Imprecise Reliability: An Introductory Overview

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10.1 Introduction

A lot of methods and models in classical reliability theory assume that all probabilities are precise, that is, that every probability involved is perfectly determinable. Moreover, it is usually assumed that there exists some complete probabilistic information about the system and component reliability behavior. The completeness of the probabilistic information means that two conditions must be fulfilled:

1) all probabilities or probability distributions are known or perfectly determinable;
2) the system components are independent, i.e., all random variables, describing the component reliability behavior, are independent, or, alternatively, their dependence is precisely known.

The precise system reliability measures can always (at least theoretically) be computed if both these conditions are satisfied (it is assumed here that the system structure is precisely defined and that there is a known function linking the system *time to failure* (TTF) and TTFs of components or some logical system function [8]). If at least one of these conditions is violated, then only interval reliability measures can be obtained. In reality, it is difficult to expect that the first condition is fulfilled. If the information we have about the functioning of components and systems is based on a statistical analysis, then a probabilistic uncertainty model should be used in order to mathematically represent and manipulate that information. However, the reliability assessments that are combined to describe systems and components may come from various

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sources. Some of them may be objective measures based on relative frequencies or on well-established statistical models. A part of the reliability assessments may be supplied by experts. If a system is new or exists only as a project, then there are often not sufficient statistical data on which to base precise probability distributions. Even if such data exist, we do not always observe their stability from the statistical point of view. Moreover, failure times may not be accurately observed or may even be missed. Sometimes, failures do not occur at all or occur partially, leading to censored observations of failure times, and the censoring mechanisms themselves may be complex and not precisely known. As a result, only partial information about reliability of system components may be available, for example, the *mean time to failure* (MTTF) or bounds for the *probability of failure* at a time. Of course, one can always assume that the TTF has a certain probability distribution, where, for example, exponential, Weibull and lognormal are popular choices. However, how should we trust the obtained results of reliability analyses if our assumptions are only based on our experiences or on those of experts. One can reply that if an expert provides an interval for the MTTF on the basis of his experience, why should we reject his assumptions concerning the probability distribution of TTFs? The fact is that judgements elicited from experts are usually imprecise and unreliable due to the limited precision of human assessments. Therefore, any assumption concerning a certain probability distribution in combination with imprecision of expert judgements may lead to incorrect results which often cannot be validated due to lack of (experimental) data.

In many situations, it is unrealistic to assume that components of systems are independent. Let us consider two programs functioning in parallel (two-version programming). If these programs were developed by means of the same programming language, then possible errors in a language library of typical functions produce dependent faults in both programs. Several experimental studies show that the assumption of independence of failures between independently developed programs does not hold. However, the main difficulty here is that the degree of dependence between components is unknown, and one typically does not get sufficient data from which to learn about such dependence in detail. Similar examples can be presented for various applications. This implies that the second condition for complete information is also violated in most practical applications, and it is difficult to obtain precise reliability measures for a system, indeed such measures are mostly based on strong assumptions.

Dependence modelling is particularly important for large systems, for example to support high reliability software testing under practical

constraints [86]. Wooff *et al* [150] present an approach based on Bayesian graphical modelling to support software testers, and thus enhance the reliability of software systems, in which dependencies are quantified precisely via elicitation of expert judgements. Due to the enormous elicitation task this is difficult to achieve completely in practice, hence imprecise probability assessments may be needed to enable wide-scale implementation of such methods, where imprecision at varying levels of model structuring and belief quantification can be used to guide efficient elicitation. This is an important research topic both from the perspective of statistical theory based on imprecise probability and reliability theory. Another possible way to model and quantify dependence structures is via Bayes linear methods [35], where expectation (`prevision') rather than probability is the central concept. In principle, due to linearity of expectation, it promises to be easier to generalize this statistical framework to allow imprecision than it is for probability theory, but this is still an open topic for research.

One of the tools to cope with imprecision of available information in reliability analysis is *fuzzy reliability theory* [17,18,48,127,129], which is based on using fuzzy and possibilistic models [51], models of fuzzy statistics [143]. However, the framework of this theory does not cover a large variety of possible judgements in reliability. Moreover, it requires to assume a certain type of possibility distributions of TTF or time to repair, and may be unreasonable in a wide scope of cases. Existing models of fuzzy reliability theory meet some difficulties from the practical point of view. Let us consider one of the most powerful models proposed by Cai [17], according to which the TTF of the *i*-th component is considered to be a fuzzy variable governed by a possibility distribution function $\mu_i(t)$ [51]. Then the reliability measure (possibility of failure before time t) of a series system consisting of n components is defined as $\max_{i=1,\ldots,n} \sup_{u \le t} \mu_i(u)$. If all components are identical, then the possibility of failure does not depend on n. This controversial result is due to the operations min and max used in calculations, which practitioners cannot accept because it is well known that system reliability decreases with n. In this approach, a similarly problematic property holds for parallel systems. Other problems with this theory are the lack of clear interpretation of the possibility function, and lack of consistent and well founded theory for relating the possibility distribution function to statistical data. Cai [16] proposed a method based on computing the possibilistic likelihood function. However, this method has a shortcoming. By increasing the number of observations, the imprecision of the obtained possibility distribution function does not decrease and may even increase,

which is not acceptable for practitioners in reliability analysis. It should be noted that the first point can be explained [107] by interpreting the possibility distribution by means of lower and upper probability distributions [52] and considering conditions of independence of random variables. Some models use fuzzy probabilities to describe the system reliability behavior. This representation can be regarded as a special type of second-order uncertainty models. However, most existing models using fuzzy probabilities also have shortcomings due to unreasonable usage of fuzzy operations and comparison indices. Moreover, the fuzzy sets and possibility theory are often used in reliability analysis as an alternative to the classical probability theory that cannot be accepted by many practitioners. In spite of these shortcomings, fuzzy reliability models can be viewed as an interesting class of models for taking incompleteness of information into account, with a variety of challenging open research problems.

Another approach to reliability analysis under incomplete information, based on the use of *random set and evidence theories* [89], has been proposed in the literature [6,65,93]. Random set theory offers an appropriate mathematical model of uncertainty when the information is not complete or when the result of each observation is not point-valued but set-valued, so that it is not possible to assume the existence of a unique probability measure. However, this approach also does not cover all possible judgements in reliability.

To overcome every difficulty of the methods considered above, the theory of imprecise probabilities [144] and its analogues (the theory of interval statistical models [77], the theory of interval probability [148,149]) can be used, which can be a general and promising tool for reliability analysis.

Coolen [28] provided an insight into imprecise reliability, discussing a variety of issues and reviewing suggested applications of imprecise probabilities in reliability. The idea of using some aspects of imprecise probability theory in reliability had already been considered in the literature. For example, Barlow and Proschan [8] studied a case of the lack of information about independence of components (Frechet bounds [55]) and nonparametric interval reliability analysis of ageing classes of TTF distributions. Barzilovich and Kashtanov [10] considered interval methods for optimal preventive maintenance under incomplete information. It has also been shown [26,27,37] how several commonly used concepts in reliability theory can be generalized, and combined with prior knowledge, through the use of imprecise probabilities in a generalized Bayesian statistical framework. Recently, nonparametric predictive inference has been developed, see Coolen *et al* [34] for an introductory overview, as a

coherent statistical framework offering exciting application opportunities in reliability in situations where sufficient data are available. In this approach, only few mathematical assumptions are made, leading to imprecision, and further sources of uncertainty such as censored observations also lead to imprecision. Applications of this approach to maintenance and replacement problems have also been presented. We discuss this approach in more detail in Sec. Imprecise probability models for inference.

Further examples of applications of imprecise probabilities to reliability analysis have been presented by Utkin and Gurov [63,133], we briefly consider some of these examples. Suppose that the following information is available about components of a two-component series system. The MTTF of the first component is 10 hours, and the probability that the second component fails before 2 hours is 0.01. Without additional assumptions, the reliability of the system cannot be determined by means conventional reliability theory because the probability distribution of TTF is unknown. Any assumption about a certain probability distribution of TTF may lead to incorrect results. The reliability can also not be determined by means of methods of fuzzy reliability theory without further assumptions. However, this problem can be solved by using imprecise probabilities, with the restricted information leading to imprecise reliability quantifications.

A main objective of imprecise reliability is the analysis of system reliability using only available information without additional assumptions or, with a minimal number of assumptions. This theory also allows clear insights into the effects of any such further assumptions, as reflected via their effect on the imprecision in the system reliability measures. The following virtues of imprecise probability theory can be pointed out:

1) It is not necessary to make assumptions about probability distributions of random variables characterizing the component reliability behavior (TTFs, numbers of failures in a unit of time, etc.).
2) Imprecise probability theory is completely based on classical probability theory and can be regarded as its generalization. Therefore, imprecise reliability models can be interpreted in terms of classical probability theory. Conventional reliability models can be regarded as a special case of imprecise models.
3) Imprecise probability theory provides a unified tool (natural extension) for computing system reliability measures under partial information about the component reliability behavior.
4) Imprecise probability theory provides a generalization of possibility theory and evidence theory, and allows us to explain and understand some results of these approaches in reliability analysis.

5) Reliability measures that are different in kind can be combined and involved into the natural extension in a straightforward way. This implies that quite different reliability measures and estimates can be combined for computing the system reliability measures.
6) Imprecise probability theory allows us to obtain the best possible bounds for the system reliability given any information about component reliability and dependence structures.
7) The possible large imprecision of resulting system reliability measures reflects the available incompleteness of initial information and can direct the search for effective additional information sources.

At the same time, we can not assert that imprecise probability theory is the best and unique tool for reliability analysis under incomplete information. Ben-Haim [12,13] developed info-gap decision theory which has been successfully applied to solving some reliability problems. Info-gap models differ from the models of possibility, random set, and probability theories using real-valued measures functions defined on the space of events, which express either a probability or a possibility for each event in the space. An info-gap model of uncertainty is a family of nested sets. Each set corresponds to a particular degree of knowledge-deficiency, according to its level of nesting. There are no measure functions in an info-gap model of uncertainty.

This introductory overview of imprecise reliability is not intended as an exhaustive and comprehensive review of the literature. Instead, its aim is to show that imprecise reliability theory offers exciting opportunities and has been developed, yet this process is still at a relatively early stage, in particular with regard to (large-scale) practical applications.

