Optimality Conditions and Duality for Multiobjective Programming Involving (C, α, ρ, d) type-I Functions *

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Summary. In this chapter, we present a unified formulation of generalized convex functions. Based on these concepts, sufficient optimality conditions for a nondifferentiable multiobjective programming problem are presented. We also introduce a general Mond-Weir type dual problem of the problem and establish weak duality theorem under generalized convexity assumptions. Strong duality result is derived using a constraint qualification for nondifferentiable multiobjective programming problems.

Key words: Multiobjective programming problem, (C, α, ρ, d)-type I function, Optimality conditions, Duality.

1 Introduction

Convexity plays an important role in the design and analysis of successful algorithms for solving optimization problems. However, the convexity assumption must be weakened in order to tackle different real-world optimization problems. Therefore, several classes of generalized convex functions have been introduced in the literature and corresponding optimality conditions and duality theorems for mathematical programming problems involving these generalized convexities have been derived. In 1981, Hanson introduced the concept of invexity in [10]. Optimality conditions and duality for different mathematical programming problems with invex functions have also been obtained

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by other researchers. For example, Bector and Bhatia [4] studied minimax programming problems and relaxed the convexity assumptions in the sufficient optimality in Schmitendorf [26] using invexity. Jeyakumar and Mond [12] introduced the concept of v-invexity, which can be seen as an extension of invexity, and derived optimality conditions and duality theorems for multiobjective programming problems involving the generalized convexity. Some other extensions of these generalized convexities can be found in [13], [5] and [22]. Other classes of generalized convex functions were defined in [27, 28, 11, 24, 27, 28, 5, 9, 18, 25, 6, 1, 32].

Liang et al. [14], [15] and [16] introduced a unified formulation of generalized convexity so called (F, α, ρ, d)-convexity. Recently, Yuan et al. [33] defined (C, α, ρ, d)-convexity, which is a generalization of (F, α, ρ, d)-convexity, and established optimality conditions and duality results for nondifferentiable minimax fractional programming problems involving the generalized convexity. Chinchuluun et al. [7] also considered nondifferentiable multiobjective fractional programming problems under (C, α, ρ, d)-convexity assumptions.

On the other hand, Hanson and Mond [11] defined two new classes of functions called type I and type II functions.

Based on type I functions and (F, α, ρ, d)-convexity, Hachimi and Aghezzaf [9] defined (F, α, ρ, d)-type I functions for differentiable multiobjective programming problems and derived sufficient optimality conditions and duality theorems.

In this chapter, motivated by [9], [11] and [33], we introduce (C, α, ρ, d)-type I functions. Based on the new concept of generalized convexity, we establish optimality conditions and duality theorems for the following nondifferentiable multiobjective programming problem:

$$\text{(VOP)} \qquad \min \ f(x) = (f_1(x), \cdots, f_l(x))$$
$$\text{s.t. } x \in S = \{x \in \mathbb{R}^n |\ g(x) = (g_1(x), \cdots, g_q(x)) \leqq 0\},$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, l$, and $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2, \ldots, q$, are Lipschitz functions on $\mathbb{R}^n$.

Throughout this chapter, we use the following notations. Let $L = \{1, \ldots, l\}$ and $Q = \{1, \ldots, q\}$ be index sets for objective and constraint functions, respectively. For $x_0 \in S$, the index set of the equality constraints is denoted by $I(x_0) = \{j|g_j(x_0) = 0\}$. If x and $y \in \mathbb{R}^n$, then

$$x \leqq y \Leftrightarrow x_i \leq y_i, i = 1, \ldots, n;$$
$$x \leqslant y \Leftrightarrow x \leqq y \text{ and } x \neq y;$$
$$x < y \Leftrightarrow x_i < y_i, i = 1, \ldots, n.$$

We denote the Clarke generalized directional derivative of f at x in the direction y and Clarke generalized gradient of f at x by $f^\circ(x; y) = (f_1^\circ(x; y), \ldots, f_l^\circ(x; y))$ and $\partial^\circ f(x) = (\partial^\circ f_1(x), \ldots, \partial^\circ f_l(x))$, respectively [8].

Definition 1. *We say that $x_0 \in S$ is an (a weak) efficient solution for problem (VOP) if and only if there exists no $x \in S$ such that $f(x) \leqslant (<) f(x_0)$.*

This chapter is organized as follows. In the next section, we introduce a unified formulation of generalized convexity. Sufficient optimality conditions for the multiobjective programming problem involving the new generalized convexity are established in Section 3. In Section 4, we extend a constraint qualification in [23] in terms of Hadamard type derivatives, relaxing some assumptions. In the last section, we present the general mixed Mond-Weir dual program for (VOP) and derive weak and strong duality results.

2 Definitions

Convexity plays a central role in mathematical programming. In addition, several problems with nonconvex functions still have properties similar to convex problems. By defining more general classes of functions, we are able to understand the structures of more general optimization problems.

In this section we introduce a unified formulation of generalized convex functions, which are extensions of (F, ρ, α, d) type-I functions presented in [9] and (C, ρ, α, d)-convex functions presented in [33].

