Learning the Nonlinear Dynamics of Cyberlearning

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Summary. The aim of this paper is to present a method to model the dynamics of a distance interactive learning process, so that the navigation through a course is achieved by optimal control techniques and the learner becomes proficient in the material in the least time possible, given his attitude, learning style, basic knowledge and propensities for study. A nonlinear dynamical system representation of the learning process, permits to enact optimal adaptive controls of the process at the general level, by defining a system which is controllable, observable and reachable in a technical sense, determined through a simultaneous estimation and optimization algorithm, which assures a correct representation.

1 Introduction

Learning is a dynamic process dependent on the quality of the teaching , the method and the instruments available. In Ancient Times, teaching was principally argumentative, in the form of a discussion, known as the Socratic method . As writing instruments were costly and not easily accessible, learning relied on memorization and teaching on Oratory . The same method lasted well into the era of printing and cheap paper (for instance, it is not known whether Galileo Galilei used the Socratic style in his major works to make his presentation didactic or to offend Pope Bonifacius VIII [11]). Eventually, the process was integrated with formal lectures and a blackboard. Many innovations have been tried, such as the Harvard case method, overhead projectors, teaching machines and projective teaching methods. In the ensuing discussion, none of the alternatives have been shown to be superior [1], because essentially, the underlying dynamic learning process is unobservable, in a technical sense, (given the history of the process, one can not determine the initial state of the process).

The process of e-Learning, that is interactive distance learning will depend on the quality of the teaching process, the method and the instruments used [21]. As such it defines a dynamic process which is affected by synergy [4] .

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Evidence suggests that learning is highly nonlinear with substantial lags and the learning dynamics form irreversible processes. Thus nonlinear dynamical modelling seems to be an appropriate representation of e-Learning knowledge acquisition process, since a linear dynamical system would be a too simple process [5].

A number of difficulties preclude identification and control by traditional methods for general nonlinear systems [12], so to avoid biases and suboptimization a simultaneous estimation and optimization method must be applied [9] [10].

The aim of this paper is to formulate a nonlinear dynamic system implementation of distance interactive learning through computer techniques (e-Learning), such that the underlying process is observable, identifiable, controllable and reachable, in a well defined technical sense. Further the system must satisfy the statistical properties for a maximum likelihood estimate. Adaptive optimal control trajectories will be formulated to guide the succession of learning frames to be studied.

The outline of the paper is the following. In the next section, the experimental set up to define an e-Learning optimal control process is described and the various aspects are examined. In the third section, the mathematical dynamic learning system is formalized and relevant mathematical properties indicated. In section four the implementation aspects of two cybercourses are discussed present on the World-Wide Web.

2 Knowledge Acquisition by e-Learning Dynamics

Distance Education is the delivery of education courses from one location to students at other locations [14], while Cyberschools are the institutions that deliver these courses, Cyberlearning is the process by which content is designed, transmitted and acquired.

The effect of Distance Education, depends on the content of what is taught and method by which it is imparted. Students have differing learning styles to acquire Knowledge, which depend on [1]:

- the method of exposition [13] :
  - a formal axiomatic or deductive exposition of the material,
  - an informal presentation, followed by its formalization,
  - an intuitive and illustrative development of the material,
- the structure of the presentation [21]:
  - a linear structure, as in a book,
  - a guided tree structure with a limited capacity of selection,
  - an adaptive feedback mechanism in a tree like decision network.
- the interaction policy envisaged [6]:
  - no interaction allowed,
  - periodical question periods,

– full interaction.

Only with an adaptive feedback structure of the method of exposition can the content be varied and adapted in real time according to the needs of the student, so here, an adaptive dynamic interactive course, based on browser technology will be examined. This type of structure will be termed a cybercourse and the learning process will be indicated as the Cyberlearning process.

The learner can start in his own time at the initial page and navigate through the course on the web, add his own notes to the material presented, do the exercises, carry out the assignments and contact by e-mail other students, the instructor or Intelligent Agents, which are just pieces of software to analyze the output of the student, verify his progress while navigating, etc. Other intelligent Agents will monitor his performance and generally supervise on the work that is being done.

The system will monitor all his actions and advise on the next actions to be taken. The learner is of course absolutely free to chose other actions. The indication of the actions to take are based on the development by the system of an optimal trajectory to the completion of the course, based on the monitored proficiency shown. After any deviation on the part of the learner for whatever reason, the system recalculates the new optimal control path to completion based on the new position reached and capabilities shown by the learner.

A cybercourse will consist of a set of units each composed of a set of frames. Each unit is composed of multiple sets of similar frames to handle different methods of exposition, while the structure of the presentation and the interaction policy adopted will be handled by the underlying nonlinear dynamical process, based on the requirements of the student.

The utilization of a client computer, or a browser, allows the system server to record for each individual connection, the succession of frames traversed, the actions that have been performed on the client system and the length of time involved. Thus, at the server, all actions performed by the student can be monitored [6]. Through the state space formulation of the dynamic representation of the learning process, the input and output sequence will result well defined, so that the optimal control trajectory can be formulated.

