifolds, we simply describe its canonical Seifert blocks and the way how they are glued together. The gluing schema can be most naturally presented by a graph. This explains once more why graph manifolds are called so.

Proposition 2.4.8 shows that the class $\mathcal{G}$ is closed with respect to cutting along essential annuli. For simplicity, we formulate and prove it for irreducible manifolds. Recall that M_F denotes a manifold obtained from a manifold M by cutting along a surface $F \subset M$.

Proposition 2.4.8. *Let A be an essential annulus in an irreducible 3-manifold M. Then $M \in \mathcal{G} \iff M_A \in \mathcal{G}$.*

Proof. Let us prove that if M_A is a graph manifold, then so is M. Denote by Y^3 the connected component of a regular neighborhood of $\partial M \cup A$ that contains A. Since the boundary curves of A are nontrivial, $\partial M \cup A$ fibers onto circles. It follows that Y^3 also fibers onto circles over a surface F (one can prove that either $Y^3 = N^2 \times S^1$ or $Y^3 = K^2 \tilde{\times} S^1$). By construction, $M = M_1 \cup Y^3$, where the manifold $M_1 = M \setminus Y^3$ is homeomorphic with M_A, and $M_1 \cap Y^3$ consists of one or two boundary tori. It follows that $M_A \in \mathcal{G} \Rightarrow M \in \mathcal{G}$.

To prove the inverse implication, we construct a canonical system $\mathcal{T}$ of essential tori in M. As we have mentioned earlier, it coincides with the JSJ-system for M. One of the properties of $\mathcal{T}$ is that A is isotopic to another annulus (still denoted by A) which lies in the complement to $\mathcal{T}$, see Sect. 6.4.4. This means that A is contained in a JSJ-chamber Q of $\mathcal{T}$. In our case all the JSJ-chambers are Seifert manifolds. Since any essential annulus in a Seifert manifold Q is saturated (with respect to a Seifert structure on Q), the manifold Q_A is Seifert. It follows that M_A, being composed of Q_A and all the other JSJ-chambers of $\mathcal{T}$, is a graph manifold. $\square$

2.4.2 Manifolds of Complexity ≤ 8

As we know from Sect. 2.3.3, all closed orientable irreducible 3-manifolds of complexity ≤ 6 are graph manifolds. By Proposition 2.4.3, the class $\mathcal{G}$ is closed with respect to connected sums. It follows that all (not necessarily irreducible) closed orientable 3-manifolds of complexity ≤ 6 are graph manifolds. The following question arises: what is the complexity of the simplest closed orientable 3-manifold not contained in the class $\mathcal{G}$? In this section we show that the first nongraph closed orientable 3-manifold has complexity 9.

Theorem 2.4.9. *All closed orientable 3-manifolds of complexity no greater than 8 are graph manifolds.*

This was initially proved by computer. Later, a purely theoretical proof was found (see [32]). The computer program is based on the following observation.

Proposition 2.4.10. *Let M be an orientable 3-manifold with $\partial M = S^1 \times S^1$. Suppose that M has an almost simple spine P whose singular graph SP is either empty or consists of one or a few disjoint nonclosed chains with ≤ 2 vertices each. Then $M \in \mathcal{G}$.*

Proof. We apply to P the same simplification moves as in the proof of Theorem 2.2.4, with the following modifications:

(1) Since M may be reducible or boundary reducible, removing an arc ℓ from the 1-dimensional part of P may produce not only another spine of M, but also a spine of a new 3-manifold M_1. Let D be a proper disc in M intersecting ℓ transversally at one point. Then M_1 can be viewed as the manifold M_D, obtained by cutting M along D. It follows from Corollary 2.4.4 (see also Remark 2.4.5) that $M \in \mathcal{G} \iff \hat{M}_D \mathcal{G}$ is a graph manifold.
(2) It may happen that the proper annulus $A \subset M$ that intersects P along a nontrivial simple closed curve l in a 2-component α of P is essential. In this case we cannot apply moves 2 or 3 from the proof of Theorem 2.2.4, but simply cut P along l and obtain a spine of the manifold M_A. By Proposition 2.4.8, $M \in \mathcal{G} \iff M_A \in \mathcal{G}$.

At any step of the simplification procedure the above assumption concerning the singular graph is preserved: We obtain an almost simple spine whose singular graph consists of nonclosed chains with ≤ 2 vertices. After terminating the procedure, we get a collection of special spines modeled on closed chains with ≤ 2 vertices such that the boundaries of the corresponding manifolds are either empty or consist of tori. There are only a few such spines. It is easy to enumerate them and verify that in all cases they determine graph manifolds. Since our simplification moves preserve the property of a manifold to belong to $\mathcal{G}$, M is also a graph manifold. $\square$

The computer works in the following way. It first looks through all the regular graphs of degree 4 with ≤ 8 vertices and, for each graph, lists all the possible spines modeled on it (see the proof of Theorem 2.1.1). Each spine P is tested for the following questions:

1. Is there a short boundary curve?
2. Is there a counterpass?
3. Is the corresponding manifold closed and orientable ?

If it obtains a positive answer to one of the first two questions, or a negative answer to the third question, the computer leaves aside P and goes on to the next spine. Otherwise it tests P for the following property:

4. Does there exist a 2-component α of P such that $P \setminus \alpha$ collapses to an almost simple polyhedron whose singular graph is either empty or consists of nonclosed chains with ≤ 2 vertices?

The main result of the computer experiment is that in all cases the answer to the last question turned out to be positive. By Proposition 2.4.10, this implies the conclusion of Theorem 2.4.9.

The complete text of the above-mentioned theoretical proof of Theorem 2.4.9 takes up nearly a 100 pages and therefore we will limit ourselves to a brief outline. The proof naturally splits up into three stages. First, we prove that any closed irreducible orientable 3-manifold of complexity ≤ 8 is obtained by attaching a solid torus to a 3-manifold of complexity ≤ 3 whose boundary is a torus. We then find out that all such 3-manifolds are graph manifolds except 14 remarkable manifolds $Q_i, 1 \leq i \leq 14$, which are hyperbolic and hence do not belong to the class $\mathcal{G}$ of graph manifolds. (In fact, Q_{12}, Q_{13}, Q_{14} are homeomorphic to Q_6, Q_1, Q_2, respectively. We distinguish them, since they have different special spines with three true vertices). This implies that all closed irreducible orientable 3-manifolds of complexity ≤ 8 are in the class $\mathcal{G}$ except possibly manifolds of the form $(Q_i)_{p,q}, 1 \leq i \leq 14$ (p, q are coprime integers) obtained by pasting solid tori to Q_i. Finally, a more specific analysis shows that any $(Q_i)_{p,q}$ is still in $\mathcal{G}$, provided that its complexity is ≤ 8.

Now let us comment on each step of the proof separately.

