15

Weights of Evidence and Internal Conflict for Support Functions

Nevin L. Zhang

Abstract. Shafer [1] defined weights of evidence and the weight of internal conflict for separable support functions. He also formulated a conjecture, the weight-of-conflict conjecture, which implies that these definitions can be extended in a natural way to all support functions. In this paper I show that the extension to support functions can be carried out whether or not the weight-of-conflict conjecture is true.

1 Prerequisites

This section reviews basic concepts and results needed for the theorems in the next section. See Shafer [1] for details.

Let Θ be a finite set, called a frame of discernment. A function $\mathrm{Bel} : 2^\Theta \to [0,1]$ is called a belief function over Θ if

(1) $\mathrm{Bel}(\varnothing) = 0,\ \mathrm{Bel}(\Theta) = 1$, and
(2) for every integer n and arbitrary subsets $A_1, A_2, \ldots, A_n$ of Θ,

$$\mathrm{Bel}\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{k=1}^{n} (-1)^{k-1} \sum \left\{ \mathrm{Bel}\left(\bigcap_{i\in I} A_i\right) \middle| \, |I| = k, I \subseteq \{1,2,\ldots,n\} \right\}.$$

Given a belief function Bel over the frame Θ, there exists a unique map $m : 2^\Theta \to [0,1]$ (called the basic probability assignment for Bel) such that for each subset A of Θ,

$$\mathrm{Bel}\,(A) = \sum \{m\,(B)\,|B \subseteq A\}\,.$$

The function $Q : 2^\Theta \to [0,1]$ defined by

$$Q\,(A) = \sum \{m\,(B)\,|B \supseteq A\}$$

for each subset A of Θ is called the commonality function for Bel.

Suppose Bel_1 and Bel_2 are belief functions, with basic probability assignments m_1 and m_2, respectively. If the number

$$K = \sum \left\{ m_1 (A_1) \, m_2 (A_2) \, | A_1, A_2 \in 2^{\Theta}, A_1 \cap A_2 \neq \varnothing \right\}$$

is not zero, then we say that the orthogonal sum of Bel_1 and Bel_2 exists. We denote this orthogonal sum by $\mathrm{Bel}_1 \oplus \mathrm{Bel}_2$; by definition, it is the function over Θ whose basic probability assignment is given by

$$m(A) = \frac{1}{K} \sum \left\{ m_1 (A_1) \, m_2 (A_2) \, | A_1 \cap A_2 = A \right\}.$$

The number $-\log K$ is called the weight of conflict between Bel_1 and Bel_2. The weight of conflict and the orthogonal sum are defined similarly for more than two belief functions. They do not depend on the order of combination, and

$$\begin{aligned} \mathrm{Con}\,(\mathrm{Bel}_1, \ldots, \mathrm{Bel}_n) = & \mathrm{Con}\,(\mathrm{Bel}_1, \ldots, \mathrm{Bel}_{n-1}) \\ & + \mathrm{Con}\,(\mathrm{Bel}_1 \oplus \cdots \oplus \mathrm{Bel}_{n-1}, \mathrm{Bel}_n)\,, \end{aligned} \tag{1}$$

where $\mathrm{Con}(\mathrm{Bel}_1, \ldots, \mathrm{Bel}_n)$ stands for the weight of conflict among $\mathrm{Bel}_1, \ldots,$ Bel_n.

A subset of Θ to which the basic probability assignment assigns a positive number is called a focal element. If $\mathrm{Bel}_1 \oplus \mathrm{Bel}_2$ exists, then its set of focal elements consists of all nonempty intersections of the form $A_1 \cap A_2$, where A_1 is a focal element of Bel_1 and A_2 is a focal element of Bel_2.

The belief function whose only focal element is Θ is called the vacuous belief function. If Bel_1 is vacuous, then $\mathrm{Bel}_1 \oplus \mathrm{Bel}_2 = \mathrm{Bel}_2$. A belief function with at most one focal element other than Θ is called a simple support function; the focal element not equal to Θ is called the focus. A belief function which can be expressed as an orthogonal sum of simple support functions is called a separable support function.

Suppose S is a simple support function focused on A. Then

$$w = -\log \left[1 - S(A) \right]$$

is called the weight of evidence focused on A.

The union of the focal elements of a belief function is called its core. If A is the core of Bel, then $\mathrm{Bel}(B) = 1$ if and only if $B \supseteq A$. If the core A of Bel is a proper subset of Θ, then it is sometimes convenient to replace the frame Θ by A or by some other set B such that $A \subset B \subset \Theta$. (This means that we work not with $\mathrm{Bel} : 2^{\Theta} \to [0,1]$ but with the restriction $\mathrm{Bel}|2^{B}$, which is a belief function over B whenever $\mathrm{Bel}(B) = 1$.)

