7

Allocations of Probability[1]

Glenn Shafer

Abstract. This paper studies *belief functions*, set functions which are normalized and monotone of order ∞. The concepts of *continuity* and *condensability* are defined for belief functions, and it is shown how to extend continuous or condensable belief functions from an algebra of subsets to the corresponding power set. The main tool used in this extension is the theorem that every belief function can be represented by an *allocation of probability*—i.e., by a $\cap$-homomorphism into a positive and completely additive probability algebra. This representation can be deduced either from an integral representation due to Choquet or from more elementary work by Revuz and Honeycutt.

Key words: Belief function, Allocation of probability, Capacity, Upper and lower probabilities, Condensability, Continuity

1 Belief Functions

In his pathbreaking "Theory of capacities," Gustave Choquet (1953) established the following definitions: a class $\mathcal{E}$ of subsets of a set Ω is a multiplicative subclass of $\mathcal{P}(\Omega)$ if $A \cap B$ is in $\mathcal{E}$ whenever A and B are in $\mathcal{E}$, an additive subclass of $\mathcal{P}(\Omega)$ if $A \cup B$ is in $\mathcal{E}$ whenever A and B are in $\mathcal{E}$. A real-valued function g on a multiplicative subclass $\mathcal{E}$ is *monotone of order n* if

$$g(A) \geqq \sum \left\{ (-1)^{|I|+1} g\left(\cap_{i \in I} A_i\right) | \varnothing \neq I \subset \{I, \cdots, n\} \right\}$$

for every collection $A, A_1, \cdots, A_n$ of elements of $\mathcal{E}$ such that $A \supset A_i$ for all i, *monotone of order* ∞ if it is monotone of order n for all $n \geqq 1$. A real-valued function g on an additive subclass $\mathcal{E}$ is *alternating of order n* if

$$g(A) \leqq \sum \left\{ (-1)^{|I|+1} g\left(\cup_{i \in I} A_i\right) | \varnothing \neq I \subset \{I, \cdots, n\} \right\}$$

[1] This research was partially supported by NSF Grant GP43248.
AMS 1970 *subject classifications.* Primary 60A05; secondary 62A99.

for every collection $A, A_1, \cdots, A_n$ of elements of $\mathcal{E}$ such that $A \subset A_i$ for all i; *alternating of order* ∞ if it is alternating of order n for all $n \geqq 1$.

We call a function f on a multiplicative subclass $\mathcal{E}$ of $\mathcal{P}(\Omega)$ a *belief function* if $\varnothing$ and Ω are in $\mathcal{E}, f(\varnothing) = 0, f(\Omega) = 1$, and f is monotone of order ∞. The condition that f be monotone of order ∞ implies in particular that f is increasing; hence a belief function always takes values in the interval [0, 1]. The name "belief function" derives from the thought that these functions might be used to represent partial belief: if Ω is interpreted as a set of "possibilities" and A is a subset of Ω, then $f(A)$ might express one's degree of belief that the truth lies in A. In a recent monograph (1976a), I argue at length that belief functions are useful and appropriate for the representation of partial belief, and I study these functions in detail in the case where Ω is finite. This paper develops tools for extending that study to the case where Ω is infinite.

We call a function f^* on an additive subclass $\mathcal{E}^*$ of $\mathcal{P}(\Omega)$ an *upper probability function* if $\varnothing$ and Ω are in $\mathcal{E}, f^*(\varnothing) = 0, f^*(\Omega) = 1$, and f^* is alternating of order ∞. Notice that if f is a belief function on $\mathcal{E}$, then the function f^* defined on the additive subclass $\mathcal{E}^* = \{\overline{A} | A \in \mathcal{E}\}$ by $f^*(A) = 1 - f(\overline{A})$ is an upper probability function.

It will be shown in Sect. 5 below that a belief function f on a multiplicative subclass $\mathcal{E}$ of $\mathcal{P}(\Omega)$ can always be extended to a belief function on $\mathcal{P}(\Omega)$. In fact, it always has a *canonical extension* to $\mathcal{P}(\Omega)$: namely, the belief function $\bar{f}$ on $\mathcal{P}(\Omega)$ given by

$$\bar{f}(A) = \sup \varSigma \left\{ (-1)^{|I|+1} f(\cap_{i \in I} A_i) | \varnothing \neq I \subset \{1, \cdots, n\} \right\},$$

where the supremum is taken over all $n \geqq 1$ and all collections $A_1, \cdots, A_n$ of elements of $\mathcal{E}$ that are subsets of A. We call this extension canonical because it is minimal; i.e., $\bar{f} \leqq g$ for any other belief function g on $\mathcal{P}(\Omega)$ that extends f. (In fact, $\bar{f} \leqq g$ for any other belief function g on $\mathcal{P}(\Omega)$ such that $f \leqq g|\mathcal{E}$.) This can also be expressed by saying that $\bar{f}$'s upper probability function $(\bar{f})^*$ is the maximal extension of f^*; i.e., $(\bar{f})^* \geqq g$ for any other upper probability function g on $\mathcal{P}(\Omega)$ that extends f^*.

In this paper we consider two regularity conditions for a belief function over an infinite set Ω: *continuity* and *condensability*. We call a belief function f on $\mathcal{P}(\Omega)$ *continuous* if it satisfies

$$f(\cap_i A_i) = \lim_{i \to \infty} f(A_i) \tag{1}$$

for every decreasing sequence $A_1 \supset A_2 \supset \cdots$ of subsets of Ω, and we call a belief function on a proper multiplicative subclass of $\mathcal{P}(\Omega)$ *continuous* if it can be extended to a continuous belief function on $\mathcal{P}(\Omega)$. We call a belief function f on $\mathcal{P}(\Omega)$ *condensable* if

$$f(\cap \mathcal{Q}) = \inf_{A \in \mathcal{Q}} f(A) \tag{2}$$

for every downward net $\mathcal{Q}$ in $\mathcal{P}(\Omega)$, and we call a belief function on a proper multiplicative subclass of $\mathcal{P}(\Omega)$ *condensable* if it can be extended to a condensable belief function on $\mathcal{P}(\Omega)$. (A subset $\mathcal{Q}$ of $\mathcal{P}(\Omega)$ is called a *downward* *net* if for every pair A_1, A_2 of elements of $\mathcal{Q}$ there exists an element A of $\mathcal{Q}$ such that $A \subset A_1 \cap A_2$.)

