A Unified Framework of Morphological Associative Memories

Naiqin Feng[1,2], Yuhui Qiu[2], Fang Wang[2], and Yuqiang Sun[1]

[1] College of Computer & Information Technology, Henan Normal University, Xinxiang 453007

[2] Faculty of Computer & information Science, South West-China University, Chongqing 400715

fengnq@swu.edu.cn

Abstract. The morphological neural network models, including morphological associative memories (MAM), fuzzy morphological associative memories (FMAM), enhanced morphological associative memories (EFMAM), etc., are extremely new artificial neural networks. They have many attractive advantages such as unlimited storage capacity, one-short recall speed and good noise-tolerance to erosive or dilative noise. Although MAM, FMAM and EFMAM are different and easily distinguishable from each other, they have the same morphological theory base. Therefore in this paper a unified theoretical framework of them is presented. The significance of the framework consists in: (1) It can help us find some new methods, definitions and theorems for morphological neural networks; (2) We have a deeper understanding of MAM, FMAM and EFMAM while having the unified theoretical framework.

1 Introduction

The theory of artificial neural networks has been successful applied to a wide variety of pattern recognition problems [3,4]. In this theory, the first step in computing the next state of a neuron or in performing the next layer neural-network computation involves the linear operation of multiplying neural values by their synaptic strengths and adding the results. A nonlinear activation function usually follows the linear operation in order to provide for non-linearity of the network and set the next state of the neuron. In recent years, a number of different morphological neural network models and applications have emerged [1,5,8,12,15,16]. First attempts in formulating useful morphological neural networks appeared in [10]. Since then, only a few papers involving morphological neural networks have appeared. Suarez-Araujo applied morphological neural networks to compute homothetic auditory and visual invariants [2]. Davidson employed morphological neural networks in order to solve template identification and target classification problems [9], [11]. All of these researchers devised multi-layer morphological neural networks for very specialized applications. A more comprehensive and rigorous basis for computing with morphological neural networks appeared in [6] where it was shown that morphological neural networks are capable of solving any conventional computational problem. In 1998, Ritter et al.

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proposed the concept of morphological associative memories (MAM) and the concept of morphological auto-associative memories (auto-MAM) [7], which constitute a class of networks not previously discussed in detail.

MAM is based on the algebraic lattice structure $(R, \wedge, \vee, +)$ or morphological operations. MAM behaves more like human associative memories than the traditional semilinear models such as the Hopfield net. Once a pattern has been memorized, recall is instantaneous when the MAM is presented with the pattern. In the absence of noise, an auto-MAM will always provide perfect recall for any number of patterns programmed into its memory. The auto-MAM $\mathbf{M}_{XX}$ is extremely robust in recalling patterns that are distorted due to dilative changes, while auto-MAM $\mathbf{W}_{XX}$ is extremely robust in recalling patterns that are distorted due to erosive changes.

In 2003, Wang and Chen presented the model of fuzzy morphological associative memories (FMAM). Originated from the basic ideas of MAM, the FMAM uses two basic morphological operations $(\wedge, \cdot)$, $(\vee, \cdot)$ instead of fuzzy operation $(\wedge, \vee)$ in fuzzy associative memory [13]. FMAM solves fuzzy rules memory problem of the MAM. Under certain conditions, FMAM can be viewed as a new encoding way of fuzzy associative memory such that it can embody fuzzy operators and the concepts of fuzzy membership value and fuzzy rules. Both auto-FMAM and auto-MAM have the same attractive advantages, such as unlimited storage capacity, one-shot recall speed and good noise-tolerance to either erosive or dilative noise. However, they suffer from the extreme vulnerability to noise of mixing erosion and dilation, resulting in great degradation on recall performance. To overcome this shortcoming, in 2005, Wang and Chen further presented an enhanced FMAM (EFMAM) based on the empirical kernel map [14].

Although MAM, FMAM and EFMAM are different and easily distinguishable from each other, we think that they have the same theoretical base, i.e. the same morphological base, therefore they can be unified together. This paper tries to establish a unified theoretical framework of MAM, FMAM and EFMAM. The more the thing is abstracted, the deeper the thing is understood. Consequently it is possible that some new methods and theorems are obtained. This is the reason why we research and propose the unified theoretical framework of MAM, FMAM and EFMAM

2 Unified Computational Base of MAM, FMAM and EFMAM

Traditional artificial neural network models are specified by the network topology, node characteristics, and training or learning rules. The underlying algebraic system used in these models is the set of real numbers R together with the operations of addition and multiplication and the laws governing these operations. This algebraic system, known as a ring, is commonly denoted by $(R, +, \times)$. The basic computations occurring in the morphological network proposed by Ritter et al. are based on the algebraic lattice structure $(R, \wedge, \vee, +)$, where the symbols $\wedge$ and $\vee$ denote the binary operations of minimum and maximum, respectively, while the basic computations used in FMAM and EFMAM are based on the algebraic lattice structure $(R_+, \wedge, \vee, \cdot)$ $(R_+ = (0, \infty))$.

In unified morphological associative memories (UMAM), the basic computations are based on the algebraic lattice structure $(U, \wedge, \vee, \mathrm{O})$, where $U=R$, or $U = R_+$; O= +, $-$, $\cdot$, or /. If $U=R$ and O=+, then $(U, \wedge, \vee, \mathrm{O}) = (R, \wedge, \vee, +)$, which is the computational base of MAM; If $U=R_+$ and O=$\cdot$, then $(U, \wedge, \vee, \mathrm{O}) = (R_+, \wedge, \vee, \cdot)$, which is the computational base of FMAM and EFMAM. Of course, the symbol O also can be other appreciated operators, for example, $-$ or /.