10.2 System Reliability Analysis

Consider a system consisting of n components. Suppose that partial information about reliability of components is represented as a set of lower and upper expectations $\underline{\mathbf{E}} f_{ij}$ and $\overline{\mathbf{E}} f_{ij}$, $i = 1,...,n$, $j = 1,...,m_i$, of functions f_{ij}. Here m_i is a number of judgements that are related to the *i*-th component; $f_{ij}(X_i)$ is a function of the random TTF X_i of the *i*-th component or some different random variable, describing the *i*-th component reliability and corresponding to the *j*-th judgement about this component. For example, the interval-valued probability that a failure is in the interval $[a,b]$ can be represented as expectations of the indicator

function $I_{[a,b]}(X_i)$ such that $I_{[a,b]}(X_i)=1$ if $X_i \in [a,b]$ and $I_{[a,b]}(X_i)=0$ if $X_i \notin [a,b]$. The lower and upper MTTFs are expectations of the function $f(X_i)=X_i$.

Denote $\mathbf{X}=(x_1,...,x_n)$ and $\mathrm{X}=(X_1,...,X_n)$. Here $x_1,...,x_n$ are values of random variables $X_1,...,X_n$, respectively. It is assumed that the random variable X_i is defined on a sample space Ω and the random vector X is defined on a sample space $\Omega^n=\Omega\times...\times\Omega$. If X_i is the TTF, then $\Omega=\mathbf{R}_+$. If X_i is a random state of a multi-state system [9], then $\Omega=\{1,...,L\}$, where L is a number of states of the multi-state system. In the case of a discrete TTF, $\Omega=\{1,2,...\}$, i.e. $\Omega=\mathbf{Z}_+$. According to Barlow and Proschan [8], the system TTF can be uniquely determined by the component TTFs. Then there exists a function $g(\mathrm{X})$ of the component lifetimes characterizing the system reliability behavior. The same holds for a multi-state system. If X_i is a random state, then a state of the multi-state system is determined by states of its components, i.e., there exists a function $g(\mathrm{X})$ called a structure function.

In terms of imprecise probability theory the lower and upper expectations can be regarded as *lower and upper previsions*. The functions f_{ij} and g can be regarded as *gambles* (the case of unbounded gambles is studied by Troffaes and de Cooman [96]). The lower and upper previsions $\underline{\mathbf{E}}f_{ij}$ and $\overline{\mathbf{E}}f_{ij}$ can be also viewed as bounds for an unknown precise prevision $\mathbf{E}f_{ij}$ which will be called a *linear prevision*. Since the function g is the system TTF, then, for computing the reliability measures (such as the probability of failure, MTTF, *k*-th moment of TTF), it is necessary to find lower and upper previsions of a gamble $h(g)$, where the function h is defined by the system reliability measure which has to be found. For example, if this measure is the probability of failure before time t, then $h(g)=I_{[0,t]}(g)$.

If we assume that the vector X is governed by some unknown joint density $\rho(\mathbf{X})$, then $\underline{\mathbf{E}}h(g)$ and $\overline{\mathbf{E}}h(g)$ can be computed by solving the following optimization problems (*natural extension*):

$$\underline{\mathbf{E}}h(g)=\min_{\mathbf{P}}\int_{\Omega^n} h(g(\mathbf{X}))\rho(\mathbf{X})\mathbf{d}\mathbf{X},$$

$$\overline{\mathbf{E}}h(g)=\max_{\mathbf{P}}\int_{\Omega^n} h(g(\mathbf{X}))\rho(\mathbf{X})\mathbf{d}\mathbf{X},$$

subject to

$$\rho(\mathbf{X}) \geq 0, \ \int_{\Omega^n} \rho(\mathbf{X})\mathbf{dX} = 1,$$

$$\underline{\mathbf{E}} f_{ij} \leq \int_{\Omega^n} f_{ij}(x_i)\rho(\mathbf{X})\mathbf{dX} \leq \overline{\mathbf{E}} f_{ij}, \ i \leq n, \ j \leq m_i.$$

Here the minimum and maximum are taken over the set $\mathbf{P}$ of all possible density functions $\{\rho(\mathbf{X})\}$ satisfying the above constraints, i.e., solutions to the problems are defined on the set $\mathbf{P}$ of densities that are consistent with partial information expressed in the form of the constraints. These optimization problems mean that we only have to find the largest and smallest possible values of $\mathbf{E}h(g)$ over all densities from the set $\mathbf{P}$.

If the considered random variables are discrete and the sample space Ω^n is finite, then integrals and densities in the optimization problems are replaced by sums and probability mass functions, respectively.

It should be noted that only joint densities are used in the above optimization problems because, in a general case, we may not be aware whether the variables $X_1,...,X_n$ are dependent or not. If it is known that components are independent, then $\rho(\mathbf{X}) = \rho_1(x_1) \times \cdots \times \rho_n(x_n)$. In this case, the set $\mathbf{P}$ is reduced and consists only of the densities that can be represented as a product of marginal densities. This results in more precise reliability assessments. The manner in which the condition of independence influences on the precision of assessments is often an interesting topic of study, as it may provide useful insights into the effect of independence assumptions.

If the set $\mathbf{P}$ is empty, this means that the set of available evidence is *conflicting* and the optimization problems become irrelevant, hence this method would not provide any solutions. For example, if two experts provide [10,12] and [14,15] as bounds for the MTTF of a component, this information is clearly conflicting because these bounds produce non-intersecting sets of probability distributions, so the set $\mathbf{P}$ of common distributions is empty. There are several ways to cope with conflicting evidence. One is to localize the conflicting evidence and discard it, another is to somehow correct the conflicting evidence making it non-conflicting [102]. A third possibility is to introduce some beliefs to every judgement and to deal with second-order hierarchical models [109,110] which will be considered below.

The dual optimization problems for computing the lower $\underline{\mathbf{E}}h(g)$ and upper $\overline{\mathbf{E}}h(g)$ previsions of $h(g)$ are [300,133]:

$$\underline{\mathbf{E}}h(g) = \max\left\{c + \sum_{i=1}^{n}\sum_{j=1}^{m_i}\left(c_{ij}\underline{\mathbf{E}}f_{ij} - d_{ij}\overline{\mathbf{E}}f_{ij}\right)\right\},$$

subject to $c_{ij}, d_{ij} \in \mathbf{R}_+$, $i = 1,...,n$, $j = 1,...,m_i$, $c \in \mathbf{R}$, and $\forall \mathbf{X} \in \Omega^n$,

$$c + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \left(c_{ij} - d_{ij} \right) f_{ij} \le h(g(\mathbf{X})).$$

The dual optimization problem for computing the upper prevision $\overline{\mathbf{E}}h(g)$ of the system function $h(g)$ is

$$\overline{\mathbf{E}}h(g) = \min \left\{ c + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \left(c_{ij} \overline{\mathbf{E}} f_{ij} - d_{ij} \underline{\mathbf{E}} f_{ij} \right) \right\},$$

subject to $c_{ij}, d_{ij} \in \mathbf{R}_+$, $i = 1,...,n$, $j = 1,...,m_i$, $c \in \mathbf{R}$, and $\forall \mathbf{X} \in \Omega^n$,

$$c + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \left(c_{ij} - d_{ij} \right) f_{ij} \ge h(g(\mathbf{X})).$$

Here c, c_{ij}, d_{ij} are optimization variables such that c corresponds to the constraint $\int_{\Omega^n} \rho(\mathbf{X}) \mathbf{dX} = 1$, c_{ij} corresponds to the constraint $\int_{\Omega^n} f_{ij}(x_i) \rho(\mathbf{X}) \mathbf{dX} \le \overline{\mathbf{E}} f_{ij}$, and d_{ij} corresponds to the constraint $\underline{\mathbf{E}} f_{ij} \le \int_{\Omega^n} f_{ij}(x_i) \rho(\mathbf{X}) \mathbf{dX}$. It turns out that dual optimization problems are simpler in comparison with primal ones in many applications, because this representation allows avoidance of situations with infinite numbers of optimization variables.

Most reliability measures (probabilities of failure, MTTFs, failure rates, moments of TTF, etc.) can be represented in the form of lower and upper previsions or expectations. Each measure is defined by a gamble f_{ij}. Precise reliability information is a special case of imprecise information when lower and upper previsions of the gamble f_{ij} coincide, i.e., $\underline{\mathbf{E}} f_{ij} = \overline{\mathbf{E}} f_{ij}$.

For example, let us consider a series system consisting of two components. Suppose that the following information about reliability of components is available. The probability of the first component failure before 10 hours is 0.01. The MTTF of the second component is between 50 and 60 hours. It can be seen from the example that the available information is heterogeneous and it is impossible to find system reliability measures on the basis of conventional reliability models without using additional assumptions about probability distributions. At the same time, this information can be formalized as follows:

$$\underline{\mathbf{E}} I_{[0,10]}(X_1) = \overline{\mathbf{E}} I_{[0,10]}(X_1) = 0.01, \ \underline{\mathbf{E}} X_2 = 50, \ \overline{\mathbf{E}} X_2 = 60,$$

or

$$0.01 \le \int_{\mathbf{R}_+^2} I_{[0,10]}(x_1)\rho(x_1,x_2)\mathbf{d}x_1\mathbf{d}x_2 \le 0.01,$$

$$50 \le \int_{\mathbf{R}_+^2} x_2\rho(x_1,x_2)\mathbf{d}x_1\mathbf{d}x_2 \le 60.$$

If it is known that components are statistically independent, then the constraint $\rho(x_1,x_2) = \rho_1(x_1)\rho_2(x_2)$ is added. The above constraints form a set $\mathbf{P}$ of possible joint densities. Suppose that we want to find the probability of system failure after time 100 hours. This measure can be regarded as previsions of the gamble $I_{[100,\infty)}(\min(X_1,X_2))$, i.e., $g(\mathrm{X}) = \min(X_1,X_2)$ and $h(g) = I_{[100,\infty)}(g)$. Then the objective functions are of the form:

$$\underline{\mathbf{E}}h(g) = \min_{\mathbf{P}} \int_{\mathbf{R}_+^2} I_{[100,\infty)}(\min(x_1,x_2))\rho(x_1,x_2)\mathbf{d}x_1\mathbf{d}x_2,$$

$$\overline{\mathbf{E}}h(g) = \max_{\mathbf{P}} \int_{\mathbf{R}_+^2} I_{[100,\infty)}(\min(x_1,x_2))\rho(x_1,x_2)\mathbf{d}x_1\mathbf{d}x_2.$$