Step 1. Let P be a minimal special spine of a closed irreducible orientable 3-manifold M of complexity ≤ 8. We wish to prove that P has a 2-component α such that after puncturing α and collapsing we get a spine with ≤ 3 true vertices. To simplify the notation, we restrict ourselves to the case when P has exactly eight true vertices. Recall that puncturing a 2-component of P corresponds to removing a solid torus from M.

Let us study in more detail what happens to P when we puncture and collapse its 2-component α. In the collapsing process α disappears completely. Suppose α is adjacent to an edge e of P twice. Then the 2-component β that is adjacent to e the third time also disappears completely. One can easily show that the boundary curves of α, β, and of all the other 2-components

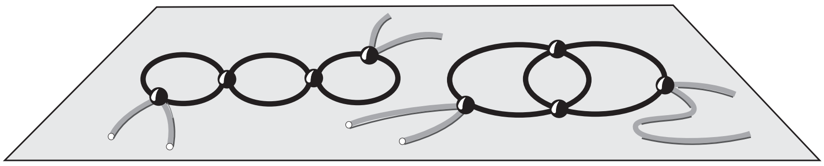

Fig. 2.18. Fragments containing boundary curves that pass through six edges and only four true vertices

that disappear under collapsing contain ≥ 5 true vertices of P together. This means that we get a spine with ≤ 3 true vertices.

Suppose now that no boundary curve passes through an edge twice. Let us call the *length* of a 2-component α of P (or of its boundary curve $c(\alpha)$) the total number of passages of $c(\alpha)$ through edges (with multiplicity taken into consideration). Since P has 16 edges, and since each of them is incident to exactly three 2-components, the total length of the 2-components is equal to 48. On the other hand, P has nine 2-components. It follows that there is a 2-component α adjacent to ≥ 6 different edges. If α contains ≥ 5 different true vertices, we may puncture it and get a spine with ≤ 3 true vertices. If not, then the singular graph SP of P contains one of the fragments shown in Fig. 2.18.

Analyzing the ways in which the boundary curves can pass through each of the fragments, one can always find another boundary curve that contains ≥ 5 different true vertices of P.

STEP 2. Let us introduce 14 remarkable special spines $P_i, 1 \leq i \leq 14$, with ≤ 3 true vertices that determine manifolds Q_i with tori as boundaries. It is convenient to do this by using Figs. 2.19 and 2.20. The manifolds Q_i are the complement spaces of knots in 3-manifolds of genus ≤ 1. For example, Q_2 and Q_{14} are homeomorphic to the complement space of **figure eight knot in S^3. One can show that Q_1 is homeomorphic to Q_{13} and Q_6 is homeomorphic to Q_{12}. All other manifolds Q_i are distinct.

Proposition 2.4.11. *Suppose that the boundary of a compact orientable 3-manifold Q is a torus and that Q has a special spine P with ≤ 3 true vertices. Then either Q is a graph manifold, or P is homeomorphic to one of the spines $P_i, 1 \leq i \leq 14$.*

The proof consists, roughly speaking, of going through all the possible special spines with three or less true vertices and analyzing the corresponding 3-manifolds. There are seven different regular graphs with ≤ 3 vertices. Only 3 of them (the closed chains with 2 and 3 vertices, and the chain with 2 vertices and with an additional loop) may produce manifolds that are not in $\mathcal{G}$. By using the symmetry of the three suspicious graphs and certain artificial tricks, the process can be kept within reasonable limits, which, however, are too large to be presented here. See [79] for details.

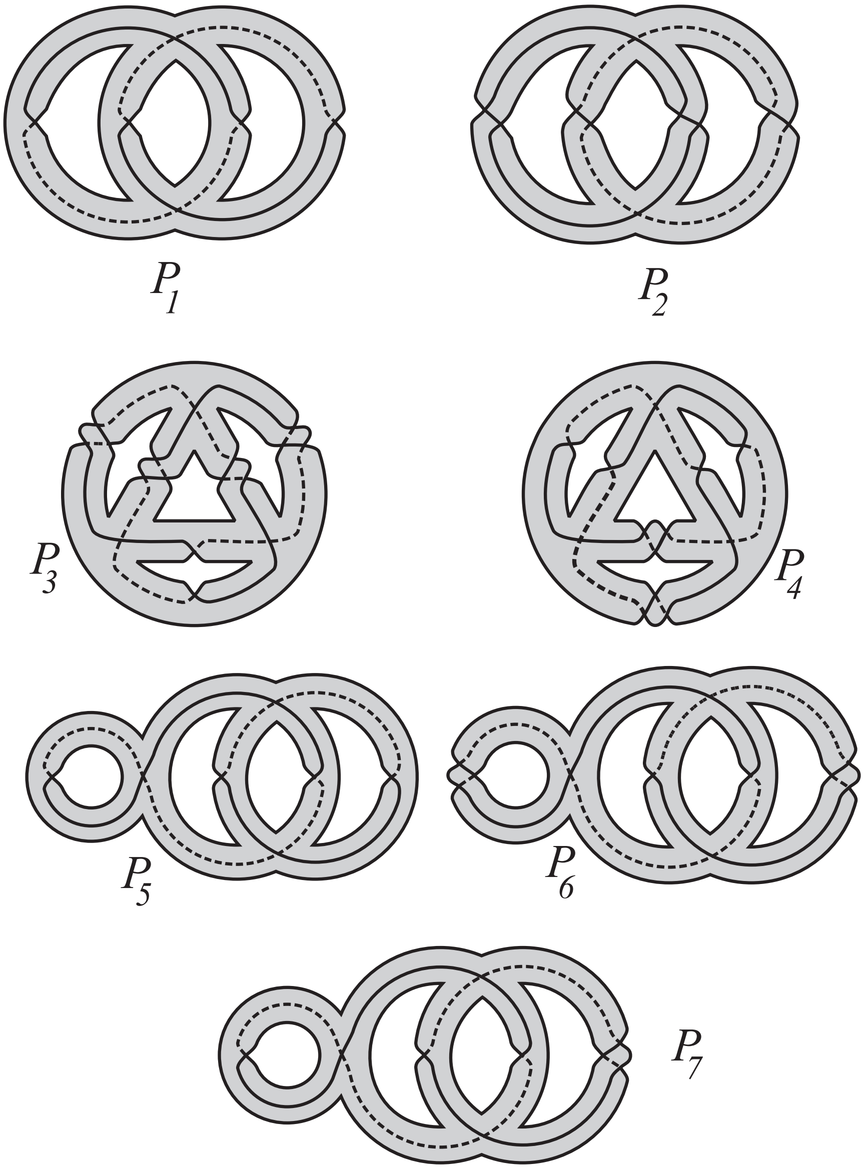

Fig. 2.19. Seven remarkable special spines with ≤ 3 true vertices

STEP 3. Now we prove that if a 3-manifold M of complexity $c(M) \leq 8$ is obtained by a Dehn filling of one of $Q_i, 1 \leq i \leq 14$, then $M \in \mathcal{G}$. Let P be a minimal special spine of M having ≤ 8 vertices. According to Step 1, one can puncture a 2-component α of P such that after collapsing we get a special polyhedron P' with ≤ 3 true vertices. By construction, P' is