3 Unified Morphological-Norm Operators

3.1 Operators in MAM, FMAM and EFMAM

As that described in the preceding section, morphological associative memories are based on the lattice algebra structure $(R, \wedge, \vee, +)$. Suppose we are given a vector pair $\mathbf{x}=(x_1,\ldots, x_n)'\in R^n$, , and $\mathbf{y}=(y_1,\ldots,y_m) \in R^m$. An associative morphological memory that will recall the vector $\mathbf{y}$ when presented the vector $\mathbf{x}$ is given by

$$\mathbf{W}=\mathbf{y}\boxed{\vee}(-\mathbf{x})'=\begin{pmatrix} y_1 - x_1 & \cdots & y_1 - x_n \\ \vdots & \ddots & \vdots \\ y_m - x_1 & \cdots & y_m - x_n \end{pmatrix} \tag{1}$$

since $\mathbf{W}$ satisfies the equation $\mathbf{W}\boxed{\vee}\mathbf{x} = \mathbf{y}$ as can be verified by the simple computation

$$\mathbf{W} \boxed{\vee} \mathbf{x} = \begin{pmatrix} \vee_{i=1}^{n} y_1 - x_i + x_i \\ \vdots \\ \vee_{i=1}^{n} y_m - x_i + x_i \end{pmatrix} = \mathbf{y} \tag{2}$$

$\mathbf{W}$ is called the max product of $\mathbf{y}$ and $\mathbf{x}$. We also can denote the min product of $\mathbf{y}$ and $\mathbf{x}$ using operator $\boxed{\wedge}$ like (1) and (2). Similarly, let $(\mathbf{x}^1, \mathbf{y}^1),\ldots,(\mathbf{x}^k, \mathbf{y}^k)$ be k vector pairs with $\mathbf{x}^\xi=(x_1^\xi,\ldots, x_n^\xi)'\in R^n$ and $\mathbf{y}^\xi=(y_1^\xi,\ldots, y_m^\xi)'\in R^m$ for $\xi=1,\ldots, k$. For a given set of pattern associations $\{(\mathbf{x}^\xi, \mathbf{y}^\xi): \xi=1,\ldots, k\}$ we define a pair of associated pattern matrices $(\mathbf{X},\mathbf{Y})$, where $\mathbf{X}=(\mathbf{x}^1,\cdots,\mathbf{x}^k)$, $\mathbf{Y}=(\mathbf{y}^1,\cdots,\mathbf{y}^k)$. Thus, $\mathbf{X}$ is of dimension $n\times k$ with i, jth entry x_i^j and $\mathbf{Y}$ is of dimension $m\times k$ with i, jth entry y_i^j. With each pair of matrices $(\mathbf{X},\mathbf{Y})$, two natural morphological $m\times n$ memories $\mathbf{W}_{XY}$ and $\mathbf{M}_{XY}$ are defined by

$$\mathbf{W}_{XY} = \bigwedge_{\xi=1}^{k}[\mathbf{y}^\xi \boxed{\wedge} (-\mathbf{x}^\xi)'] \text{ and } \mathbf{M}_{XY} = \bigvee_{\xi=1}^{k}[\mathbf{y}^\xi \boxed{\vee} (-\mathbf{x}^\xi)']. \tag{3}$$

Obviously, $\mathbf{y}^\xi\boxed{\wedge}(-\mathbf{x}^\xi)'= \mathbf{y}^\xi\boxed{\vee}(-\mathbf{x}^\xi)'$. It therefore follows from this definition that

$$\mathbf{W}_{XY}\le\mathbf{y}^\xi\boxed{\wedge}(-\mathbf{x}^\xi)'=\mathbf{y}^\xi\boxed{\vee}(-\mathbf{x}^\xi)'\le\mathbf{M}_{XY}, \forall\ \xi=1,\ldots, k. \tag{4}$$

In view of equations (2) and (3), this last set of inequalities implies that

$$\mathbf{W}_{XY}\boxed{\vee}\ \mathbf{x}^\xi\le[\mathbf{y}^\xi\boxed{\wedge}(-\mathbf{x}^\xi)']\ \boxed{\vee}\mathbf{x}^\xi=\mathbf{y}^\xi= [\mathbf{y}^\xi\boxed{\vee}(-\mathbf{x}^\xi)']\ \boxed{\wedge}\ \mathbf{x}^\xi\le\mathbf{M}_{XY}\boxed{\wedge}\ \mathbf{x}^\xi \tag{5}$$

$\forall\ \xi=1,\ldots, k$ or, equivalently, that

$$\mathbf{W}_{XY} \boxed{\vee} \mathbf{X} \le\mathbf{Y} \le\mathbf{M}_{XY} \boxed{\wedge} \mathbf{X}. \tag{6}$$

If $\mathbf{W}_{XY} \boxtimes \mathbf{X} = \mathbf{Y}$, then $\mathbf{W}_{XY}$ is called a $\boxtimes$-perfect memory for (**X**, **Y**); if $\mathbf{M}_{XY} \boxdot \mathbf{X} = \mathbf{Y}$, then $\mathbf{M}_{XY}$ is called a $\boxdot$-perfect memory for (**X**, **Y**).