3 The Mathematical Algorithm

The representation of phenomena by dynamic systems is more general than their representation by a static system, since the latter will always constitute a special case of the former.

Modelling a phenomenon by a dynamic system means imposing the structure of the phenomenon on the system variables. It also means that the dated variables, which represent the phenomenon, must agree with the dated estimated values of the system and the mathematical properties of the system variables must apply to the variables of the phenomenon. In dynamic modelling, this requirement is not just that the variables in the two systems be defined compatibly (both integer variables etc.), but their compatibility must extend to their dynamic structure . In short, both must posses compatible properties in their controllability, observability and stability: important properties which will be examined. If a stable phenomenon is modelled by an unstable system, the realisation may agree over certain limited intervals, but it is bound to diverge. Properties of the latter cannot be used to represent the former, since they are different.

The functional form of the dynamic system must be identified and the relevant parameters estimated. Since the system is nonlinear the value of the parameters will depend on the point considered. This precludes the application of standard system identification techniques for dynamic systems [22] and requires an algorithm that will solve simultaneously the estimation and the optimal control problem, the former in the space of parameters, the latter in the space of the decision variables.

To this end, the aim of this section is to present such an algorithm. First the dynamic system will be characterized, then it is examined how to ensure that the correct statistical properties of the estimates be obtained by solving an optimization problem which will also determine the optimal control strategy. Finally the solution of the optimization problem is discussed.

3.1 The Dynamical System Formulation

Mathematical System Theory deals essentially with the study of the dynamical relationships of systems under various conditions. A Dynamical System is a precise mathematical object [15]. Not every relationship can be modelled by mathematical system theory, since a representation which is non anticipatory is required [15].

Dynamical Systems have been defined at a high level of generality, to refine concepts and perceive unity in a diversity of applications and by appropriate modelling, whole hierarchies of phenomena can be represented as systems defined at different levels.

Definition 1. *[15]: A Dynamical System is a composite mathematical object defined by the following axioms:*

1. *There is a given time set T, a state set X, a set of input values U, a set of acceptable input functions $\Omega = \omega : \Omega \rightarrow U$, a set of output values Y and a set of output functions $\Gamma = \gamma : \Gamma \rightarrow Y$.*
2. *(Direction of time). T is an ordered subset of the reals.*
3. *The input space Ω satisfies the following conditions.*
   a) *(Nontriviality). Ω is nonempty.*

   b) *(Concatenation of inputs) An input segment* $\omega_{(t_1,t_2]}$, $\omega \in \Omega$ *restricted to* $(t_1, t_2] \cap T$. *If* $\omega, \omega' \in \Omega$ *and* $t_1 < t_2 < t_3$ *there is an* $\omega'' \in \Omega$ *such that* $\omega''_{(t_1,t_2]} = \omega_{(t_1,t_2]}$ *and* $\omega''_{(t_2,t_3]} = \omega'_{(t_2,t_3]}$.
4. *There is a state transition function* $\varphi : T \times T \times X \times \Omega \to X$ *whose value is the state* $x(t) = \varphi(t; \tau, x, \omega) \in X$ *resulting at time* $t \in T$ *from the initial state* $x = x(\tau) \in X$ *at the initial time* $\tau \in T$ *under the action of the input* $\omega \in \Omega$. φ *has the following properties:*
   a) *(Direction of time).* φ *is defined for all* $t \geq \tau$, *but not necessarily for all* $t < \tau$.
   b) *(Consistency).* $\varphi(t; t, x, \omega) = x$ *for all* $t \in T$, *all* $x \in X$ *and all* $\omega \in \Omega$.
   c) *(Composition property). For any* $t_1 < t_2 < t_3$ *there results:*

$$\varphi(t_3; t_1, x, \omega) = \varphi(t_3; t_2, \varphi(t_2; t_1, x, \omega), \omega)$$

   for all $x \in X$ *and all* $\omega \in \Omega$.
   d) *(Causality). If* $\omega, \omega' \in \Omega$ *and* $\omega_{(\tau,t]} = \omega'_{(\tau,t]}$ *then* $\varphi(t; \tau, x, \omega) = \varphi(t; \tau, x, \omega')$.
5. *There is a given readout map* $\eta : T \times X \to Y$ *which defines the output* $y(t) = \eta(t, x(t))$. *The map* $(\tau, t] \to Y$ *given by* $\sigma \mapsto \eta(\sigma, \varphi(\sigma, \tau, x, \omega))$, $\sigma \in (\tau, t]$, *is an output segment, that is the restriction* $\gamma_{(\tau,t]}$ *of some* $\gamma \in \Gamma$ *to* $(\tau, t]$.

The following mathematical structures in definition 1 will be indicated by:

- the pair $(t, x), t \in T, x \in X \quad \forall t$ is called an event,
- the state transition function $\varphi(x_t, u_t)$ is called a trajectory.

Phenomena can be modelled by a dynamical systems in the input/output sense.