Solutions to the problems are $\underline{\mathbf{E}}h(g) = 0$ and $\overline{\mathbf{E}}h(g) = 0.59$, which are the sharpest bounds for the probability of system failure after time 100 hours based solely on the given information. If there is no information about independence, then optimization problems for computing $\underline{\mathbf{E}}h(g)$ and $\overline{\mathbf{E}}h(g)$ can be written as

$$\underline{\mathbf{E}}h(g) = \max\{c + 0.01c_{11} - 0.01d_{11} + 50c_{21} - 60d_{21}\},$$

subject to $c_{11}, d_{11}, c_{21}, d_{21,} \in \mathbf{R}_+$, $c \in \mathbf{R}$, and $\forall(x_1,x_2) \in \mathbf{R}_+^2$,

$$c + (c_{11} - d_{11})I_{[0,10]}(x_1) + (c_{21} - d_{21})x_2 \le I_{[100,\infty)}(\min(x_1,x_2)),$$

and

$$\overline{\mathbf{E}}h(g) = \min\{c + 0.01c_{11} - 0.01d_{11} + 60c_{21} - 50d_{21}\},$$

subject to $c_{11}, d_{11}, c_{21}, d_{21,} \in \mathbf{R}_+$, $c \in \mathbf{R}$, and $\forall(x_1,x_2) \in \mathbf{R}_+^2$,

$$c + (c_{11} - d_{11})I_{[0,10]}(x_1) + (c_{21} - d_{21})x_2 \ge I_{[100,\infty)}(\min(x_1,x_2)),$$

The solutions to these problems are $\underline{\mathbf{E}}h(g) = 0$ and $\overline{\mathbf{E}}h(g) = 0.99$. This example clearly shows the possible influence of independence assumptions.

Another method for computing $\underline{\mathbf{E}}h(g)$ and $\overline{\mathbf{E}}h(g)$ is based on an assertion that optimal densities in the primal optimization problems are the weighted sums of Dirac functions [138] which have unit area concentrated in the immediate vicinity of some point. In this case, the infinite dimensional optimization problems are reduced to a problem with a finite number of variables equal to the number of constraints (pieces of

evidence). The optimization problems, unfortunately, become non-linear, but it turns out that in some special cases [108,114,117,140] their solution is rather simple. If there is no information about independence of components, then

$$\underline{\mathbf{E}}h(g) = \min_{c_k, \mathbf{X}_k} \sum_{k=1}^{N+1} c_k h(g(\mathbf{X}_k)),$$

$$\overline{\mathbf{E}}h(g) = \max_{c_k, \mathbf{X}_k} \sum_{k=1}^{N+1} c_k h(g(\mathbf{X}_k)),$$

subject to

$$\sum_{k=1}^{N+1} c_k = 1,\ c_k \geq 0,\ k = 1,...,N+1,$$

$$\underline{\mathbf{E}} f_{ij} \leq \sum_{k=1}^{N+1} c_k f_{ij}(x_i^{(k)}) \leq \overline{\mathbf{E}} f_{ij},\ j \leq m_i,\ i \leq n,$$

where $\mathbf{X}_k = (x_1^{(k)},...,x_n^{(k)}) \in \mathbf{R}_+^n$, $c_k \in \mathbf{R}_+$, $N = \sum_{i=1}^n m_i$.

Here $\mathbf{X}_k$, c_k are optimization variables. If components are independent, then

$$\underline{\mathbf{E}}h(g) = \min_{c_j, \mathbf{X}_j} \sum_{l_1=1}^{m_1+1} ... \sum_{l_n=1}^{m_n+1} h(g(x_1^{(l_1)},...,x_n^{(l_n)})) \prod_{v=1}^{n} c_{l_v}^{(v)},$$

$$\overline{\mathbf{E}}h(g) = \max_{c_j, \mathbf{X}_j} \sum_{l_1=1}^{m_1+1} ... \sum_{l_n=1}^{m_n+1} h(g(x_1^{(l_1)},...,x_n^{(l_n)})) \prod_{v=1}^{n} c_{l_v}^{(v)},$$

subject to

$$\sum_{k=1}^{m_l+1} c_k^{(l)} = 1,\ c_k^{(l)} \geq 0,\ l = 1,...,n,$$

$$\underline{\mathbf{E}} f_{ij} \leq \sum_{l=1}^{m_i+1} f_{ij}(x_i^{(l)}) c_l^{(i)} \leq \overline{\mathbf{E}} f_{ij},\ j \leq m_i,\ i \leq n.$$

Let us introduce the notion of the *imprecise reliability model* of the i-th component as a set of m_i available lower and upper previsions and corresponding gambles

$$\mathbf{M}_i = \langle \underline{\mathbf{E}}_{ij}, \overline{\mathbf{E}}_{ij}, f_{ij}(X_i), j = 1,...,m_i \rangle = \wedge_{j=1}^{m_i} \mathbf{M}_{ij} = \wedge_{j=1}^{m_i} \langle \underline{\mathbf{E}}_{ij}, \overline{\mathbf{E}}_{ij}, f_{ij}(X_i) \rangle.$$

Our aim is to get the imprecise reliability model $\mathbf{M} = \langle \underline{\mathbf{E}}, \overline{\mathbf{E}}, h(g(\mathrm{X})) \rangle$ of the system. This can be done by using the natural extension which will be regarded as a transformation of the component imprecise models to the system model and denoted $\wedge_{i=1}^n \mathbf{M}_i \to \mathbf{M}$. The models in the above considered example are $\mathbf{M}_1 = \langle 0.01, 0.01, I_{[0,10]}(X_1) \rangle$, $\mathbf{M}_2 = \langle 50, 60, X_2 \rangle$,

$\mathbf{M} = \langle \underline{\mathbf{E}}, \overline{\mathbf{E}}, I_{[100,\infty)}(\min(X_1, X_2)) \rangle$.

Different forms of optimization problems for computing system reliability measures are studied by Utkin and Kozine [138]. However, if the number of judgements about component reliability behavior, $\sum_{i=1}^{n} m_i$, and the number of components, n, are large, optimization problems for computing $\underline{\mathbf{E}}h(g)$ and $\overline{\mathbf{E}}h(g)$ cannot be practically solved due to their extremely large dimensionality. This fact restricts the application of imprecise calculations to reliability analysis. Therefore, simplified algorithms for approximate solutions to such optimization problems must be developed, together with analytical solutions for some special types of systems and initial information. Some efficient algorithms are proposed by Utkin and Kozine [115,137]. The main idea underlying these algorithms is to decompose the difficult non-linear optimization problems into several linear programming problems which are easy to solve. For example, in terms of the introduced imprecise reliability models, an algorithm given in [115] allows us to replace the complex transformation $\wedge_{i=1}^{n} \mathbf{M}_i \to \mathbf{M}$ by a set of $n+1$ simple transformations

$$\mathbf{M}_i \to \mathbf{M}_i^0 = \langle \underline{\mathbf{E}}, \overline{\mathbf{E}}, h(X_i) \rangle, \; i = 1,...,n,$$
$$\wedge_{i=1}^{n} \mathbf{M}_i^0 \to \mathbf{M}.$$

10.3 Judgements in Imprecise Reliability

The judgements considered above can be related to *direct* ones, which are a straightforward way to elicit the imprecise reliability characteristics of interest. Moreover, the condition of independence of components can be related to *structural* judgements. However, there is a wide variety of possible judgements [76] that imprecise reliability theory can deal with, and other types of initial information have to be pointed out.

Comparative judgements are based on comparison of reliability measures concerning one or two components [76,99]. An example of a comparative judgement related to one component is the probability of the i-th component failure before time t is less than the probability of the same component failure in time interval $[t_1, t_2]$. This judgement can be formally represented as $\underline{\mathbf{E}}(I_{[t_1,t_2]}(X_i) - I_{[0,t]}(X_i)) \geq 0$. An example of a comparative judgement related to two components is the MTTF of the i-th component is less than the k-th component MTTF, which can be rewritten as $\underline{\mathbf{E}}(X_k - X_i) \geq 0$. By using the property of previsions $\overline{\mathbf{E}}X = -\underline{\mathbf{E}}(-X)$,

for instance, the last comparative judgement can be rewritten as $\overline{\mathbf{E}}(X_i - X_k) \leq 0$.

Many reliability measures are based on *conditional probabilities* or *conditional previsions*, for example, failure rate, mean residual TTF, probability of residual TTF, etc. Moreover, experts sometimes find it easier to quantify uncertainties using probabilities of outcomes conditionally on the occurrence of other events. The lower and upper residual MTTFs can be formally represented as $\underline{\mathbf{E}}(X - t \,|\, I_{[t,\infty)}(X))$ and $\overline{\mathbf{E}}(X - t \,|\, I_{[t,\infty)}(X))$, where $X - t$ is the residual lifetime. The lower and upper probabilities of residual TTF after time z (lower and upper residual survivor functions) are similarly written as $\underline{\mathbf{E}}(I_{[z,\infty)}(X - t) \,|\, I_{[t,\infty)}(X))$ and $\overline{\mathbf{E}}(I_{[z,\infty)}(X - t) \,|\, I_{[t,\infty)}(X))$. It should be noted that the imprecise conditional reliability measures may be computed from unconditional ones by using the generalized Bayes rule [144]. For example, if lower $\underline{\mathbf{E}}X$ and upper $\overline{\mathbf{E}}X$ MTTFs are known, then the lower and upper residual MTTFs produced by the generalized Bayes rule are $\max\{0, \underline{\mathbf{E}}X - t\}$ and $\overline{\mathbf{E}}X$, respectively. A more detailed description of conditional judgements in reliability analysis can be found in [136].

It is often reasonable to assume that lifetime probability distribution functions are unimodal. Therefore, additional information about *unimodality* of lifetime probability distributions may be taken into account for imprecise reliability calculations [104,106]. Implementing an unimodality condition on discrete probability distributions into imprecise reliability calculations has been studied in [106].

