Mediative Fuzzy Logic: A Novel Approach for Handling Contradictory Knowledge

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Abstract. In this paper we are proposing a novel fuzzy method that can handle imperfect knowledge in a broader way than Intuitionistic fuzzy logic does (IFL). This fuzzy method can manage non-contradictory, doubtful, and contradictory information provided by experts, providing a mediated solution, so we called it Mediative Fuzzy Logic (MFL). We are comparing results of MFL, with IFL and traditional Fuzzy logic (FL).

1 Introduction

Uncertainty affects all decision making and appears in a number of different forms. The concept of information is fully connected with the concept of uncertainty; the most fundamental aspect of this connection is that uncertainty involved in any problem-solving situation is a result of some information deficiency, which may be incomplete, imprecise, fragmentary, not fully reliable, vague, *contradictory*, or deficient in some other way [1]. The general framework of fuzzy reasoning allows handling much of this uncertainty.

Nowadays, we can handle much of this uncertainty using Fuzzy logic type-1 or type-2 [2,3], also we are able to deal with hesitation using Intuitionistic fuzzy logic, but what happens when the information collected from different sources is somewhat or fully contradictory. What do we have to do if the knowledge base changes with time, and non-contradictory information becomes into doubtful or contradictory information, or any combination of these three situations? What should we infer from this kind of knowledge? The answer to these questions is to use a fuzzy logic system with logic rules for handling non-contradictory, contradictory or information with a hesitation margin. Mediative fuzzy logic is a novel

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approach presented for the first time in [4] which is able to deal with this kind of inconsistent information providing a common sense solution when contradiction exists, this is a mediated solution.

There are a lot of applications where information is inconsistent. In economics for estimating the Gross Domestic Product (GDP), it is possible to use different variables; some of them are distribution of income, personal consummation expenditures, personal ownership of goods, private investment, unit labor cost, exchange rate, inflation rates, and interest rates. In the same area for estimating the exportation rates it is necessary to use a combination of different variables, for example, the annual rate of inflation, the law of supply and demand, the dynamic of international market, etc. [5]. In medicine information from experiments can be somewhat inconsistent because living being might respond different to some experimental medication. Currently, randomized clinical trials have become the accepted scientific standard for evaluating therapeutic efficacy, and contradictory results from multiple randomized clinical trials on the same topic have been attributed either to methodological deficiencies in the design of one of the trials or to small sample sizes that did not provide assurance that a meaningful therapeutic difference would be detected [6]. In forecasting prediction, uncertainty is always a factor, because to obtain a reliable prediction it is necessary to have a number of decisions, each one based on a different group, in [7] says: Experts should be chosen "whose combined knowledge and expertise reflects the full scope of the problem domain. Heterogeneous experts are preferable to experts focused in a single specialty".

The aim of this paper is to present MFL as new fuzzy method for going around from traditional, intuitionistic, and now from meditative fuzzy logic. This is a transparent way from the point of view of the inference system. This paper is organized as follows. In section 2, we are giving some historical antecedent about different logic systems. In section 3, we are explaining Mediative Fuzzy Logic (MFL). In section 4, we are showing some experimental results, and finally we have the conclusions.

2 Historical Background

Throughout history, distinguish good from bad arguments has been of fundamental importance to ancient philosophy and modern science. The Greek philosopher Aristotle (384 BC – 322 BC) is considered a pioneer in the study of logic and, its creator in the traditional way. The Organon is his surviving collected works on logic [8]. Aristotelian logic is centered in the syllogism. In Traditional logic, a syllogism (deduction) is an inference that basically consists of three things: the major and minor premises, and the proposition (conclusion) which follows logically from the major and minor premises [9]. Aristotelian logic is “bivalent” or “two-valued”, that is, the semantics rules will assign to every sentence either the value "True" or the value "False". Two basic laws in this logic are the law of contradiction (*p cannot be both p and not p*), and the law of the excluded middle (*p must be either p or not p*).