Definition 2. *A Dynamical System in an input/output sense is a composite mathematical object defined as follows:*

1. *There are given sets* T, U, Ω, Y *and* Γ *satisfying all the properties required by definition 1*
2. *There is a set* A *indexing a family of functions*

$$\mathcal{F} = \{f_\alpha : T \times \Omega \to Y, \alpha \in A\}$$

   each member of $\mathcal{F}$ *is written explicitly as* $f_\alpha(t, \omega) = y(t)$ *which is the output resulting at time* t *from the input* ω *under the experiment* α. *Each* f_α *is called an input/output function and has the following properties:*
   a) *(Direction of time). There is a map* $\iota : A \to T$ *such that* $f_\alpha(t, \omega)$ *is defined for all* $t \geq \iota(\alpha)$.
   b) *(Casuality) Let* $\tau, t \in T$ *and* $\tau < t$ *If* $\omega, \omega' \in \Omega$ *and* $\omega_{(\tau,t]} = \omega'_{(\tau,t]}$, *then* $f_\alpha(t, \omega) = f_\alpha(t, \omega')$ *for all* α *such that* $\tau = \iota(\alpha)$.

While the input/output approach may determine a family of functions, the state space approach represents the trajectories in the way indicated, through a unique function, so the latter approach is intuitively more appealing, especially in applications. However, both representations show the relationships of the time series of the single inputs on the state and the outputs. The first representation defines a unique mapping, while the second representation, restricted to a subspace, does not.

The representations are equivalent. It is easy to transform a given system from a state space formulation to a input/output formulation and vice versa [2] [15], so each may be used as convenience suggests.

A sufficiently general representation of a dynamic system may be formulated by applying definition 1, recalling the equivalence of an input-output system and a system in state form:

$$x_{t+1} = \varphi(x_t, u_t) \tag{1}$$

$$y_t = \eta(x_t) \tag{2}$$

where $x_t \in X \subseteq R^r$ may simply be taken as a r-dimensional vector in an Euclidean space X, indicating the state of the system at time t, $u_t \in U \subseteq R^q$ may be taken as a q-dimensional vector in an Euclidean subspace U of control variables and $y_t \in Y \subseteq R^p$ is a p-dimensional vector in an Euclidean space Y of output variables, in line with the definitions 1, 2.

The definition of a dynamical system is based on an intermediary set of states and a transition function or a family of functions. Neither of these constructions are unique, so if it is desired to represent a system by such structures, equivalence of the possible structures must be shown.

Definition 3. *Given two states x_{t_0} and $\hat{x}_{t_0}$ belonging to systems S and $\hat{S}$ which may not be identical, but have a common input space Ω and output space Y, the two states are said to be equivalent if and only if for all input segments $\omega_{[t_0,t)} \in \Omega$ the response segment of S starting in state x_{t_0} is identical with the response segment of $\hat{S}$ starting in state $\hat{x}_{t_0}$; that is*

$$x_{t_0} \cong \hat{x}_{t_0} \Leftrightarrow \eta(t, \varphi(x_{t_0}, \omega_{[t_0,t)})) = \hat{\eta}(t, \hat{\varphi}(\hat{x}_{t_0}, \omega_{[t_0,t)})) \\ \forall t \in T, t_0 \leq t, \forall \omega_{[t_0,t)} \in S, \hat{S} \tag{3}$$

Definition 4. *A system is in reduced form if there are no distinct states in its state space which are equivalent to each other.*

Definition 5. *Systems S and $\hat{S}$ are equivalent $S \equiv \hat{S}$ if and only if to every state in the state space of S there corresponds an equivalent state in the state space of $\hat{S}$ and vice versa.*

A number of important questions must be asked of the system description of the cyberlearning representation:

- can a certain state $x^* \in S$ be reached from the present state, or if the dynamical system attains a given state x_0 at time 0 can it also be made to reach a certain state x^*. Evidently it is required to determine the set of states reachable from a specific state x_t.
- can a dynamical system be driven to a given state by an input u. Thus controllability is concerned with the connectedness properties of the system representation.
- Reachability and controllability lead naturally to the determination of a dynamical system's observability, which provides the conditions to determine the given actual state uniquely.
- The stability of the system is important since it provides conditions on the way the trajectories will evolve, given a perturbation or an admissible control.

These conditions are very important, since they allow trajectories to be defined, the initial point of trajectories to be determined and their stability properties to be derived. Moreover they can be applied at any moment in time to determine if the goals of the cyberlearning system are still attainable.

Definition 6. *Given a state $x^* \in M \subseteq X$, it is reachable from the event (t_0, x_0) at time T if there exists a bounded measurable input $u_t \in \Omega$ such that the trajectory of the system satisfies:*

$$x_{t_0} = x_0 \tag{4}$$
$$x_T = x^* \qquad \forall x_{t_0} \in M, \quad 0 \leq t \leq T \tag{5}$$

The sets of states reachable from x_{t_0} is denoted by:

$$\Re(x_{t_0}) = \bigcup_{0 \leq T \leq \infty} \{x_T | x_T \quad reachable \quad at \quad time \quad T\} \tag{6}$$

the system is reachable at x_{t_0} if $\Re(x_{t_0}) = M$ and reachable if $\Re(x_{t_0}) = M \quad \forall x \in M$.