Some qualitative or quantitative judgements about *kurtosis*, *skewness*, and *variance* can also be taken into account in imprecise reliability calculations [104]. For example, we may know that the component TTF typically has a flat density function, which is rather constant near zero, and very small for larger values of the variable (negative kurtosis). This qualitative judgement can be represented by the lower and upper previsions $\underline{\mathbf{E}}X^2 = \overline{\mathbf{E}}X^2 = h$ together with $\underline{\mathbf{E}}(X^4 - 3h^2) \leq 0$, where $h \in [\inf X^2, \sup X^2]$. If, for instance, we know that data are skewed to the right (positive skewness), then this information can be formalized by the lower and upper previsions $\underline{\mathbf{E}}X = \overline{\mathbf{E}}X = h$ together with $\overline{\mathbf{E}}(3hX^2 - X^3) \leq 2h^3$. If we know that the variance of the component TTF is less than the expectation squared, then additional constraints to optimization problems for computing lower and upper previsions are of the

form: $\underline{\mathbf{E}}X = \overline{\mathbf{E}}X = h$ and $\overline{\mathbf{E}}X^2 \le 2h^2$. In such cases, the natural extension can be conveniently formulated as a parametric linear optimization problem with the parameter h.

Experts are often asked about k%-*quantiles* of the TTF X, i.e., they supply points x_i such that $\Pr\{X \le x_i\} = k/100$. As pointed out by Dubois and Kalfsbeek [50], experts are often more confident at supplying intervals rather than point-values, because their knowledge is often restricted. So experts may provide intervals for quantiles in the form $[\underline{x}_i, \overline{x}_i]$. This information can be written as

$$\Pr\{X \le [\underline{x}_i, \overline{x}_i]\} = q_i,$$

and it can be interpreted as I do not know the true value of the quantile exactly, but I belief one of the values in the interval $[\underline{x}_i, \overline{x}_i]$ to be its true value. It is worth noting that the considered model of uncertainty differs from standard uncertainty models used in the imprecise probability theory, where there exists an interval of previsions of a certain gamble. In the models of quantiles, the gamble is viewed as a set of gambles for which the same previsions are defined. The model is represented as the union of imprecise models

$$\vee_{t \in [\underline{x}_i, \overline{x}_i]} \langle q_i, q_i, I_{[0,t]}(X) \rangle.$$

The symbol $\vee_{t \in [\underline{x}_i, \overline{x}_i]}$ means that at least one of the models $\langle q_i, q_i, I_{[0,t]}(X) \rangle$ is true. Then arbitrary reliability measures may be computed by using the natural extension. For example, if there are n judgements about imprecise quantiles ($q_1 \le ... \le q_n$) and a sample space of TTF of a component is bounded by values x_0 and x_N, then the lower and upper MTTFs of the component are

$$\underline{\mathbf{E}}X = q_1 x_0 + \sum_{i=1}^{n} (q_{i+1} - q_i) \max_{k=1,...,i} \underline{x}_k, \quad q_{n+1} = 1,$$

$$\overline{\mathbf{E}}X = (1 - q_n) x_N + \sum_{i=1}^{n} (q_i - q_{i-1}) \min_{k=i,...,n} \overline{x}_k, \quad q_0 = 0.$$

10.4 Imprecise Probability Models for Inference

Standard models for inference usually require a large number of observations of events, e.g. failures, or assume that an appropriate precise prior probability distribution is available (for Bayesian models). A possible

way to avoiding these assumptions is by use of imprecise probability models or models with imprecise prior distribution for statistical inference [37]. As an alternative to the kind of models also used in robust Bayesian analysis [14], which provide useful models in the imprecise probability context although requiring a different interpretation of the lower and upper bounds for inferences, Coolen [24] presents a generalization by including a further parameter which explicitly controls the level of imprecision in case of updating with newly available data. For all these models, computation of lower and upper previsions, as required for many reliability applications, may seem to involve complex nonlinear optimization problems, in particular if multi-dimensional parameters are involved. Coolen [25] shows how these optimization problems can be replaced by relatively straightforward one-dimensional search problems, independent of the dimensionality of the original parameter space, which makes such methods far more readily available for use in imprecise reliability.

The *imprecise Dirichlet model* (IDM) was introduced by Walley [145] as a model for objective statistical inference from multinomial data. In the IDM, prior or posterior uncertainty about the multinomial distribution parameter θ are described by sets of Dirichlet distributions, and inferences about events are summarized by lower and upper probabilities which depend on the choice of a hyperparameter s. The hyperparameter determines how quickly upper and lower probabilities of events converge as statistical data accumulate. There are several arguments [145] in favour of $1 \le s \le 2$. The IDM avoids some shortcomings of alternative objective models, either frequentist or Bayesian. Coolen [27] presented a generalization of the IDM, suitable for lifetime data including right-censored observations, which are common in reliability theory and survival analysis. The resulting imprecise inferences typically encompass frequentist results for the same setting. For statistical inference on interval-valued data, Utkin [116,119] considered a set of IDMs produced by these data. The set of IDMs in this case does not require to divide the time-axis into a number of intervals for constructing the multinomial model. These intervals are produced by bounds of interval-valued data. The following example illustrates the above. Suppose that we observe $N = 5$ intervals of TTF $A_1 = [10,14]$, $A_2 = [12,16]$, $A_3 = [9,11]$, $A_4 = [12,14]$, $A_5 = [13,\infty)$. Then the lower and upper probabilities of an arbitrary interval A are of the form:

$$\underline{P}(A \mid s) = \frac{\sum_{i\,:\,A_i \subseteq A} 1}{N+s}, \quad \overline{P}(A \mid s) = \frac{\sum_{i\,:\,A_i \cap A \neq \emptyset} 1+s}{N+s}.$$

Let $A = [0,14]$ and $s = 1$. Then $\underline{P}([0,14] \mid 1) = 3/6$, $\overline{P}([0,14] \mid 1) = 1$. It

can be seen from the above expressions that the lower and upper probabilities do not depend on the division of the time-axis into intervals and right-censored observations (A_5) can be analyzed by the set of IDMs.

Quaeghebeur and de Cooman [88] applied the main ideas underlying the IDM to all distributions belonging to the exponential family, and constructed similar imprecise probability models for sampling from these distributions. Although the IDM has been established as an attractive model for statistical inference using imprecise probabilities, in reliability and other application areas, it has several serious shortcomings that were raised both by Walley himself in the paper introducing the IDM and by several discussion contributors to this paper [145]. These shortcomings were mostly apparent in situations where one has few observations, as is regularly the case in reliability problems. Recently, Coolen and Augustin [31] presented an alternative imprecise probability model for inference in case of multinomial data, which overcomes the reported shortcomings of the IDM. Applications of this model to reliability problems have not yet been presented.

Walley [145] proposed a *bounded derivative model* for statistical inference about a real-valued parameter in problems where there is little or no prior information. Prior ignorance about the parameter is modelled by a set of all continuous probability density functions for which the derivative of the log-density is bounded by a positive constant. This is also a promising model, which as far as we are aware has not yet been applied to reliability problems.

For restricting to a set of possible distribution functions of TTF, and for formalizing judgements about the *ageing* aspects of lifetime distributions, various nonparametric or semi-parametric classes of probability distributions can be used. In particular, the classes of all IFRA (increasing failure rate average) and DFRA (decreasing failure rate average) distributions have been studied by Barlow and Proschan [8]. In order to formalize judgements about the ageing aspects of lifetime distributions, new flexible classes of distributions, denoted as $\mathbf{H}(r,s)$ *classes* [64,131,132,134], have been proposed and investigated. The probability distribution of the component (or system) lifetime X can be written as $H(t) = \Pr(X \geq t) = \exp(-\Lambda(t))$, where $\Lambda(t) = \int_0^t \lambda(x)\mathbf{d}x$ and $\lambda(t)$ is the time-dependent failure rate, also known as the hazard rate. Let r and s be numbers such that $0 \leq r \leq s \leq +\infty$. A probability distribution belongs to a class $\mathbf{H}(r,s)$ with parameters r and s if $\Lambda(t)/t^r$ increases and $\Lambda(t)/t^s$ decreases as t increases. In particular, $\mathbf{H}(1,+\infty)$ is the class of all IFRA distributions; $\mathbf{H}(r,s)$ with $1 \leq r < s$ is the class of all IFRA distributions

whose failure rate increases with rate bounded by r and s; $\mathbf{H}(0,1)$ is the class of all DFRA distributions; $\mathbf{H}(r,s)$ with $r < s \leq 1$ is the class of all DFRA distributions whose failure rate decreases with rate bounded by r and s; and $\mathbf{H}(r,s)$, $r < 1 < s$ is a class containing distributions whose failure rate is non-monotone. Inferences for such classes, and solutions to corresponding computational problems, were presented by Utkin and others in the papers referred to above. To make these promising distributional classes available for imprecise reliability analysis in practice, a number of interesting research problems are still open, including the important question of how to fit such classes to available data.

From statistical perspective, imprecise probability enables inferential methods based on relatively few mathematical assumptions, in particular in situations where data are available. During the last decade, Coolen, with a number of co-authors, has developed *nonparametric predictive inference* (NPI), where inferences are directly on future observable random quantities, e.g. the random time to failure of the next system. In this approach, imprecision depends in an intuitively logical way on the available data, as it decreases if information is added, yet aspects as censoring or grouping of data result in an increase of imprecision. Foundations of NPI, including proofs of its consistency in theory of interval probability, are presented by Augustin and Coolen [4]. An introduction to NPI in reliability is presented in [34], and theory for dealing with right-censored observations in NPI in [41], with applications to some specific reliability problems presented in [39,40]. This framework is also suitable for guidance on high reliability demonstration, answering the important question of how many failure-free observations are required in order to accept a system in a critical operation [32]. The fact that, in such situations, imprecise reliability theory allows decisions to be based on the more pessimistic one of the lower and upper probabilities, e.g. lower probability of failure-free operation over a period of specified length, is an intuitively attractive manner for dealing with indeterminacy. Recently, Coolen also considered probability safety assessment from similar perspective [30].

In early work, Coolen and Newby [38] showed how NPI can also be applied for support of replacement decisions for technical systems, which is often a core reliability activity. Along such lines, Coolen-Schrijner and Coolen [32,42,43,44,45] investigated NPI-based alternatives to established replacement strategies based on the length of time a system has been in operation. These methods are fully adaptive to available failure data, and imprecision is reflected in bounds of cost functions. In addition, their results provide clear insights into the influence of a variety of assumptions

which are often used for the more established methods, and which may frequently be rather unrealistic if considered in detail. Hence, the fact that their NPI-based method can do without most of such assumptions and still be useful under quite a reasonable data requirement is interesting, and suggests that further development of NPI-based methods for imprecise reliability is an interesting topic of research.