In the Hellenistic period, the stoics work on logic was very wide, but in general, one can say that their logic is based on propositions rather than in logic of terms, like the Aristotelian logic. The Stoic treatment of certain problems about modality and bivalence are more significant for the shape of Stoicism as a whole. Chrysippus (280BC-206BC) in particular was convinced that bivalence and the law of excluded middle apply even to contingent statements about particular future events or states of affairs. The law of excluded middle says that for a proposition, p, and its contradictory, $\neg p$, it is necessarily true, while bivalence insists that the truth table that defines a connective like ‘or’ contains only two values, true and false [10].

In the mid-19th century, with the advent of symbolic logic, we had the next major step in the development of propositional logic with the work of the logicians Augustus DeMorgan (1806-1871) [11] and, George Boole (1815-1864). Boole was primarily interested in developing special mathematical to replace Aristotelian syllogistic logic. His work rapidly reaps benefits, he proposed "Boolean algebra” that was used to form the basis of the truth-functional propositional logics utilized in computer design and programming [12,13]. In the late 19th century, Gottlob Frege (1848-1925) claimed that all mathematics could be derived from purely logical principles and definitions and he considered verbal concepts to be expressible as symbolic functions with one or more variables [14].

L. E. J. Brouwer (1881-1966) published in 1907 in his doctoral dissertation the fundamentals of intuitionism [15], his student Arend Heyting (1898-1980) did much to put intuitionism in mathematical logic, he created the Heyting algebra for constructing models of intuitionistic logic [16]. Gerhard Gentzen (1909-1945), in (1934) introduces systems of natural deduction for intuitionist and classical pure predicate calculus [17], his cornerstone was cut-elimination theorem which implies that we can put every proof into a (not necessarily unique) normal form. He introduces two formal systems (sequent calculi) LK and LJ. The LJ system is obtained with small changes into the LK system and it is suffice for turning it into a proof system for intuitionistic logic.

Nowadays, Intuitionistic logic is a branch of logic which emphasizes that any mathematical object is considered to be a product of a mind, and therefore, the existence of an object is equivalent to the possibility of its construction. This contrasts with the classical approach, which states that the existence of an entity can be proved by refuting its non-existence. For the intuitionist, this is invalid; the refutation of the non-existence does not mean that it is possible to find a *constructive* proof of existence. Intuitionists reject the *Law of the Excluded Middle* which allows proof by contradiction. Intuitionistic logic has come to be of great interest to computer scientists, as it is a constructive logic, and is hence a logic of what computers can do.

Bivalent logic was the prevailing view in the development of logic up to XX century. In 1917, Jan Łukasiewicz (1878-1956) developed the three-value propositional calculus, inventing ternary logic [18]. His major mathematical work centered on mathematical logic. He thought innovatively about traditional propositional logic, the principle of non-contradiction and the law of excluded middle. Łukasiewicz worked on multi-valued logics, including his own three-valued propositional calculus, the first non-classical logical calculus. He is responsible for one of the most elegant axiomatizations of classical propositional logic; it has just three axioms and is one of the most used axiomatizations today [19].

Paraconsistent logic is a logic rejecting the principle of non-contradiction, a logic is said to be *paraconsistent* if its relation of logical consequence is not explosive. The first paraconsistent calculi was independently proposed by Newton C. da Costa (1929-) [20] and Ja kowski, and are also related to D. Nelson's ideas [21]. Paraconsistent logic was proposed in 1976 by the Peruvian philosopher Miró Quesada, it is a non-trivial logic which allows inconsistencies. The modern history of paraconsistent logic is relatively short. The expression "paraconsistent logic" is at present time well-established and it will make no sense to change it. It can be interpreted in many different ways which correspond to the many different views on a logic which permits to reason in presence of contradictions. There are many different paraconsistent logics, for example, *non-adjunctive*, *non-truth-functional*, *many-valued*, and *relevant*.