Though condensability is a rather restrictive condition it is intimately related to the idea of "weights of evidence" (see Shafer (1976a)) and to Dempster's rule for combining belief functions (see Shafer (1978)), and hence it seems intuitively appropriate for belief functions that purport to represent empirical knowledge. The weaker condition of continuity seems appropriate in the case of partial beliefs arising from theoretical knowledge; it applies in particular to the partial beliefs arising from knowledge of chances or "objective probabilities."

The conditions of continuity and condensability can also be stated in terms of the upper probability function. A belief function f on $\mathcal{P}(\Omega)$ is continuous if

$$f^*(\cap_i A_i) = \lim_{i\to\infty} f^*(A_i)$$

for every increasing sequence $A_1 \subset A_2 \subset \cdots$ of subsets of Ω; it is condensable if

$$f^*(\cup\, \mathcal{Q}) = \sup_{A\in\mathcal{Q}} f^*(A)$$

for every upward net $\mathcal{Q} \subset \mathcal{P}(\Omega)$, or equivalently, if

$$f^*(A) = \sup\{f^*(B) | B \subset A; B \text{ is finite}\} \tag{3}$$

for all $A \subset \Omega$. This last expression shows how strong a condition condensability is; a condensable belief function on a power set is completely determined by its upper probabilities for finite subsets.

Suppose f is a belief function on an algebra $\mathcal{E}$ of subsets of Ω or, more generally, on a subset $\mathcal{E}$ of $\mathcal{P}(\Omega)$ that is both a multiplicative and an additive subclass. Then, as we see in Sect. 5 below, f is continuous if and only if it satisfies (1) for every decreasing sequence $A_1 \supset A_2 \supset \cdots$ of elements of $\mathcal{E}$ such that $\cap_i A_i$ is in $\mathcal{E}$. And f is condensable if and only if for every $A \in \mathcal{E}$ and every $\varepsilon > 0$ there exists a cofinite subset B of Ω such that $A \subset B$ and $f(C) - f(A) < \varepsilon$ for all $C \in \mathcal{E}$ such that $A \subset C \subset B$. These theorems are proven by showing how to extend a belief function satisfying one of these conditions to a continuous (or condensable) belief function on $\mathcal{P}(\Omega)$; the extensions exhibited are *canonical* in the sense that they award each subset of Ω the minimal degree of belief that is compelled by the adoption of f on $\mathcal{E}$ and by the hypothesis of continuity (or condensability).

The most important tool we use in our study of the extension of belief functions is the representation theorem presented in Sect. 3. This theorem is a direct consequence of an integral representation due to Choquet (1953), and it can also be deduced from more elementary work by Revuz (1955) and Honeycutt (1971). (These scholars' results are reviewed in Sect. 2.) The theorem says that every belief function can be represented by an *allocation of probability*: i.e., that for every belief function $f : \mathcal{E} \to [0, 1]$ there exists a complete Boolean algebra $\mathcal{M}$, a positive and completely additive measure μ on $\mathcal{M}$, and a mapping $\rho : \mathcal{E} \to \mathcal{M}$ that preserves finite meets and satisfies $f = \mu \circ \rho$. Notice the intuitive interpretation of this representation: the elements of $\mathcal{M}$ are portions of one's belief or "probability," and $\rho(A)$ is the portion of one's probability that is "allocated" or committed to A.

In addition to helping us extend belief functions, the representation of belief functions by allocations of probability also helps give intuitive content to the idea of condensability. It is also useful in the study of Dempster's rule of combination and in the study of particular belief functions that arise in connection with statistical inference.

2 $\cap$-homomorphisms

Suppose $\mathcal{E}$ is a multiplicative subclass of $\mathcal{P}(\Omega)$ containing both $\varnothing$ and Ω, and suppose $\mathcal{F}$ is a multiplicative subclass of $\mathcal{P}(\mathcal{X})$ containing both $\varnothing$ and $\mathcal{X}$. We call $r : \mathcal{E} \to \mathcal{F}$ a $\cap$-*homomorphism* if $r(\varnothing) = \varnothing, r(\Omega) = \mathcal{X}$, and $r(A \cap B) = r(A) \cap r(B)$ for all $A, B \in \mathcal{E}$. (Cf. Choquet (1953), p. 197.) It is easily seen that if f is a belief function and r is a $\cap$-homomorphism, then $f \circ r$ is also a belief function.

Since a finitely additive probability measure qualifies as a belief function, this implies in particular that $\mu \circ r$ is a belief function whenever $r : \mathcal{E} \to \mathcal{F}$ is a $\cap$-homomorphism, $\mathcal{F}$ is an algebra, and μ is a finitely additive probability measure on $\mathcal{F}$. Probability measures being abundant and $\cap$-homomorphisms being easy to construct, this fact enables us to construct an abundance of belief functions. In fact, all belief functions can be obtained in this way:

Theorem 1. *Suppose $\mathcal{E}$ is a multiplicative subclass of $\mathcal{P}(\Omega)$ and f is a belief function on $\mathcal{E}$. Then there exists a set $\mathcal{X}$, an algebra $\mathcal{F}$ of subsets of $\mathcal{X}$, a finitely additive probability measure μ on $\mathcal{F}$, and a $\cap$-homomorphism r: $\mathcal{E} \to \mathcal{F}$ such that $f = \mu \circ r$.*

This theorem is due to Choquet; it is a direct consequence of his integral representation theorem. It is also a direct consequence of a construction due to Revuz (1955) and Honeycutt (1971).