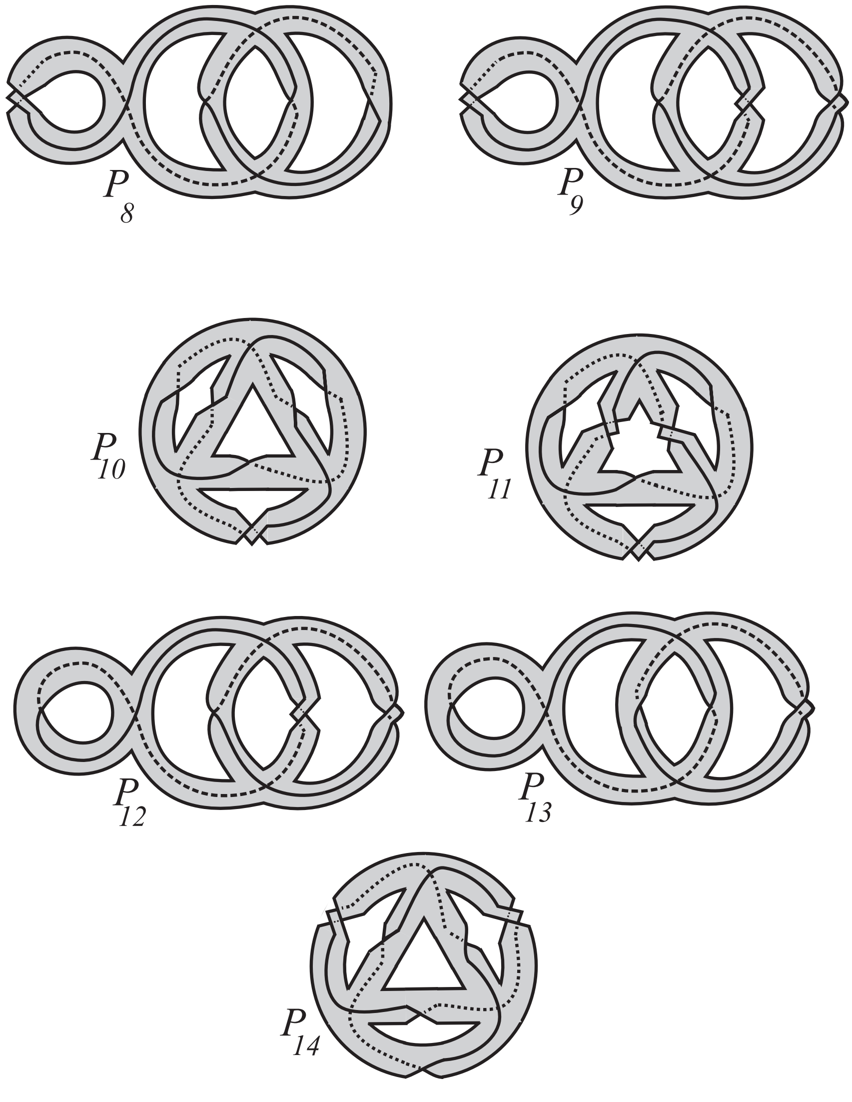

Fig. 2.20. Another seven remarkable special spines with ≤ 3 true vertices

a special spine of a 3-manifold Q such that M is a Dehn filling of Q. It follows from Proposition 2.4.11 that if P' is not homeomorphic to a polyhedron $P_i, 1 \leq i \leq 14$, then Q and M are graph manifolds. It remains to investigate the case when P' is one of P_i.

Proposition 2.4.12. *Suppose a special spine P of a closed orientable 3-manifold M has no more than eight true vertices and suppose that after puncturing one of its 2-components and collapsing we obtain the spine $P_i, 1 \leq i \leq 14$. Then M is a graph manifold.*

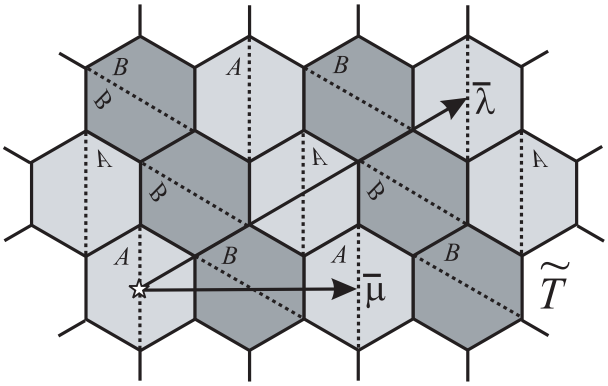

Fig. 2.21. Cell decomposition of $\tilde{T}$

The proof of Proposition 2.4.12 should be carried out in all 14 cases but it follows the same outline and uses the same tricks. Let us carry it out once for the case $i = 1$.

Let us identify the manifold Q_1 with a regular neighborhood of P_1 in M. Denote by T the boundary torus of Q_1. Then the natural collapse of Q_1 onto P_1 induces a locally homeomorphic map $T \to P_1$ such that the inverse image of each 2-component of P_1 consists of two 2-cells. Since P_1 contains two 2-components, this map determines a decomposition of T onto four 2-cells. Construct the universal covering $\tilde{T}$ of T. It can be presented as a plane decomposed into hexagons, see Fig. 2.21. The group of covering translations is isomorphic to the group $\pi_1(T) = H_1(T, Z)$. We choose a basis $\bar{\mu}, \bar{\lambda}$ as shown in Fig. 2.21. The corresponding elements μ, λ of $\pi_1(T)$ (which can be also viewed as oriented loops) form a coordinate system on T.

Since Q_1 is a regular neighborhood of P_1 in M, the difference $V = M \setminus \operatorname{Int} Q_1$ is a solid torus. This means that M has the form $M = (Q_1)_{p,q}$, where coprime integers p, q are determined by the requirement that the curve $\mu^p \lambda^q$ is homotopic to the meridian of V.

Denote by X the part of P_1 that disappears after puncturing and collapsing. Assume that X is an open 2-cell. In other words, the spine P of M is obtained from P_1 by attaching the 2-cell $\bar{X}$ that disappears under puncturing and collapsing. Denote by ℓ the boundary curve of $\bar{X}$. All the intersection points of ℓ with the graph SP_1, as well as all the self-intersection points of ℓ, are true vertices of P. The number of such points must not be greater than 6, since the total number of true vertices is ≤ 8, and two of them are the true vertices of P_1.