Let $C : X \times X \times \mathbb{R}^n \to \mathbb{R}$ be convex with respect to the third argument such that $C_{(x,x_0)}(0) = 0$ for any $(x, x_0) \in S \times S$. Let $\rho = (\rho^1, \rho^2)$, where $\rho^1 = (\rho^1_1, \ldots, \rho^1_l) \in \mathbb{R}^l$, $\rho^2 = (\rho^2_1, \ldots, \rho^2_q) \in \mathbb{R}^q$. Let $\alpha = (\alpha^1, \alpha^2)$, where $\alpha^1 = (\alpha^1_1, \ldots, \alpha^1_l)$, $\alpha^2 = (\alpha^2_1, \ldots, \alpha^2_q)$, and $\alpha^i_j(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \backslash \{0\}$, $i = 1, 2$, $j \in L$ or Q. $d = (d^1, d^2)$ is a vector function, where $d^1 = (d^1_1, \ldots, d^1_l)$, $d^2 = (d^2_1, \ldots, d^2_q)$, and $d^i_j(\cdot,\cdot)$ is pseudometric on $\mathbb{R}^n$, $i = 1, 2$, $j \in L$ or Q. We assume that, for any $a, b, c \in \mathbb{R}^s$, the symbol $\frac{ab}{c}$ denotes $\left(\frac{a_1 b_1}{c_1}, \ldots, \frac{a_s b_s}{c_s}\right)$, and the symbol $\frac{a+b}{c}$ denotes $\left(\frac{a_1+b_1}{c_1}, \ldots, \frac{a_s+b_s}{c_s}\right)$. If $\xi = (\xi_1, \ldots, \xi_l) \in \partial^\circ \varphi(x_0)$, then $C_{(x,x_0)}(\xi)$ denotes the vector $(C_{(x,x_0)}(\xi_1), \ldots, C_{(x,x_0)}(\xi_l))$. We are now ready to present the new classes of functions.

(φ, ψ) is (C, α, ρ, d)*-type I* at x_0, if for all $x \in S$ we have

$$\frac{\varphi(x) - \varphi(x_0)}{\alpha^1(x, x_0)} \geqq C_{(x,x_0)}(\xi) + \frac{\rho^1 d^1(x, x_0)}{\alpha^1(x, x_0)}, \forall \xi \in \partial^\circ \varphi(x_0)$$
$$\frac{-\psi(x_0)}{\alpha^2(x, x_0)} \geqq C_{(x,x_0)}(\eta) + \frac{\rho^2 d^2(x, x_0)}{\alpha^2(x, x_0)}, \forall \eta \in \partial^\circ \psi(x_0)$$

(φ, ψ) is *pseudoquasi* (*strictly pseudoquasi*) (C, α, ρ, d)*-type I* at x_0, if for all $x \in S$ we have

$$\varphi(x) < (\leqq) \varphi(x_0) \quad \Rightarrow \quad C_{(x,x_0)}(\xi) + \frac{\rho^1 d^1(x, x_0)}{\alpha^1(x, x_0)} < 0, \forall \xi \in \partial^\circ \varphi(x_0) \quad (1)$$
$$-\psi(x_0) \leqq 0 \quad \Rightarrow \quad C_{(x,x_0)}(\eta) + \frac{\rho^2 d^2(x, x_0)}{\alpha^2(x, x_0)} \leqq 0, \forall \eta \in \partial^\circ \psi(x_0)$$

(φ, ψ) is *weak strictly-pseudoquasi* (C, α, ρ, d)*-type I* at x_0, if for all $x \in S$ we have

$$\varphi(x) \leqslant \varphi(x_0) \quad \Rightarrow \quad C_{(x,x_0)}(\xi) + \frac{\rho^1 d^1(x,x_0)}{\alpha^1(x,x_0)} < 0, \forall \xi \in \partial^\circ \varphi(x_0)$$
$$-\psi(x_0) \leqq 0 \quad \Rightarrow \quad C_{(x,x_0)}(\eta) + \frac{\rho^2 d^2(x,x_0)}{\alpha^2(x,x_0)} \leqq 0, \forall \eta \in \partial^\circ \psi(x_0)$$

(φ, ψ) is *strong pseudoquasi(weak pseudoquasi)* (C, α, ρ, d)*-type I* at x_0, if for all $x \in S$ we have

$$\varphi(x) \leqslant (<) \varphi(x_0) \quad \Rightarrow \quad C_{(x,x_0)}(\xi) + \frac{\rho^1 d^1(x,x_0)}{\alpha^1(x,x_0)} \leqslant 0, \forall \xi \in \partial^\circ \varphi(x_0) \quad (2)$$
$$-\psi(x_0) \leqq 0 \quad \Rightarrow \quad C_{(x,x_0)}(\eta) + \frac{\rho^2 d^2(x,x_0)}{\alpha^2(x,x_0)} \leqq 0, \forall \eta \in \partial^\circ \psi(x_0)$$