Definition 7. *A system is locally reachable at x_{t_0} if for every neighbourhood $N(x_{t_0}, h)$ of x_{t_0}, $\Re(x_{t_0}) \cap N_{x_0}$ is also a neighbourhood of x_{t_0} with the trajectory from the event (t_0, x_{t_0}) to $\Re(x_{t_0}) \cap N_{x_0}$ lying entirely within N_{x_0}. The system is locally reachable if it is locally reachable for each $x \in M$.*

These definitions lead to an important property for many systems, namely that reachability may not be symmetric, that is: if x_T is reachable from x_{t_0} the converse may not hold. Thus a weaker notion of reachability is opportune.

Definition 8. *Two states x^* and $\hat{x}$ are weakly reachable from each other if and only if there exist states $x^0, x^1, ..., x^k \in M$ such that $x^0 = x^*, x^k = \hat{x}$ and either x^i is reachable from x^{i-1} or x^{i-1} is reachable from x^i $(\forall 1 = 1, 2, ..., k)$. The system is weakly reachable if it is weakly reachable from every $x \in M$.*

Theorem 1. *The following implications apply:*

- *If the system is locally reachable then it is reachable,*
- *if the system is reachable then it is weakly reachable,*

Proof: Immediate from the definitions.

Definition 9. *State x_{t_0} of a system is controllable if and only if there exists a $u \in \Omega$ such that:*

$$\varphi(t; t_0, x_{t_0}, u) = \emptyset \tag{7}$$

The system is said to be controllable if and only if every state of the system is controllable.

Theorem 2. *A system which is controllable and in which every state is reachable from the zero state ($\emptyset$) is strongly connected*

Proof: Follows from definition 9 and 6, see [15].

Definition 10. *Simple and Multiple experiments:*

- *A simple experiment is an input/output pair $(u_{[t_0,t)}, y_{[t_0,t)})$ that is, given the system in an unknown state an input $u_{[t_0,t)}$ is applied over the interval of time (t, t_0) and the output $y_{[t_0,t)}$ is observed.*
- *A multiple experiment of size N consists of N input/output pairs $(u^i_{[t_0,t)}, y^i_{[t_0,t)} \quad i = 1, 2, ..., N$ where on applying on the i-th realization of the N systems the input $(u^i_{[t_0,t)})$ the i-th output $y^i_{[t_0,t)}$ is observed.*

Definition 11. *A system is simply (multiply) observable at state x_{t_0} if and only if a simple experiment (a multiple experiment) permits the determination of that state uniquely.*

Definition 12. *Equivalence of Systems:*

- *Two systems are simply equivalent if it is impossible to distinguish them by any simple experiment,*
- *Two systems are multiply equivalent if it is impossible to distinguish them by any multiple experiment.*

Theorem 3. *If two systems are simply equivalent and strongly connected, then they are multiply equivalent.*

Theorem 4. *If two systems are multiply equivalent then they are equivalent, (definition 5).*

Definition 13. *A system is initial-state determinable if the initial state x_0 can be determined from an experiment on the system started at x_0.*

Theorem 5. *A system is in reduced form if and only if it is initial-state determinable by an infinite multiple experiment.*

The definitions 10 - 13 and the theorems 3 - 5 formally justify the possibility of defining one or more representations of the dynamical system considered at a chosen level of detail. Notice however the distinction between systems that are simply equivalent and multiply equivalent. This distinction is crucial, if dynamical systems are considered, while with comparative static models, the distinction does not apply and the latter are consequently limited.

It is usual, since stability for nonlinear systems is an equilibrium concept, to examine autonomous systems in continuous time.

Thus consider:

$$\dot{x} = \varphi(x, t) \tag{8}$$

$$x(t_0) = x_0 \tag{9}$$

an autonomous nonlinear system, while x_0 is the initial state of the system.

Definition 14. *The equilibrium point $x = 0$ is called a stable equilibrium point of the system (8) if for all t_0, $\epsilon > 0$, there exists $\delta(t_0, \epsilon)$ such that:*

$$\mid x_0 \mid < \delta(t_0, \epsilon) \Rightarrow \mid x(t) \mid < \epsilon \quad \forall t \geq t_0 \tag{10}$$

The solution of the dynamic system given in equations (1) - (2) may be determined in a number of different ways, depending on the structure of the functions that are given [15].

3.2 Simultaneous Estimation and Optimization

A given finite dimensional estimation and optimization problem is considered, which is nonlinear and dynamic to determine simultaneously the maximum likelihood parameter estimates and the optimal control trajectory to the dynamic system.

It is important to apply a suitable data driven statistical method to determine the most appropriate statistical form and precise values of the parameters, which should have the following properties [16]:

1. the parameter estimates are unbiased, this means that:
   - as the size of the data set grows larger, the estimated parameters tend to their true values,
2. the parameter estimates are consistent, which require the following conditions to be satisfied:
   - the estimated parameters are asymptotically unbiased,
   - the variance of the parameter estimate must tend to zero as the data set tends to infinity.
3. the parameter estimates are asymptotically efficient,

- the estimated parameters are consistent,
- the estimated parameters have smaller asymptotic variance as compared to any other consistent estimator,

4. the residuals have minimum variance, which will require to ensure that this is so::
   - the variance of the residuals must be minimum,
   - the residuals must be homoscedastic,
   - the residuals must not be serially correlated.
5. the residuals are unbiased (have zero mean),
6. the residuals have a noninformative distribution (usually, the Gaussian distribution). If the distribution of the residuals is informative, the extra information can be used to reduce the variance of the residuals to yield better estimates.