Recall that if we factor this covering by the translations $\bar{\mu}, \bar{\lambda}$ corresponding to μ and λ, we recover T. If we additionally identify the hexagons marked by the letter A with respect to the composition of the symmetry in the dotted diagonal of the hexagon and the translation by $-\bar{\mu} + \bar{\lambda}/2$, and do the same for hexagons marked by the letter B, we obtain P_1. T is shown in Fig. 2.22 as

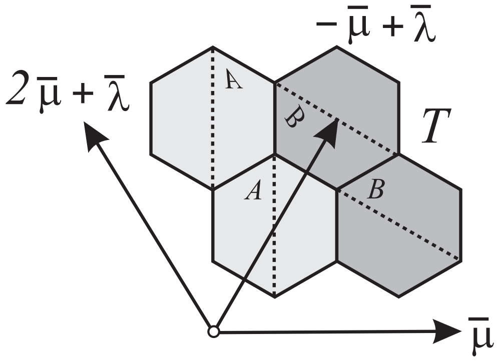

Fig. 2.22. Cell decomposition of T

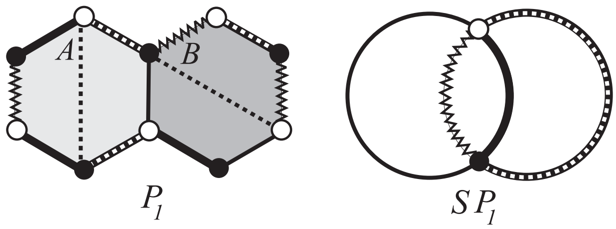

Fig. 2.23. Cell decomposition of P_1 and decorated SP_1

a polygonal disc D composed of four hexagons. Each side of D is identified with another one via the translation along one of the three vectors $\bar{\mu}$, $-2\bar{\mu}+\bar{\lambda}$, and $-\bar{\mu}+\bar{\lambda}$. P_1 can be presented as the union of two hexagons, see Fig. 2.23. The edges of the hexagons are oriented and decorated with four different patterns. To recover P_1, one should identify the edges having the same pattern. Figure 2.23 shows also the singular graph SP_1 of P_1 equipped with the same decoration.

To the curve ℓ on P_1 (the boundary curve of the attached 2-cell) there corresponds a curve $\bar{\ell}$ of type (p,q) on the torus T and an arc $\tilde{\ell}$ on $\tilde{T}$. One end of $\tilde{\ell}$ is obtained from the other by translation on the vector $p\bar{\mu}+q\bar{\lambda}$. Since ℓ crosses the edges of P_1 in ≤ 6 points, $\tilde{\ell}$ does the same with respect to the edges of $\tilde{T}$. Choosing one hexagon in $\tilde{T}$ as the initial one, and successively marking off those cells which may be reached at the expense of 1, 2, 3, 4, 5, or 6 intersections, one can select all the possible pairs of coprime parameters (p,q) that potentially may produce a spine with ≤ 8 true vertices. In our case they are the following: (1,0), (0,1), (1,1), (-1,1), (-2,1), (-3,1), (-4,1), (-1,2), (-3,2), (-5,2), (-4,3), (-5,3) (up to simultaneous change of signs). See Fig. 2.24.

Let us investigate these pairs. The pairs (-4,1), (-5,2), (-5,3), (-4,3), (-1,2), (1,1) are actually impossible, since in all these cases ℓ intersects at least six edges of P_1 and has at least one self-intersection. For example, if $\tilde{\ell}$ joins the hexagons (0,0) and (5,-2) as shown in Fig. 2.25a, it crosses the edges six times

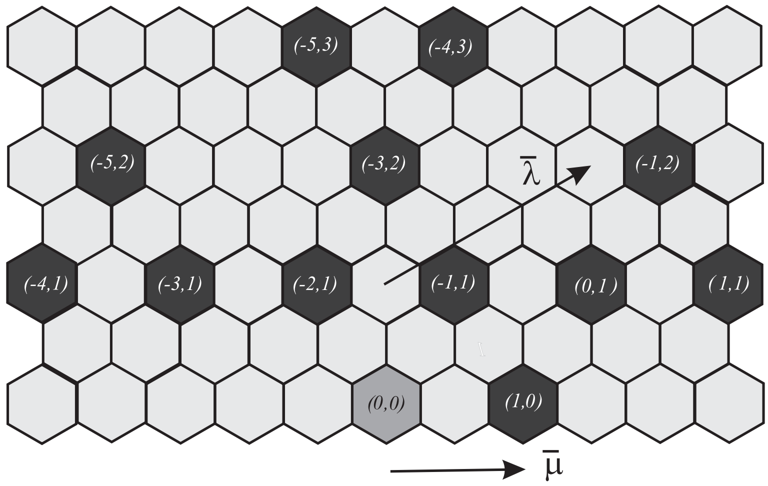

Fig. 2.24. Suspicious hexagons are shown in *black*

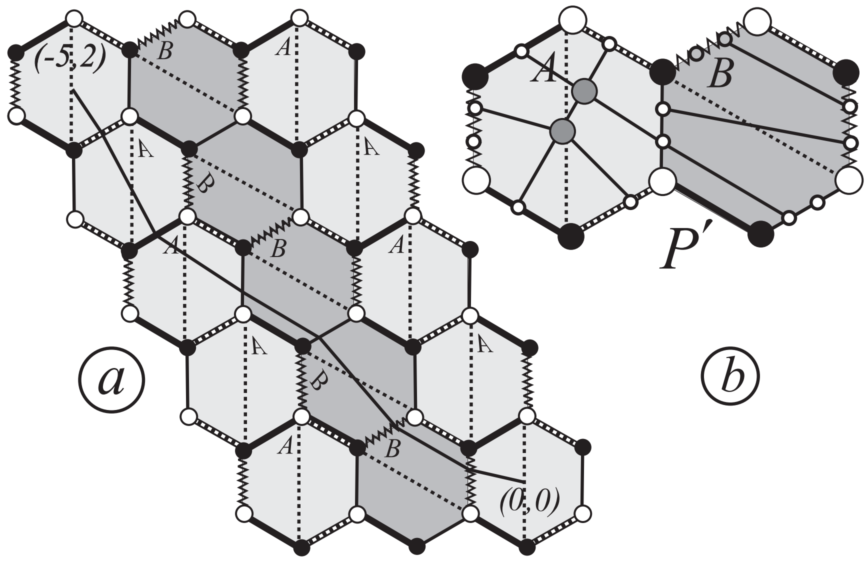

Fig. 2.25. A spine P' of $(Q_1)_{-5,2}$ having ten true vertices

and its projection $\ell \subset P_1$ has two self-intersections, see Fig. 2.25b, where the self-intersections are indicated with gray fat dots.

It can be checked directly that for all the remaining pairs (p, q) (i.e., for (-3,1), (-2,1), (-3,2), (-1,1), (1,0), (0,1)) we get graph manifolds. Indeed, let us attach a 2-cell to P_1 so as to obtain a special spine P' of $(Q_1)_{p,q}$ with ≤ 8 true vertices. It turns out that in all these cases one can find a 2-component of P' so that after puncturing and collapsing we get a spine satisfying the assumption of Proposition 2.4.10. It follows that the corresponding manifold $(Q_1)_{p,q}$ belongs to the class $\mathcal{G}$.