In its simplest version Choquet's integral representation theorem is merely a sharpening of the Krein-Milman theorem (see Choquet (1969), Vol. II, p. 117). It states that if $\mathcal{L}$ is a locally convex Hausdorff topological vector space, $\mathcal{U}$ is a compact convex subset of $\mathcal{L}$, and $f \in \mathcal{U}$, then there exists a Radon probability measure μ on $\mathcal{U}$ such that the support of $\mathcal{U}$ is contained in the closure $\mathcal{X}$ of the extreme points of $\mathcal{U}$ and f is the resultant of μ. (In other words, $\alpha(f) = \int_{\mathcal{X}} \alpha(g) d\mu(g)$ for every continuous linear function $\alpha : \mathcal{U} \to R$.) If we take $\mathcal{L}$ to be the vector space of all real-valued functions on $\mathcal{E}$, endowed with the topology of simple convergence, and let $\mathcal{U} \subset \mathcal{L}$ be the set of all belief functions on $\mathcal{E}$, then the set of extreme points of $\mathcal{U}$ consists of the *two-valued* belief functions—those that take only the values zero and one. (See Choquet (1953), pp. 260–261. Notice that the two-valued belief functions on $\mathcal{E}$ are in a one-to-one correspondence with the filters in $\mathcal{E}$; a filter $\mathcal{F} \subset \mathcal{E}$ corresponds to the belief function which assigns degree of belief one to all elements of $\mathcal{F}$ and degree of belief zero to all elements of $\mathcal{E}-\mathcal{F}$.) And this set is compact and hence equal to its closure $\mathcal{X}$. For each $A \in \mathcal{E}$, the mapping $\alpha_A : \mathcal{L} \to R : g \to g(A)$ is continuous and linear, and hence

$$\begin{aligned} f(A) = \alpha_A(f) &= \int_{\mathcal{X}} g(A)\ d\mu(g) \\ &= \mu\left(\{g \in \mathcal{X}\,|g(A) = 1|\}\right). \end{aligned}$$

That is to say, $f = \mu \circ r$, where r is the $\cap$-homomorphism given by $r(A) = \{g \in \mathcal{X}|g(A) = 1\}$.

In order to relate Theorem 1 to Revuz' construction, set $\mathcal{X} = \mathcal{P}(\mathcal{E}) - \varnothing$, define $r : \mathcal{E} \to \mathcal{P}(\mathcal{X})$ by $r(A) = \{B \in \mathcal{E}|\varnothing \neq B \subset A\}$, and let $\mathcal{F}$ be the algebra of subsets of $\mathcal{X}$ generated by the image $r(\mathcal{E})$. Revuz' work, as emended by Honeycutt, shows how to construct, for a given belief function f on $\mathcal{E}$, a unique finitely additive probability measure μ on $\mathcal{F}$ such that $f = \mu \circ r$.

The measure μ obtained in Choquet's proof is countably additive (in fact, it is a Radon measure), but the $\cap$-homomorphism r obtained in this proof need not preserve infinite intersections. In the Revuz-Honeycutt construction, on the other hand, the $\cap$-homomorphism r preserves arbitrary intersections (provided these intersections are in $\mathcal{E}$), but the measure μ need not be countably additive.

3 Allocations of Probability

As it turns out, it is both useful and intuitively appealing to replace the measure space $(\mathcal{X}, \mathcal{F}, \mu)$ of the preceding representation by a *probability* algebra: i.e., a complete Boolean algebra that has associated with it a positive and completely additive probability measure. In this section we show that every belief function can be represented by a $\cap$-homomorphism into a probability algebra. We call such $\cap$-homomorphisms *allocations of probability.*

Some notation and nomenclature: we denote a probability algebra $\mathcal{M}$'s zero by Λ, its unit by V. We use the symbols $\wedge, \vee$ and $\leqq$ to denote meet, join and majorization in $\mathcal{M}$, reserving the analogous symbols $\cap, \cup$ and $\subset$ for their set-theoretic roles. To say that the measure μ on $\mathcal{M}$ is positive is to say that $\mu(M) > 0$ for every nonzero element M of $\mathcal{M}$. To say that it is completely additive is to say that $\mu(\vee\mathcal{B}) = \Sigma_{M\in\mathcal{B}}\mu(M)$ whenever $\mathcal{B}$ is a collection of pairwise disjoint elements of $\mathcal{M}$. And when we say $\rho : \mathcal{E} \to \mathcal{M}$ is a $\cap$-homomorphism, we mean, of course, that $\rho(\varphi) = \Lambda, \rho(\Omega) = \mathrm{V}$, and $\rho(A \cap B) = \rho(A) \wedge \rho(B)$.

The condition that the measure μ on a probability algebra $\mathcal{M}$ be both positive and completely additive implies in particular that $\mathcal{M}$ must satisfy the *countable chain condition*: every collection of pairwise disjoint elements of $\mathcal{M}$ is countable. And using this statement one can further deduce that every subset $\mathcal{B}$ of $\mathcal{M}$ must have a countable subset $\mathcal{C}$ such that $\vee\mathcal{B} = \vee\mathcal{C}$, that $\mu(\vee\mathcal{B}) = \sup_{M\in\mathcal{B}} \mu(M)$ for every upward net $\mathcal{B}$ in $\mathcal{M}$, and that $\mu(\wedge\mathcal{B}) = \inf_{M\in\mathcal{B}} \mu(M)$ for every downward net $\mathcal{B}$ in $\mathcal{M}$. (See pp. 61–69 of Halmos (1963).)

Theorem 2. *Suppose f is a belief function on a multiplicative subclass $\mathcal{E}$. Then there exists an allocation of probability $\rho : \mathcal{E} \to \mathcal{M}$ such that $f = \mu \circ \rho$, where μ is the measure associated with the probability algebra $\mathcal{M}$.*

Proof. Recall that if $\mathcal{M}_0$ is a σ-algebra of subsets and μ_0 is a countably additive probability measure on $\mathcal{M}_0$, then a probability algebra can be constructed by taking the quotient of $\mathcal{M}_0$ by the σ-ideal $\mathcal{I}$ consisting of all sets in $\mathcal{M}_0$ of μ_0-measure zero; this quotient $\mathcal{M} = \mathcal{M}_0/\mathcal{I}$ is a complete Boolean algebra and the measure μ that μ_0 induces on $\mathcal{M}$ is positive and completely additive. The projection $\pi : \mathcal{M}_0 \to \mathcal{M}$ satisfies $\mu_0 = \mu \circ \pi$; and since it is a Boolean homomorphism, it is in particular a $\cap$-homomorphism. (For details, again see Halmos (1963).)

Since f is a belief function, Choquet's integral representation supplies us a σ-algebra $\mathcal{M}_0$, a countably additive probability measure μ_0 on $\mathcal{M}_0$, and a $\cap$-homomorphism $r : \mathcal{E} \to \mathcal{M}_0$ satisfying $f = \mu_0 \circ r$. Let $\mathcal{M}$ and π be defined as in the preceding paragraph, and set $\rho = \pi \circ r$. Then $f = \mu \circ \rho$, and ρ, being the composition of two $\cap$-homomorphisms, is itself a $\cap$-homomorphism and hence an allocation of probability.