Given a subset A of Θ, let Bel_A denote the belief function whose only focal element is A; this means that $\mathrm{Bel}_A(B) = 1$ whenever $B \supseteq A$ and $\mathrm{Bel}_A(B) = 0$ otherwise. [The corresponding basic probability assignment m_A satisfies $m_A(A) = 1$ and $m_A(B) = 0$ when $B \neq A$.] If Bel is another belief function over Θ with $\mathrm{Bel}(\bar{A}) < 1$, then $\mathrm{Bel} \oplus \mathrm{Bel}_A$ exists and $(\mathrm{Bel} \oplus \mathrm{Bel}_A)(A) = 1$. The belief function $\mathrm{Bel} \oplus \mathrm{Bel}_A$ can be thought of as a belief function over Θ or as a belief function over A; in either case its values are given by

$$(\mathrm{Bel} \oplus \mathrm{Bel}_A)(B) = \frac{\mathrm{Bel}\left(B \cup \bar{A}\right) - \mathrm{Bel}\left(\bar{A}\right)}{1 - \mathrm{Bel}\left(\bar{A}\right)}. \tag{2}$$

Changing Bel to $\mathrm{Bel} \oplus \mathrm{Bel}_A$ is called conditioning Bel on A.

The belief function Bel_A is idempotent with respect to the operation $\oplus : \mathrm{Bel}_A \oplus \mathrm{Bel}_A = \mathrm{Bel}_A$. This fact, together with the commutivity and associativity of $\oplus$, allows us to write

$$(\mathrm{Bel}_1 \oplus \cdots \oplus \mathrm{Bel}_n) \oplus \mathrm{Bel}_A = (\mathrm{Bel}_1 \oplus \mathrm{Bel}_A) \oplus \cdots \oplus (\mathrm{Bel}_n \oplus \mathrm{Bel}_A). \tag{3}$$

In words: combining and then conditioning on A gives the same result as conditioning on A and then combining.

If we condition a simple support function on A, then the result, considered as a belief function over A, is again a simple support function. (Suppose S is a simple support function focused on B. Then S has at most two focal elements, B and Θ. Since Bel_A has only one focal element, A, the orthogonal sum $S \oplus \mathrm{Bel}_A$ has at most two focal elements, $B \cap A$ and $\Theta \cap A = A$. If $B \cap A = \varnothing$ or $B \cap A = A$, then A is $S \oplus \mathrm{Bel}_A$'s only focal element, and therefore $S \oplus \mathrm{Bel}_A$ is the vacuous belief function over A.) It follows from this and (3) that if we condition a separable support function on A, then the result, considered as a belief function over A, is again a separable support function.

Suppose S is a separable support function over the frame Θ; we assume, without loss of generality, that Θ is the core of S. In this case there exists a unique set $S_1, \ldots, S_n$ of nonvacuous simple support functions with distinct foci such that $S = S_1 \oplus \cdots \oplus S_n$. The weight of conflict among these S_i is called the weight of internal conflict in S and is denoted by W_S. If we denote the focus of S_i by A_i and denote the weight of evidence focused on A_i by w_i, then the function $V_s : 2^\Theta \to [0, \infty)$ defined by

$$V_S(A) = \sum \{w_i | A_i \not\supseteq A\}$$

is called the impingement function for $S; V(A)$ is the total weight of evidence impinging on A. It turns out that the commonality function Q_S for S satisfies

$$Q_S(A) = \exp\left[W_S - V_S(A)\right],$$

or

$$V_S(A) = W_S - \log Q_S(A), \tag{4}$$

for every nonempty subset A of Θ.

Suppose M is a field of subsets of the frame Θ, and suppose A is a subset of Θ. Since Θ is finite, there is a smallest element of M containing A and a largest element of M contained in A. We denote these elements of M by A^+ and A^-, respectively:

$$A^{+} = \bigcup \{B | B \text{ is an atom of } M; B \cap A \neq \varnothing\}$$

or

$$A^{-} = \bigcup \{B | B \text{ is an atom of } M; B \subseteq A\}.$$

We say that Bel is carried by the field M if all the focal elements of Bel are in M. This is equivalent to the requirement that $\mathrm{Bel}(A) = \mathrm{Bel}(A^{-})$ for all $A \subseteq \Theta$.