However, the part X of P which disappears after puncturing and collapsing is a priori not necessarily a cell. One can represent it as a simple polyhedron $\bar{X}$ attached to P_1 along $\partial\bar{X}$. In this case $\partial\bar{X}$ is a regular graph of degree 3 and hence has a nonzero even number of vertices.

Suppose it has two vertices, which are joined by three edges, i.e., it is what is usually called a *θ-curve*. The case of *spectacles* (two circles joined by a segment) is excluded since P would have a counterpass and hence could be simplified. The case of four or more vertices is even simpler. So we restrict ourselves to considering only $\bar{X}$ such that $\partial\bar{X}$ is a θ-curve.

Denote by $\bar{a}, \bar{b}, \bar{c}$ the edges of $\bar{X}$. We can think of $\bar{X}$ as being contained in the solid torus $V = M \setminus \mathrm{Int}\ Q_1, \partial V = T$, such that $\partial\bar{X} \subset T = \partial Q_1$ and the complement of $\partial V \cup X$ in V is an open 3-ball. When we attach $\bar{X}$ to P_1, the vertices of $\partial\bar{X}$ become true vertices of P. Since P has not more than eight true vertices, the images a, b, c of $\bar{a}, \bar{b}, \bar{c}$ under gluing may intersect the edges of the singular graph SP_1 in ≤ 4 points, and if in 4, then they cannot intersect each other or possess self-intersections. If we lift $\bar{a}, \bar{b}, \bar{c}$ to the arcs $\tilde{a}, \tilde{b}, \tilde{c}$ on $\tilde{T}$ having a common endpoint, we get a triode (a wedge of three arcs) intersecting the edges in ≤ 4 points. The free ends of the wedge lie on three different hexagons which are obtained from each other by translations on nontrivial integer linear combinations of $\bar{\mu}, \bar{\lambda}$. A simple analysis of the covering shows that there exist a few such triodes, but the projections onto P_1 of their edges have at least one additional intersection point. This finishes the proof of Proposition 2.4.12 and Theorem 2.4.9.

2.5 Hyperbolic Manifolds

2.5.1 Hyperbolic Manifolds of Complexity 9

As we have shown in the earlier section, all closed irreducible orientable 3-manifolds up to complexity 8 are graph manifolds. Is this result sharp? The answer is affirmative. We describe here a remarkable closed orientable 3-manifold M_1 of complexity 9 which is hyperbolic and thus does not belong to the class $\mathcal{G}$ of all graph manifolds. This manifold was discovered by Weeks [46, 133] and independently by Fomenko and Matveev [32]. We have called it “remarkable,” since it is twice minimal. First, it has the minimal complexity among all closed orientable hyperbolic 3-manifolds. Second, its hyperbolic volume $V(M_1) \approx 0.94272$ is also minimal among all the *known* 3-manifolds of the same class. Conjecture 2.5.1 is motivated by these facts, together with an as experimental observation made in [32] that the growth of the volume correlates (in some sense) with the growth of the complexity.

Conjecture 2.5.1 ([32]). M_1 has the least volume of any closed orientable hyperbolic 3-manifold.

No counterexamples to this conjecture have appeared and the difference between the known lower estimates of the volumes and $V(M_1)$ remains substantial. Personally, I believe that the conjecture is true.

Let Q_1 be the 3-manifold represented by its special spine P_1 with two true vertices, see Fig. 2.19. Its boundary $T = \partial Q_1$ is a torus with coordinate system (μ, λ). One can think of μ and λ as oriented simple closed curves on T which are images of the oriented segments $\bar{\mu}, \bar{\lambda}$ in $\tilde{T}$ under the universal covering projection map $\tilde{T} \to T$, see Fig. 2.21. We define the above-mentioned remarkable manifold M_1 as the manifold $(Q_1)_{5,-2}$ obtained from Q_1 by attaching a solid torus such that the image of its meridian has the type $\mu^5\lambda^{-2}$.

To show that M_1 is hyperbolic, we recall briefly the Thurston method, one of the most important and successful methods for understanding finite-volume hyperbolic 3-manifolds by considering their decomposition into ideal tetrahedra. This technique was introduced in [120] and was used for computation of volumes in [46, 133] and [32]. It forms also the basis for the SNAPPEA computer program, written by Weeks, which allows one to determine the hyperbolic structure and volume of a large number of hyperbolic manifolds.

First, we use Corollary 1.1.28 to decompose Int Q_1 into two topological ideal tetrahedra. Next to each ideal tetrahedron we associate a complex variable that determines its geodesic shape. A system of complex polynomial equations is generated, the equations coming from the need for the tetrahedra to glue together correctly at the edges. This system of *consistency conditions* for Q_1 has many solutions parameterized by elements of U, where U is the set of all complex numbers z such that $\mathrm{Im} z > 0$ and $z \neq 1/2(1+t\mathbf{i})$, $\sqrt{15} < t < \infty$. If both ideal tetrahedra are regular, then the hyperbolic structure on Q_1 thus obtained is complete. This means that Q_1 is hyperbolic. By the way, all other manifolds Q_2–Q_{14} are also hyperbolic and, as shown in [32], are the only orientable hyperbolic 3-manifolds of complexity ≤ 3 having one cusp.

It turns out that the parameter z and hence the geometric shape of the ideal tetrahedra can be chosen so that z satisfies the consistency conditions and that the completion of the corresponding hyperbolic structure on Q_1 is a closed hyperbolic manifold homeomorphic to M_1, see [46, 133] and [32]. This means that M_1 is hyperbolic.

Remark 2.5.2. The value of z that produces the hyperbolic structure of M_1 has irrational real and imaginary parts. So there may arise the question whether approximate values of z that can be found by computer (and that satisfy the consistency and the completion conditions only approximately) are sufficient for proving that M_1 is hyperbolic. This difficulty can be overcome, since z lies strictly inside U. On the other hand, M_1 is arithmetic [22, 23], so one can prove the existence of a hyperbolic structure without using computers.

Theorem 2.5.3. *The complexity of M_1 is equal to 9.*

Proof. Let us construct a special spine of M_1 having nine true vertices. Since $c(M_1) > 8$ by Theorem 2.4.9, this would be sufficient for proving that $c(M_1) =$

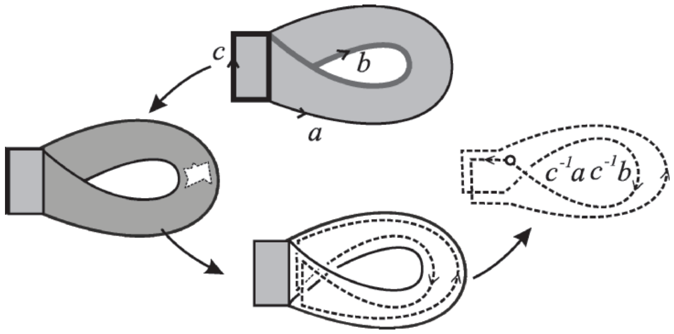

Fig. 2.26. Möbius triplet

9. The hexagon (5,-2) is one of "suspicious" hexagons, since it can be joined with the initial hexagon (0,0) by an arc $\tilde{l}$ in $\tilde{T}$ intersecting six edges, see Fig. 2.25. So it would be natural to look for $\tilde{l}$ such that its projection l onto P_1 has not more than one self-intersection point. If we find one, then a special spine of M_1 with 9=2+6+1 vertices can be obtained by attaching to P_1 a new disc 2-component along l. Unfortunately, the projections of all arcs on $\tilde{T}$ that join (0,0) with (5,-2) and intersect not more than six edges have at least two self-intersection points. Therefore, the maximum we can get by attaching a disc is a spine of $(Q_1)_{-5,2}$ having ten true vertices.