(Notice that the appeal to Choquet's integral representation could be replaced by a more elementary approach based on Revuz' construction. That construction yields a $\cap$-homomorphism $r : \mathcal{E} \to \mathcal{M}_1$, where $\mathcal{M}_1$ is merely an algebra with a finitely additive probability measure μ_1. But the Stone representation theorem could be used to construct a σ-algebra $\mathcal{M}_0$, a countably additive measure μ_0, and a Boolean homomorphism $g : \mathcal{M}_1 \to \mathcal{M}_0$ such that $\mu_1 = \mu_0 \circ g$.)

The representation of a belief function f by an allocation ρ can be much more useful in theoretical discussions than the representation by a $\cap$-homomorphism into the algebra of a measure space, particularly if one is concerned with the conditions of continuity and condensability. For example:

Theorem 3. *Suppose $\rho : \mathcal{P}(\Omega) \to \mathcal{M}$ is an allocation for the belief function f. Then f is continuous if and only if*

$$\rho(\cap_i A_i) = \wedge_i \rho(A_i) \tag{4}$$

for every sequence $A_1, A_2, \cdots$ of subsets of Ω. And f is condensable if and only if

$$\rho(\cap\, \mathcal{Q}) = \wedge_{A\in\mathcal{Q}} \rho(A) \tag{5}$$

for every nonempty subset $\mathcal{Q}$ of $\mathcal{P}(\Omega)$.

The proof of this theorem is straightforward and directly yields a generalization to the case of an allocation ρ for a belief function on an arbitrary multiplicative subclass $\mathcal{E}$ of $\mathcal{P}(\Omega)$: in this case we may say that (1) holds for every decreasing sequence $A_1 \supset A_2 \supset \cdots$ of elements of $\mathcal{E}$ whose intersection is in $\mathcal{E}$ if and only if (4) holds for every sequence $A_1, A_2, \cdots$ of elements of $\mathcal{E}$ whose intersection is in $\mathcal{E}$; and that (2) holds for every downward net $\mathcal{Q} \subset \mathcal{E}$ whose intersection is in $\mathcal{E}$ if and only if (5) holds for every subset $\mathcal{Q}$ of $\mathcal{E}$ whose intersection is in $\mathcal{E}$.

The representation of a belief function by an allocation of probability ρ into a probability algebra $\mathcal{M}$ is intuitively meaningful because nonzero elements of $\mathcal{M}$ can be thought of as "probability masses" or "portions of belief," and $\rho(A)$ can be thought of as the (total) portion of belief one commits to A. The defining characteristics of an allocation of probability suit this interpretation; it seems reasonable to require that the measure of a portion of belief should always be positive, that the measures of disjoint portions should add, and that the portion committed to $A \cap B$ should include all of what is committed both to A and to B.

The notion of an allocation also lends itself to a geometric intuition. Suppose, for example, that ρ is an allocation from a power set $\mathcal{P}(\Omega)$ into a probability algebra $\mathcal{M}$. Then think of the probability represented by $\mathcal{M}$ as spread over the set Ω. But instead of distributing this probability in a fixed way, allow it a limited freedom of movement: require that a probability mass $M \in \mathcal{M}$ be constrained to remain inside a set $A \subset \Omega$ if and only if $M \leqq \rho(A)$. This makes geometric sense: if we write "M ct A" to indicate that M is constrained to A, then we find that M ct A and M ct B imply M ct $A \cap B$, that M ct A and N ct A imply $M \vee N$ ct A, etc.

Occasionally, it is convenient to shift our attention from an allocation $\rho : \mathcal{E} \to \mathcal{M}$ to the mapping $\zeta : \mathcal{E}^* \to \mathcal{M}$ defined by $\zeta(A) = \overline{\rho(\bar{A})}$. We call ζ an *allowment of probability* for $f = \mu \circ \rho$; it is dual to ρ in that it satisfies $f^* = \mu \circ \zeta$ and preserves joins rather than meets. Notice that in terms of the geometric intuition associated with an allocation, $\zeta(A) = \overline{\rho(\bar{A})}$ is the total probability mass that is not constrained to $\bar{A}$; i.e., the total probability mass that is *allowed* to move into A.

4 Condensability

The intuition associated with an allocation of probability on a power set $\mathcal{P}(\Omega)$ acquires its full force only when that allocation is condensable, for it is only in that case that a probability mass committed to each of a collection $\mathcal{B}$ of subsets of Ω is necessarily committed to the intersection $\cap\, \mathcal{B}$. Indeed, if f is a belief function on $\mathcal{P}(\Omega)$ with allocation $\rho : \mathcal{P}(\Omega) \to \mathcal{M}$ and allowment $\zeta : \mathcal{P}(\Omega) \to \mathcal{M}$, then the following conditions are all equivalent to the statement that f is condensable:

(1) $\rho(\cap \mathcal{B}) = \wedge_{\mathrm{B} \in \mathcal{B}}(B)$ for all $\mathcal{B} \subset \mathcal{P}(\Omega)$.

(2) If $\mathcal{B} \subset \mathcal{P}(\Omega), M \in \mathcal{M}$, and M ct B for each $B \in \mathcal{B}$, then M ct $\cap \mathcal{B}$.
(3) $\zeta(\cup\mathcal{B}) = \vee_{B\in\mathcal{B}}\zeta(B)$ for all $\mathcal{B} \subset \mathcal{P}(\Omega)$.
(4) If $\varnothing \neq A \subset \Omega$, then there exists a sequence $\omega_1, \omega_2, \cdots$ of elements of A and a countable disjoint partition $M_1, M_2, \cdots$ of $\zeta(A)$ such that $M_i \leqq \zeta(\{\omega_i\})$ for each i.
(5) There exists a mapping $\lambda : \mathcal{M} \to \mathcal{P}(\Omega)$ such that an element M of $\mathcal{M}$ and a subset A of Ω satisfy M ct A if and only if $\lambda(M) \subset A$.

Notice the geometric interpretation of (4) and (5). For each $M \in \mathcal{M}, \lambda(M)$ is the smallest subset of Ω to which all of M is constrained. And (4) demands sufficient freedom of movement for the probability mass $\zeta(A)$ to allow any diffusion, or "continuous" distribution, to be reversed: it must be possible for $\zeta(A)$ to "condense" into a countable number of discrete probability masses, each still located within A.