(φ, ψ) is *weak quasi-strictly-pseudo* (C, α, ρ, d)*-type I* at x_0, if for all $x \in S$ we have

$$\varphi(x) \leqslant \varphi(x_0) \quad \Rightarrow \quad C_{(x,x_0)}(\xi) + \frac{\rho^1 d^1(x,x_0)}{\alpha^1(x,x_0)} \leqq 0, \forall \xi \in \partial^\circ \varphi(x_0)$$
$$-\psi(x_0) \leqq 0 \quad \Rightarrow \quad C_{(x,x_0)}(\eta) + \frac{\rho^2 d^2(x,x_0)}{\alpha^2(x,x_0)} \leqslant 0, \forall \eta \in \partial^\circ \psi(x_0)$$

We note that we can derive many different classes of generalized convex functions by changing the inequalities of these conditions.

3 Sufficient Optimality

Aghezzaf and Hachimi [1, 9] considered multiobjective programming problems with (F,ρ)-convex functions and (F,α, ρ,d)-type I functions, and established a number of sufficient optimality conditions. We adapt these results to the classes of generalized (C,α, ρ,d)-type I functions.

Theorem 1. *Assume that there exist a feasible solution x_0 for (VOP) and vectors $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_l) \in \mathbb{R}^l$ and $\bar{v} = (\bar{v}_1, \ldots, \bar{v}_q) \in \mathbb{R}^q$ such that*

$$0 \in \bar{u}^T \partial^\circ f(x_0) + \bar{v}^T \partial^\circ g(x_0), \quad (3)$$
$$\bar{v}^T g(x_0) = 0, \quad (4)$$
$$\bar{u} > 0, \bar{v} \geqq 0. \quad (5)$$

If (f, g_I) is strong pseudoquasi (C, α, ρ, d)-type I at x_0, and

$$\bar{u}^T \frac{\rho^1 d^1(x,x_0)}{\alpha^1(x,x_0)} + \bar{v}_I^T \frac{\rho_I^2 d_I^2(x,x_0)}{\alpha_I^2(x,x_0)} \geq 0, \quad (6)$$

then x_0 is an efficient solution of (VOP).

Proof. Suppose to the contrary that x_0 is not an efficient solution of (VOP). Then there exists a feasible solution x such that

$$f(x) \leqslant f(x_0) \text{ and } g_I(x_0) = 0.$$

Hence,

$$f(x) \leqslant f(x_0) \text{ and } -g_I(x_0) \leqq 0.$$

Since (f, g_I) is strong pseudoquasi (C, α, ρ, d)-type I at x_0, we can write

$$C_{(x,x_0)}(\xi) + \frac{\rho^1 d^1(x,x_0)}{\alpha^1(x,x_0)} \leqslant 0, \forall\, \xi \in \partial^\circ f(x_0),$$
$$C_{(x,x_0)}(\eta_I) + \frac{\rho_I^2 d_I^2(x,x_0)}{\alpha_I^2(x,x_0)} \leqq 0, \forall\, \eta_I \in \partial^\circ g_I(x_0).$$

Let us denote $\tau = \sum_{i=1}^{l} \bar{u}_i + \sum_{j\in I} \bar{v}_j$. Multiplying the above inequalities with $\frac{1}{\tau}\bar{u}$ and $\frac{1}{\tau}\bar{v}_I$, respectively, and using the convexity assumption of C, we have

$$C_{(x,x_0)}\left(\frac{1}{\tau}\bar{u}^T\xi + \frac{1}{\tau}\bar{v}_I^T\eta_I\right) + \frac{1}{\tau}\bar{u}^T\frac{\rho^1 d^1(x,x_0)}{\alpha^1(x,x_0)} + \frac{1}{\tau}\bar{v}_I^T\frac{\rho_I^2 d_I^2(x,x_0)}{\alpha_I^2(x,x_0)} < 0,$$
$$\forall \xi \in \partial^\circ f(x_0), \eta_I \in \partial^\circ g_I(x_0),$$

since $\bar{u} > 0$. From the last inequality, using (3) and (4), we have

$$\bar{u}^T\frac{\rho^1 d^1(x,x_0)}{\alpha^1(x,x_0)} + \bar{v}^T\frac{\rho^2 d^2(x,x_0)}{\alpha^2(x,x_0)} < 0,$$

which contradicts (6). □

The next theorems will be presented without proofs since they can be proven using the similar argument as in the proof of Theorem (1).

We can weaken the strict inequality requirement that $\bar{u} > 0$ in the above theorem but we require different convexity conditions on (f, g_I). This adjustment is given by the following theorem.