It turns out that one can save one true vertex by attaching to P_1 not a disc, but a so-called *Möbius triplet* Y, see Fig. 2.26. One can think of Y as being contained in the solid torus $V = M_1 \setminus \operatorname{Int} Q_1$ such that $\partial Y \subset T = \partial Q_1$ and the complement of $\partial V \cup Y$ in V is an open 3-ball.

If we cut out a disc D from the Möbius 2-component of Y, then the rest collapses onto ∂Y. We need to track the behavior of ∂D under the collapse. Observe that ∂D is deformed into the curve $c^{-1}ac^{-1}b$, where a, b, c are the three coherently oriented edges of the θ-curve ∂Y. Let us attach Y to P_1 to obtain a special polyhedron $P' = P_1 \cup Y$ as shown in Fig. 2.27. The arcs $\tilde{a}, \tilde{b}, \tilde{c} \subset \tilde{T}$ that corresponds to a, b, c form a triode such that its branch point is in the hexagons $(-1/2, 0)$ while the free ends of $\tilde{a}, \tilde{b}, \tilde{c}$ are in the hexagons $(1/2, 0), (-1/2, 0), (-5/2, 1)$, respectively.

Therefore, the curve $c^{-1}ac^{-1}b$ has type (3,-1)+(2,-1)=(5,-2). It follows that P' is a special spine of $(Q_1)_{5,-2}$. The images in P_1 of the arcs a, b, c have only one intersection point (shown in Fig. 2.27 as a fat gray dot). Therefore, P' has nine vertices. A regular neighborhood of its singular graph is represented in Fig. 2.28. □

Remark 2.5.4. The manifold Q_2 (the complement of the figure eight knot) is a twin of Q_1: Just as Q_1, it admits a decomposition into two regular ideal tetrahedra. It was used by W. Thurston to illustrate his method [120]. Just as Q_1, it admits a decomposition into two regular ideal tetrahedra. The manifold $M_2 = (Q_2)_{5,1}$ has complexity 9 and its volume is the second one among all

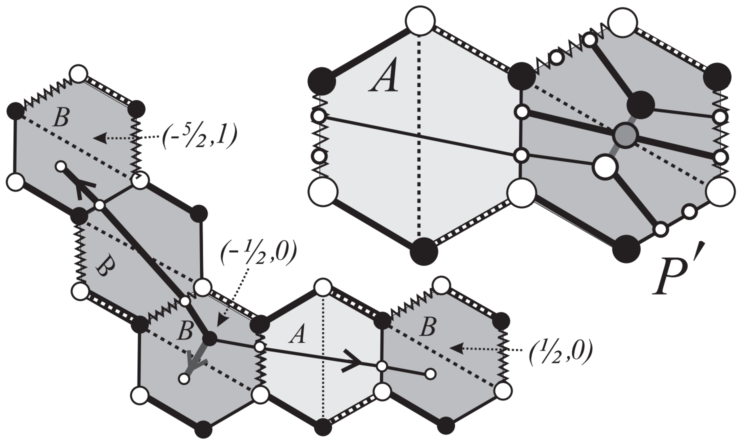

Fig. 2.27. A special spine P of $(Q_1)_{-5,2}$ obtained by attaching a Möbius triplet to P_1

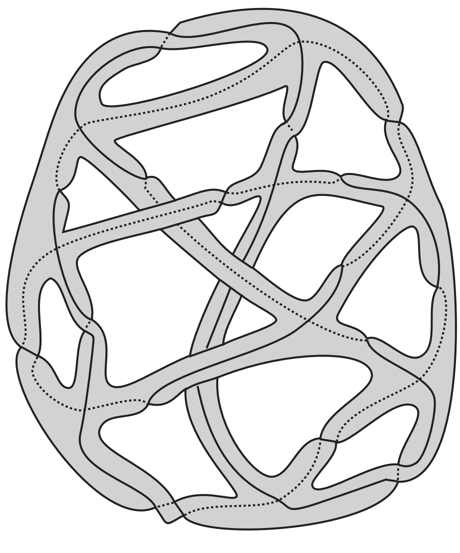

Fig. 2.28. A special spine of M_1

known volumes of closed hyperbolic 3-manifolds. A special spine of M_2 is shown on Fig. 2.29.

2.6 Lower Bounds of the Complexity

As we have shown in Sect. 2.1.2, it is relatively easy to obtain upper bounds for complexity. However, the problem of finding lower bounds is quite difficult. Of course, we know the exact value of the complexity for all the manifolds

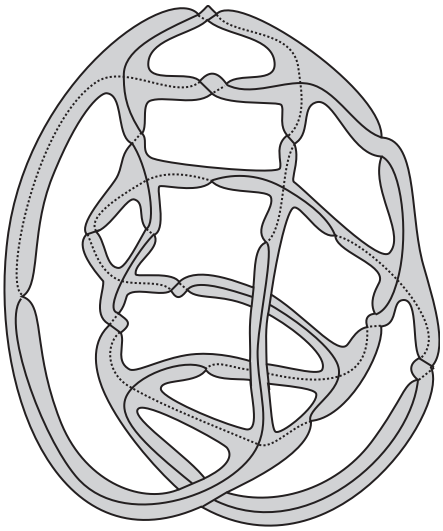

Fig. 2.29. A special spine of M_2

from the table (see Appendix), but there are only finitely many of them. In this section we present several lower bounds for the complexity of arbitrary 3-manifolds and describe two infinite series of hyperbolic 3-manifolds whose complexity is known exactly.

2.6.1 Logarithmic Estimates

The first bound is based on the evident observation that if the first homology group of a 3-manifold M is large, then $c(M)$ cannot be too small.