5 The Canonical Extension of Belief Functions

Given a belief function f on a multiplicative subclass $\mathcal{E}$ of $\mathcal{P}(\Omega)$, we define $\bar{f}$ on $\mathcal{P}(\Omega)$ by setting

$$\bar{f}(A) = \sup\left\{\Sigma(-1)^{|I|+1} f(\cap_{i\in I} A_i) | \varnothing \neq I \subset \{1, \cdots, n\}\right\}, \tag{6}$$

where the supremum is taken over all $n \geqq 1$ and all collections $A_1, A_2, \cdots A_n$ of elements of $\mathcal{E}$ that are subsets of A.

Notice that if $\mathcal{E}$ is an additive as well as a multiplicative subclass, then (6) reduces to

$$\bar{f}(A) = \sup\{f(B) | B \in \mathcal{E}; B \subset A\}. \tag{7}$$

In this case we define $\tilde{f}$ and $\hat{f}$ on $\mathcal{P}(\Omega)$ by

$$\tilde{f}(A) = \sup\{\lim_{i\to\infty} f(A_i) | A_1 \supset A_2 \supset \cdots \in \mathcal{E}; \cap A_i \subset A\} \tag{8}$$

and

$$\hat{f}(A) = \inf\left\{\bar{f}(B) | B \subset \Omega \text{ is cofinite; } A \subset B\right\}. \tag{9}$$

Theorem 4. *Suppose f is a belief function on a multiplicative subclass $\mathcal{E}$ of $\mathcal{P}(\Omega)$.*

(1) *$\bar{f}$ is a belief function, and $f = \bar{f}|\mathcal{E}$. Furthermore,*

$$\bar{f} = \inf\{g | g \textit{ is a belief function on } \mathcal{P}(\Omega) \textit{ and } g|\mathcal{E} = f\}.$$

(2) *Suppose $\mathcal{E}$ is an additive as well as a multiplicative subclass. Then f is continuous if and only if*

$$f(\cap_i A_i) = \lim_{i\to\infty} f(A_i) \tag{10}$$

for every decreasing sequence $A_1 \supset A_2 \supset \cdots$ *of elements of* $\mathcal{E}$ *such that* $\cap_i A_i \in \mathcal{E}$. *If* f *is continuous, then* $\tilde{f}$ *is a continuous belief function,* $f = \tilde{f}|\mathcal{E}$, *and*

$$\tilde{f} = \inf \{g | g \text{ is a continuous belief function on } \mathcal{P}(\Omega) \text{ and } g|\mathcal{E} = f\}.$$

If f *is continuous and* $\mathcal{E}$ *is closed under countable intersections, then* $\tilde{f} = \bar{f}$.

(3) *Suppose* $\mathcal{E}$ *is an additive as well as a multiplicative subclass. Then* f *is condensable if and only if for every* $A \in \mathcal{E}$ *and every* $\varepsilon > 0$ *there exists a cofinite subset* B *of* Ω *such that* $A \subset B$ *and* $f(C) - f(A) < \varepsilon$ *for all* $C \in \mathcal{E}$ *such that* $A \subset C \subset B$. *If* f *is condensable, then* $\hat{f}$ *is a condensable belief function,* $f = \hat{f}|\mathcal{E}$, *and*

$$\hat{f} = \inf \{g | g \text{ is a condensable belief function on } \mathcal{P}(\Omega) \text{ and } g|\mathcal{E} = f\}.$$

If f *is condensable and* $\mathcal{E}$ *is closed under arbitrary unions and intersections, then* $\hat{f} = \bar{f}$.

Proof. Let $\rho : \mathcal{E} \to \mathcal{M}$ be an allocation of probability for f, and let μ denote the measure on $\mathcal{M}$.

(1) Define $\bar{\rho} : \mathcal{P}(\Omega) \to \mathcal{M}$ by $\bar{\rho}(A) = \vee\{\rho(B) | B \in \mathcal{E}; B \subset A\}$. It is easily verified that $\bar{\rho}$ is an allocation and that $\bar{f} = \mu \circ \bar{\rho}$; hence $\bar{f}$ is a belief function. The other assertions in (1) are then obvious.
(2) It is clear that if f is continuous, then (10) holds. Suppose, on the other hand, that (10) holds.

For each $A \subset \Omega$, define $\mathcal{D}(A) \subset \mathcal{M}$ by

$$\mathcal{D}(A) = \{\wedge_{B \in \mathcal{B}} \rho(B) | \mathcal{B} \text{ is a countable subset of } \mathcal{E}; \cap \mathcal{B} \subset A\}.$$

Notice that $\mathcal{D}(A)$ is an upward net in $\mathcal{M}$. (If M_1 and M_2 are the elements of $\mathcal{D}(A)$ corresponding to subsets $\mathcal{B}_1$ and $\mathcal{B}_2$ of $\mathcal{E}$, then $\mathcal{B} \equiv \{B_1 \cup B_2 | B_1 \in \mathcal{B}_1; B_2 \in \mathcal{B}_2\}$ will also be countable subset of $\mathcal{E}$ with $\cap \mathcal{B} \subset A$, and the element of $\mathcal{D}(A)$ corresponding to $\mathcal{B}$ will majorize both M_1 and M_2.) Define $\tilde{\rho} : \mathcal{P}(\Omega) \to \mathcal{M}$ by $\tilde{\rho}(A) = \vee\mathcal{D}(A)$. We will show that $\tilde{\rho}$ is a continuous allocation, that $\tilde{f} = \mu \circ \tilde{\rho}$, and that $\tilde{\rho}|\mathcal{E} = \rho$; the assertions of (2) will then be obvious.