10.5 Second-order Reliability Models

Natural extension is a powerful tool for analyzing the system reliability on the basis of available partial information about the component reliability. However, it has a disadvantage. Let us imagine that two experts provide the following judgements about the MTTF of a component: (1) MTTF is not greater than 10 hours; (2) MTTF is not less than 10 hours. The natural extension produces the resulting MTTF $[0,10]\cap[10,\infty)=10$. In other words, the absolutely precise MTTF is obtained from extremely imprecise initial data. This is unrealistic in the practice of reliability analysis. The reason of such results is that probabilities of judgements are assumed to be 1. If we assign some different probabilities to judgements, then we obtain more realistic assessments. For example, if the belief to each judgement is 0.5, then, according to [73], the resulting MTTF is greater than 5 hours. Let us consider another example. Suppose that many experts, say 1000, provide the same interval for some probability of failure, say [0.9, 0.99] and one expert provides the interval [0, 0.89]. Clearly, these judgements are conflicting and the set of probability distributions produced by these intervals is empty. As a result, we can not use the natural extension. Of course, we can use the so called *unanimity rule* defined as the envelope of the expert previsions [97], which is guaranteed to exist, but leads to extremely imprecise results (in the considered example, the resulting interval is [0, 0.99]). On the other hand, it is intuitively obvious that our belief to the judgement supplied by the last expert is rather low in comparison with our belief to the judgement provided by 1000 experts, and the unreliable judgement could be removed from consideration. One might say that this example is highly artificial. Of course, the example is given here only for illustration purposes. However, what to do if only 2 experts instead of 1000 provide the interval [0.9, 0.99] and one expert provides the interval [0, 0.99]. In this case, it is difficult to remove the contradictory interval. Of course, the inconsistency of the assessments in this artificial example were trivial, but in practice, with a variety of assessments on possibly different random variables, it may actually be

difficult to discover whether or not the assessments are inconsistent, providing a further difficulty. However, in case of precise judgements, it is extremely unlikely that different assessments, even when made by a single expert, are consistent, so the generalization to interval reliability offers powerful methods for checking and dealing with realistic uncertainty judgements.

The above examples imply that in order to obtain accurate and realistic system reliability assessments it is necessary to take into account some vagueness of information about the component reliability measures, i.e., to assume that expert judgements and statistical information about reliability of a system or its components may be unreliable. one possible solution is the use of *second-order uncertainty models,* also known as *hierarchical uncertainty models*, on which much attention has been focused in recent years, particularly in the statistics literature. These models describe the uncertainty of a random quantity by means of two levels. For example, suppose that an expert provides a judgement about the mean level of component performance [131]. If this expert sometimes provides incorrect judgements, we have to take into account some degree of belief to this judgement. In this case, the information about the mean level of component performance can be considered on the first level of the hierarchical model (first-order information) and the degree of belief to the expert judgements is considered on the second level (second-order information). Many papers are devoted to the theoretical [62,81,147] and practical [58,82] aspects of second-order uncertainty models. Lindqvist and Langseth [79] investigated monotone multi-state systems under the assumption that probabilities of the component states (first-order probabilities) can be regarded as random variables governed by the Dirichlet probability distribution (second-order probabilities). A comprehensive review of hierarchical models is given in [49], where it is argued that Bayesian hierarchical models are most common [61]. However, the use of Bayesian hierarchical models may be unrealistic in problems where only partial information is available about the system behavior.

Troffaes and de Cooman [97] specify and discuss two general ways for approaching the problem of aggregating expert opinions: *axiomatic* and *ad hoc*. Axiomatic approaches aim at deriving a preferably unique rule of aggregation from axioms or properties that this rule should satisfy. Ad hoc approaches are not as much concerned with axioms: one simply proposes or derives a mathematical formula, together with some form of justification. Both approaches have shortcomings and virtues, but axiomatic ones can be justified for various applications and initial data, whereas ad hoc approaches depend on specific applications and data.

Various methods of the pooling of assessments, taking into account the quality of experts, are available in the literature [23,57,80]. These methods use the concept of precise probabilities for modelling uncertainty, and the quality of experts is modelled by means of *weights* assigned to each expert in accordance with some rules. It should be noted that most of these rules use some available information about correctness of previous expert opinions. This might meet several difficulties. First, the behavior of experts is unstable, i.e., exact judgements related to a system elicited from an expert do not mean that this expert will provide results of the same quality for new systems. Second, when experts provide imprecise values of an evaluated quantity, the weighted rules can lead to controversial results. For instance, if an expert with a small weight, say 0.1, provides a very large interval, say [0, 10], for a quantity (covering its sample space), it is obvious that this expert is too cautious and the interval he supplies is non-informative, although this interval covers a true value of the quantity. On the other hand, if an expert with a large weight, say 0.9, supplies a very narrow interval, say [5, 5.01], the probability that true value of the quantity lies in this interval is rather small. We can see that the values of weights contradict with the probabilities of provided intervals. It should be noted that sometimes we do not know anything about the quality of experts, and assignment of weights might meet some psychological difficulties. This implies that weights for experts as measures of the quality of their expertise should not normally be interpreted as measures of the quality of provided opinions [113,116,119].

Most axiomatic second-order uncertainty models assume that there is a precise second-order probability distribution (or possibility distribution). Moreover, most models use precise probabilities for the first-level uncertainty quantification. Unfortunately, such information is often absent in many applications and additional assumptions may lead to some inaccuracy in results. A study of some tasks related to the homogeneous second-order models without any assumptions about probability distributions has been presented by Kozine and Utkin [73,75]. However, these models are of limited use due to homogeneity of gambles considered on the first-order level. A hierarchical uncertainty model for combining different types of evidence was proposed by Utkin [103,109], where the second-order probabilities can be regarded as confidence weights and the first-order uncertainty is modelled by lower and upper previsions of different gambles. However, the proposed model [103,109] supposes that the second-order initial information is analyzed only for one random variable. At the same time, the reliability applications suppose that there is a set of random variables (component TTFs) described by a second-order uncertainty model, and it is necessary to find a model for some function of

these variables (system TTF). Suppose that we have a set of weighted expert judgements related to some measures $\mathbf{E}f_{ij}(X_i)$ of the component reliability behavior, $i = 1,...,n$, $j = 1,...,m_i$, i.e., there are lower and upper previsions $\underline{\mathbf{E}}f_{ij}$ and $\overline{\mathbf{E}}f_{ij}$. Here n is the number of components, m_i is the number of judgements that are related to the *i*-th component. Suppose that each expert is characterized by an interval of probabilities $[\underline{\gamma}_{ij}, \overline{\gamma}_{ij}]$. Then the judgements can be represented as

$$\Pr\left\{\underline{\mathbf{E}}f_{ij} \le \mathbf{E}f_{ij} \le \overline{\mathbf{E}}f_{ij}\right\} \in [\underline{\gamma}_{ij}, \overline{\gamma}_{ij}],\ i \le n,\ j \le m_i.$$

Here the set $\{\underline{\mathbf{E}}f_{ij}, \overline{\mathbf{E}}f_{ij}\}$ contains the *first-order previsions*, the set $\{\underline{\gamma}_{ij}, \overline{\gamma}_{ij}\}$ contains the *second-order probabilities*. Our aim is to produce new judgements which can be regarded as combinations of available ones. In other words, the following tasks can be solved:

1) Computing the probability bounds $[\underline{\gamma}, \overline{\gamma}]$ for some new interval $[\underline{\mathbf{E}}g, \overline{\mathbf{E}}g]$ of the system linear prevision $\mathbf{E}g$.
2) Computing an average interval $[\underline{\mathbf{E}}\mathbf{E}g, \overline{\mathbf{E}}\mathbf{E}g]$ for the system linear prevision $\mathbf{E}g$ (reduction of the second-order model to first-order one).

An imprecise hierarchical reliability model of systems has been studied by Utkin [111]. This model supposes that there is no information about independence of components. A model taking into account the possible independence of components leads to complex non-linear optimization problems. However, this difficulty can be overcome by means of approaches proposed in [112,121]. Some hierarchical models of reliability taking into account the imprecision of parameters of known lifetime distributions are investigated in [118,120].

10.6 Reliability of Monotone Systems

A system is called *monotone* if it does not become better by a failure of one or more components. Various results have been obtained for computing imprecise reliability measures of typical monotone systems based on some particular types of initial information.

Some results concerning the reliability of typical systems are given in [70,71]. If initial information about reliability of components is restricted by lower and upper MTTFs, then the lower and upper system MTTFs have been obtained in explicit form for *series* and *parallel* systems [98,128].

The MTTFs of *cold standby* systems have been obtained by Utkin and Gurov [63,133]. The cold standby systems do not belong to a class of monotone systems. Nevertheless, we consider these systems as typical ones. It is worth noticing that expressions in the explicit form have been derived for the cases of independent components and complete lack of information about independence.

Suppose that the probability distribution functions of the component TTFs X_i are known only at some points t_{ij}, i.e., the available initial information is represented in the form of lower $\underline{\mathbf{E}}I_{[0,t_{ij}]}(X_i)$ and upper $\overline{\mathbf{E}}I_{[0,t_{ij}]}(X_i)$ previsions, $i=1,...,n$, $j=1,...,m_i$. Here t_{ij} is the j-th point of the i-th component TTF. Explicit expressions for lower and upper probabilities of the system failures before some time t have been obtained for series, parallel [114], *m-out-of-n* [117], cold standby [108] systems. For example, the lower and upper probabilities of the n-component parallel system failure before time t, for independent components, are

$$\underline{\mathbf{E}}I_{[0,t]}\left(\max_{i=1,...,n} X_i\right) = \prod_{i=1}^{n} \underline{\mathbf{E}}I_{[0,t_{iw_i}]}(X_i),$$

$$\overline{\mathbf{E}}I_{[0,t]}\left(\max_{i=1,...,n} X_i\right) = \prod_{i=1}^{n} \overline{\mathbf{E}}I_{[0,t_{iv_i}]}(X_i),$$

and, in case of complete lack of knowledge about independence,

$$\underline{\mathbf{E}}I_{[0,t]}\left(\max_{i=1,...,n} X_i\right) = \max_{i=1,...,n} \underline{\mathbf{E}}I_{[0,t_{iw_i}]}(X_i),$$

$$\overline{\mathbf{E}}I_{[0,t]}\left(\max_{i=1,...,n} X_i\right) = \min\left(1, \sum_{i=1}^{n} \overline{\mathbf{E}}I_{[0,t_{iv_i}]}(X_i)\right),$$

where $v_i = \min\{j \ : \ t_{ij} \geq t\}$ and $w_i = \max\{j \ : \ t_{ij} \leq t\}$.