Lemma 2.6.1. *Suppose that a special spine P of a closed orientable 3-manifold M contains a 2-component α whose boundary curve passes along some edge e three times. Then M has an almost simple spine having a smaller number of true vertices.*

Proof. If the boundary curve $\partial\alpha$ of α has a counterpass on the edge e, then by Proposition 2.3.3 P can be simplified. Suppose $\partial\alpha$ passes along e all three times in the same direction. Then P contains a simple closed curve ℓ such that ℓ intersects SP at two points $A, B \in e$ and visits all three wings adjacent to e, see Fig. 2.30. ℓ can be easily constructed by thinking of α as an attached disc and considering two disjoint proper arcs in the disc that join points in distinct preimages of e. One can easily see that ℓ can be shifted away from P into the boundary of a regular neighborhood N of P. Since M is closed, ∂N is a 2-sphere. It follows that l bounds a disc $D \subset M$ such that $D \cap P = \ell$, and we can simplify the spine by adding D to P and piercing another 2-cell. □

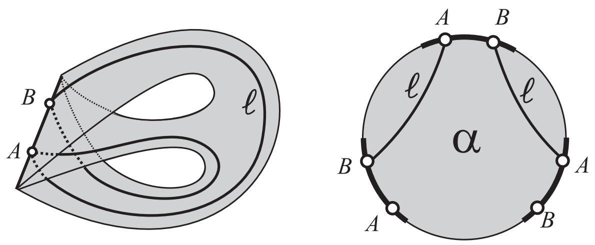

Fig. 2.30. ℓ can be shifted from P

Denote by $|\mathrm{Tor}(H_1(M))|$ the order of the torsion subgroup of the first homology group $H_1(M; Z)$ and by β_1 the first Betti number of M, i.e., the rank of the free part of $H_1(M; Z)$.

Theorem 2.6.2. *[90] Let M be a closed irreducible orientable 3-manifold different from $L_{3,1}$. Then $c(M) \geq 2\log_5 |Tor(H_1(M))| + \beta_1 - 1$.*

Proof. Since for $H_1(M) = 0, Z_2$ the right-hand side of the above inequality is negative, the conclusion of the theorem holds for $M = S^3$ and $M = RP^3$. So we can assume that M is not one of these manifolds. Choose an almost simple spine P of M having $c(M)$ true vertices. By Theorem 2.2.4, we may assume that P is special. Let $A(P)$ be the relation matrix of the presentation corresponding to P and n be the number of generators in that presentation. Then $n = c(M) + 1$, and, as M is closed, $A(P)$ is a square matrix of order n. Since P has the smallest number of true vertices, Lemma 2.6.1 implies that it has no edges along which some component passes three times. There are no counterpasses either, therefore each column of the matrix contains either two nonzero elements (one of them is equal to ± 2, and the other is equal to ± 1), or three elements (each is equal to ± 1).

The matrix $A(P)$ has a minor A' of order $n - \beta_1$ whose determinant is nonzero and is divisible by $|\mathrm{Tor}(H_1(M))|$. On the other hand, the absolute value of the determinant is equal to the volume of the parallelepiped whose base vectors are the columns of A'. The volume does not exceed the product of the lengths of those vectors. It is clear that the length of each vector is not greater than $\sqrt{5}$. Hence $|\det A'| \leq (\sqrt{5})^{n-\beta_1}$, which implies that $n \geq 2\log_5 |\det A'| + \beta_1$. Since $|\det A'|$ is greater than any its divisor, and $n = c(M) + 1$, we have $c(M) \geq 2\log_5 |\mathrm{Tor}(H_1(M))| + \beta_1 - 1$. □

In some cases (for instance, for $L_{5,2}$) this bound is sharp. Let us show that for an infinite series of lens spaces this bound is *almost sharp* (in certain sense). Let $u_i, 1 \leq i < \infty$, be the Fibonacci numbers given by the initial values $u_1 = u_2 = 1$ and the recurrence relation $u_{i+1} = u_i + u_{i-1}$. Denote by L_n the lens space $L(p, q)$ with parameters $p = u_n, q = u_{n-2}$.

Corollary 2.6.3. *If $n > 4$, then $nC_n - 2 \leq c(L_n) \leq n - 4$, where $C_n = (2/n)\log_5(\sqrt{5}u_n)$.*

Proof. It follows from Theorem 2.6.2 that $c(L_n) \geq 2\log_5 u_n - 1 = nC_n - 2$, so we get the first inequality. To get the second one, we recall that all partial quotients in the expansion of u_n/u_{n-2} as a regular continued fraction are 1, so their sum $S(u_n, u_{n-2})$ is $n-1$. Therefore, by item E of Sect. 2.3.3, we have $c(L_n) \leq S(u_n, u_{n-2}) - 3 = n - 4$ for all $n > 4$. $\square$

Remark 2.6.4. One can easily show that $C_n > 0.5$ for all $n > 4$ and that C_n tends to $2\log_5(\frac{1+\sqrt{5}}{2}) \approx 0.59798$ as $n \to \infty$.

Corollary 2.6.3 shows that for an infinite series of 3-manifolds complexity depends logarithmically on the order of the torsion subgroup of the first homology group. This remarkable fact was first observed by Pervova.

The bound in Theorem 2.6.2 has the following shortcoming: It is trivial for closed manifolds having zero first homology group, i.e., for homology spheres. It would be natural to attempt to find a bound depending on the fundamental group.

Definition 2.6.5. *Let a group G be given by a presentation*

$$G = \langle g_1, \ldots, g_n \mid r_1, \ldots, r_m \rangle.$$

Then the length of that presentation *is the number $|r_1| + \ldots + |r_m|$, where $|r_i|$ denotes the length of the word r_i with respect to $g_1, \ldots, g_n$. The* presentation complexity $\hat{c}(G)$ *of G is the minimum of the lengths of all its finite presentations.*

Let us consider some examples of estimates of complexity of groups. Evidently, the complexity of the cyclic group Z_n does not exceed n. It is interesting to note that it may be much less than n. For example, the presentation $\langle a, b, c \mid a^4b, b^5c, c^2 \rangle$ of Z_{40} has length 13. It may appear that this small value (compared with the order of the group) is due to the fact that 40 has nontrivial divisors. However, the group Z_{47}, which has prime order, can be given by the presentation $\langle a, b, c \mid a^4b, b^4c, c^3a^{-1} \rangle$ of length 14.

Proposition 2.6.6. *Let M be a closed irreducible orientable 3-manifold, different from S^3, RP^3, and $L_{3,1}$. Then $c(M) \geq -1 + \hat{c}(\pi_1(M))/3$.*

Proof. Let P be a special spine of M having $k = c(M)$ true vertices. Then the length of the presentation of $\pi_1(M)$ that corresponds to P is $3(k+1)$. It follows that $\hat{c}(\pi_1(M)) \leq 3(k+1)$ and thus $k \geq -1 + \hat{c}(\pi_1(M))/3$. $\square$

Sometimes this proposition allows to obtain better bounds than those given by Theorem 2.6.2. For example, it can be shown that the complexity of any nontrivial finitely presented group G that coincides with its commutator subgroup is at least 10. It follows from Proposition 2.6.6 that the complexity of any homology sphere cannot be less than 3. This agrees with the fact that the complexity of the first nontrivial homology sphere (the dodecahedron space) equals 5, see [35, 91].