Fuzzy sets, and the notions of inclusion, union, intersection, relation, etc, were introduced in 1965 by Dr. Lofti Zadeh [2], as an extension of Boolean logic. Fuzzy logic deals with the concept of partial truth, in other words, the truth values used in Boolean logic are replaced with degrees of truth. Zadeh is the creator of the concept Fuzzy logic type-1 and type-2. Type-2 fuzzy sets are fuzzy sets whose membership functions are themselves type-1 fuzzy sets; they are very useful in circumstances where it is difficult to determine an exact membership function for a fuzzy set [22].

K. Atanassov in 1983 proposed the concept of Intuitionistic fuzzy sets (IFS) [23], as an extension of the well-known Fuzzy sets defined by Zadeh. IFS introduces a new component, degree of nonmembership with the requirement that the sum of membership and nonmbership functions must be less than or equal to 1. The complement of the two degrees to 1 is called the hesitation margin. George Gargov proposed the name of intuitionistic fuzzy sets with the motivation that their fuzzification denies the law of excluded middle, wish is one of the main ideas of intuitionism [24].

3 Mediative Fuzzy Logic

Since knowledge provided by experts can have big variations and sometimes can be contradictory, we are proposing to use a Contradiction fuzzy set to calculate a mediation value for solving the conflict. Mediative Fuzzy Logic is proposed as an extension of Intuistionistic fuzzy Logic [23,25]. Mediative fuzzy logic (MFL) is based in traditional fuzzy logic with the ability of handling contradictory and doubtful information, so we can say that also it is an intutitionistic and paraconsistent fuzzy system.

A traditional fuzzy set in X [25], given by

$$A = \{(x, \mu_A(x)) | \, x \in X\} \tag{1}$$

where $\mu_A : X \to [0, 1]$ is the membership function of the fuzzy set A.

An intuitionistic fuzzy set B is given by

$$B = \{(x, \mu_B(x), \nu_B(x)) \mid x \in X\} \tag{2}$$

where $\mu_B : X \to [0, 1]$ and $\nu_B : X \to [0, 1]$ are such that

$$0 \leq \mu_B(x) + \nu_B(x) \leq 1 \tag{3}$$

and $\mu_B(x)$; $\nu_B(x) \in [0, 1]$ denote a degree of membership and a degree of non-membership of $x \in A$, respectively.

For each intuitionistic fuzzy set in X we have a "hesitation margin" $\pi_B(x)$, this is an intuitionistic fuzzy index of $x \in B$, it expresses a hesitation degree of whether x belongs to A or not. It is obvious that $0 \leq \pi_B(x) \leq 1$, for each $x \in X$.

$$\pi_B(x) = 1 - \mu_B(x) - \nu_B(x) \tag{4}$$

Therefore if we want to fully describe an intuitionistic fuzzy set, we must use any two functions from the triplet [10].

1. Membership function
2. Non-membership function
3. Hesitation margin

The application of intuitionistic fuzzy sets instead of fuzzy sets, means the introduction of another degree of freedom into a set description, in other words, in addition to μ_B we also have ν_B or π_B. Fuzzy inference in intuitionistic has to consider the fact that we have the membership functions μ as well as the non-membership functions ν. Hence, the output of an intuitionistic fuzzy system can be calculated as follows:

$$IFS = (1-\pi)FS_{\mu} + \pi FS_{\nu} \tag{5}$$

where FS_{μ} is the traditional output of a fuzzy system using the membership function μ, and FS_{ν} is the output of a fuzzy system using the non-membership function ν. Note in equation (6), when $\pi = 0$ the *IFS* is reduced to the output of a traditional fuzzy system, but if we take into account the hesitation margin of π the resulting IFS will be different.

In similar way, a contradiction fuzzy set *C* in *X* is given by:

$$\zeta_C(x) = \min(\mu_C(x), \nu_C(x)) \tag{6}$$

where $\mu_C(x)$ represents the agreement membership function, and for the variable $\nu_C(x)$ we have the non-agreement membership function.