General expressions for the reliability of arbitrary monotone systems under the same conditions are given by Utkin [110]. Moreover, it is proved that the lower (upper) bound for the system reliability of arbitrary monotone systems by given lower and upper points of probability distributions of the component TTFs depends only on these upper (lower) points. This result allows us to simplify the system reliability analysis.

It is interesting to study a case when the initial information about reliability of components is given in the form:

$$\underline{p}_{ij} \leq \Pr\{\underline{\alpha}_{ij} \leq X_i \leq \overline{\alpha}_{ij}\} \leq \overline{p}_{ij}, \ i=1,...,n, \ j=1,...,m_i,$$

where

$$[\underline{\alpha}_{i1}, \overline{\alpha}_{i1}] \subset [\underline{\alpha}_{i2}, \overline{\alpha}_{i2}] \subset ... \subset [\underline{\alpha}_{im_i}, \overline{\alpha}_{im_i}], \ i=1,...,n.$$

So there are nested intervals $[\underline{\alpha}_{ij}, \overline{\alpha}_{ij}]$, with interval probabilities $[\underline{p}_{ij}, \overline{p}_{ij}]$ for the event that the failure of the i-th component is inside these intervals. If we denote $v_i = \max\{j : \underline{\alpha}_{ij} \geq t\}$ and $w_i = \max\{j : \overline{\alpha}_{ij} \leq t\}$, then the lower and upper probabilities, for instance, of the n-component series system failure before time *t*, under the assumption of independent components, are

$$\underline{\mathbf{E}} I_{[0,t]}\left(\min_{i=1,\ldots,n} X_i\right) = 1 - \prod_{i=1}^{n} (1 - \underline{p}_{iw_i}),$$

$$\overline{\mathbf{E}} I_{[0,t]}\left(\min_{i=1,\ldots,n} X_i\right) = 1 - \prod_{i=1}^{n} \underline{p}_{iv_i}.$$

If there is no information about independence, then

$$\underline{\mathbf{E}} I_{[0,t]}\left(\min_{i=1,\ldots,n} X_i\right) = \max_{i=1,\ldots,n} \underline{p}_{iw_i},$$

$$\overline{\mathbf{E}} I_{[0,t]}\left(\min_{i=1,\ldots,n} X_i\right) = 1 - \max\left(0, \sum_{i=1}^{n} \underline{p}_{iv_i} - (n-1)\right).$$

It can be seen that the lower and upper bounds for the system unreliability depend only on the lower probabilities of the nested intervals. This implies that knowledge of upper probabilities does not give any useful information in this case. The same is valid for arbitrary monotone systems. Moreover, the initial information can be regarded as the possibility and necessity measures [51]. It is proved that the system reliability measures $\underline{\mathbf{E}} I_{[0,t]}(\cdot)$ and $\overline{\mathbf{E}} I_{[0,t]}(\cdot)$ also can be regarded as the possibility and necessity measures. This result allows us to obtain and to explain the reliability measures of systems by fuzzy initial data.

10.7 Multi-state and Continuum-state Systems

The reliability behavior of many system can be formalized by means of multi-state and continuum-state models which can be viewed as an extension of binary-state models [78]. Let L be a set representing levels of component performance ranging from perfect functioning, $\sup L$, to complete failure, $\inf L$. A *general* model of the structure function of a system consisting of n multi-state components can be written as $S : L^n \to L$. If $L = \{0,1\}$, we have a classical *binary* system; if $L = \{0,1,\ldots,m\}$, we have a *multi-state* system; if $L = [0,T]$, $T \in \mathbf{R}^+$, we

have a *continuum* system. The i-th component may be in a state $x_i(t)$ at arbitrary time t. This implies that the component is described by the random process $\{X_i(t), t \geq 0\}$, $X_i(t) \in L$. Then the probability distribution function of the i-th component states at time t is defined as the mapping $F_i : L \to [0,1]$ such that $F_i(r,t) = \Pr\{X_i(t) \geq r\}$, $\forall r \in L$. The state of the system at time t is determined by states of its n components, i.e., $S(\mathbf{X}) = S(X_1,\ldots,X_n) \in L$.

The *mean level of component performance* is defined as $\mathbf{E}\{X_i(t)\}$. For a system, we write the *mean level of system performance* $\mathbf{E}\{S(\mathbf{X})\}$. Suppose that probability distributions of the component states are unknown and we have only partial information in the form of lower $\underline{\mathbf{E}}\{X_i(t)\}$ and upper $\overline{\mathbf{E}}\{X_i(t)\}$ mean levels of component performance. It is proved by Utkin and Gurov [131], that the number of states in this case does not influence on the mean level of system performance which is defined only by boundary states $\inf L$ and $\sup L$. This implies that reliability analysis of multi-state and continuum-state systems by such initial data is reduced to analysis of a binary system. A number of expressions for these systems have been obtained in explicit form [131].

At the same time, incomplete information about reliability of the multi-state and continuum-state components can be represented as a set of reliability measures (precise or imprecise) defined for different time moments. For example, interval probabilities of some states of a multi-state unit at time t_1 may be known. How to compute the probabilities of states at time t_2 without any information about the probability distribution of time to transitions between states? This problem has been solved by using the imprecise probabilities models [139].

10.8 Fault Tree Analysis

Fault tree analysis (FTA) is a logical and diagrammatic method to evaluate the probability of an accident resulting from sequences and combinations of faults and failure events. Fault tree analysis can be regarded as a special case of event tree analysis. A comprehensive study of event trees by representing initial information in the framework of convex sets of probabilities has been proposed by Cano and Moral [22]. Therefore, this work may be a basis for investigating fault trees. One of the advantages of imprecise fault tree analysis is a possibility to consider

dependent events in a straightforward way, although complete lack of knowledge about the level of dependence is likely to lead to too much imprecision for practical use of such methods. However, the influence of any additional assumptions about dependence will then easily show in the final results, which in itself may provide valuable information, as well as guidance on the information requirement for practically useful conclusions.

Other substantial topics include the influence of events in a fault tree on a top event, and the influence of uncertainty of the event description on uncertainty of the top event description. This may be done by introducing and computing importance measures of events and uncertainty importance measures of their description. However, we are not aware of any reported study on this topic within the framework of interval reliability, which clearly suggests an important area of research.

10.9 Repairable Systems

Reliability analysis of *repairable* systems often involves difficult computational tasks, even when based on precise initial information. In addition, such analyses tend to require a substantial information input, often not in line with practical experience as many state descriptors are typically not observable or directly measurable without serious effort, if at all. A simple repairable process with instantaneous repair (*the time to repair* (TTR) is equal to 0), and under complete lack of information about dependence of random TTFs X_i, has been studied in [133]. According to this work, if the lower and upper MTTFs of a system are known, then the time-dependent lower $\underline{B}(t)$ and upper $\overline{B}(t)$ *mean time between failures* (MTBF) before time t are $\underline{B}(t) = 0$,

$$\overline{B}(t) = \min_{1 \le k < +\infty} \left(\overline{\mathbf{E}} X \sum_{i=1}^{k} \frac{1}{i} + \min \left(\frac{t - k\overline{\mathbf{E}}X}{k+1}, \frac{t - k\overline{\mathbf{E}}X}{k} \right) \right).$$

These bounds are of limited interest because $\underline{B}(t) = 0$ and $\overline{B}(t)$ becomes very large for large values of t (with $\overline{B}(t) \to \infty$ for $t \to \infty$), due to the lack of information about dependence.

Another basic and interesting model for repairable systems, based on interval-valued Markov chains, has been considered by Kozine and Utkin [72,74]. Some results on optimal preventive maintenance under incomplete information are presented in [10]. Useful preventive replacement guidelines for situations where failure data are available are presented

within the NPI framework, as discussed in Sec. Imprecise probability models for inference. A quite general approach for reliability analysis of repairable systems, proposed by Gurov and Utkin, is to substitute the optimal density functions of TTF and time to repair, which are weighted sums of Dirac functions [138], into integral equations modelling arbitrary repairable systems, and to solve the obtained optimization problems. Let us illustrate this approach for computing the lower and upper probabilities of the working state at time t (the time-dependent availability) under condition that the distributions of TTF and TTR are unknown and only the precise MTTF, denoted a, and the precise *mean time to repair*, denoted b, are specified. For the component, the following system of integral equations holds:

$$\begin{cases} y_0(s,t) = \int_0^t f(x+s) y_1(0,t-x)\mathbf{d}x + f(t+s) \\ y_1(\tau,t) = \int_0^t g(x+\tau) y_0(0,t-x)\mathbf{d}x \end{cases}.$$

Here $f(x)$ and $g(x)$ are unknown densities of the TTF and TTR such that $\int_0^\infty xf(x)\mathbf{d}x = a$ and $\int_0^\infty xg(x)\mathbf{d}x = b$. The probability of the working state $p_0(t)$ at time t is computed as

$$p_0(t) = \int_0^\infty y_0(s,t)\mathbf{d}s.$$

The optimal densities $f_o(x)$ and $g_o(x)$ are in the classes of densities of the form (the weighted sums of Dirac functions $\delta(x-c)$, see Sec. System reliability analysis):

$$f_o(x) = \frac{x_2 - a}{x_2 - x_1}\delta(x - x_1) + \frac{a - x_1}{x_2 - x_1}\delta(x - x_2),$$

$$g_o(x) = \frac{z_2 - b}{z_2 - z_1}\delta(x - z_1) + \frac{b - z_1}{z_2 - z_1}\delta(x - z_2).$$

Here $x_1, x_2, z_1, z_2 \in \mathbf{R}_+$ are optimization variables. Then the lower (upper) bound for $p_0(t)$ is computed by minimizing (maximizing) $p_0(t)$ over all possible values of x_1, x_2, z_1, z_2 after substituting the densities $f_o(x)$ and $g_o(x)$ into integral equations.

Although it is possible, in principle, to analyze arbitrary systems in this manner, this approach requires extremely complex non-linear optimization problems. An efficient and practical approach for imprecise reliability analysis of repairable systems remains an open problem.

