Chapter 9

Diagnosis and reconfiguration of quantised systems

Quantised systems are continuous-variable systems whose sensor and actuator signals can only be accessed through quantisers that produce symbolic state or event sequences. Hence, quantised systems have a discrete-event behaviour. This chapter shows how quantised systems can be represented by stochastic automata and how state observation, diagnostic and control problems can be solved. First a stochastic automaton is set up so as to represent the discrete-event behaviour of the quantised system completely. Second the given analysis and design problems are solved for the automaton by means of the methods that have been developed in Chapter 8.

9.1 Introduction to quantised systems

9.1.1 Supervision of hybrid systems

The preceding chapters have considered either continuous-variable systems, which have real-valued signals and can be described by differential or difference equations, or discrete-event systems, which have signals with symbolic values and can be described by automata, Petri nets or similar models. This chapter is devoted to an important class of systems, in which both continuous and discrete phenomena have to be taken into account. Such systems are called *hybrid systems*.

The mixture of discrete and continuous signals and discrete and continuous forms of the models used is typical for supervisory control tasks and plays a particular role in diagnosis and fault-tolerant control. Nevertheless, hybrid systems have attracted substantial interest only during the last ten years and only preliminary results are available for their supervision and control.

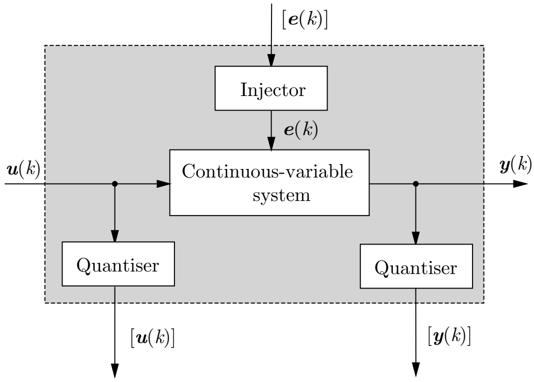


Fig. 9.1. Quantised system

Structure of quantised systems. The main problems in dealing with hybrid systems result from the different ranges of the signals. These problems are investigated in this chapter for quantised systems depicted in Fig. 9.1. The relation between the different signal ranges are represented by quantisers and injectors. The *quantiser* transforms a real-valued signal into a sequence of symbols, where the real-valued signal or signal vector is denoted by a lower-case letter like $\boldsymbol{u}$ or y and the corresponding quantised signals by $[\boldsymbol{u}]$ or $[y]$, respectively. If, in the simplest case, the quantiser decides to which real interval of a given set of intervals the current value $y(t)$ belongs, the value of the quantised signal $[y(t)]$ at the time instant t is the number of the corresponding interval. This interval can be associated with symbolic names like "normal", "high" or "low", which give a semantic signal value. As long as the signal does not leave a given interval, the quantised value remains the same. Hence, a continuous change of $y(t)$ is transformed into a sequence of discrete changes of $[y(t)]$, which shows that the quantiser can be used as an interface between real-valued and symbolic signals.

The *injector* carries out the inverse mapping. Its input is a symbolic signal like $[\boldsymbol{e}]$, which is associated with a real-valued signal $\boldsymbol{e}$. An example is given in Fig. 9.1, where the injector associates to a symbolic fault $[\boldsymbol{e}]$ the real-valued fault input $\boldsymbol{e}$.

The relation between $[\boldsymbol{e}]$ and $\boldsymbol{e}$ can be either deterministic where every symbolic value is associated with a unique real value or non-deterministic where the associated real value is randomly selected from a given set of signal values or may vary within this set as long as the symbolic value does not change. In any case, the injector is the interface from symbolic to real-valued signals.

Reasons for introducing quantisers and injectors. The question why quantisers or injectors occur in the system has many answers:

- **Measurement uncertainties:** Many physical quantities cannot be precisely measured as, for example, the biomass concentration in bioreactors, substance concentrations in the liquid or the gaseous phase, the temperature in cement kilns or blast furnaces. Then, quantisers are introduced as a representation of systematic measurement errors.

- **Alarms:** The abnormal behaviour of industrial plants is signalled by means of alarm messages, which represent quantised signal values.

- **Discrete actuators:** Many industrial actuators can only be switched among a set of discrete values rather than be varied continuously. For example, gas burners are used in an on/off mode. This fact necessitates the introduction of an injector that transforms discrete values $[u]$ into the associated real input values u.

- **Discrete control:** Many industrial processes are controlled by programmable logic controllers, which react on quantised measurements by prescribing discrete input values. For example, the controller of an elevator does not know the lift position precisely, but has only the information between or at which floor the lift currently is.