We are using the agreement and non-agreement instead membership and non-membership, because we think these names are more adequate when we have contradictory fuzzy sets.

We are proposing three expressions for calculating the inference at the system's output, these are

$$MFS = \left(1 - \pi - \frac{\zeta}{2}\right)FS_{\mu} + \left(\pi + \frac{\zeta}{2}\right)FS_{\nu} \tag{7}$$

$$MFS = \min\left(\left((1-\pi) * FS_{\mu} + \pi * FS_{\nu}\right), \left(1 - \frac{\zeta}{2}\right)\right) \tag{8}$$

$$MFS = \left((1-\pi) * FS_{\mu} + \pi * FS_{\nu}\right) * \left(1 - \frac{\zeta}{2}\right) \tag{9}$$

In this case, when the contradictory index ζ is equal to zero, the system's output can be reduced to an intituionistic fuzzy output or, in case that $\pi = 0$, it can be reduced to a traditional fuzzy output.

4 Experimental Results

For testing the system we dealt with the problem of population control. This is an interesting problem that can be adapted to different areas. We focused in controlling the population size of an evolutionary algorithm by preserving, killing or creating individuals in the population. Dynamic population size algorithms attempt to optimize the balance between efficiency and the quality of solutions discovered by varying the number of individuals being investigated over the course of the evolutionary algorithm's run. We used Sugeno Inference system to calculate FS_μ and FS_ν, so the system is divided in two main parts: the inference system of the agreement function side, and the inference system of the non-agreement function side.

At the FS_μ side, we defined the variable percentage of cycling (pcCycling) with three terms, Small, Medium and Large. The universe of discourse is in the range [0,100]. We used a Sugeno Inference System, which in turn have three variables for the outputs: MFSCreate, MFSKill and MFSPreserve. They correspond to the amount of individuals that we have to create, kill and preserve in the population. Each output variable has three constant terms, so we have:

1. For the MFSCreate variable, the terms are: Nothing=0, Little=0.5, and Many=1.
2. For FSKill we have: Nothing=0, Little=0.5, All=1.
3. For MFSPreserve we have: Nothing=0, More or Less=0.5, All=1.

The rules for the FS_μ side are:

if (pcCycled is small) then (create is nothing)(kill is nothing)(preserve is all)
if (pcCycled is medium) then (create is little)(kill is little)(preserve is moreOrLess)
if (pcCycled is large) then (create is many)(kill is all)(preserve is nothing)

At the side FS_ν, we defined the input variable NMFpcCycled with three terms: NoSmall, NoMedium, and NoLarge, they are shown in Fig. 2. They are applied to a Sugeno Inference System with three output variables: nCreate, nKill, and nPreserve. In similar way, they are contributing to the calculation of the amount of individuals to create, kill and preserve, respectively. Each output variable has three constant terms, they are:

1. For nCreate we have the output terms: Nothing=0, Little=0.5, Many=1.
2. For nKill we have: Nothing=0, Little=0.5, and All=1.
3. For nPreserve we have: Nothing=0, More or Less=0.5, and All =1.

The corresponding rules are:

if (NMFpcCycled is Nsmall) then (create is nothing)(kill is nothing)(preserve is all)

if (NMFpcCycled is Nmedium) then (create is little)(kill is nothing)(preserve is moreOrLess)

if (NMFpcCycled is Nlarge) then (create is many)(kill is all)(preserve is nothing)

Using the agreement function (FS_{μ}) and the non-agreement functions (FS_{ν}) we obtained the hesitation fuzzy set and the contradictory fuzzy set.

We performed experiments for the aboventioned problem obtaining results for traditional and intuitionistic fuzzy systems. Figures 5, 6, and 7 show results of a traditional fuzzy system (*FS*), and in figures 8, 9, and 10 we have the intuitionistic fuzzy outputs (*IFS*). Moreover, we did experiments for calculating the meditative fuzzy output using equations (7), (8), and (9). Next we are commenting about them.