The basic computations used in FMAM and EFMAM are based on the algebraic lattice structure $(R_+, \wedge, \vee, \cdot)$ $(R_+ = (0, \infty))$. If the input vector $\mathbf{x}^l = (x_1^l, \cdots, x_n^l)'$ is defined in R_+^n, and the output vector $\mathbf{y}^l = (y_1^l, \cdots, y_m^l)'$ is defined in R_+^m, by using some transformation, for example, exp (x) and exp (y) (acting on each component of ***x, y***), the input vectors and output vectors can be transformed into R_+^n and R_+^m, respectively. Set $\mathbf{X} = (\mathbf{x}^1, \cdots, \mathbf{x}^k)$, $\mathbf{Y} = (\mathbf{y}^1, \cdots, \mathbf{y}^k)$, with each pair of matrices (**X**, **Y**), two new morphological $m \times n$ memories A_{XY} and B_{XY} are as follows:

$$\mathbf{A}_{XY} = (\wedge_{l=1}^{k} (\mathbf{y}^l \circledcirc (\mathbf{x}^l)^{-1}), \ \mathbf{B}_{XY} = (\vee_{l=1}^{k} (\mathbf{y}^l \circledcirc (\mathbf{x}^l)^{-1}) \tag{7}$$

$$(\mathbf{x}^l)^{-1} = \left(\frac{1}{x_1^l}, \cdots, \frac{1}{x_n^l} \right)' \qquad x_i^l > 0, \ \ \forall i = 1, \cdots, n \tag{8}$$

$$\mathbf{y}^l \circledcirc (\mathbf{x}^l)^{-1} = \mathbf{y}^l \circledcirc (\mathbf{x}^l)^{-1} = \begin{pmatrix} \frac{y_1^l}{x_1^l} & \cdots & \frac{y_1^l}{x_n^l} \\ \vdots & \ddots & \vdots \\ \frac{y_m^l}{x_1^l} & \cdots & \frac{y_m^l}{x_n^l} \end{pmatrix} \tag{9}$$

where $\boxdot$ and $\boxtimes$ denote fuzzy composite operation $(\wedge, \cdot)$ and $(\vee, \cdot)$ often used in fuzzy set theory, respectively. In FMAM and EFMAM, the recall is given by

$$\mathbf{A}_{XY} \boxtimes \mathbf{x}^l = (\bigwedge_{l=1}^{k} \mathbf{y}^l \boxdot (\mathbf{x}^l)^{-1}) \boxtimes \mathbf{x}^l \text{ and } \mathbf{B}_{XY} \boxdot \mathbf{x}^l = (\bigvee_{l=1}^{k} \mathbf{y}^l \boxtimes (\mathbf{x}^l)^{-1}) \boxdot \mathbf{x}^l \tag{10}$$

With analyzing for MAM, FMAM and EFMAM, we can easily see that there exist reversible operators in memory and recall. For MAM, the reversible operators in memory and recall are – and +, respectively; for FMAM and EFMAM, they are / and ×, respectively. We unify them with the following definitions.

3.2 Unified Morphological-Norm Operators

Definition 1. For an $m \times p$ matrix **A** and a $p \times n$ matrix **B** with entries from U, the matrix product $\mathbf{C} = \mathbf{A} \overset{\circ}{\vee} \mathbf{B}$, also called the morphological max product norm of **A** and **B**, is defined by

$$c_{ij} = \bigvee_{k=1}^{p} a_{ik} \mathrm{O}\, b_{kj} = (a_{i1} \mathrm{O} b_{1j}) \vee (a_{i2} \mathrm{O} b_{2j}) \vee \cdots \vee (a_{ip} \mathrm{O}\, b_{pj}). \tag{11}$$

Where, $\overset{\circ}{\vee}$ is a unified morphological operator, which represents one of the $\overset{+}{\vee}, \overset{-}{\vee}, \overset{\cdot}{\vee}$, and $\overset{/}{\vee}$. The symbol O represents a reversible operation, such as +, —, ×, or /.

Definition 2. For an $m \times p$ matrix **A** and a $p \times n$ matrix **B** with entries from U, the matrix product $\mathbf{C} = \mathbf{A} \overset{\circ}{\wedge} \mathbf{B}$, also called the morphological min product norm of **A** and **B**, is defined by

$$c_{ij} = \bigwedge_{k=1}^{p} a_{ik} \mathrm{O}\, b_{kj} = (a_{i1} \mathrm{O} b_{1j}) \wedge (a_{i2} \mathrm{O} b_{2j}) \wedge \cdots \wedge (a_{ip} \mathrm{O}\, b_{pj}). \tag{12}$$

Where, $\overset{\mathrm{o}}{\wedge}$ is a unified morphological operator, which represents one of the $\overset{+}{\wedge}$, $\overset{-}{\wedge}$, $\overset{\cdot}{\wedge}$, and $\overset{/}{\wedge}$. The symbol O represents a reversible operation, such as +, $-$, ×, or /.

Definition 3. For an *m*×*p* matrix **A** and a *p*×*n* matrix **B** with entries from *U* and the max product **C** =**A** $\overset{+}{\vee}$ **B**, the morphological operator $\overset{+}{\vee}$ is defined by:

$$c_{ij} = \bigvee_{k=1}^{p} a_{ik} + b_{kj} = (a_{i1} + b_{1j}) \vee (a_{i2} + b_{2j}) \vee \cdots \vee (a_{ip} + b_{pj}). \tag{13}$$

Similarly, we can define the morphological operators $\overset{-}{\vee}$, $\overset{\cdot}{\vee}$, and $\overset{/}{\vee}$.