In short, through correct implementation of statistical estimation techniques the estimates are as close as possible to their true values, all the information that is available is applied and the uncertainty surrounding the estimates and the data fit is reduced to the maximum extent possible. Thus the estimates of the parameters, which satisfy all these conditions, are the 'best' possible in a 'technical sense' [16].

By setting up the statistical properties, that a given estimate must fulfil, as constraints to the maximum likelihood problem to be solved, the parameters are defined implicitly by this optimization problem. The latter can be inserted into the optimal control system for the policy determination, so that statistically correct estimates will always result. The solution yielding the best policy can be chosen, where $N+1, ..., \mathcal{T}$ is the forecast period, by solving the optimization problem given below. By recursing on the specifications, better and better fits can be derived. At each iteration, the best combination of parameterization and policy is obtained.

The unknowns to be determined are the input and output variables considered and the parameters of the functional form specified in the current iteration.

The mathematical program is formulated with respect to the residual variables, but it is immediate that for a given functional form, the unknown parameters will be specified and thus the unknowns of the problem will also be defined and available. Hence the mathematical program is fully specified for each functional form to be considered.

Consider the data set of a phenomenon consisting of measurements (y_i, x_i, u_i) over $(i = 1, 2, .., N)$ periods, where it is assumed, that $y_i \in R^p$ is a p-dimensional vector, while $x_i \in R^r$ is a r-dimensional vector of explanatory or state variables of the dynamic process of dimension. Also, u_i is a q-dimensional vectors of control variables. Let $w_i \in R^r$, $v_i \in R^p$ be stochastic processes also to be determined.

The optimization problem to be solved is the following:

$$\begin{aligned}
Min \quad J &= \sum_{i=N+1}^{\mathcal{T}} c(x_i, u_i, y_i) && (11)\\
\varphi(x_i, u_i, y_i, w_i : \theta_1) &= x_{i+1} && (12)\\
\eta(x_i, u_i, v_i : \theta_2) &= y_{i+1} && (13)\\
\frac{1}{N}\sum_{i=1}^{N} w_i &= 0 && (14)\\
\frac{1}{N}\sum_{i=1}^{N} v_i &= 0 && (15)\\
\frac{1}{N}\sum_{i=1}^{N} w_i^2 &\leq k_w && (16)\\
\frac{1}{N}\sum_{i=1}^{N} v_i^2 &\leq k_v && (17)\\
-\epsilon_0 \leq \frac{1}{N}\sum_{i=1}^{N} v_i w_i &\leq \epsilon_0 && (18)\\
-\epsilon_1 \leq \frac{1}{N}\sum_{i=1}^{N} w_i w_{i-1} &\leq \epsilon_1 && (19)\\
-\epsilon_2 \leq \frac{1}{N}\sum_{i=1}^{N} v_i v_{i-1} &\leq \epsilon_2 && (20)\\
-\epsilon_3 \leq \frac{1}{N}\sum_{i=1}^{N} v_i w_{i-1} &\leq \epsilon_3 && (21)\\
-\epsilon_4 \leq \frac{1}{N}\sum_{i=1}^{N} w_i v_{i-1} &\leq \epsilon_4 && (22)\\
&\cdots\cdots\cdots\cdots\cdots\cdots &&\\
-\epsilon_{2s} \leq \frac{1}{N}\sum_{i=1}^{N} v_{i-s} w_{i-s} &\leq \epsilon_{2s} && (23)\\
-\epsilon_{2s+1} \leq \frac{1}{N}\sum_{i=1}^{N} w_i w_{i-s} &\leq \epsilon_{2s+1} && (24)\\
-\epsilon_{2s+2} \leq \frac{1}{N}\sum_{i=1}^{N} v_i v_{i-s} &\leq \epsilon_{2s+2} && (25)\\
-\epsilon_{2s+3} \leq \frac{1}{N}\sum_{i=1}^{N} v_i w_{i-s} &\leq \epsilon_{2s+3} && (26)
\end{aligned}$$

$$-\epsilon_{2s+4} \leq \frac{1}{N}\sum_{i=1}^{N} w_i v_{i-s} \leq \epsilon_{2s+4} \quad (27)$$

$$\frac{1}{2}g_w^T \Psi(\Psi^T\Psi)^{-1}\Psi^T g_w - \frac{N}{2} \leq \chi^2_{1-\alpha:p-1} \quad (28)$$

$$\frac{1}{2}g_v^T \Psi(\Psi^T\Psi)^{-1}\Psi^T g_v - \frac{N}{2} \leq \chi^2_{1-\alpha:p-1} \quad (29)$$

$$-\epsilon_{2r+1} \leq \frac{1}{N}\sum_{i=1}^{N} w_i^{2r+1} \leq \epsilon_{2r+1}; r = 3, 4, ... \quad (30)$$