Theorem 2. *Assume that there exist a feasible solution x_0 for (VOP) and vectors $\bar{u} \in \mathbb{R}^l$ and $\bar{v} \in \mathbb{R}^q$ such that*

$$0 \in \bar{u}^T\partial^\circ f(x_0) + \bar{v}^T\partial^\circ g(x_0), \tag{7}$$
$$\bar{v}^T g(x_0) = 0, \tag{8}$$
$$\bar{u} \geqslant 0, \bar{v} \geqq 0.$$

If (f, g_I) is weak strictly-pseudoquasi (C, α, ρ, d)-type I at x_0, and

$$\bar{u}^T\frac{\rho^1 d^1(x,x_0)}{\alpha^1(x,x_0)} + \bar{v}_I^T\frac{\rho_I^2 d_I^2(x,x_0)}{\alpha_I^2(x,x_0)} \geq 0, \tag{9}$$

then x_0 is an efficient solution of (VOP).

Since an efficient solution is also weak efficient, the above formulated theorems are still valid for weak efficiency, however, we can weaken the convexity assumptions for weak efficient solutions. Therefore, the following theorems can be formulated.

Theorem 3. *Assume that there exist a feasible solution x_0 for (VOP) and vectors $\bar{u} \in \mathbb{R}^l$ and $\bar{v} \in \mathbb{R}^q$ such that the triplet $(x_0, \bar{u}, \bar{v})$ satisfies (3), (4) and (5). If $(f,\, g_I)$ is weak pseudoquasi $(C,\, \alpha,\, \rho,\, d)$-type I at x_0, and*

$$\bar{u}^T \frac{\rho^1 d^1(x, x_0)}{\alpha^1(x, x_0)} + \bar{v}_I^T \frac{\rho_I^2 d_I^2(x, x_0)}{\alpha_I^2(x, x_0)} \geqslant 0,$$

then x_0 is a weak efficient solution of (VOP).

Theorem 4. *Assume that there exist a feasible solution x_0 for (VOP) and vectors $\bar{u} \in \mathbb{R}^l$ and $\bar{v} \in \mathbb{R}^q$ such that the triplet $(x_0, \bar{u}, \bar{v})$ satisfies (7), (8) and (9). If $(f,\, g_I)$ is pseudoquasi $(C,\, \alpha,\, \rho,\, d)$-type I at x_0 with*

$$\sum_{i=1}^{l} \bar{u}_i \rho_i^1 \frac{d_i^1(x, x_0)}{\alpha_i^1(x, x_0)} + \sum_{j \in I} \bar{v}_j \rho_j^2 \frac{d_j^2(x, x_0)}{\alpha_j^2(x, x_0)} \geqslant 0,$$

then x_0 is a weak efficient solution for (VOP).

4 A Constraint Qualification

For some necessary optimality conditions of multiobjective programming problems, constraint qualifications are used in order to avoid the situation where some of the Lagrange multipliers vanish [17, 23]. In this section, we weaken assumptions of constraint qualification in Preda [23] in terms of Hadamard type derivatives, relaxing some assumptions. The Hadamard derivative of f at x_0 in the direction $v \in \mathbb{R}^n$ is defined by

$$df(x_0, v) = \lim_{(t,u) \to (0^+, v)} \frac{f(x_0 + tu) - f(x_0)}{t}.$$

f is said to be Hadamard differentiable at x_0 if $df(x_0, v)$ exists for all $v \in \mathbb{R}^n$. Obviously, $df(x_0, 0) = 0$.

Following Preda and Chitescu [23] we use the following notations. The tangent cone to a nonempty set W at point $x \in \mathrm{cl}W$ is defined by

$$T(W; x) = \{v \in \mathbb{R}^n \mid \exists \{x^m\} \subset W :\ x = \lim_{m \to \infty} x^m,\ \exists \{t^m\}, t^m > 0 :\ v = \lim_{m \to \infty} t^m (x^m - x)\},$$

where $\mathrm{cl}W$ is the closure of W.

Let x_0 be a feasible solution of Problem (VOP). For each $i \in L$, let $L^i = L \backslash \{i\}$, and let the nonempty sets $W^i(x_0)$ and $W(x_0)$ be defined as follows: $W(x_0) = \{x \in S | f(x) \leq f(x_0)\}$, $W^i(x_0) = \{x \in S | f_k(x) \leq f_k(x_0),\ \text{for}\ k \in L^i\} (l > 1)$, and $W^i(x_0) = W(x_0)(l = 1)$. Then, we give the following definition.

Definition 2. *The almost linearizing cone to $W(x_0)$ at x_0 is defined by*

$$H\left(W(x_0); x_0\right) = \{v \in \mathbb{R}^n | df_i(x_0, v) \leq 0,\ i \in L, \mathit{and}\ dg_j(x_0, v) \leq 0,\ j \in I(x_0)\}$$

Proposition 1. *If $df_i(x_0, \cdot)$ $i \in L$, and $dg_j(x_0, \cdot)$ $j \in I(x_0)$ are convex functions on $\mathbb{R}^n$, then $H\left(W(x_0); x_0\right)$ is a closed convex cone.*

Proof. The proof is very similar to that of Proposition 3.1 in [23]. So we omit this. □

The following lemma illustrates the relationship between the tangent cones $T\left(W^i(x_0); x_0\right)$ and the almost linearizing cone $H\left(W(x_0); x_0\right)$.