Definition 4. For an *m*×*p* matrix **A** and a *p*×*n* matrix **B** with entries from *U* and the min product C =A $\overset{+}{\wedge}$ B, the morphological operator $\overset{+}{\wedge}$ is defined by:

$$c_{ij} = \bigwedge_{k=1}^{p} a_{ik} + b_{kj} = (a_{i1} + b_{1j}) \wedge (a_{i2} + b_{2j}) \wedge \cdots \wedge (a_{ip} + b_{pj}). \tag{14}$$

Similarly, we can define the morphological operators $\overset{-}{\wedge}$, $\overset{\cdot}{\wedge}$, or $\overset{/}{\wedge}$.

Definition 5. Let ($\mathbf{x}^1$, $\mathbf{y}^1$),…,($\mathbf{x}^k$, $\mathbf{y}^k$) be k vector pairs with $\mathbf{x}^\xi=(x_1^\xi,\ldots,x_n^\xi)'\in R^n$ and $\mathbf{y}^\xi=(y_1^\xi,\ldots,y_m^\xi)'\in R^m$ for ξ=1,…, k. For a given set of pattern associations {($\mathbf{x}^\xi$, $\mathbf{y}^\xi$): ξ=1,…, k} and a pair of associated pattern matrices (**X**,**Y**), where $\mathbf{X} = (\mathbf{x}^1,\cdots,\mathbf{x}^k)$, $\mathbf{Y} = (\mathbf{y}^1,\cdots,\mathbf{y}^k)$, the morphological min-product memory $\mathbf{W}_{XY}$ is defined by

$$\mathbf{W}_{XY} = \mathbf{Y} \overset{\mathrm{o}}{\wedge} \mathbf{X}' = \bigwedge_{\xi=1}^{k} [\mathbf{y}^\xi \overset{\mathrm{o}}{\wedge} (\mathbf{x}^\xi)'] = \bigwedge_{\xi=1}^{k} \begin{pmatrix} y_1^\xi \mathrm{O}\, x_1^\xi & \cdots & y_1^\xi \mathrm{O}\, x_n^\xi \\ \vdots & \ddots & \vdots \\ y_m^\xi \mathrm{O}\, x_1^\xi & \cdots & y_m^\xi \mathrm{O}\, x_n^\xi \end{pmatrix} \tag{15}$$

And the morphological max-product memory $\mathbf{M}_{XY}$ is defined by

$$\mathbf{M}_{XY} = \mathbf{Y} \overset{\mathrm{o}}{\vee} \mathbf{X}' = \bigvee_{\xi=1}^{k} [\mathbf{y}^\xi \overset{\mathrm{o}}{\vee} (\mathbf{x}^\xi)'] = \bigvee_{\xi=1}^{k} \begin{pmatrix} y_1^\xi \mathrm{O}\, x_1^\xi & \cdots & y_1^\xi \mathrm{O}\, x_n^\xi \\ \vdots & \ddots & \vdots \\ y_m^\xi \mathrm{O}\, x_1^\xi & \cdots & y_m^\xi \mathrm{O}\, x_n^\xi \end{pmatrix} \tag{16}$$

Since $\mathbf{y}^\xi \overset{\mathrm{o}}{\wedge} (\mathbf{x}^\xi)' = \mathbf{y}^\xi \overset{\mathrm{o}}{\vee} (\mathbf{x}^\xi)'$, $\mathbf{W}_{XY}$ and $\mathbf{M}_{XY}$ follow from this definition that

$$\mathbf{W}_{XY} \le \mathbf{y}^\xi \overset{\mathrm{o}}{\wedge} (\mathbf{x}^\xi)' = \mathbf{y}^\xi \overset{\mathrm{o}}{\vee} (\mathbf{x}^\xi)' \le \mathbf{M}_{XY} \quad \forall \xi = 1,\cdots,k \tag{17}$$

Let $\overset{\Theta}{\wedge}$ represents the reverse of $\overset{\mathrm{o}}{\wedge}$, and $\overset{\Theta}{\vee}$ represents the reverse of $\overset{\mathrm{o}}{\vee}$, that is, O and Θ are reversible each other. If O=+ or ×, then Θ=$-$ or ÷; on the contrary, if O=$-$ or ÷, then Θ=+ or ×. Then, $\mathbf{W}_{XY}$ and $\mathbf{M}_{XY}$ satisfy that

$$\mathbf{W}_{XY} \overset{\Theta}{\vee} \mathbf{x}^{\xi} \le [\mathbf{y}^{\xi} \overset{o}{\wedge} (\mathbf{x}^{\xi})'] \overset{\Theta}{\vee} \mathbf{x}^{\xi} = \mathbf{y}^{\xi} = [\mathbf{y}^{\xi} \overset{o}{\vee} (\mathbf{x}^{\xi})'] \overset{\Theta}{\wedge} \mathbf{x}^{\xi} \le \mathbf{M}_{XY} \overset{\Theta}{\wedge} \mathbf{x}^{\xi} \tag{18}$$

$\forall \xi = 1, \cdots, k$ or equivalently, that

$$\mathbf{W}_{XY} \overset{\Theta}{\vee} \mathbf{X} \le \mathbf{Y} \le \mathbf{M}_{XY} \overset{\Theta}{\wedge} \mathbf{X} \tag{19}$$

Definition 6. A matrix $\mathbf{A}=(a_{ij})_{m\times n}$ is said to be a $\overset{o}{\vee}$-perfect memory for (**X**,**Y**) if and only if $\mathbf{A} \overset{o}{\vee} \mathbf{X} = \mathbf{Y}$. The matrix $\mathbf{A}=(a_{ij})_{m\times n}$ is said to be a $\overset{o}{\wedge}$-perfect memory for (**X**,**Y**) if and only if $\mathbf{A} \overset{o}{\wedge} \mathbf{X} = \mathbf{Y}$.