$$\frac{1}{N}\sum_{i=1}^{N} w_i^{2r} \leq \frac{2r!}{r!2^r}\sigma_w^{2r}; r = 3, 4, ... \quad (31)$$

$$-\epsilon_{2r+1} \leq \frac{1}{N}\sum_{i=1}^{N} v_i^{2r+1} \leq \epsilon_{2r+1}; r = 3, 4, ... \quad (32)$$

$$\frac{1}{N}\sum_{i=1}^{N} v_i^{2r} \leq \frac{2r!}{r!2^r}\sigma_v^{2r}; r = 3, 4, ... \quad (33)$$

$$x_i \in X, y_i \in Y, u_i \in U, w_i \in W, v_i \in V \quad (34)$$

The conditions indicated above are met for an optimal solution of the program (11)-(34). The formal proof of these properties are presented in [10]. Here we shall show the connection between the constraints and the statistical properties indicated above which must be satisfied.

The abstract model of the dynamical system is to be optimized with regard to a given merit function(11) such that the sum of squares of the residuals to be less than a critical value k_w, k_v which can be decreased by dichotomous search at every iteration, until the problem does not yield a feasible solution.

The least values obtained for these parameters, while retaining a feasible solution to the whole problem, are equivalent to a minimization of the statistical estimation error and the maximum likelihood estimate of the parameters, under appropriate distributional assumptions concerning the residuals.

All the serial correlations between the residual are not significantly different from zero, as enforced by the constraints (18) - (27).

Moreover to ensure that these conditions hold throughout the possible variation of the independent variables, the residuals must be homoscedastic and thus satisfy (28) - (29). The homoscedasticity condition on the residuals is obtained by regressing the original variables of the problem, indicated by the data matrix Ψ, on the normalised square of the residuals, which are indicated respectively by: g_w, g_v. This leads to a set of nonlinear equations in the squared residuals. The χ^2 test is applied at a confidence level of $(1-\alpha)$ with $m-1$ degrees of freedom and a significance level of α, [3].

The conditions 4 and 5 hold at the solution of the optimization problem. Conditions 2 and 1 also hold because of the following consideration.

The constraints (14) - (27) as well as (30) - (33) are sample moments, so they will converge in probability to their population values. The ones indicated by (14) - (27) will converge to zero. For the second group, those representing the odd moments of the distribution, indicated by (30) and (32), will converge in probability to their population value of zero, while the even moments will converge in probability to their population values. These constraints enforce the residuals to have a noninformative distribution, here a Gaussian.

Thus the condition 6 is also met. Condition 3, which is also very important will hold in all cases that the constrained minimization problem (11) - (34) has a solution.

Finally it is easy to show that this constrained minimization problem (11) - (34) will dominate the solutions obtainable by the traditional three phase procedure, since whenever the latter has a statistically correct solution, the new procedure will also have such a solution, but not conversely.

3.3 Solving the Optimization Problem

An iterative procedure is here specified to minimize a given function subject to equality and inequality constraints, by solving a linear complementarity problem at each iteration, subject to a suitable trust region defined by a set of inequalities [18]. The detailed procedure and convergence results have been presented in [10] to which the reader is referred.

Consider the following optimization problem:

$$Min \quad Z = f(w) \qquad f : R^n \to R \tag{35}$$

$$g(w) \geq 0 \qquad g : R^n \to R^p \tag{36}$$

$$h(w) = 0 \qquad h : R^n \to R^q \tag{37}$$

The proposed algorithm consists in defining a quadratic approximation to the objective function, a linear approximation to the constraints and determining a critical point of the approximation by solving a linear complementarity problem (LCP), as given in [18].

Expanding the functions in a Taylor series, at the given iteration point w^k, the equality constraints may be eliminated simply by converting them into $p + 1$ inequality constraints.

Unconstrained variables can be transformed into nonnegative variables for the LCP algorithm, by defining a suitable offset.

A set of trust region constraints can be imposed on the problem as a system of linear inequalities centered around the iteration point, to limit the change in the possible solution. Thus an LCP results which, it can be shown, has a solution either on a trust region constraint or inside. If the solution point occurs on a trust region constraint and the solution is feasible while a reduction in the objective function has occurred, the solution point is taken as the new starting point and a new iteration is started. Otherwise, if the new point is infeasible, the trust region is reduced. Finally if there has been an increase in the objective function, the trust region is enlarged and the iteration is repeated, with suitable safeguards to force a reduction in the objective function. If instead the solution point occurs inside the trust region, it can be considered an approximate stationary point. If the objective function is bounded from below, for all values of the variables which satisfy the constraints, a local minimum point will be found. The minimum time free end optimal control problem will therefore be solved, which specifies the optimal learning trajectory. A great advantage of this algorithm is that variables restricted to binary values can be considered and the solutions determined will respect these conditions [7].

4 Implementation of Cybercourses

A practical application of the relevant methodology, succinct enough to fit the limited space available, would constitute a toy application, which would not reveal the possibility and the advantages of using Cyberlearning, as defined here, for actual courses. Thus the aim of this section is to describe briefly two cybercourses realized with the methodology **I**nteractive **D**istance **E**lectronic and **A**daptive **L**earning **S**ystem (**I.D.E.A.L.S.**) described in the previous sections, which the interested reader can find on the web, so that they can be analyzed in detail and examine the scientific methodology which lies behind the logical and numerical constructs, which ensures the correctness.