Lemma 1. *Let x_0 be a feasible solution of Problem* (VOP). *If $df_i(x_0, \cdot)$ $i \in L$, and $dg_j(x_0, \cdot)$ $j \in I(x_0) (\neq \varnothing)$ are convex functions on $\mathbb{R}^n$, then*

$$\bigcap_{i \in L} \text{clco}\ T\left(W^i(x_0); x_0\right) \subseteq H\left(W(x_0); x_0\right) \tag{10}$$

Proof. Here, we give a proof for only part $l > 1$ since the proof for part $l = 1$ is similar. For $i \in L$, let us define

$$H\left(W^i(x_0); x_0\right) = \{v \in \mathbb{R}^n |\ df_k(x_0, v) \leq 0,\ k \in L^i, \text{and}\ dg_j(x_0, v) \leq 0,\ j \in I(x_0)\}$$

According to Proposition 1, $H\left(W^i(x_0); x_0\right)$ is closed and convex for all $i \in L$. We know that

$$\bigcap_{i \in L} H\left(W^i(x_0); x_0\right) \subseteq H\left(W(x_0); x_0\right)$$

Next, we show that, for every $i \in L$,

$$T\left(W^i(x_0); x_0\right) \subseteq H\left(W^i(x_0); x_0\right). \tag{11}$$

Let $i \in L$ and $v \in T\left(W^i(x_0); x_0\right)$. If $v = 0$, it is obvious that $v = 0 \in H\left(W^i(x_0); x_0\right)$. Now, we assume $v \neq 0$. Therefore, we have a sequence $\{x^m\} \subseteq W^i(x_0)$ and a sequence $\{t^m\} \subseteq \mathbb{R}$, with $t^m > 0$, such that

$$\lim_{m \to \infty} x^m = x_0,\ \lim_{m \to \infty} t^m(x^m - x_0) = v.$$

Let us take $v^m = t^m(x^m - x_0)$. Then, $\frac{v^m}{t^m} \to 0$ as $m \to \infty$. Since $v^m \to v$ and $v \neq 0$, for any positive real number ε, there exists a positive integer number

N such that $v^m \in B(v, \varepsilon)$ for all $m > N$. Therefore $\|v\| - \varepsilon \le \|v^m\| \le \|v\| + \varepsilon$ for all $m > N$. Hence, for all $m > N$, we have

$$\frac{\|v\| - \varepsilon}{t^m} \le \frac{\|v^m\|}{t^m} \to 0.$$

Since ε is an arbitrary positive number, selecting ε as a sufficiently small number, we can deduce that $\frac{1}{t^m} \to 0$. Then for all $j \in I(x_0)$ and for all sufficiently large m, we have

$$g_j\left(x_0 + \frac{1}{t^m}v^m\right) = g_j(x^m) \le 0 = g_j(x_0),\ j \in I(x_0), \tag{12}$$

$$f_k\left(x_0 + \frac{1}{t^m}v^m\right) = f_k(x^m) \le f_k(x_0),\ k \in L^i. \tag{13}$$

By definition of Hadamard derivative, we have

$$dg_j(x_0, v) \le 0,\ j \in I(x_0), \tag{14}$$
$$df_k(x_0, v) \le 0,\ k \in L^i. \tag{15}$$

This shows $v \in H\left(W^i(x_0); x_0\right)$ or (11) is true. Hence, due to the fact that every $H\left(W^i(x_0); x_0\right)$ is convex and closed, one obtains

$$\text{clco}\ T\left(W^i(x_0); x_0\right) \subseteq H\left(W^i(x_0); x_0\right),\ \forall\ i \in L.$$

Thus (10) holds. □

Definition 3. *We say that Problem* (VOP) *satisfies the generalized Guignard constraint qualification* (GGCQ) *at x_0 if*

$$\bigcap_{i \in L} \text{clco}\ T\left(W^i(x_0); x_0\right) \supseteq H\left(W(x_0); x_0\right). \tag{16}$$

holds.

Theorem 5. *Let $x_0 \in S$ be an efficient solution of Problem* (VOP)*. Suppose that $l > 1$, and*
(A1) *constraint qualification* (GGCQ) *holds at x_0;*
(A2) *there exists $i \in L$ such that $df_i(x_0, \cdot)$ is a concave function on $\mathbb{R}^n$*
(A3) *$df_k(x_0, \cdot), k \in L^i$ and $dg_j(x_0, \cdot), j \in I(x_0)$ are convex function on $\mathbb{R}^n$.*
Then the system

$$df_k(x_0, v) \le 0, k \in L^i \tag{17}$$
$$df_i(x_0, v) < 0 \tag{18}$$
$$dg_j(x_0, v) \le 0, j \in I(x_0) \tag{19}$$

has no solution $v \in \mathbb{R}^n$.