- **Switching system dynamics:** The system dynamics switches if the input or state exceeds certain bounds. An example is given by the tank system described in Section 10.1 where the dynamical properties depend on whether the liquid levels are above or below the height of the connecting pipes. According to this quantisation of the levels h_1 and h_2 the equations given on page 506 are valid in one of the four possible configurations of existing or nonexisting flows through the upper valves. Here, signal quantisation occur internally in the system. If brought into the hybrid system structure depicted in Fig. 3.6, the discrete-event part of the tank model switches the continuous model among the four different equations.

These arguments show that quantised systems occur naturally in the engineering practice. However, in addition to the situations described above, injectors or quantisers may be deliberately introduced for the following reasons:

- **Uniformity of the system description:** The mixture of differential equations for the continuous-variable part and automata for the discrete-event part makes the model of hybrid systems very complex. Therefore, it is reasonable to deal with all signals uniformly as discrete-valued signals by introducing additional injectors and quantisers. The considerable simplification of observation and diagnostic problem due to this uniformity of the signals will become obvious in Sections 9.5 and 9.6.

- **Information reduction:** If the control aim concerns a global assessment of the system behaviour, it is reasonable to use models that have direct reference to these assessments. In the diagnostic problem considered in Section 9.6 the faults occurring in the system changes the behaviour qualitatively. Therefore, quantised information about the system behaviour is sufficient to identify the fault.

- **Reference to heuristic models:** The experience of human operators refers to subsets of the signal space rather than to specific signal values. Hence, this knowledge considers continuous-variable systems as systems with quantised signal spaces.

Quantised systems in fault-tolerant control. For systems subject to faults an additional motivation comes from the fact that faults are, in general, quantised phenomena. Obvious examples of faults concern broken wires, a leakage in a pipe, or a valve that is stuck open or closed. However, even if the fault concerns a change of some parameter, the controlled system will show an abnormal behaviour only if the parameter change is large enough and cannot be compensated by the control loops installed. Hence, also in this case a quantisation of the parameter changes into faulty and non-faulty ones is reasonable.

In Fig. 9.1 the fault is, therefore, described by its qualitative value $[\boldsymbol{e}]$, which is transformed into the actual real-valued fault parameter or fault signal by an injector. Diagnosis has only to find the qualitative value $[e]$ rather than the real value $\boldsymbol{e}$. Compared with the diagnosis of discrete-event systems investigated in Chapter 8 the qualitative value $[e(k_h)]$ of the fault corresponds to a symbolic fault $f(k_h)$ that occurs at the given time k_h. Therefore, in the following, the qualitative fault $[e]$ will alternatively be denoted by the fault symbol f.

In summary, in many practical situations the relevant information used in supervisory control is included in the quantised signal. The introduction of the injectors and the quantisers aims at reducing the information and, in this way, at simplifying the control task. It follows the guideline:

> Many process supervision tasks can be solved with reasonable effort only if as much information about the system as possible is ignored.

If a more global information about the system is sufficient to solve a given task, then this global information should be used rather than the more detailed one. In the quantised system approach the resolution of the injector and the quantiser can be used to adapt the "granularity" of the information used by the supervisor to the task to be solved.

9.1.2 The quantised system approach to supervisory control

A quantised system represents a dynamical system with real-valued signals that can only be measured through quantisers (Fig 9.1). Instead of the real-valued input $\boldsymbol{u}(k)$ and output $\boldsymbol{y}(k)$ only the quantised signal values $[\boldsymbol{u}(k)]$ and $[\boldsymbol{y}(k)]$ are available. All tasks that will be considered in this chapter should be solved by using the quantised signals only.

With respect to the general hybrid system shown in Fig. 3.6, the discrete-event subsystem is missing here. This simplification is made in order to emphasise the main problems that occur in the situation that a continuous-variable system has to be supervised by using symbolic information only. It will be shown that the quantised system can be described by a discrete-event model, which includes the continuous-variable subsystem together with the quantisers and injector. Therefore, the extension to hybrid systems can obviously be made by combining this discrete-event model of the quantised system with the model of the discrete subsystem.

The way how process supervision tasks can be solved for quantised systems will be explained in this chapter by considering two important problems: the observation of the quantised state of the system and the diagnosis of faults. The following presents the problems to be solved together with a brief outline of the way of solution, which will be explained in Sections 9.5 and 9.6. The quantised systems considered here are discrete-time systems where the time k refers to the k-th sampling time.

State observation. State observation concerns the problem of determining the internal state of a dynamical system from the input and output measurements. For continuous-variable systems the main idea is to use a LUENBERGER observer that determines an approximation $\hat{\boldsymbol{x}}$ of the continuous state $\boldsymbol{x}$. However, the application of these results is possible only if the systems input and output are measured quantitatively and if the model has the form of differential or difference equations.