2.6.2 Complexity of Hyperbolic 3-Manifolds

Now we turn our attention to hyperbolic 3-manifolds. As we have mentioned earlier, there is a correlation between their complexities and volumes. A very nice partial case of this observation was found by Anisov [6]. Recall that all regular ideal tetrahedra in the hyperbolic space H^3 are congruent and have the same volume $V_0 \approx 1.0149$. The volumes of all other ideal tetrahedra in H^3 are less than V_0.

Lemma 2.6.7. *Let M be a hyperbolic 3-manifold with nonempty boundary. Then $c(M) \geq V(M)/V_0$, where $V(M)$ is the hyperbolic volume of M.*

Proof. Since M is hyperbolic, it is irreducible and boundary irreducible, and contains no essential annuli. By Theorem 2.2.4, its minimal almost simple spine is special. It follows from Corollary 1.1.28 that M can be decomposed into $k = c(M)$ topological ideal tetrahedra $\Delta_i, 1 \leq i \leq k$. Further we follow Thurston's arguments [120]. These tetrahedra can be lifted to H^3, straightened inside H^3 to hyperbolic ideal tetrahedra and projected back into M. The new tetrahedra Δ_i' can overlap, but they still cover M. It follows that $V(M) \leq \sum_{i=1}^{k} V(\Delta_i) \leq kV_0$, where $V(\Delta_i)$ is the volume of Δ_i. We can conclude that $k \geq V(M)/V_0$. □

In some cases Lemma 2.6.7 is sufficient for exact computation of complexity [6].

Corollary 2.6.8. *Suppose that a hyperbolic 3-manifold M can be decomposed into k straight regular ideal tetrahedra. Then $c(M) = k$.*

Proof. Since $V(M) = kV_0$, we have $c(M) \geq k$ by Lemma 2.6.7. The inequality $c(M) \leq k$ follows from Corollary 1.1.28. □

There are not many 3-manifolds satisfying the assumption of Corollary 2.6.8. Q_1 and Q_2 (see Remark 2.5.4) as well as all their finite coverings are among them. Since $H_1(Q_1; Z) = Z \oplus Z_5$ and $H_1(Q_2; Z) = Z$, there are infinitely many such coverings. These manifolds form the first nontrivial infinite series of 3-manifolds with known complexities: if a 3-manifold M is a k-sheeted covering of Q_1 or Q_2, then $c(M) = 2k$.

Remark 2.6.9. Q_2 can be represented as a Stallings manifold fibered into punctured tori over S^1 with the monodromy matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Let W_1 be a closed Stallings manifold with fiber $S^1 \times S^1$ and the same monodromy matrix. Consider the k-sheeted covering W_k of W_1 corresponding to the kernel of the superposition of the abelinization map $\pi_1(W_1) \to H_1(W_1; Z) = Z$ and the mod k reduction $Z \to Z_k$. One can easily construct a special spine P of W_k with $2k + 5$ true vertices. Indeed, it suffices to take a k-sheeted covering of the spine P_2 of Q_2 and attach an additional 2-cell D which fills up the puncture of the fiber. It follows that $c(W_k) \leq 2k+5$ (see [5]).

On the other hand, a short calculation shows that $|\mathrm{Tor}(H_1(W_n))| = u_{2n+1} + u_{2n-1} - 2$. Taking into account that the first Betti number of W_n is 1 and applying Theorem 2.6.2, we get $c(W_n) \geq 2C'_n n$, where $C'_n = (1/n)\log_5(u_{2n+1} + u_{2n-1} - 2)$. It follows that $2C'_n n \leq c(W_n) \leq 2n + 5$, so we have another example of a logarithmic growth of the complexity (compare with Corollary 2.6.3). It is interesting to note that C'_n has exactly the same limit as C_n from Corollary 2.6.3.

2.6.3 Manifolds Having Special Spines with One 2-Cell

We describe for every $n \geq 2$ an interesting class $\mathcal{M}_n$ of orientable 3-manifolds having complexity n. The manifolds from $\mathcal{M}_n, 2 \leq n < \infty$ form the second infinite set of manifolds with known complexity (the first such set was described in the earlier section). This class was introduced in [33], see also [40].

Definition 2.6.10. *An orientable 3-manifold M belongs to the class $\mathcal{M}_n$, if it has a special spine with n true vertices and exactly one 2-cell.*

Examples of spines with one 2-cell are shown in Fig. 2.31. Each $\mathcal{M}_n$ contains a manifold presented either by the upper spine (if $n \neq 3k+1$) or by the lower one (if $n = 3k+1$). The only exception is the case $n = 1$ when $\mathcal{M}_n$ is empty. As proved in [33], the number of manifolds in $\mathcal{M}_n$ grows exponentially as $n \to \infty$.

Theorem 2.6.11. *[33] Let $M \in \mathcal{M}_n$. Then $c(M) = n$.*

Proof. By definition of the class $\mathcal{M}_n$, the manifold M has a special spine P with n vertices. Therefore, $c(M) \leq n$. To prove that $c(M) = n$, consider a handle decomposition ξ_P of M that corresponds to P. Since P has only one 2-cell, the set of all normal surfaces in M can be easily described. All closed normal surfaces are normally parallel to ∂M. Since $\chi(\partial M) = 2(1-n)$ and $n > 0$, there are no normal spheres among them. Therefore, M is irreducible. All nonclosed normal surfaces are contained in the union of all balls and

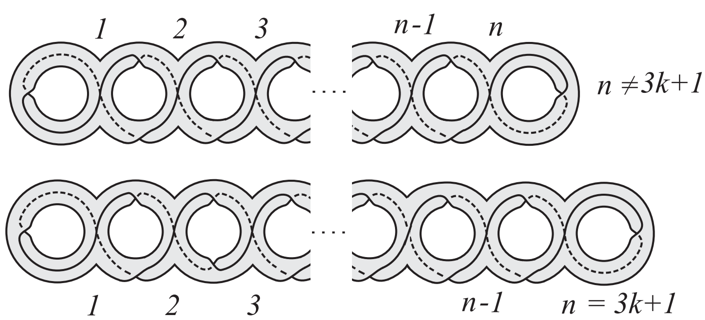

Fig. 2.31. Spines with one 2-cell

beams of ξ_P. The set of such surfaces contains no discs, thus M is boundary irreducible. This set can contain annuli, but all of them are compressible. It follows from Theorem 2.2.4 that M has a special spine P' with $m = c(M)$ true vertices.

Let us prove that $m \geq n$. Denote by k the number of 2-cells of P'. Counting the Euler characteristic of M, we get $2(k - m) = \chi(M) = 2(1 - n)$ and $n + k - 1 = m$. It follows that $m \geq n$. □

Manifolds from $\mathcal{M}_n$ possess many other good properties. They are hyperbolic manifolds with totally geodesic boundary and have Heegaard genus $n + 1$. Moreover, each manifold $M \in \mathcal{M}_n$ has a unique special spine with n vertices, which is homeomorphic to the cut locus of M (the set of points of M having more than one shortest geodesic to ∂M). See [33] for the proof.