In fact, in the existing MAM there are only two memories $\mathbf{W}_{XY}$ and $\mathbf{M}_{XY}$ defined by using operators $\overset{-}{\wedge}$ and $\overset{-}{\vee}$, respectively. In the existing FMAM and EFMAM, it is also the same, i.e. there are only two memories $\mathbf{W}_{XY}$ and $\mathbf{M}_{XY}$ defined by using operators $\overset{/}{\wedge}$ and $\overset{/}{\vee}$, respectively. But according to the definitions 1 to 6, there will be four memories in MAM, FMAM or EFMAM, respectively. The two additional memories defined by using operators $\overset{+}{\wedge}$ and $\overset{+}{\wedge}$ (for MAM), and by using $\overset{\cdot}{\wedge}$ and $\overset{\cdot}{\vee}$ (for FMAM or EFMAM), respectively. That is to say, there are more methods in the unified framework than there are in MAM, FMAM and EFMAM.

4 Unified Morphological Theorems

Ritter gave 7 theorems with respect to MAM in [7]. Wang et al. also proved 6 theorems with respect to FMAM in [13] and 4 theorems with respect to EFMAM in [14]. Our research result shows that these theorems can be unified. We give two of them and their proofs as two examples.

Theorem 1: If **A** is $\overset{o}{\vee}$-perfect memory for (**X**, **Y**) and **B** is $\overset{o}{\wedge}$-perfect memory for (**X**, **Y**), then

$$\mathbf{A} \le \mathbf{W}_{XY} \le \mathbf{M}_{XY} \le \mathbf{B} \quad \text{and} \quad \mathbf{W}_{XY} \overset{O}{\vee} \mathbf{X} = \mathbf{Y} = \mathbf{M}_{XY} \overset{O}{\wedge} \mathbf{X}.$$

Proof of Theorem 1: If **A** is $\overset{o}{\vee}$-perfect memory for (**X**, **Y**), then $(\mathbf{A} \overset{o}{\vee} \mathbf{x}^{\xi})_i = y_i^{\xi}$ for all $\xi = 1,\dots, k$ and all $i=1,\dots, m$. Equivalently

$$\bigvee_{j=1}^{n} (a_{ij} \mathrm{O} x_j^{\xi}) = y_i^{\xi} \quad \forall \xi = 1, \cdots, k \quad \text{and} \quad \forall i = 1, \cdots, m \ .$$

For MAM, $U=R$, $\mathrm{O}=\pm$, $\Theta=\mp$, it follows that for an arbitrary index $j \in \{1, \cdots, n\}$ we have

$$\begin{aligned} a_{ij} \mathrm{O} x_j^{\xi} \le y_i^{\xi} \quad \forall \xi = 1, \cdots, k \quad &\Leftrightarrow a_{ij} \le y_i^{\xi} \Theta x_j^{\xi} \quad \forall \xi = 1, \cdots, k \\ &\Leftrightarrow a_{ij} \le \bigwedge_{\xi=1}^{k} (y_i^{\xi} \Theta x_j^{\xi}) = w_{ij} \end{aligned} \tag{20}$$

For FMAM and EFMAM, $U=R_+$, O=× or ／, Θ=／ or ×, the set of inequalities (20) also can be derived.

This shows that $\mathbf{A}\leq\mathbf{W}_{\mathrm{XY}}$. In view of (19), we now have $\mathbf{Y}=\mathbf{A}\overset{\mathrm{o}}{\vee}\mathbf{X}\leq\mathbf{W}_{\mathrm{XY}}\overset{\mathrm{o}}{\vee}\mathbf{X}\leq\mathbf{Y}$, and therefore, $\mathbf{W}_{\mathrm{XY}}\overset{\mathrm{o}}{\vee}\mathbf{X}=\mathbf{Y}$. A similar argument shows that if **B** is $\overset{\mathrm{o}}{\wedge}$-perfect memory for (**X**, **Y**), then $\mathbf{M}_{\mathrm{XY}}\leq\mathbf{B}$ and $\mathbf{M}_{\mathrm{XY}}\overset{\mathrm{o}}{\wedge}\mathbf{X}=\mathbf{Y}$. Consequently we have $\mathbf{A}\leq\mathbf{W}_{\mathrm{XY}}\leq\mathbf{M}_{\mathrm{XY}}\leq\mathbf{B}$ and $\mathbf{W}_{\mathrm{XY}}\overset{\mathrm{o}}{\vee}\mathbf{X}=\mathbf{Y}=\mathbf{M}_{\mathrm{XY}}\overset{\mathrm{o}}{\wedge}\mathbf{X}$. □

Theorem 2: $\mathbf{W}_{\mathrm{XY}}$ is $\overset{\mathrm{o}}{\vee}$-perfect memory for (**X**, **Y**) if and only if for each ξ = 1,…, k, each row of the matrix $[\mathbf{y}^{\xi}\overset{\Theta}{\vee}(\mathbf{x}^{\xi})']-\mathbf{W}_{\mathrm{XY}}$ contains a zero entry. Similarly, $\mathbf{M}_{\mathrm{XY}}$ is $\overset{\mathrm{o}}{\wedge}$-perfect memory for (**X**, **Y**) if and only if for each ξ = 1,…, k, each row of the matrix $\mathbf{M}_{\mathrm{XY}}-[\mathbf{y}^{\xi}\overset{\Theta}{\vee}(\mathbf{x}^{\xi})']$ contains a zero entry.