The relation $\tilde{\rho}|\mathcal{E} = \rho$ follows from the fact that

$$\rho(\cap\mathcal{B}) = \wedge_{B \in \mathcal{B}} \rho(B)$$

whenever $\mathcal{B} \in \mathcal{E}$ is countable and $\cap \mathcal{B} \in \mathcal{E}$. (See the comment following Theorem 3.2.) For in the case where $\mathcal{B} \subset \mathcal{E}$ and $\cap \mathcal{B} \subset A \in \mathcal{E}$, we therefore have

$$\wedge_{B \in \mathcal{B}} \rho(B) \leqq \wedge_{B \in \mathcal{B}} \rho(A \cup B) = \rho(\cap_{B \in \mathcal{B}}(A \cup B)) = \rho(A).$$

To verify that $\tilde{f} = \mu \circ \tilde{\rho}$, we must notice that for any sequence $A_1, A_2, \cdots$ in $\mathcal{E}$ there is a decreasing sequence $B_1, B_2, \cdots$, defined by

$$B_i = A_1 \cap \cdots \cap A_i,$$

which satisfies both $\cap_i B_i = \cap_i A_i$ and $\wedge_i \rho(B_i) = \wedge\ _i\rho(A_i)$. Hence

$$\mathcal{D}(A) = \{\wedge_i \rho(A_i) \,|A_1, A_2, \cdots \in \mathcal{E}; A_1 \supset A_2 \supset \cdots; \cap_i A_i \subset A\}.$$

And since $\mathcal{D}(A)$ is an upward net, it follows that

$$\begin{aligned}\mu(\tilde{\rho}(A))) &= \mu(\vee \mathcal{D}(A)) \\ &= \sup\nolimits_{M \in \mathcal{D}(A)} \mu(M) \\ &= \sup\{\mu(\wedge_i \rho(A_i)) \,|A_1, A_2, \cdots \in \mathcal{E}; A_1 \supset A_2 \supset \cdots; \cap_i A_i \subset A\} \\ &= \sup\{\lim\nolimits_{i\to\infty} f(A_i) | A_1, A_2, \cdots \in \mathcal{E}; A_1 \supset A_2 \supset \cdots; \cap_i A_i \subset A\} \\ &= \tilde{f}(A).\end{aligned}$$

The fact that $\tilde{\rho}|\mathcal{E} = \rho$ means in particular that $\tilde{\rho}(\varnothing) = \Lambda$ and $\tilde{\rho}(\Omega) = \mathrm{V}$. So in order to show that $\tilde{\rho}$ is a continuous allocation, we need only show that it preserves countable meets; i.e., that

$$\tilde{\rho}(\cap_i A_i) = \wedge_i \tilde{\rho}(A_i),$$

or

$$\vee \mathcal{D}(\cap_i A_i) = \wedge \vee \mathcal{D}(A_i)$$

for any sequence $A_1, A_2, \cdots$ of subsets of Ω. To this end, we fix the sequence $A_1, A_2, \cdots$ and simplify our notation by setting $\mathcal{D} \equiv \mathcal{D}(\cap_i A_i), \mathcal{D}_i \equiv \mathcal{D}(A_i)$ and $M \equiv \wedge_i \vee \mathcal{D}_i$. Our task is then to show that $\vee\mathcal{D} = M$. And since $\mathcal{D} \subset \mathcal{D}_i$ for each i, the relation $\vee\mathcal{D} \leqq \wedge_i \vee \mathcal{D}_i = M$ is immediate, and it remains only to show that $\vee\mathcal{D} \geqq M$.

Since $\mathcal{D}_i$ is an upward net, it will include an element that arbitrarily nearly covers its meet $\vee\mathcal{D}_i$. In particular, if $\varepsilon > 0$ then we can choose $M_i \in \mathcal{D}_i$ such that

$$\mu(\vee\mathcal{D}_i - M_i) \leqq \frac{\varepsilon}{2i}.$$

(PROOF. By the countable chain condition, $\mathcal{D}_i$ has a countable subset $\mathcal{E}_i$ such that $\vee\mathcal{E}_i = \vee\mathcal{D}_i$. Since $\mathcal{D}_i$ is an upward net, $\mathcal{E}_i$ may be taken as an increasing sequence, and then the continuity of μ assures that an element sufficiently far along in this sequence will have measure within $\varepsilon/2i$ of the measure of $\vee\mathcal{D}_i$.) Since $M \leqq \vee\mathcal{D}_i$, we also have

$$\mu(M - M_i) \leqq \frac{\varepsilon}{2i}.$$

Fix $\varepsilon > 0$ and choose such an $M_i \in \mathcal{D}_i$ for each i. And let $\mathcal{B}_i$ be a countable subset of $\mathcal{E}$ such that $\cap\, \mathcal{B}_i \subset A_i$ and $M_i = \wedge_{B \in \mathcal{B}_i} \rho(B)$. Set $\mathcal{B}_\varepsilon = \cup_i \mathcal{B}_i$ and $M_\varepsilon = \wedge_i M_i$. Then $\cap\, \mathcal{B}_\varepsilon \subset \cap_i A_i$, and

$$M_\varepsilon = \wedge_i (\wedge_{B \in \mathcal{B}_i} \rho(B)) = \wedge_{B \in \mathcal{B}_\epsilon} \rho(B);$$

thus $M_\varepsilon \in \mathcal{D}$, so that $M_\varepsilon \leqq \vee \mathcal{D}$. Since

$$\mu\,(M - M_\varepsilon) = \mu\,(\vee_i (M - M_i)) \leqq \varepsilon$$

it follows that $\vee \mathcal{D}$ includes all but at most ε of M. And since ε is arbitrary, this yields the conclusion that $\vee \mathcal{D} \geqq M$.

(3) Suppose f is condensable. Then there exists a condensable belief function g on $\mathcal{P}(\Omega)$ such that $f = g|\mathcal{E}$. Since g is condensable,

$$g(A) = \inf \{g(B) | B \subset \Omega \text{ is cofinite;} \quad A \subset B\}. \tag{11}$$

(Cf. (3).) It follows that for all $A \in \mathcal{E}$ and all $\varepsilon > 0$, there exists a cofinite subset B of Ω such that $A \subset B$ and $f(C) - f(A) < \varepsilon$ for all $C \in \mathcal{E}$ such that $A \subset C \subset B$.

Suppose, on the other hand, that the condition of the preceding sentence is met. Then we define $\hat{\rho} : \mathcal{P}(\Omega) \to \mu$ by

$$\hat{\rho}(A) = \wedge \{\bar{\rho}(B) | B \subset \Omega \text{ in confinite}; A \subset B\}.$$

It is clear that $\hat{f} = \mu \circ \hat{\rho}$. We will show that $\hat{\rho}|\mathcal{E} = \rho$ and that $\hat{\rho}$ is a condensable allocation.