2 Main Results

Theorem 1. *If* Bel_1 *and* Bel_2 *are both belief functions over* Θ*, and if* Bel_1 *and* Bel_2 *agree on the field* M *generated by the focal elements of* Bel_1*, then their commonality functions* Q_1 *and* Q_2 *satisfy* $Q_1 \geq Q_2$*.*

Proof. For any subset A of Θ,

$$\begin{aligned} \mathrm{Bel}_1(A) = \mathrm{Bel}_1\left(A^{-}\right) &= \mathrm{Bel}_2\left(A^{-}\right) \\ &= \sum \left\{m_2(B) \,|B \subseteq A^{-}\right\} \\ &= \sum \left\{m_2(B) \,|B^{+} \subseteq A\right\}, \end{aligned}$$

where m_2 is the basic probability assignment for Bel_2. This implies that the basic probability assignment for Bel_1 is given by

$$m_1(A) = \sum \left\{m_2(B) \,|B^{+} = A\right\}.$$

Therefore

$$\begin{aligned} Q_1(A) &= \sum \{m_1(B) \,|B \supseteqq A\} \\ &= \sum \left\{m_2(B) \,|B^{+} \supseteqq A\right\} \\ &\geq \sum \{m_2(B) \,|B \supseteqq A\} = Q_2(A). \end{aligned}$$

Theorem 2. *If* S *and* T *are both separable support functions over* Θ*, and if* S *and* T *agree on the field* M *generated by the focal elements of* S*, then*

$$W_S \leq W_T \tag{5}$$

and

$$V_S \leq V_T. \tag{6}$$

Proof. We will assume, without loss of generality, that Θ is the core of S.

Let $S = S_1 \oplus \cdots \oplus S_n$ be the unique decomposition of S into nonvacuous simple support functions with distinct foci, and let $A_1, \ldots, A_n$ denote these foci. We will prove (5) by induction on n.

If $n = 1$, then (5) is immediate, because $W_S = 0$.

Suppose (5) is true for all $k < n$.

Consider the belief functions $S \oplus \mathrm{Bel}_{A_1}$ and $T \oplus \mathrm{Bel}_{A_1}$. Since A_1 is in M, it follows from (2) and from the agreement of S and T on M that $S \oplus \mathrm{Bel}_{A_1}$ and $T \oplus \mathrm{Bel}_{A_1}$ agree on M. In particular, they agree on

$$M' = \{A \cap A_1 | A \in M\} ,$$

which is a subset of M. When $S \oplus \mathrm{Bel}_{A_1}$ and $T \oplus \mathrm{Bel}_{A_1}$ are considered as belief functions over A_1, they are both separable support functions, and M' is the field of subsets generated by the focal elements of $S \oplus \mathrm{Bel}_{A_1}$. Moreover, the number of nonvacuous simple support functions with distinct foci in the decomposition of $S \oplus \mathrm{Bel}_{A_1}$ is less than n. To see this, use (3) to write

$$S \oplus \mathrm{Bel}_{A_1} = (S_1 \oplus \mathrm{Bel}_{A_1}) \oplus \cdots \oplus (S_n \oplus \mathrm{Bel}_{\mathrm{A}_1}) , \tag{7}$$

and recall that the $S_i \oplus \mathrm{Bel}_{A_1}$, considered as belief functions over A_1, are simple support functions. Since A_1 is the focus of S_1, $S_1 \oplus \mathrm{Bel}_{A_1}$ is vacuous, and others of the $S_i \oplus \mathrm{Bel}_{A_1}$ may also be vacuous. If we omit these from the right-hand side of (7), and if we then combine any of the $S_i \oplus \mathrm{Bel}_{A_1}$ that have a common focus to obtain a single simple support function with that focus, then we will have reduced (7) to the unique decomposition of $S \oplus \mathrm{Bel}_{A_1}$ into nonvacuous simple support functions with distinct foci, and the number of these simple support functions will be less than n.

It follows from the inductive hypothesis that

$$W_{S \oplus \mathrm{Bel}_{\mathrm{A}_1}} \leq W_{T \oplus \mathrm{Bel}_{A_1}} .$$

But by (1),

$$W_{S \oplus \mathrm{Bel}_{A_1}} = W_S + \mathrm{Con}\,(S, \mathrm{Bel}_{A_1})$$

and

$$W_{T \oplus \mathrm{Bel}_{A_1}} = W_T \oplus \mathrm{Con}\,(T, \mathrm{Bel}_{A_1}) .$$

And

$$\begin{aligned} \mathrm{Con}\,(S, \mathrm{Bel}_{A_1}) &= -\log\left[1 - S\left(\bar{A}_1\right)\right] \\ &= -\log\left[1 - T\left(\bar{A}_1\right)\right] \\ &= -\mathrm{Con}\,(T, \mathrm{Bel}_{A_1}) . \end{aligned}$$

So $W_S \leq W_T$.

From (5), (4), and Theorem 1, we immediately obtain (6).

3 Support Functions

In this section we show how the weight of internal conflict and the impingement function can be defined for support functions.