Proof of Theorem 2: We only prove the theorem in one domain for either the memory $\mathbf{W}_{\mathrm{XY}}$ or the memory $\mathbf{M}_{\mathrm{XY}}$. The result of proof for the other memory can be derived in an analogous fashion.

$\mathbf{W}_{\mathrm{XY}}$ is $\overset{\mathrm{o}}{\vee}$-perfect memory for (**X**, **Y**)

$$\Leftrightarrow (\mathbf{W}_{\mathrm{XY}}\overset{\mathrm{o}}{\vee}\mathbf{x}^{\xi})_i = y_i^{\xi} \quad \forall\xi=1,\dots,k \text{ and } \forall i=1,\dots,m$$

$$\Leftrightarrow y_i^{\xi}-(\mathbf{W}_{\mathrm{XY}}\overset{\mathrm{o}}{\vee}\mathbf{x}^{\xi})_i = 0 \quad \forall\xi=1,\dots,k \text{ and } \forall i=1,\dots,m$$

$$\Leftrightarrow y_i^{\xi}-\bigvee_{j=1}^{n}(w_{ij}\mathrm{O}x_j^{\xi}) = 0 \quad \forall\xi=1,\dots,k \text{ and } \forall i=1,\dots,m$$

$$\Leftrightarrow \bigwedge_{j=1}^{n}(y_i^{\xi}-(w_{ij}\mathrm{O}x_j^{\xi})) = 0 \quad \forall\xi=1,\dots,k \text{ and } \forall i=1,\dots,m$$

$$\Leftrightarrow \bigwedge_{j=1}^{n}(y_i^{\xi}\Theta x_j^{\xi}-w_{ij}) = 0 \quad \forall\xi=1,\dots,k \text{ and } \forall i=1,\dots,m$$

$$\Leftrightarrow \bigwedge_{j=1}^{n}([\mathbf{y}^{\xi}\overset{\Theta}{\vee}(\mathbf{x}^{\xi})']-\mathbf{W}_{\mathrm{XY}})_{ij} = 0 \quad \forall\xi=1,\dots,k \text{ and } \forall i=1,\dots,m$$

This last set of equations is true if and only if for each ξ=1,…,k and each integer i =1,…,m, each column entry of the *i*th row of $[\mathbf{y}^{\xi}\overset{\Theta}{\vee}(\mathbf{x}^{\xi})']-\mathbf{W}_{\mathrm{XY}}$ contains at least one zero entry. □

We need to note that the conditions the equation set given above holds are different for MAM and for FMAM or EFMAM. For MAM, it holds in $U=R$; for FMAM or EFMAM, it holds in $U=R_+$.

5 Discussions

What are the advantages of the unified framework of morphological associative memories? We think that there are at least three benefits in it:

Firstly, the unified theoretical framework is beneficial to understanding the MAM, FMAM and EFMAM. This paper analyzes the common properties of MAM, FMAM and EFMAM, and establishes the theoretical framework of unified morphological associative memory (UMAM) by extracting these common properties. The more the thing is abstracted, the deeper the thing is understood. Therefore the UMAM is of great benefit to us in research and applications with respect to MAM, FMAM and EFMAM.

Secondly, the UMAM can help us find some new methods. In fact, the method of defining the morphological memory $\mathbf{W}_{\mathrm{XY}}$ or $\mathbf{M}_{\mathrm{XY}}$ in MAM, FMAM or EFMAM is not unique. For example, according to (15) and (16), the $\mathbf{W}_{XY}$ and $\mathbf{M}_{XY}$ also can be defined by:

$$ {}^{1}\mathbf{W}_{XY}=\wedge_{\xi=1}^{K}(\mathbf{y}^{\xi}\overset{+}{\wedge}(\mathbf{x}^{\xi})') \text{ or } {}^{2}\mathbf{W}_{XY}=\wedge_{\xi=1}^{K}(\mathbf{y}^{\xi}\overset{\cdot}{\wedge}(\mathbf{x}^{\xi})') \tag{21} $$

And

$$ {}^{1}\mathbf{M}_{XY}=\vee_{\xi=1}^{K}(\mathbf{y}^{\xi}\overset{+}{\vee}(\mathbf{x}^{\xi})') \text{ or } {}^{2}\mathbf{M}_{XY}=\vee_{\xi=1}^{K}(\mathbf{y}^{\xi}\overset{\cdot}{\vee}(\mathbf{x}^{\xi})') \tag{22} $$

Consequently, there are more methods defining the memories $\mathbf{W}_{XY}$ and $\mathbf{M}_{XY}$ in the UMAM.

Finally, the methods in the UMAM are complementary rather than competitive. For this reason, it is frequently advantageous to use these methods in combination rather than exclusively.