4.1 Cybercourses

The courses develop interactive adaptive tutorials, based a common set of general structural and organizational principles.

An array containing indications of the frames successively examined, the time spent on studying each, the collateral actions performed: e-mail sent and received, library information systems consulted, and how the tasks assigned were executed is generated frame by frame. Intelligent Agents can be queried and questions addressed to the instructor levied. On the output side, the next period's predicted state vector is determined, which contains the next frame to be studied, and an output array indicating the expected time taken to study the next frame, the tasks which are expected to be tackled and the performance which will result on the that frame. All this information as well as the results are saved in a database for each individual, which will be then used to update the system.

Similar proposals have been indicated [21] [6], but in these papers the next frame is either chosen by the learner or on the basis of a weighting function of the results. There is no adaptive optimal control exercised nor are special techniques used to avoid excessive oscillations between learning styles, (see section 4.2).

An initial estimate to the individual's learning style can be obtained from the characteristics indicated and from his performance over the first few exemplary frames. Here use is made of a pattern recognition technique called **T.R.A.C.E.** (**T**otal **R**ecognition by **A**daptive **C**lassification **E**xperiments) [17], which has given good results. Also initial representative trajectories can also be defined experimentally. As the individual moves through the frames, pertinent information is obtained, which is used to improve the predictive accuracy of the system representation. A cybercourse should be equipped with a cultural dimension to promote conceptual knowledge and purposeful reflection [13]. To this end full interaction between the learner, the instructor and the Cyberlearning process is envisaged.

Thus a number of instruments are incorporated in the frames structure so as to encourage this cultural extension of the material in many directions.

Any part of a frame may be highlighted and one of five action buttons can be clicked, leading to different actions. Clicking on:

- The demonstration button: a new window is presented with a detailed proof of the highlighted material,
- The reference button: a new window appears with a set of references on the matter highlighted. To this end the **S.I.B.I.L.L.A.** (**S**istema **I**nterattivo **BI**b **L**iografico con **L**iste **A**utomatiche), which is a bibliographic referential system [19], may be used.
- The example button: a new window appears and a worked example is shown step by step.
- The apropos button: a new window appears and a set of links appear with indications to more general information.
- The history button: a new window appears and a short history of the development of the concepts is given.

Each window opened also contains the five buttons and a label for that window so that the interaction process can continue. Other command buttons are also provided, such as repeat buttons, path visualization and e-mail connectivity.

The Cyberlearning Course

Cyberlearning (http://banach.sta.uniroma1.it/ideals/cyberlearning.html) is an application of the **I.D.E.A.L.S.** methodology which is explained in the tutorial in the form of an interactive distance learning course. The interactive course provides an adaptive and interactive tutorial on how it works and how to formulate the frames at the various levels of learning style adopted. As such it consists of three units:

1. Introduction and general principles of the **I.D.E.A.L.S.** methodology,
2. The framework for the dynamic adaptive control and the justification of its correctness and adequacy.

3. the rules and the method to construct the frames.

Because of the aim of this course, some additional instruments are provided, which are not usually part of the **I.D.E.A.L.S.** methodology, which may be characterised as follows:

- backward branching: at any frame the user may return to a previous frame and re-execute it in a debugging mode. Thus he may alter in some way his interaction and compare the optimal control that had been formulated before and the new formulation of the optimal control. This will be useful to study the consequences of different decisions on the optimal path formulated by the system,
- The user can force a choice of a frame, so as to inspect the frames that would have been provided with other outcomes. In short he can vary the state of the system and examine the effects of such a modification.
- The recursive estimation of the relationships, which define the nonlinear dynamic system can be checked at each frame, so that the precise mechanism that governs the transitions studied.

Thus the interested reader can study the pedagogical reasons behind this approach, experiment with the various alternatives, test himself for a better understanding and determine the time taken to accomplish a certain task under the different learning styles. The principles governing the nonlinear dynamical system can be studied in the most suitable way, based on his prior knowledge and preferred style and through the exercises he will obtain a deep knowledge of the subject matter. Finally the principles to construct frames are specified, again under different learning styles.

The Mathematical Programming Course

A cybercourse (http://banach.sta.uniroma1.it/proma06/initial.html) is a mathematical programming course with applications in decision making, imparted to senior undergraduates.

Different learning styles characterize the material presented: from an axiomatic approach distinguished by formal analysis and results to an approach built up by examples with development of intuitive explanations to justify rules, principles and algorithms and of course the tasks to be carried out in each frame will be very different.

The course itself is structured in 18 units (chapters) from traditional mathematical programming methods to more advanced methods such as nonlinear complementarity theory and variational inequalities, as well as the techniques to handle dynamic optimization problems and simultaneous estimation and optimization problem as indicated in section 3. Special evidence is given to interior point methods and a particular emphasis is given to modeling principles and the relevant methodology, since this aspect is considered an important element in a course on mathematical programming, while less so in an optimization course where the concern is only with the solution techniques of such problems.