Proof. Suppose to the contrary that there exists $v \in \mathbb{R}^n$ such that (17)–(19) hold. Obviously, $v \neq 0$. Thus, we have $0 \neq v \in H\left(W(x_0); x_0\right)$. Using Assumption (A1), we have $v \in \operatorname{clco} T\left(W^i(x_0); x_0\right)$. Therefore, there exists a sequence $\{v_s\} \subseteq T\left(W^i(x_0); x_0\right)$ such that

$$\lim_{s \to \infty} v_s = v \tag{20}$$

For any v_s, $s = 1, 2, \ldots$, there exist numbers $k_s, \lambda_{sr} \geq 0$, and $v_{sr} \in T\left(W^i(x_0); x_0\right), r = 1, 2, \ldots, k_s$, such that

$$\sum_{r=1}^{k_s} \lambda_{sr} = 1, \quad \sum_{r=1}^{k_s} \lambda_{sr} v_{sr} = v_s \tag{21}$$

Since $v_{sr} \in T\left(W^i(x_0); x_0\right)$, by definition, there exist sequences $\{x_{sr}^m\} \subseteq W^i(x_0)$ and $\{t_{sr}^m\} \subseteq \mathbb{R}$, $t_{sr}^m > 0$ for all n, such that, for any s and r,

$$\lim_{m \to \infty} x_{sr}^m = x_0, \lim_{m \to \infty} t_{sr}^m (x_{sr}^m - x_0) = v_{sr} \tag{22}$$

Let us denote $v_{sr}^m = t_{sr}^m (x_{sr}^m - x_0)$. Similarly to the corresponding part of the proof in Lemma 1, we know that $\frac{1}{t_{sr}^m} \to 0$ as $m \to \infty$. Then for any sufficiently large m, we have

$$g_j\left(x_0 + \frac{1}{t_{sr}^m} v_{sr}^m\right) = g_j(x_{sr}^m) \leq 0 = g_j(x_0), \; j \in I(x_0), \tag{23}$$

$$f_k\left(x_0 + \frac{1}{t_{sr}^m} v_{sr}^m\right) = f_k(x_{sr}^m) \leq f_k(x_0), \; k \in L^i. \tag{24}$$

and

$$f_i\left(x_0 + \frac{1}{t_{sr}^m} v_{sr}^m\right) = f_i(x_{sr}^m) \geq f_i(x_0), \tag{25}$$

since x_0 is an efficient solution to Problem (VOP). Using (22)–(25), by definition of Hadamard derivative, we can have

$$dg_j(x_0, v_{sr}) \leq 0, \; j \in I(x_0), \tag{26}$$

$$df_k(x_0, v_{sr}) \leq 0, \; k \in L^i, \tag{27}$$

$$df_i(x_0, v_{sr}) \geq 0. \tag{28}$$

From this system, (20), (21) and Assumptions (A2), (A3), it follows that

$$dg_j(x_0, v) \leq 0, \; j \in I(x_0),$$
$$df_k(x_0, v) \leq 0, \; k \in L^i,$$
$$df_i(x_0, v) \geq 0.$$

This contradicts the system (17)–(19). □

Theorem 6. *Suppose that the assumptions of Theorem 5 hold. Then, there exist vectors* $\lambda \in \mathbb{R}^l$ *and* $\mu \in \mathbb{R}^q$ *such that, for any* $v \in \mathbb{R}^n$,

$$\lambda^T df(x_0, v) + \mu^T dg(x_0, v) \geqq 0 \tag{29}$$

$$\mu^T g(x_0) = 0 \tag{30}$$

$$\lambda = (\lambda_1, \dots, \lambda_l)^T > 0, \mu = (\mu_1, \dots, \mu_q)^T \geqq 0 \tag{31}$$

Proof. The proof is similar to that of Theorem 3.2 in [23]. □

Remark 1. It is easy to check that if f is Hadamard differentiable then f is also directional differentiable at x_0, but, we do not need the assumption that f and g are quasiconvex at x_0 of [23].

Theorem 7. *Suppose that the assumptions of Theorem 5 hold, and suppose that* $df_i(x_0, v) = f_i^\circ(x_0; v)$ *and* $dg_j(x_0, v) = g_j^\circ(x_0; v)$ *for all* $i \in L, j \in I(x_0)$. *Then, there exist vectors* $\lambda \in \mathbb{R}^l$ *and* $\mu \in \mathbb{R}^q$ *such that*

$$0 \in \lambda^T \partial^\circ f(x_0) + \mu^T \partial^\circ g(x_0)$$
$$\mu^T g(x_0) = 0,$$
$$\lambda = (\lambda_1, \dots, \lambda_l)^T > 0, \ \mu = (\mu_1, \dots, \mu_q)^T \geqq 0.$$