6 Experiments

A number of experiments are conducted to demonstrate the advantages of the methods in UMAM. Three typical of experiments are as follows:

Experiment 1. Let

$$ \mathbf{x}^{1}=\begin{pmatrix}0\\0\\0\end{pmatrix},\ \mathbf{y}^{1}=\begin{pmatrix}0\\1\\0\end{pmatrix};\ \mathbf{x}^{2}=\begin{pmatrix}0\\-2\\-4\end{pmatrix},\ \mathbf{y}^{2}=\begin{pmatrix}-1\\-1\\0\end{pmatrix};\ \mathbf{x}^{3}=\begin{pmatrix}0\\-3\\0\end{pmatrix},\ \mathbf{y}^{3}=\begin{pmatrix}0\\-2\\0\end{pmatrix} $$

Then both

$$ {}^{1}\mathbf{W}_{XY}=\bigwedge_{\xi=1}^{3}(\mathbf{y}^{\xi}\overset{-}{\wedge}(\mathbf{x}^{\xi})')=\begin{pmatrix}0&0&0\\1&1&1\\0&0&0\end{pmatrix}\wedge\begin{pmatrix}-1&1&3\\-1&1&3\\0&2&4\end{pmatrix}\wedge\begin{pmatrix}0&3&0\\-2&1&-2\\0&3&0\end{pmatrix}=\begin{pmatrix}-1&0&0\\-2&1&-2\\0&0&0\end{pmatrix} $$

and

$$ {}^{1}\mathbf{M}_{XY}=\bigvee_{\xi=1}^{3}(\mathbf{y}^{\xi}\overset{-}{\vee}(\mathbf{x}^{\xi})')=\begin{pmatrix}0&0&0\\1&1&1\\0&0&0\end{pmatrix}\vee\begin{pmatrix}-1&1&3\\-1&1&3\\0&2&4\end{pmatrix}\vee\begin{pmatrix}0&3&0\\-2&1&-2\\0&3&0\end{pmatrix}=\begin{pmatrix}0&3&3\\1&1&3\\0&3&4\end{pmatrix} $$

are perfect recall memories, because they satisfy the definition 6, respectively. But both

$$ {}^{2}\mathbf{W}_{XY}=\bigwedge_{\xi=1}^{3}(\mathbf{y}^{\xi}\overset{+}{\wedge}(\mathbf{x}^{\xi})')=\begin{pmatrix}0&0&0\\1&1&1\\0&0&0\end{pmatrix}\wedge\begin{pmatrix}-1&-3&-5\\-1&-3&-5\\0&-2&-4\end{pmatrix}\wedge\begin{pmatrix}0&-3&0\\-2&-5&-2\\0&-3&0\end{pmatrix}=\begin{pmatrix}-1&-3&-5\\-2&-5&-5\\0&-3&-4\end{pmatrix} $$

and

$$ {}^{2}\mathbf{M}_{XY}=\bigvee_{\xi=1}^{3}(\mathbf{y}^{\xi}\overset{+}{\vee}(\mathbf{x}^{\xi})')=\begin{pmatrix}0&0&0\\1&1&1\\0&0&0\end{pmatrix}\vee\begin{pmatrix}-1&-3&-5\\-1&-3&-5\\0&-2&-4\end{pmatrix}\vee\begin{pmatrix}0&-3&0\\-2&-5&-2\\0&-3&0\end{pmatrix}=\begin{pmatrix}0&0&0\\1&1&1\\0&0&0\end{pmatrix} $$

are not.

Experiment 2. Set

$$\mathbf{x}^1=\begin{pmatrix}1\\2\\3\end{pmatrix},\ \mathbf{y}^1=\begin{pmatrix}1\\0\\0\end{pmatrix};\ \mathbf{x}^2=\begin{pmatrix}2\\3\\4\end{pmatrix},\ \mathbf{y}^2=\begin{pmatrix}0\\-1\\-1\end{pmatrix};\ \mathbf{x}^3=\begin{pmatrix}3\\4\\1\end{pmatrix},\ \mathbf{y}^3=\begin{pmatrix}0\\1\\0\end{pmatrix}$$

If the Ritter's method is used, then both

$$^1\mathbf{W}_{XY}=\bigwedge_{\xi=1}^{3}(\mathbf{y}^\xi \overset{-}{\wedge} (\mathbf{x}^\xi)')=\begin{pmatrix}0&-1&-2\\-1&-2&-3\\-1&-2&-3\end{pmatrix}\wedge\begin{pmatrix}-2&-3&-4\\-3&-4&-5\\-3&-4&-5\end{pmatrix}\wedge\begin{pmatrix}-3&-4&-1\\-2&-3&0\\-3&-4&-1\end{pmatrix}=\begin{pmatrix}-3&-4&-4\\-3&-4&-5\\-3&-4&-5\end{pmatrix}$$

and

$$^1\mathbf{M}_{XY}=\bigvee_{\xi=1}^{3}(\mathbf{y}^\xi \overset{-}{\vee} (\mathbf{x}^\xi)')=\begin{pmatrix}0&-1&-2\\-1&-2&-3\\-1&-2&-3\end{pmatrix}\vee\begin{pmatrix}-2&-3&-4\\-3&-4&-5\\-3&-4&-5\end{pmatrix}\vee\begin{pmatrix}-3&-4&-1\\-2&-3&0\\-3&-4&-1\end{pmatrix}=\begin{pmatrix}0&-1&-1\\-1&-2&0\\-1&-2&-1\end{pmatrix}$$

are not perfect recall memories. But if the method in UMAM is used, then both

$$^2\mathbf{W}_{XY}=\bigwedge_{\xi=1}^{3}(\mathbf{y}^\xi \overset{+}{\wedge} (\mathbf{x}^\xi)')=\begin{pmatrix}2&3&4\\1&2&3\\1&2&3\end{pmatrix}\wedge\begin{pmatrix}2&3&4\\1&2&3\\1&2&3\end{pmatrix}\wedge\begin{pmatrix}3&4&1\\4&5&2\\3&4&1\end{pmatrix}=\begin{pmatrix}2&3&1\\1&2&2\\1&2&1\end{pmatrix}$$

and

$$^2\mathbf{M}_{XY}=\bigvee_{\xi=1}^{3}(\mathbf{y}^\xi \overset{+}{\vee} (\mathbf{x}^\xi)')=\begin{pmatrix}2&3&4\\1&2&3\\1&2&3\end{pmatrix}\vee\begin{pmatrix}2&3&4\\1&2&3\\1&2&3\end{pmatrix}\vee\begin{pmatrix}3&4&1\\4&5&2\\3&4&1\end{pmatrix}=\begin{pmatrix}3&4&4\\4&5&3\\3&4&3\end{pmatrix}$$

are perfect recall memories.