Suppose $A \in \mathcal{E}$. Clearly $\hat{\rho}(A) \geqq \rho(A)$. In order to show that $\hat{\rho}(A) = \rho(A)$, we fix $\varepsilon > 0$ and choose a cofinite subset B_ε of Ω such that $A \subset B_\varepsilon$ and $f(C) - f(A) < \varepsilon/2$ for all $C \in \mathcal{E}$ such that $A \subset C \subset B_\varepsilon$. Then

$$\begin{aligned}
\rho(A) &= \wedge \{\vee \{\rho(C) | C \in \mathcal{E}; C \subset B\} \,|B \subset \Omega \text{ is confinite; } A \subset B\} \\
&= \wedge \{\vee \{\rho(C) | C \in \mathcal{E};\; A \subset C \subset B\} \,|B \subset \Omega \text{ is confinite; } A \subset B\} \\
&\leqq \vee \{\rho(C) | C \in \mathcal{E};\; A \subset C \subset B_\varepsilon\}.
\end{aligned}$$

Denote this last element of $\mathcal{M}$ by M_ε. Since $\{\rho(C) | C \in \mathcal{E}; A \subset C \subset B_\varepsilon\}$ is an upward net, we may choose $C_\varepsilon \in \mathcal{E}$ such that $A \subset C \subset B_\varepsilon$ and $\mu(M_\varepsilon) - f(C_\varepsilon) = \mu(M_\varepsilon - \rho(C_\varepsilon)) < \varepsilon/2$. Since $M_\varepsilon \geqq \hat{\rho}(A) \geqq \rho(A)$, we have

$$\begin{aligned}
\mu(\hat{\rho}(A)) - \rho(A)) &\leqq \mu(M_\varepsilon - \rho(A)) = |\mu(M_\varepsilon) - f(A)| \\
&= |\mu\,(M_\varepsilon) - f\,(C_\varepsilon) + f\,(C_\varepsilon) - f\,(A)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

And since ε may be chosen arbitrarily small, this means $\mu(\hat{\rho}(A) - \rho(A)) = 0$, or $\hat{\rho}(A) = \rho(A)$. So $\hat{\rho}|\mathcal{E} = \rho$.

The fact that $\hat{\rho}|\mathcal{E} = \rho$ means in particular that $\hat{\rho}(\varnothing) = \Lambda$ and $\hat{\rho}(\Omega) = V$. So in order to show that $\hat{\rho}$ is a condensable allocation, we need only show that it preserves arbitrary meets. Fix a subset $\mathcal{B}$ of $\mathcal{E}$. A cofinite subset of Ω contains $\cap\, \mathcal{B}$ if and only if it contains some finite intersection of elements of $\mathcal{B}$, and it does this if and only if it itself is the intersection of a finite number of cofinite subsets of Ω, each of which contains some element of $\mathcal{B}$. Hence

$$\begin{aligned}
\hat{\rho}(\cap\mathcal{B}) &= \wedge\{\bar{\rho}(C)|C \subset \Omega \text{ is cofinite}; \cap\, \mathcal{B} \subset C\} \\
&= \wedge\{\bar{\rho}(C_1 \cap \cdots \cap C_n)|n \geqq 1; C_1, \cdots, C_n \text{ are} \\
&\qquad \text{confinite subsets of } \Omega\text{, each containing some} \\
&\qquad \text{element of } \mathcal{B}\} \\
&= \wedge\{\bar{\rho}(C_1) \wedge \cdots \wedge \bar{\rho}(C_n)|n \geqq 1; C_1, \cdots, C_n \\
&\qquad \text{are confinite subsets of } \Omega\text{, each containing} \\
&\qquad \text{some element of } \mathcal{B}\} \\
&= \wedge\{\bar{\rho}(C)|C \subset \Omega \text{ is confinite; } C \text{ containing some} \\
&\qquad \text{element of } \mathcal{B}\} \\
&= \wedge_{B\in\mathcal{B}}\hat{\rho}(B).
\end{aligned}$$

So $\hat{\rho}$ is a condensable allocation.

Suppose g is a condensable belief function on $\mathcal{P}(\Omega)$ and $g|\mathcal{E} = f$. Then $g \geqq \bar{f}$ by (1), and comparison of (9) and (11) shows that $g \geqq \hat{f}$.

Finally, suppose $\mathcal{E}$ is closed under arbitrary unions and intersections. Then a mapping $\theta : \mathcal{P}(\Omega) \to \mathcal{E}$ may be defined by $\theta(A) = \cup\{B|B \in \mathcal{E}, B \subset A\}$. This mapping satisfies $\bar{f} = f \circ \theta$ and preserves arbitrary intersections. So if $\mathcal{B}$ is a downward net in $\mathcal{P}(\Omega)$, then $\{\theta(B)|B \in \mathcal{B}\}$ is a downward net in $\mathcal{E}$. Using all these facts, together with the condensability of f, we obtain

$$\begin{aligned}
\bar{f}(\cap\, \mathcal{B}) = f(\theta(\cap\, \mathcal{B})) &= f(\cap_{B\in\mathcal{B}}\theta(B)) \\
&= \inf_{B\in\mathcal{B}} f(\theta(B)) = \inf_{B\in\mathcal{B}} \bar{f}(B)
\end{aligned}$$

for any downward net $\mathcal{B}$ in $\mathcal{P}(\Omega)$. Thus $\bar{f}$ is condensable. It follows that $\bar{f} = \hat{f}$.

The belief function $\bar{f}$ assigns to each subset of Ω only the degree of belief that f forces it to assign, and it is therefore the belief function on $\mathcal{P}(\Omega)$ that we will adopt if our knowledge about Ω is limited to what f says about $\mathcal{E}$. (See Chap. 6 of Shafer (1976a) for further discussion.) Hence we may call $\bar{f}$ the *canonical extension* of f to $\mathcal{P}(\Omega)$.

Similarly, let us call a continuous belief function h on $\mathcal{P}(\Omega)$ the *canonical continuous extension* of f to $\mathcal{P}(\Omega)$ in the case where f is continuous and

$$h = \inf\{g|g \text{ is a continuous belief function on } \mathcal{P}(\Omega) \text{ and } g|\mathcal{E} = f\}.$$

And let us call a condensable belief function h on $\mathcal{P}(\Omega)$ the *canonical condensable extension* of f to $\mathcal{P}(\Omega)$ in the case where f is condensable and

$$h = \inf \{g | g \text{ is a condensable belief function on } \mathcal{P}(\Omega) \text{ and } g|\mathcal{E} = f\} .$$

Theorem 4 tells us that canonical continuous and condensable extensions always exist when $\mathcal{E}$ is an additive as well as a multiplicative subclass; it is an interesting open question whether they always exist when $\mathcal{E}$ is merely a multiplicative subclass.