4.2 Correctness and Adequacy

The justification why such a complex construction will work in reality, are of course implicit in the results presented in section 3. If the derivations are correct and if the concepts can be adequately interpreted by observable measures taken from the behaviour exhibited by the learner, given an acceptable indeterminacy level, then the results obtained by the learner will coincide, modulo the acceptable indeterminacy level, with the results indicated by the nonlinear dynamic system.

To see this, consider two nonlinear dynamical systems which interact as the course proceeds:

- The individual's learning procedure can be regarded as a nonlinear dynamic system, which is not observable as indicated in definition 11, since no single or set of experiments can determine uniquely what he has learnt.
- a representation of certain aspects of his behaviour may be realized by a specification of a nonlinear dynamical system which does not consider the unobservable aspects of the individual's learning process, but just the optimal sequence of frames and learning tasks to achieve the specific desired result.

Certainly, if there were a sufficient set of identical students learning the same subject matter, a design of experiments could be set up to choose for this group the optimal sequence of frames and tasks.

This would require many copies of that individual's learning procedure to determine the best set of tasks, since each task administered alters his learning capabilities. This is not possible, since the individual is unique.

Alternatively, a nonlinear dynamical system in the input/output sense, as indicated in definition 2, could be considered and a number of experiments conducted on the individuals' learning capabilities, so as to determine which sets of tasks are optimal to achieve the desired result.

Rather than experimenting on the individual, as in the traditional design of experiments, a representative nonlinear dynamical system can be used, defined just with the required properties to determine the sequence of tasks to administer. Multiple copies of this system can be generated, if simulations are to be performed, or just a single copy is required, if the optimal sequence is to be determined by an optimal adaptive control algorithm. In each period the optimal policy is determined to completion, the selected task is fed to the learner and the results noted, so that the nonlinear dynamical system can be updated if warranted.

If the results of the section 3 are implemented correctly, the simultaneous estimation and optimization algorithm will converge to a dynamical system representation, which cannot be distinguished by simple and multiple experiments, modulo the acceptable level of indeterminacy, thus ensuring that they are equivalent within the level of acceptable indeterminacy, as derived in theorem 3 and 4.

To verify the optimality of the policy enacted it is not possible to perform experiments, again because of the uniqueness of the individual, but optimality must follow through the properties of the results if these has been derived correctly [10].

The adequacy of the representation must also be evaluated. This means that the aspects of the learning behaviour of the individual, which are measured, satisfy a number of properties which ensures that at each period an appropriate frame is indicated and that the trajectory to completion of the course is well defined.

Reachability of the Cybercourse Representation

A set of states, say M, will lead to the final frame indicating satisfactory completion of the course. This set of states must be reachable from intermediate states as otherwise the optimal control problem will not have a feasible solution.

Once the dynamic system has been identified, System techniques can be applied to ensure that the system is reachable with respect $\Re(x_{t_0}) = M \quad \forall x \in M$, see definition 6 [20]. Also by theorem 1, the system will be weakly reachable. Thus the learner will not be left in the lurch, with no indication on how to progress.

Controllability of the Cybercourse Representation

The learner starts from a given frame, so that it must be ensured that in all circumstances a new state is formulated which leads to a new frame. Thus every state should be reachable from the initial state. Obviously, if the trajectory under given circumstances goes into a loop, the system is not strongly connected and it must be altered to render it controllable. Thus for **I.D.E.A.L.S.** representation, definition 9 and theorem 2 must apply and there are Dynamic system techniques to check and ensure that this is so [20].

Observability of the Cybercourse Representation

It must be ensured that the two systems: certain aspects of the learning process of the student and their dynamic system representation are equivalent, so that the optimal policy determined for the latter applies to those aspects of the learner and so ensure an optimal sequence of frames.

For this to hold, the dynamic system and the real system must be simply observable (definition 11) so that the two systems be simply equivalent (see definition 12). If the two systems are controllable and reachable, see above, then the systems will be, by theorem 2, strongly connected. By theorem 3 the two systems will be multiple equivalent and therefore equivalent by theorem 4.

Stability of the Cybercourse Representation

Any path through the cybercourse must be stable, which means that the path must not oscillate or cycle so that the sequence of frames will converge to the final frame of the course. Opportune constraints are added to problem specification to ensure the stability of every optimal trajectory [8].

5 Conclusion

Open loop policies can be specified for every student which desires to enroll in the course. Special recursive estimation and optimization techniques ensure the identification of the nonlinear dynamic system formulation of the required aspects of the learning process and its optimal control to completion. The properties of the results derived ensure the equivalence of the two systems, so that the model can be used to determine the optimal control.

A close scrutiny of the structure of this paper makes it evident that the method of exposition applied in each section reflects differences in learning styles, so as to appeal to a wide audience. Thus from an intuitive description in section 2, a formal and axiomatic exposition is used in section 3 and an informal but structured presentation in section 4 to allow interested readers to understand the material to completion.

The **I.D.E.A.L.S.** methodology improves the given structure by interactively adapting presentations to the desires of the reader and ensuring that the material is presented as efficiently as possible.

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