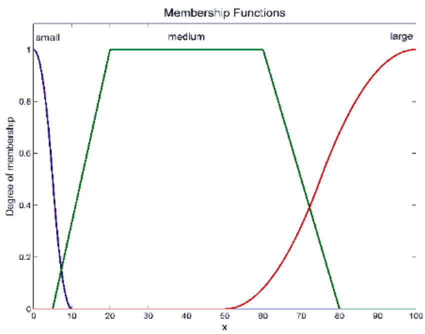

Fig. 1. Membership functions in a traditional fuzzy system (FS).

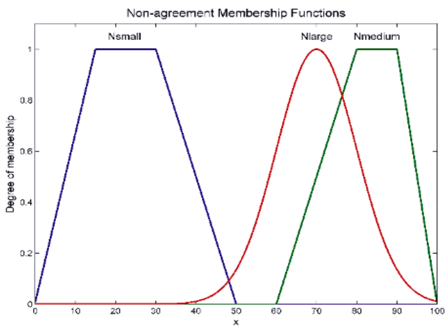

Fig. 2. Non-agreement membership functions for Mediative Fuzzy Inference System (MFS).

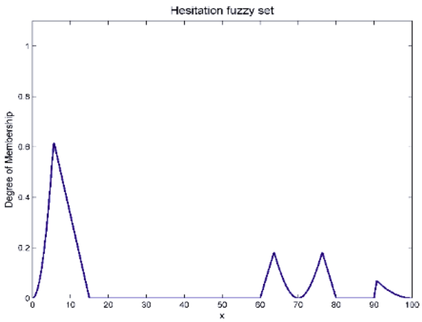

Fig. 3. Hesitation fuzzy set. We applied equation (4) to each complementary subset of membership and non-membership functions, in this case agreement and non-agreement membership functions.

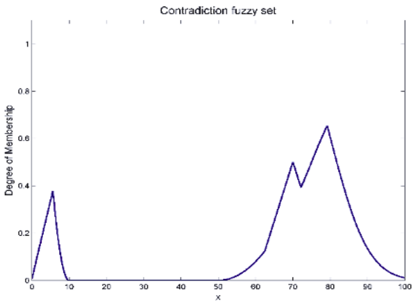

Fig. 4. Contradiction fuzzy set. We obtained this set applying equation (6) to each subset of agreement and non-agreement membership functions.

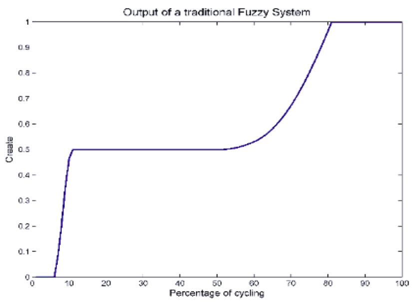

Fig. 5. Traditional FS for the output Create. We can observe that the system is inferring that we have to create 50% more individual in the actual population size when we have a percentage of cycling between 12 and 55.

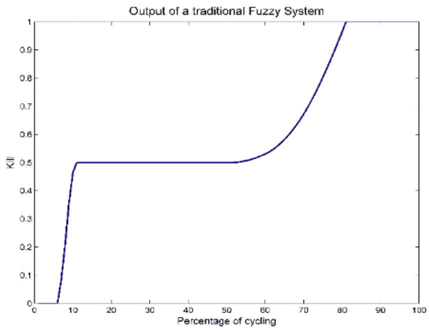

Fig. 6. The output Kill of FS, says that we have to remove 50% of the less fit individuals when we have more or less a percentage of cycling between 12 and 55.