Experiment 3. Let

$$\mathbf{X}=\begin{pmatrix}1&2&2\\4&2&2\\4&2&4\end{pmatrix},\ \ \mathbf{Y}=\begin{pmatrix}1&2&1\\2&2&2\\1&1&1\end{pmatrix}.$$

If the Ritter's method in MAM is used, then

$$^1\mathbf{W}_{XY}=\mathbf{Y}\overset{-}{\wedge}\mathbf{X}'=\begin{pmatrix}-1&-3&-3\\0&-2&-2\\-1&-3&-3\end{pmatrix},\ \ ^1\mathbf{W}_{XY}\overset{+}{\vee}\mathbf{X}=\begin{pmatrix}1&1&1\\2&2&2\\1&1&1\end{pmatrix}\neq\mathbf{Y};$$

$$^1\mathbf{M}_{XY}=\mathbf{Y}\overset{-}{\vee}\mathbf{X}'=\begin{pmatrix}0&0&0\\1&0&0\\0&-1&-1\end{pmatrix},\ \ ^1\mathbf{M}_{XY}\overset{+}{\wedge}\mathbf{X}=\begin{pmatrix}1&2&2\\2&2&2\\1&1&1\end{pmatrix}\neq\mathbf{Y}.$$

If the method in FMAM or EFMAM is used, then

$$ {}^{2}\mathbf{W}_{XY} = \mathbf{Y} \overset{/}{\wedge} \mathbf{X}' = \begin{pmatrix} 0.5 & 0.25 & 0.25 \\ 1 & 0.5 & 0.5 \\ 0.5 & 0.25 & 0.25 \end{pmatrix}, \quad {}^{2}\mathbf{W}_{XY} \overset{\cdot}{\vee} \mathbf{X} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \neq \mathbf{Y}; $$

$$ {}^{2}\mathbf{M}_{XY} = \mathbf{Y} \overset{/}{\vee} \mathbf{X}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 0.5 & 0.5 \end{pmatrix}, \quad {}^{2}\mathbf{M}_{XY} \overset{\cdot}{\wedge} \mathbf{X} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \neq \mathbf{Y}. $$

They make not perfect memory for (**X**, **Y**). But if the method in UMAM is used, then

$$ {}^{3}\mathbf{W}_{XY} = \mathbf{Y} \overset{+}{\wedge} \mathbf{X}' = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 4 \\ 2 & 3 & 3 \end{pmatrix}, \quad {}^{3}\mathbf{W}_{XY} \overset{-}{\vee} \mathbf{X} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \mathbf{Y}; $$

$$ {}^{3}\mathbf{M}_{XY} = \mathbf{Y} \overset{+}{\vee} \mathbf{X}' = \begin{pmatrix} 4 & 5 & 5 \\ 4 & 6 & 6 \\ 3 & 5 & 5 \end{pmatrix}, \quad {}^{3}\mathbf{M}_{XY} \overset{-}{\wedge} \mathbf{X} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \mathbf{Y}. $$

$$ {}^{4}\mathbf{W}_{XY} = \mathbf{Y} \overset{\cdot}{\wedge} \mathbf{X}' = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 4 \\ 1 & 2 & 2 \end{pmatrix}, \quad {}^{4}\mathbf{W}_{XY} \overset{/}{\vee} \mathbf{X} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \mathbf{Y}; $$

$$ {}^{4}\mathbf{M}_{XY} = \mathbf{Y} \overset{\cdot}{\vee} \mathbf{X}' = \begin{pmatrix} 4 & 4 & 4 \\ 4 & 8 & 8 \\ 2 & 4 & 4 \end{pmatrix}, \quad {}^{4}\mathbf{M}_{XY} \overset{/}{\wedge} \mathbf{X} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \mathbf{Y}. $$

The three experiments given above show that the methods in UMAM are complementary, and therefore the UMAM can solve more associative memory problems, especially to hetero-MAM, hetero-FMAM and hetero-EFMAM.

7 Conclusions

This paper introduces a new unified theoretical framework of neural-network computing based on lattice algebra. The main emphasis of this paper was on the unification of morphological associative memories, fuzzy morphological associative memories, and enhanced fuzzy morphological associative memories. Our research and experiments showed that the MAM, FMAM and EFMAM could be unified in the same theoretical framework. The significance of the unified framework consisted in: on the one hand we got a better and deeper understanding of the MAM, FMAM and EFMAM from the unified framework UMAM; on the other hand we obtained some new methods from it. Therefore the UMAM can solve more problems of the associative memories than the MAM, FMAM, and EFMAM do.

The lattice algebraic approach to neural-network theory is new and a multitude of open questions await exploration. For example, new methods of morphological associative memory need further investigation; the application base of the unified framework needs expanding, etc. It is our hope that these problems will be better solved in the future.

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