10.10 Structural Reliability

A probabilistic model of *structural reliability* and safety has been introduced by Freudenthal [56]. Following his work, a number of studies have been carried out to compute the probability of failure under different assumptions about initial information. The problem of structural reliability can be stated as follows. Let Y represent a random variable describing the strength of a system and let X represent a random variable describing the stress or load placed on the system. By assuming that X and Y are defined on $\mathbf{X}$ and $\mathbf{Y}$, respectively, system failure occurs when the stress on the system exceeds the strength of the system: $\Phi = \{(x \in \mathbf{X}, y \in \mathbf{Y}) : x \geq y\}$. Here Φ is a region where the combination of system parameters leads to an unacceptable or unsafe system response. Then the reliability of the system is determined as $R = \Pr\{X \leq Y\}$, and the unreliability is determined as $Q = \Pr\{X > Y\} = 1 - R$.

Uncertainty of parameters in engineering design was successfully modelled by means of interval analysis [84]. Several authors [7,85] used fuzzy set and possibility theories to cope with a lack of complete statistical information about stress and strength. The main idea of their approaches is to consider the stress and strength as fuzzy variables or fuzzy random variables. Another approach to structural reliability analysis based on using random set and evidence theories has been proposed by several authors [6,65,92]. Several solutions to structural problems by means of random set theory have been presented in [93,94,95].

A more general approach to structural reliability analysis using imprecise probabilities was proposed by Utkin and Kozine [140,141]. This approach allows us to utilize a wider class of partial information about structural parameters, which includes possible data about probabilities of arbitrary events, expectations of the random stress and strength and their functions. At the same time, this approach allows us to avoid additional assumptions about probability distributions of the random parameters because the identification of precise probability distributions requires more information than what experts or limited statistical data are able to supply.

For example, if interval-valued probabilities

$$\underline{p}_i \leq \Pr\{X \leq \alpha_i\} \leq \overline{p}_i, \ \underline{q}_j \leq \Pr\{Y \leq \beta_j\} \leq \overline{q}_j,$$

of the stress X and strength Y are known at points α_i, $i = 1,...,n$, and β_j, $j = 1,...,m$, then the interval-valued stress-strength reliability, based on complete lack of information about dependence of X and Y, is

$$\underline{R} = \max_{i=1,\ldots,n} \max\left(0, \underline{p}_i - \overline{q}_{j(i)}\right),\ j(i) = \min\{j\ :\ \alpha_i \le \beta_j\},$$

$$\overline{R} = 1 - \max_{k=1,\ldots,m} \max\left(0, \underline{q}_k - \overline{p}_{l(k)}\right),\ l(k) = \min\{l\ :\ \beta_k \le \alpha_l\}.$$

If X and Y are independent, then

$$\underline{R} = \sum_{i=1}^{n} (\underline{p}_i - \underline{p}_{i-1})(1 - \overline{q}_{j(i)}),\ j(i) = \min\{j\ :\ \alpha_i \le \beta_j\},$$

$$\overline{R} = 1 - \sum_{k=1}^{m} (\underline{q}_k - \underline{q}_{k-1})(1 - \overline{p}_{l(k)}),\ l(k) = \min\{l\ :\ \beta_k \le \alpha_l\}.$$

Utkin [112] investigated stress-strength reliability analysis based on unreliable information about statistical parameters of stress and strength in the form of a second-order hierarchical uncertainty model. However, there are cases when properties of probability distributions of the stress and strength are known, for example, from their physical nature, but some parameters of the distributions must be assigned by experts. If experts provide intervals of possible parameter values, and these experts are considered to be absolutely reliable, then the problem of structural reliability analysis is solved by standard interval arithmetic. Often, however, it will be necessary to take into account the available information about the quality of experts, to obtain more credible assessments of the stress-strength reliability. An approach for computing the stress-strength reliability under these conditions is considered in [118].

10.11 Software Reliability

Software reliability has been studied extensively in the literature with the objective of improving software performance [19,90,151]. In the last decades, various *software reliability growth models* have been developed based on testing or debugging processes, but no model can be accurate for all situations. This fact is due to the unrealistic assumptions in each model. A comprehensive critical review on probabilistic software reliability models (PSRMs) was proposed by Cai *et al* [20]. Authors argued that fuzzy software reliability models (FSRMs) should be developed in place of PSRMs because the software reliability behavior is fuzzy in nature as a result of the uniqueness of software. This point is explained in three ways. First, any two copies of software exhibit no differences. Second, software never experiences performance deterioration without external intervention. Third, a software debugging process is never replicated. Due to the uniqueness of software and the environment of its use, frequentist

statistical methods are rarely suitable for software reliability inferences. In addition, a large variety of factors contribute to the lack of success of existing PSRMs. To predict software reliability from debugging data, it is necessary to simultaneously take into account the test cases, characteristics of software, human intervention, and debugging data. It is impossible to model all four aspects precisely because of the extremely high complexity behind them [20].

To take into account the problems described above, Cai *et al* [21] proposed a simple FSRM and validated it. Central in this FSRM are the random time intervals between software failures, which are considered to be fuzzy variables governed by membership functions. Extensions of Cai's FSRMs taking into account the programmer's behavior (possibility of error removal and introduction) and combined fuzzy-probabilistic models have been investigated by Utkin *et al* [135].

Available PSRMs and FSRMs can be incorporated into more general imprecise software reliability models (ISRMs) [105], by application of the theory of imprecise probabilities. A family of non-countably many probability distributions constrained by some lower and upper distributions is constructed and analyzed in the ISRM. Let X_i be the random time interval between the $(i-1)$-th and i-th software failures. It is supposed that there exist lower and upper $\overline{P}_i(x \mid \theta_i)$ probability distributions of the random variable X_i with parameters θ_i and these distributions produce a set $\mathbf{R}_i$ of distributions such that $\underline{P}_i(x \mid \theta_i) = \min_{\mathbf{R}_i} P_i(x)$, $\overline{P}_i(x \mid \theta_i) = \max_{\mathbf{R}_i} P_i(x)$. Let $\{x_1, ..., x_n\}$ be the successive intervals between failures. It is assumed that $\theta_i = f(i, \theta)$, where f is some function characterizing the software reliability growth. The main aim is to find the function $f(i)$ and its parameters θ. It is proved that the maximum of the likelihood function by the lack of information about independence of random times between software failures is determined as follows:

$$\max_{\theta} \max_{\mathbf{R}_1 \cup \mathbf{R}_2 \cup ... \cup \mathbf{R}_n} L(x_1, ..., x_n \mid \theta) = \max_{\theta} \min_{i=1,...,n} \left\{ \overline{P}_i(x_i \mid \theta) - \underline{P}_i(x_i \mid \theta) \right\}.$$

If random variables are independent, then

$$\max_{\theta} \max_{\mathbf{R}_1 \cup \mathbf{R}_2 \cup ... \cup \mathbf{R}_n} L(x_1, ..., x_n \mid \theta) = \max_{\theta} \prod_{i=1}^{n} \left\{ \overline{P}_i(x_i \mid \theta) - \underline{P}_i(x_i \mid \theta) \right\}.$$

It is also proved that in the case of right-censoring times for software failures the upper probabilities in the above expressions are replaced by 1.

ISRMs can be regarded as a generalization of the well known

probabilistic and possibilistic models. Moreover, they allow us to explain some peculiarities of known models, for example, taking into account the condition of independence of times to software failures, which are often hidden or can be explained intuitively. For example, the ISRM explains why FSRMs, as stated in [20], allow us to take into account a lot of factors influencing the software reliability. At the same time, PSRMs and FSRMs can be regarded as some boundary cases. Indeed, too rigid and often unrealistic assumptions are introduced in PSRMs, namely, times to software failure are independent and governed by a certain distribution. In FSRMs, it is assumed that the widest class of possible distributions of times to software failure is considered and there is no information about independence. It is obvious that the functions $\overline{P}_i(x)$, $\underline{P}_i(x)$ in the ISRM cannot be chosen arbitrarily because maximization of $L(\cdot\,|\,\theta)$ over parameters θ would give $\overline{P}_i(x)=1$, $\underline{P}_i(x)=0$. This implies that the functions $\overline{P}_i(x)$, $\underline{P}_i(x)$ must be constrained. For example, they may be connected by means of common parameters. Another possibility is to restrict the degree of imprecision $\max_x\left\{\overline{P}_i(x)-\underline{P}_i(x)\right\}\le\varepsilon$. It should be noted that such constraints can also be used to arrive again at the PSRM and the FSRM. The PSRM assumes $\overline{P}_i(x)=\underline{P}_i(x)$ and $\varepsilon=0$. In the FSRM, we have identical parameters θ for lower and upper distributions.

Although quantification of software reliability metrics can provide useful insights into both the likely software performance and the quality of its development, uptake of such mathematical models has remained limited. This is mostly due to the crucial practical circumstances under which software developers and testers operate, typically with short turn-around times and huge time pressures. In addition, many models which have been suggested for supporting their activities, are based on unrealistic assumptions, e.g. independence assumptions underlying partition testing. Rees *et al.* [86] and Coolen *et al.* [36] have described such practical circumstances in detail, and report on a method employing Bayesian graphical models to support software testing of large-scale systems that require high reliability, with complex tasks and huge time pressures, technical details of the statistical aspects are described in [150]. This approach is fully subjective, with the testers' activities central to the model. As such, building the models requires substantial subjective inputs, which provides a bottle-neck to wide-scale practical application due to the enormous time pressures. So far, case studies have used a variety of methods to limit the elicitation effort, and the effect of assumptions has been studied by sensitivity analyses. It is recognized that imprecise

probabilistic methods can offer much benefit to this approach in future, putting less pressure on experts to provide reasonably coherent information on very many variables. In addition, the effects of differing levels of imprecision, at different input places of the models, on the overall test strategies that result from such exercises, can be studied in order to decide where best to focus detailed elicitation effort. This is an exciting area of future research, requiring algorithms for manipulating Bayesian graphical models with imprecise probabilities. Although research on this latter issue has been ongoing in the statistical and computer science literatures for several years, it is not yet at the stage that it can be implemented to realistic large-scale software reliability models, due to the often complex dependence structures in these models. A possible way around this problem might be the use of Bayes linear methods [59], where useful in combination with full Bayesian models, to model complex dependence structures. This would have the benefit of the fact that previsions, the core concept in Bayes linear methods, are linear functionals, which would make inclusion of imprecision more straightforward, both in principle and from computational perspective. Coolen *et al* [35] present a first approach for such Bayes linear modelling for software reliability, Goldstein and Shaw [60] have shown how Bayes linear and Bayesian methods can be combined. Generalizing these approaches to include imprecision, hence further reducing elicitation effort and more clearly reporting levels of indeterminacy, is also an exciting topic for future research.