Proof. By Theorem 6, we have

$$\lambda^T f^\circ(x_0, v) + \mu^T g^\circ(x_0, v) \geq 0, \tag{32}$$

for all $v \in \mathbb{R}^n$. If f_i, g_j are Hadamard differentiable, then they are directional differentiable at x_0, and

$$f_i^\circ(x_0; v) = df_i(x_0, v) = f_i'(x_0, v),$$

$$g_j^\circ(x_0; v) = dg_j(x_0, v) = g_j'(x_0, v),$$

Thus, f_i and g_j are regular for all $i \in L$ and $j \in I(x_0)$. By assumption, for any $v \in \mathbb{R}^n$, we have

$$0 \leq \lambda^T f^\circ(x_0, v) + \mu^T g^\circ(x_0, v) = \left(\lambda^T f + \mu^T g\right)^\circ (x_0, v),$$

or

$$0^T v \leq \left(\lambda^T f + \mu^T g\right)^\circ (x_0, v).$$

So, according to the definition of Clarke's generalized gradient, we have

$$0 \in \lambda^T \partial^\circ f(x_0) + \mu^T \partial^\circ g(x_0).$$

□

5 General Mixed Mond-Weir Type Dual

Duality theory plays a central role in mathematical programming. In this section, we introduce a general mixed Mond-Weir dual program of Problem (VOD) and establish the corresponding dual theorems under the generalized convexity assumptions. However, in order to derive strong duality result, we use the constraint qualification discussed in the previous section. Weakening the assumptions of constraint qualification would be helpful to establish more general strong duality result.

Let $M_0, M_1, \ldots, M_r$ be a partition of Q, i.e., $\bigcup\limits_{k=0}^{r} M_k = Q$, $M_{k_1} \bigcap M_{k_2} = \emptyset$ for $k_1 \neq k_2$. Let e_l be the vector of $\mathbb{R}^l$ whose components are all ones. Motivated by [3, 16, 9], we define the following general mixed Mond-Weir dual of (VOP).

$$
\begin{aligned}
\text{(VOD)} \qquad & \max f(y) + {\mu_{M_0}}^T g_{M_0}(y) e_l \\
& \text{s.t. } 0 \in \sum_{i=1}^{l} \lambda_i \partial f_i(y) + \sum_{k=0}^{r} \partial \left(\mu_{M_k}^T g_{M_k} \right)(y), \qquad (33) \\
& h_k(y) \triangleq \left({\mu_{M_k}}^T g_{M_k} \right)(y) \geq 0,\ k = 1, 2, \ldots, r, \\
& \sum_{i=1}^{l} \lambda_i = 1, \lambda_i > 0\ (i = 1, 2, \ldots, l), \lambda = (\lambda_1, \ldots, \lambda_l)^T, \\
& \mu = (\mu_1, \mu_2, \ldots, \mu_q)^T \in \mathbb{R}_+^q, y \in \mathbb{R}^n, \mu_{M_k} \in \mathbb{R}_+^{|M_k|}.
\end{aligned}
$$

Theorem 8 (`Weak Duality`). *Let x_0 be a feasible solution of (VOP), $(y_0, \bar{\lambda}, \bar{\mu})$ be a feasible solution of (VOD) and $h_0(y) \triangleq \bar{\mu}_{M_0}^T g_{M_0}(y)$. Let us use the following notations: $h(y) = (h_1(y), \ldots, h_r(y))$. Suppose that any of the following holds:*

(a) $\left(f + \bar{\mu}_{M_0}^T g_{M_0} e_l, h\right)$ is (C, α, ρ, d)-type I at y_0, f_i $(i = 1, \ldots, l)$ and h_0 are regular at y_0 and

$$
\bar{\lambda}^T \frac{\rho^1 d^1(x_0, y_0)}{\alpha^1(x_0, y_0)} + e_r^T \frac{\rho^2 d^2(x_0, y_0)}{\alpha^2(x_0, y_0)} \geq 0, \qquad (34)
$$

(b) $\left(f + \bar{\mu}_{M_0}^T g_{M_0} e_l, h\right)$ is strong pseudoquasi (C, α, ρ, d)-type I at y_0, f_i $(i = 1, \cdots, l)$ and h_0 are regular at y_0 and (34) is true

(c) $(\bar{\lambda}^T f + \bar{\mu}_{M_0}^T g_{M_0}, \sum_{k=1}^{r} \mu_{M_k}^T g_{M_k})$ is pseudoquasi (C, α, ρ, d)-type I at y_0, f_i $(i = 1, \cdots, p)$ and h_k $(k = 0, 1, \ldots, r)$ are regular at y_0 and

$$
\frac{\rho^1 d^1(x_0, y_0)}{\alpha^1(x_0, y_0)} + \frac{\rho^2 d^2(x_0, y_0)}{\alpha^2(x_0, y_0)} \geq 0.
$$

Then the following cannot hold.

$$
f(x_0) \leqslant f(y_0) + \bar{\mu}_{M_0}^T g_{M_0}(y_0) e_l. \qquad (35)
$$

Proof. Here we give the proofs of (a) and (b) since (c) can be proven similarly. Suppose to the contrary that (35) holds. Since x_0 is feasible for (VOP) and $\bar{\mu} \geqq 0$, (35) implies that

$$f(x_0) + \bar{\mu}_{M_0}^T g_{M_0}(x_0) e_l \leqslant f(y_0) + \bar{\mu}_{M_0}^T g_{M_0}(y_0) e_l \tag{36}$$

holds.