Suppose Θ and Ω are two frames of discernment. We call a function $\omega : 2^{\Theta} \to 2^{\Omega}$ a refining if $\{\omega(\theta)|\theta \in \Theta\}$ constitutes a disjoint partition of Ω, and $\omega(A) = \cup\{\omega(\theta)|\theta \in A\}$ for all subsets A of Θ. If $\omega : 2^{\Theta} \to 2^{\Omega}$ is a refining, then we say that Ω is a refinement of Θ and Θ is a coarsening of Ω.

If Bel is a belief function over Ω and $\omega : 2^{\Theta} \to 2^{\Omega}$ is a refining, then the function $\mathrm{Bel} \circ \omega$ is a belief function over Θ. If Bel_1 is a belief function over Ω, Bel_2 is a belief function over Θ, and $\mathrm{Bel}_2 = \mathrm{Bel}_1 \circ \omega$ for some refining ω, we say that Bel_1 is an extension of Bel_2.

If Bel is a belief function over Θ, m is the basic probability assignment for Bel, and $\omega : 2^{\Theta} \to 2^{\Omega}$ is a refining, then the belief function Bel^{ω} over Ω which is given by the basic probability assignment

$$m^{\omega}(A) = \begin{cases} m\,(B) & \text{if} \quad B \subseteq \Theta \text{ and } \omega\,(B) = A, \\ 0 & \text{if} \quad \text{there is no } B \subseteq \Theta \text{ such that } \omega\,(B) = A \end{cases} \tag{8}$$

is an extension of Bel to Ω. It is called the vacuous extension of Bel to Ω. It is obviously carried by the image $\omega(2^{\Theta})$, which is a field of subsets of Ω.

As (8) makes clear, a belief function and its vacuous extension have the same structure, except that the vacuous extension is embedded in a finer frame. In general, any operation on belief functions on a given frame gives the same result when carried out on the vacuous extensions to a finer frame. For example,

$$\mathrm{Con}(\mathrm{Bel}_1, \ldots, \mathrm{Bel}_n) = \mathrm{Con}(\mathrm{Bel}_1^{\omega}, \ldots, \mathrm{Bel}_n^{\omega}, \tag{9}$$

and

$$(\mathrm{Bel}_1 \oplus \cdots \oplus \mathrm{Bel}_n)^{\omega} = \mathrm{Bel}_1^{\omega} \oplus \cdots \oplus \mathrm{Bel}_n^{\omega}. \tag{10}$$

A belief function is called a support function if it can be extended to a separable support function over some refinement. Given a support function S, we let $\mathcal{S}_s$ denote the set of all separable support functions which are extensions of S. We set

$$W_S' - \inf \{W_T | T \in \mathcal{S}_S\},$$

and we define a function V_s' on 2^{Θ} by

$$V_S'(A) = \inf\{V_T(\omega(A)) | T \in \mathcal{S}_S,\ S = T \circ \omega\}.$$

We would like to call W_s' and V_s' the weight of internal conflict in S and the impingement function for S, respectively. Doing so is justified by the following theorem.

Theorem 3. *If S is a separable support function over Θ, then $W_s' = W_s$, and $V_s' = V_s$.*

Proof. Consider an arbitrary extension T of S. Let $\omega : 2^\Theta \to 2^\Omega$ be the corresponding refining, and let S^ω denote the vacuous extension of S to Ω. Since $S = T \circ \omega = S^\omega \circ \omega$, T and S^ω agree on the field of subsets $\omega(2^\Theta)$. Since the focal elements of S^ω are all in $\omega(2^\Theta)$, it follows that T and S^ω agree on the field M generated by the focal elements of S^ω. Therefore, by Theorem 2, $W_{S^\omega} \leq W_T$ and $V_{S^\omega} \leq V_T$. Since T was an arbitrary element of $\mathcal{S}_S$, and since S^ω is in $\mathcal{S}_S$, it follows that $W'_S = W_{S^\omega}$ and $V'_S = V_{S^\omega}$. On the other hand, it is clear from (9) and (10) that $W_{S^\omega} = W_S$ and $V_{S^\omega} = V_S$.

4 The Weight-of-Conflict Conjecture

Shafer [1] was unable to prove Theorem 3 because he did not have Theorem 2 available. His attempt to prove Theorem 3 led him to formulate the weight-of-conflict conjecture: if the commonality functions Q_1 and Q_2 for two separable support functions S_1 and S_2 satisfy $Q_1 \geq Q_2$, then $W_{S_1} \leq W_{S_2}$. By reasoning equivalent to that in the proof of Theorem 1, he showed that this conjecture implied Theorem 3.

The results in this paper do not tell us whether Shafer's conjecture is true. They do show, however, that the conjecture is not needed for Shafer's purposes.

Acknowledgment

The author would like to thank Glenn Shafer for his help in revising this paper.

Reference

1. Glenn Shafer, *A Mathematical Theory of Evidence*, Princeton U.P.