10.12 Human Reliability

Human reliability [66,67] is defined as the probability for a human operator to perform correctly required tasks in required conditions and not to assume tasks which may degrade the controlled system. Human reliability analysis aims at assessing this probability. Fuzzy or possibilistic descriptions of human reliability behavior are presented in [83]. Human behaviour has been described also by means of evidence theory [91]. Cai [18] noted the following factors of human reliability behavior contributing to the fuzziness:

1) inability to acquire and process an adequate amount of information about systems;
2) vagueness of the relationship between people and working environments;
3) vagueness of human thought process;
4) human reliability behavior is unstable and vague in nature because it

depends on human competence, activities, and experience.

These factors can also be addressed via imprecision, so imprecise probability theory might be successfully applied to human reliability analysis. Moreover, the behavioural interpretation of lower and upper previsions may well be suitable for describing human behavior. However, we are not aware of any research reported on imprecise human reliability, suggesting another stream of interesting research topics.

10.13 Risk Analysis

Risk of an unwanted event happening is often defined as the product of the probability of the occurrence of this event multiplied by its consequences, assuming that these consequences can be combined into a simple metric. The consequences may include financial cost, elapsed time, etc. One of the main objectives of performing risk analyses is to support decision-making processes. Risk analysis provides a basis for comparing alternative concepts, actions or system configurations under uncertainty [5,11]. A variety of methods has been developed for estimating losses and risks. When events occur frequently and when they are not very severe, it is relatively simple to estimate the risk exposure of an organization, as well as a reasonable premium when, for instance, an insurance transaction is made [53]. Commonly used methods rely on variations of the principle of maximizing expected utility, tacitly assuming that all underlying uncertainty can adequately be described by a precise and completely known probability measure. However, when the uncertainty is complex and the quality of the estimates is poor, the customary use of such rules together with over-precise data could be harmful as well as misleading. Therefore, it is necessary to extend the principle of maximizing expected utility to deal with complex uncertainty. Imprecise probability theory provides an efficient way for realizing such an extension.

The imprecision of information about unwanted events leads to consideration of minimal and maximal values of risk, which can be regarded as lower and upper previsions of consequences whose computation by complex events is studied in [101]. Some methods of handling partial information in risk analysis have been investigated by several authors [53,54]. Risk analysis under hierarchical imprecise uncertainty models has been studied by Utkin and Augustin [122], where two types of the second-order uncertainty models of states of nature are considered. The first type assumes that first-order uncertainty is modelled by lower and upper previsions of different gambles and the second-order

probabilities can be regarded as confidence weights of judgements on the first-order level. The second type assumes that some aspects of the probability distribution of the states of nature is known, for example, from their physical nature, but (some) parameters of the probability distribution must be defined by experts, and there is some degree of our belief to each expert's judgement whose value is determined by experience and competence of the expert. New procedures for risk analysis under different conditions of partial information about states of nature in the framework of imprecise probabilities have been studied by Utkin and Augustin [123,124,125,126].

In situations where risk can be assessed via experiments, with the emphasis on low risk situations where systems are only released for practical operation following a number of tests without failures, the NPI framework (see Sec. Imprecise probability models for inference) provides useful guidelines on required test effort, in particular via the use of lower probabilities of corresponding future successful operation, to take indeterminacy into account (`to err on the side of safety', so to say). Some initial results in this area have been presented [30,33]. A further interesting topic, which has remained largely neglected as far as we are aware, is the fact that consequences, and their impact on life, are often not known in great detail. In particular where random features are studied with information occurring at different moments in time, it is natural to also take learning about such consequences and impacts into account. This also typically suggests that, at least at early stages (e.g. when designing a new chemical process), indeterminacy about such risks may well be modelled via imprecision, and it should be possible to take adaptive metrics for such risks into account. It is possible that the Bayesian adaptive utility framework [47], which was developed in the seventies within economics contexts, may provide an attractive solution to this problem. However, adaptive utility has not yet been generalized to allow imprecision, even more its uptake has been almost nonexistent, quite possibly due to both the computational complexities involved and the foundational aspects. Work in this direction has recently been initiated, and we hope to report on progress in the near future, where we will also particularly focus on applications in risk and reliability.

10.14 Security Engineering

Security engineering is concerned with whether a system can survive accidental or intentional attacks on it from outside (e.g. from users or virus

intruders). In particular, computer security deals with the social regulations, managerial procedures and technological safeguards applied to computer hardware, software and data, to assure against accidental or deliberate unauthorized access to, and dissemination of, computer system resources (hardware, software, data) while they are in storage, processing or communication [68]. An important problem in security engineering is the quantitative evaluation of security efficiency. An interesting and valuable approach to measuring and predicting the operational security of a system was proposed by Brocklehurst *et al.* [15]. According to this approach, the behavior of a system should be considered from owner's and attacker's points of view. From the attacker's point of view, it is necessary to consider the *effort* (E) expended by the attacking agent and the *reward* (R) an attacker would get from breaking into the system. Effort includes financial cost, elapsed time, experience, ability of attacker, and could be expressed in such terms as mean effort to next security breach, probability of successfully resisting an attack, etc. Examples of rewards are personal satisfaction, gain of money, etc. From the owner's point of view, it is necessary to consider the system *owner's loss* (L) which can be interpreted as an *infimum selling price for a successful attack*, and the *owner's expenses* (Z) on the security means which include, for instance, anti-virus programs, new passwords, encoding, etc. The expenses come out in terms of time used for system verification, for maintenance of anti-virus software, as well as in terms of money spent on the protection. The expenses can be interpreted as a *supremum buying price for a successful attack*. Brocklehurst *et al.* [15] proposed to consider also the viewpoint of an all-knowing, all-seeing *oracle*, as well as the owner and attacker. This viewpoint could be regarded as being in a sense the true security of the system in the testing environment.

From the above, we can say that four variables are the base for obtaining security measures: effort, rewards, system owner's loss, owner's expenses. Moreover, their interpretation coincides with the behavioural interpretation of lower (expenses) and upper (system owner's loss) previsions and linear previsions (the all-knowing oracle). Therefore, imprecise probability theory provides an interesting and logical framework for quantifying such security measures [100,142]. Because of the increasing importance of security engineering, this also provides exciting opportunities for (research into) theory and application of imprecise methods.

10.15 Concluding Remarks and Open Problems

In recent years, many results have been presented which enable application of imprecise probability theory to reliability analyses of various systems, many of such results have been discussed here. Imprecise reliability theory is being developed step-by-step, mostly addressing problems from the existing reliability literature. However, the state-of-the-art is only a visible top of the iceberg called the imprecise reliability theory and there are many open theoretical and practical problems, which should be solved in future. Several exciting areas for future research have been indicated in the earlier sections, let us now say a bit more on this, and mention some further related topics of research.

It is obvious that modern systems and equipment are characterized by complexity of structures and variety of initial information. This implies that, on the one hand, it is impossible to adjust all features of a real system to the considered framework. On the other hand, introduction of some additional assumptions for constructing a reasonable model of a system may cancel all advantages of imprecise probabilities. Where are limits for introducing additional assumptions (simplification) in construction of a model? How do possible changes of initial information and assumptions influence the results of system reliability calculations? It is obvious that such questions relate to the informational aspect of imprecise reliability. The same can be said about necessity of studying the effects of possible estimation errors of initial data on resulting reliability measures. This leads to introducing and determining uncertainty importance measures.

Another important point is how to solve the optimization problems if the function $h(g(\mathbf{X}))$ is not expressed analytically in explicit form and can be computed only numerically. For example, this function may be a system of integral equations (repairable system). One of the ways to solve the corresponding optimization problems is the well-known simulation technique. However, the development of effective simulation procedures for solving the considered optimization problems is an open problem.

Many results of imprecise reliability are based either on the assumption of independence of components, or complete lack of information about independence. However, the imprecise probability theory allows us to take into account more subtle types of dependence [46,77] and, thereby, to make reliability analysis more flexible and adequate. Therefore, a clear interpretation and development of dependence concepts imprecise reliability theory is also an open problem, which has to be solved in future.

In spite of the fact that many algorithms and methods for reliability analysis of various systems have been developed, they are rather

theoretical and cover some typical systems, typical initial evidence, and typical situations. At the same time, real systems are more complex. Therefore, practical approaches to analyze real systems by imprecise reliability methods have to be developed, which is likely to require development of appropriate approximate computational methods.

In order to achieve a required level of system reliability by minimal costs, it is possible to include redundant components in systems. To optimize cost and reliability metrics, the number of redundant components in a system can be determined, together with optimal system structures. Various algorithms for determining the optimal number of redundant components are available in the literature. However, most results assume that there exists complete information about reliability. Therefore, the development of efficient algorithms of optimization by partial information is also an open problem.

A similar problem is the *product quality control* which needs a trade-off between a better product quality and lower production costs by system constraints related to operating feasibility, product specifications, safety and environmental issues. Here results obtained by Augustin [2,3], concerning decision making under partial information about probabilities of states of nature, and results by Quaeghebeur and de Cooman [87], extending some aspects of game theory, might be a basis for investigating this problem. Quality control, in particular the use of control charts, has also been considered within the nonparametric predictive inferential framework [1]. This is also an exciting research area with many open problems, and with imprecision appearing naturally related to limited information. Clearly, ensuring high quality output in production processes can greatly enhance reliability. Even earlier than that, reliability often depends on the actual design of components and systems. At such an early stage, modelling uncertainties via precise probabilities is often extremely restricted, in particular when the designs involve revolutionary products. This is another area where imprecise reliability theory may offer exciting opportunities.

It should be noted that the list of open problems can be extended. However, most problems include at least some optimization problems (natural extension), which are often very complex. This may well be the reason why imprecise probability and reliability was not greatly developed earlier in the twentieth century. Nowadays, with the ever increasing computer power, complex optimization problems do not need to stop further development of appropriate methods for dealing with uncertainty, even though such problems still may need detailed consideration, and the need to develop approximate methods will remain. We believe that these are exciting times for imprecise reliability theory, as so much more can be

achieved now than before. Therefore, the time is also right to take on challenges of actual applications, with all careful modelling and complex computational aspects involved. We look forward to these challenges, and hope that many fellow researchers also take up some of these challenges.

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