(a) By (35), (36) and the hypothesis (a), we can write the following statement for any $\bar{\xi}_i \in \partial f_i(y_0)$ and $\bar{\eta}_k \in \partial h_k(y_0)$.

$$\sum_{i=1}^{l} \frac{\bar{\lambda}_i}{\bar{\tau}} \frac{(f_i(x_0) + h_0(x_0)) - (f_i(y_0) + h_0(y_0))}{\alpha_i^1(x_0, y_0)} + \sum_{k=1}^{r} \frac{1}{\bar{\tau}} \frac{-h_k(y_0)}{\alpha_k^2(x_0, y_0)}$$
$$\geqslant C_{(x_0,y_0)}\left(\frac{1}{\bar{\tau}}(\bar{\lambda}^T \bar{\xi} + e_{r+1}^T \bar{\eta})\right) + \frac{1}{\bar{\tau}}\left(\bar{\lambda}^T \frac{\rho^1 d^1(x_0, y_0)}{\alpha^1(x_0, y_0)} + e_r^T \frac{\rho^2 d^2(x_0, y_0)}{\alpha^2(x_0, y_0)}\right),$$

where $\bar{\tau} = r + 2$. From (33), (34) and the above inequality, it follows that

$$\sum_{i=1}^{l} \frac{\bar{\lambda}_i}{\bar{\tau}} \frac{(f_i(x_0) + h_0(x_0)) - (f_i(y_0) + h_0(y_0))}{\alpha_i^1(x_0, y_0)} + \sum_{k=1}^{r} \frac{1}{\bar{\tau}} \frac{-h_k(y_0)}{\alpha_k^2(x_0, y_0)} \geq 0 \tag{37}$$

Since $(y_0, \bar{\lambda}, \bar{\mu})$ is a feasible solution of (VOD), it follows that $-h(y_0) \leqq 0$. Therefore, by (36), we have

$$\sum_{i=1}^{l} \frac{\bar{\lambda}_i}{\bar{\tau}} \frac{(f_i(x_0) + h_0(x_0)) - (f_i(y_0) + h_0(y_0))}{\alpha_i^1(x_0, y_0)} + \sum_{k=1}^{r} \frac{1}{\bar{\tau}} \frac{-h_k(y_0)}{\alpha_k^2(x_0, y_0)} < 0,$$

which is a contradiction to (37).

(b) By (36), $-h(y_0) \leqq 0$, the hypothesis (b) and the convexity of C, we obtain

$$C_{(x_0,y_0)}\left(\frac{1}{\bar{\tau}}(\bar{\lambda}^T \bar{\xi} + e_{r+1}^T \bar{\eta})\right) + \frac{1}{\bar{\tau}}\left(\bar{\lambda}^T \frac{\rho^1 d^1(x_0, y_0)}{\alpha^1(x_0, y_0)} + e_r^T \frac{\rho^2 d^2(x_0, y_0)}{\alpha^2(x_0, y_0)}\right) < 0.$$

Therefore, $C_{(x_0,y_0)}\left(\frac{1}{\bar{\tau}}(\bar{\lambda}^T \bar{\xi} + e_{r+1}^T \bar{\eta})\right) < 0$, which is a contradiction to (33).

□

Theorem 9 (Strong Duality). *Let the assumptions of Theorem 7 be satisfied. If $x_0 \in S$ is an efficient solution of (VOP), then there exist $\bar{\lambda} \in \mathbb{R}^l$, $\bar{\mu} \in \mathbb{R}^q$ such that $(x_0, \bar{\lambda}, \bar{\mu})$ is a feasible solution of (VOD) and the objective function values of (VOP) and (VOD) at the corresponding points are equal. Furthermore if the assumptions about the generalized convexity and the inequality (34) in Theorem 8 are also satisfied, then $(x_0, \bar{\lambda}, \bar{\mu})$ is an efficient solution of (VOD).*

Proof. By Theorem 7, it is obvious that $(x_0, \bar{\lambda}, \bar{\mu})$ is a feasible solution of (VOD). Moreover the objective function values of (VOP) and (VOD) at the corresponding points are equal since the objective functions are the same. Therefore $(x_0, \bar{\lambda}, \bar{\mu})$ is an efficient point of (VOD) due to the weak duality result in Theorem 8. □

6 Conclusions and Future Work

In this chapter we have defined some generalized convex functions. For mathematical programming problems with such functions, we have established sufficient optimality conditions for nonconvex nondifferentiable multiobjective programming problems with the generalized convex functions. We have also introduced a general mixed Mond-Weir type dual program of a multiobjective program and proved a weak duality theorem under the generalized convexity assumptions. Therefore, a strong duality theorem has been proved using a constraint qualification, which was derived after relaxing some assumptions of the constraint qualfication in [23] in terms of the Hadamard derivative, for nondifferentiable multiobjective programming. Weakening the assumptions of constraint qualification would be helpful to establish more general strong duality result. The chapter mainly focuses on theoretical aspects of the generalized convexity. We have not discussed any applications. Future work will include the solutions of real world engineering problems associated with the generalized convexities.

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