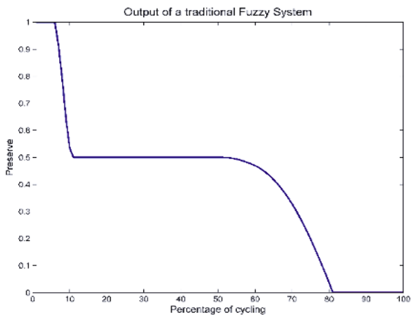

Fig. 7. The output Preserve of traditional FS says how many individuals we have to preserve, this is depending on the degree of cycling. Note that this result is in accordance with Figs. 6 and 7.

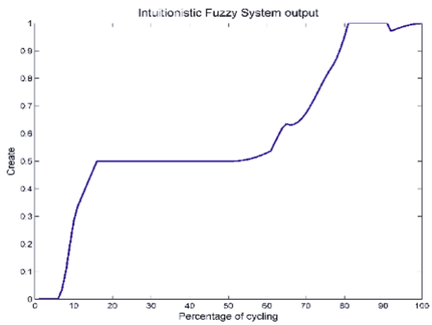

Fig. 8. We can observe that although there is contradictory knowledge, we have a softener transition in the lower part of Percentage of cycling, but when contradiction increases we cannot say the same. We used eq. (5), with FS_{Create} y FSn_{Create}.

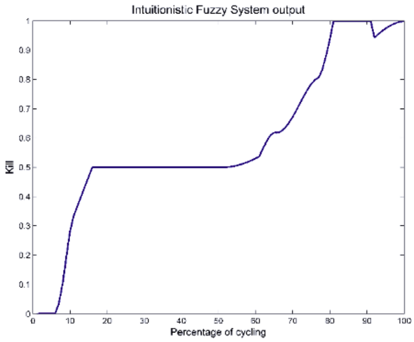

Fig. 9. In fact, a comparison using contradictory knowledge in IFS is not fear since the idea of this logic is not to use this kind of knowledge, but it is interesting to plot the inference output to compare results with MFS.

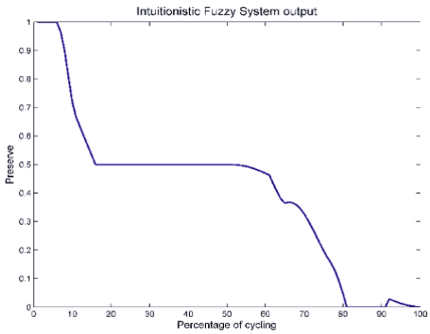

Fig. 10. IFS do not reflect contradictory knowledge at the systems' output.

Experiment #1. Using equation (7).

Equation (7) is transformed in equations (10), (11), and (12) for the three different outputs *MFSCreate*, *MFSKill*, and *MFSPreserve*. Figures 11, 12 and 13 correspond to these outputs.

$$MFSCreate = \left(1 - \pi - \frac{\zeta}{2}\right) FS_{Create} + \left(\pi + \frac{\zeta}{2}\right) FS_{nCreate} \tag{1}$$

$$MFSKill = \left(1 - \pi - \frac{\zeta}{2}\right) FS_{Kill} + \left(\pi + \frac{\zeta}{2}\right) FS_{nKill} \tag{2}$$

$$MFSPreserve = \left(1 - \pi - \frac{\zeta}{2}\right) FS_{\Pr eserve} + \left(\pi + \frac{\zeta}{2}\right) FS_{n\Pr eserve} \tag{3}$$

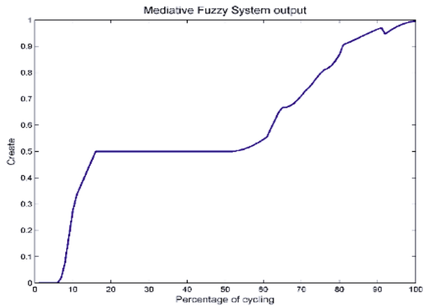

Fig. 11. MFS reflects contradictory knowledge at the systems' output. Note that here we have a softener transition values in the range between 50 and 100. We used equation (7) for plotting Figs. 11, 12 and 13. Experiment #1.