The notion of canonical extension generalizes to the case of larger multiplicative subclasses that fall short of the whole power set; if $\mathcal{E}_1 \subset \mathcal{E}_2$ are both multiplicative subclasses of $\mathcal{P}(\Omega)$ and f is a belief function on $\mathcal{E}_1$, then it is evident that

$$\bar{f}|\mathcal{E}_2 = \inf \{g | g \text{ is a belief function on } \mathcal{E}_2 \text{ and } g|\mathcal{E}_1 = f\} ,$$

and hence we may call $\bar{f}|\mathcal{E}_2$ the canonical extension of f to $\mathcal{E}_2$.

Notice that this process of canonical extension is consistent: if $\mathcal{E}_2 \subset \mathcal{E}_3$, then the canonical extension to $\mathcal{E}_3$ of f is the canonical extension to $\mathcal{E}_3$ of the canonical extension to $\mathcal{E}_2$ of f. If $\mathcal{E}_1 \subset \mathcal{E}_2$ and a belief function f on $\mathcal{E}_2$ is the canonical extension to $\mathcal{E}_2$ of its restriction $f|\mathcal{E}_1$, we say that f is *discerned* by $\mathcal{E}_1$.

It should be pointed out that the "possibilities" in a set Ω can always be split into more fully described possibilities, so that $\mathcal{P}(\Omega)$ is rendered merely a complete subalgebra of a larger power set. (See Chap. 6 of Shafer (1976a).) Thus power sets must share with all complete algebras any special status they can claim as domains for belief functions. It is reassuring, therefore, that the canonical extension of a belief function f from a complete algebra coincides with the canonical continuous extension if f is continuous and with the canonical condensable extension if f is condensable.

As the reader may have noticed, the formula for $\tilde{f}$ in Theorem 4 gives the usual inner measure when applied to a continuous (i.e., countably additive) probability measure f on an algebra $\mathcal{E}$, and in particular gives the *unique* extension of f to a continuous probability measure on the σ-algebra $\tilde{\mathcal{E}}$ generated by $\mathcal{E}$. But the canonical continuous extension of a continuous belief function on an algebra $\mathcal{E}$ is not in general its only continuous extension, even to $\tilde{\mathcal{E}}$. To see that this is true, choose an algebra $\mathcal{E} \subset \mathcal{P}(\Omega)$ that contains no singletons, but such that $\tilde{\mathcal{E}}$ contains all the singletons in $\mathcal{P}(\Omega)$. (For example, set $\Omega = [0, 1)$ and let $\mathcal{E}$ consist of all finite unions of left-closed, right-open subintervals on Ω.) And let f be the *vacuous belief function* on $\mathcal{E}$; i.e., the belief function that assigns degree of belief zero to every proper subset of Ω in $\mathcal{E}$. Then the canonical continuous extension of $\mathcal{E}$ to $\tilde{\mathcal{E}}$ is simply the vacuous belief function on $\tilde{\mathcal{E}}$. But for every $\omega \in \Omega$, the two-valued belief function on $\tilde{\mathcal{E}}$ corresponding to the principal filter $(\Omega, \Omega - \{\omega\}) \subset \tilde{\mathcal{E}}$ is also a continuous extension of f.

The method of defining $\hat{f}$ will appear familiar to some readers; it is analogous to Choquet's method of extending a capacity. It does not appear, however, that (3) of Theorem 4 can be cast as a special case of Choquet's results on the extension of capacities. (See pp. 158–164 of Choquet (1969).)

If the multiplicative subclass $\mathcal{E}$ is not closed under countable intersections, then we can easily construct a continuous two-valued belief function f on $\mathcal{E}$ such that $\tilde{f} \neq \bar{f}$. We simply choose a sequence $A_1, A_2, \cdots$ in $\mathcal{E}$ such that $\cap_i A_i \notin \mathcal{E}$ and let f be the two-valued belief function corresponding to the principal filter $\{A \in \mathcal{E} | \cap_i A_i \subset A\}$, so that $\bar{f}(\cap_i A_i) = 0$ but $\tilde{f}(\cap_i A_i) = 1$. If $\mathcal{E}$ is not closed under arbitrary intersections, then one can similarly construct a condensable two-valued belief function f such that $\hat{f} \neq \bar{f}$.

Acknowledgment

I thank the many friends and colleagues who have encouraged the research for this paper. I am especially grateful to G. A. Hunt, Terry Shafer, and Hale Trotter.

References

[1] Choquet, Gustave (1953). Theory of capacities. *Ann. Inst. Fourier* (*Grenoble*) **5** 131–295.
[2] Choquet, Gustave (1969). *Lectures on Analysis.* Benjamin, New York.
[3] Dempster, A. P. (1967). Upper and lower probabilities induced by a multivalued mapping. *Ann. Math. Statist.* **38** 325–339.
[4] Dempster, A. P. (1968). A generalization of Bayesian inference. *J. Roy. Statist. Soc. Ser. B* **30** 205–247.
[5] Halmos, Paul R. (1963). *Lectures on Boolean Algebras.* Van Nostrand-Reinhold, London.
[6] Honeycutt, James E., Jr. (1971). On an abstract Stieltjes measure. *Ann. Inst. Fourier* (*Grenoble*) **21** 143–154.
[7] Revuz, Andre (1955). Fonctions croissantes et mesures sur les espaces topologiques ordonnés. *Ann. Inst. Fourier* (*Grenoble*) **6** 187–269.
[8] Shafer, Glenn (1976a). *A Mathematical Theory of Evidence.* Princeton Univ. Press.
[9] Shafer, Glenn (1976b). A theory of statistical evidence. In *Foundations of Probability Theory, Statistical Inference and Statistical Theories of Science* (W. L. Harper and C. A. Hooker, eds.) **2**, pp. 365–436.
[10] Shafer, Glenn (1978). Dempster's rule of combination. Unpublished manuscript.
[11] Sikorski, Roman (1969). *Boolean Algebras*, 3rd ed. Springer-Verlag, New York.