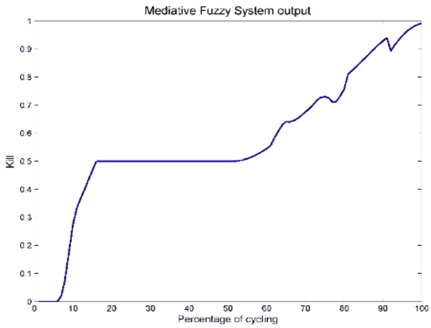

Fig. 12. Although, we have the highest degree of contradiction around the value 80, inference gives, for this region, reasonably good output values. Experiment #1.

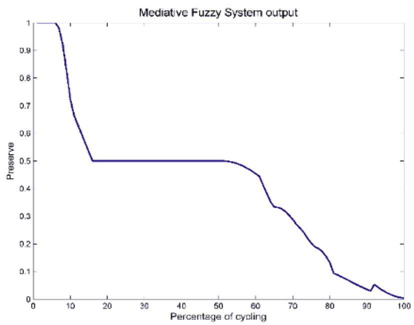

Fig. 13 Comparing results of MFS against FS and IFS, we can see that MFS can gives a softener transition when we have hesitation and contradiction fuzzy sets. We used equation (7) in experiment #1.

Experiment #2. Using equation (8).

Similar than experiment #1, we can calculate the three corresponding outputs for the system using equation (8). Figures 14, 15 and, 16 correspond to the calculated output for the variables: *MFSCreate*, *MFSKill*, and *MFSPreserve*.

Experiment #3. Using equation (9).

In the same way than experiment #1, we used equation (9) to calculate the three meditative fuzzy outputs of the system. Figures 17, 18, and 19 corresponds to this experiment.

In Fig. 3 we are showing the hesitiation fuzzy set for the Membership functions of Figs. 1 and 2, they are the agreement and non-agreement membership function respectively. Figure 3 shows the hesitation fuzzy set obtained using equation (4). Figure 4 shows the contradiction fuzzy set obtained using equation (6). Figures 5, 6, and 7 correspond to the outputs *MFSCreate*, *MFSKill*, and *MFSPreserve*, we can see in these figures that the hesitation and contradiction fuzzy set did not impact the output. Figs. 8, 9 and 10 show that the corresponding outputs were impacted by the hesitation fuzzy set and they were calculated using the IFS given in (5). The outputs in Figs. 11 to 19 were impacted by the hesitation and contradiction fuzzy set, they were calculated using MFL. We used equation (7) to calculate the output in Figs. 11, 12, and 13. Equation (8) was used to obtain Figs. 14, 15 and 16. Finally, Figs. 17, 18, and 19 were obtained using equation (9). In general, we observed that the best results were obtained using equation (7), with this equation we obtained a softener inference output in all the test that we made, this can be observed comparing Fig. 13 against Figs. 16 and 19.

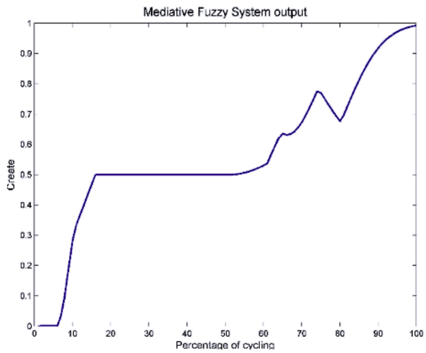

Fig. 14. MFS for the output *MFSCreate*. Figs. 14, 15 and 16 were plotted using equation (8) as base. Experiment #2.

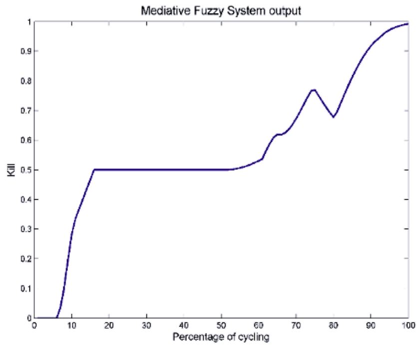

Fig. 15. MFS for the output *MFSKill.* Experiment #2.

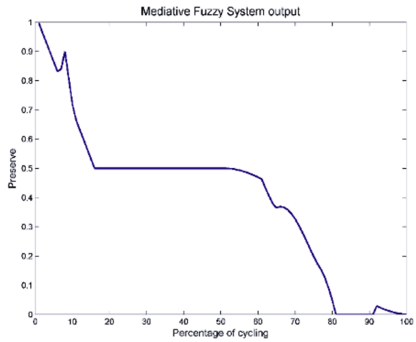

Fig. 16. Output for the variable *MFSPreserve.* Experiment #2.

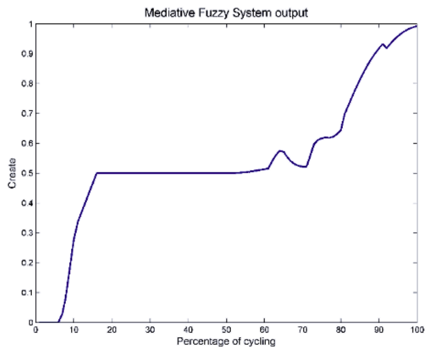

Fig. 17. Output for the variable *MFSCreate*. Figs. 17, 18, and 19 were plotted using equation (9) as base. Experiment #3.

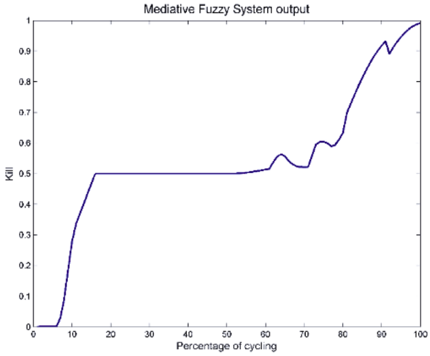

Fig. 18. Output for the variable *MFSKill*. Experiment #3.

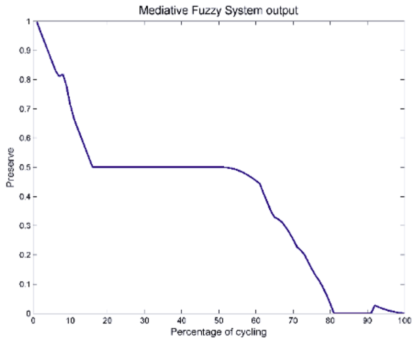

Fig. 19 Output for the variable Preserve in Experiment #3.

5 Conclusions

Through time fuzzy logic type-1 and type-2 have demonstrated their usefulness for handling uncertainty in uncountable applications. Intuitionistic fuzzy logic is relatively a new concept which introduces the degree of nonmbership as a new component, this technology also have found several application niches. Mediative fuzzy logic is a novel approach that enables us to handle imperfect knowledge in a broader way than traditional and intuitionistic fuzzy logic do. MFL is a sort of paraconsistent fuzzy logic because it can handle contradictory knowledge using fuzzy operators. MFL provides a mediated solution in case of a contradiction, moreover it can be reduced automatically to intuitionistic and traditional fuzzy logic in an automatized way, this is depending on how the membership functions (agreement and non-agreement functions) are established. We introduced three equations to perform the meditative inference. In this experiment we found the best results using equation (7) that is an extension of equation (5), i.e. it is an extension of the formula to calculate the intuitionistic fuzzy output. MFL is a good option when we have knowledge from different human experts, because it is common that experts do not fully agree all the time, so we can obtain contradiction fuzzy sets to represent the amount of disagree with the purpose of impacting the inference result. Traditional FL, and IFL will not impact the output when we have contradictory knowledge.

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