For the quantised system, only symbolic input and output information is available, but a similar state observation problem can be posed. The measurement information yield the sequence of quantised input values

$$[\boldsymbol{U}(0 \ldots k_h)] = ([\boldsymbol{u}(0)],\ \ldots,\ [\boldsymbol{u}(k_h)])$$

and the sequence of quantised output values

$$[\boldsymbol{Y}(0 \ldots k_h)] = ([\boldsymbol{y}(0)],\ \ldots,\ [\boldsymbol{y}(k_h)]).$$

Due to the more abstract measurement information the task is to reconstruct the qualitative state $[\boldsymbol{x}]$ rather than the real-valued state $\boldsymbol{x}$. The observation problem can be stated as follows:

Problem 9.1 (Observation problem for quantised systems)

Given: *Sequence of quantised input values.*
Sequence of quantised output values.
Model $\mathcal{M}$ of the quantised system.

Find: *Current qualitative state* $[\boldsymbol{x}(k_h)]$.

The algorithm presented in Section 9.5 estimates the probability

$$\mathrm{Prob}([\boldsymbol{x}(k_h)] \mid [\boldsymbol{U}(0 \ldots k_h)], [\boldsymbol{Y}(0 \ldots k_h)])$$

that the qualitative state has the value $[\boldsymbol{x}(k_h)]$ under the condition that the given input and output sequences occurred. The current qualitative state belongs to the set

$$\boldsymbol{\mathcal{X}}(k_h \mid k_h) = \{[\boldsymbol{x}(k_h)] \ : \ \mathrm{Prob}([\boldsymbol{x}(k_h)] \mid [\boldsymbol{U}(0 \ldots k_h)], [\boldsymbol{Y}(0 \ldots k_h)]) > 0\}\,, \quad (9.1)$$

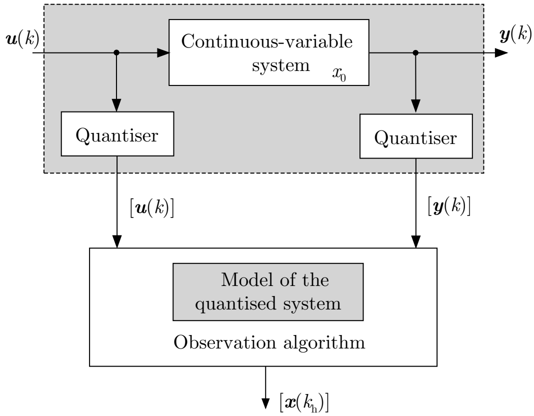


Fig. 9.2. State observation of quantised systems

where the notation $\mathcal{X}(k_h \mid k_h)$ means that the state at time k_h is reconstructed for given qualitative input and output sequences up to time k_h.

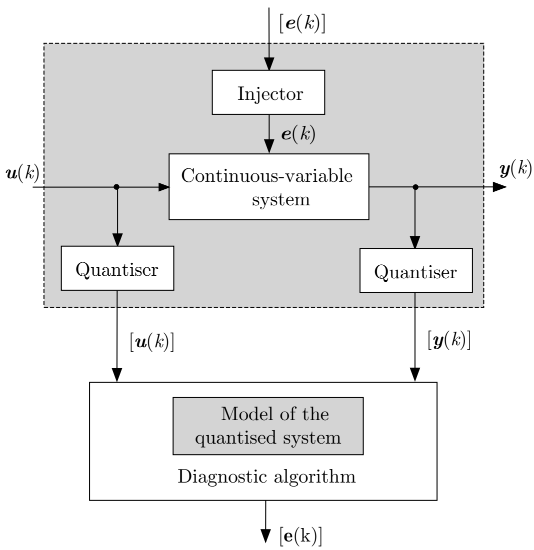


Fig. 9.3. Diagnosis of quantised systems

Process diagnosis. The diagnostic task for quantised systems concerns the problem of finding the symbolic fault value f from the quantised measurement sequences:

Problem 9.2 (Diagnostic problem for quantised systems)

Given: *Sequence of quantised input values.*

Sequence of quantised output values.

Model $\mathcal{M}$ of the quantised system.

Find: *Fault $f(k_h) = [\boldsymbol{e}(k_h)]$.*

The algorithm presented in Section 9.6 estimates the probability

$$\text{Prob}(f(k_h) \,|\, [\boldsymbol{U}(0 \dots k_h)], [\boldsymbol{Y}(0 \dots k_h)])$$

that a fault $f(k_h) = [\boldsymbol{e}(k_h)]$ has occurred provided that the quantised system with qualitative input sequence $[\boldsymbol{U}(0 \dots k_h)]$ has produced the qualitative output sequence $[\boldsymbol{Y}(0 \dots k_h)]$. In on-line applications, this task is solved for increasing time horizon $k_h = 1, 2, \dots$ and leads to the set

$$\mathcal{F}(k_h \mid k_h) = \{f(k_h) \,:\, \text{Prob}(f(k_h) \mid [\boldsymbol{U}(0 \dots k_h)], [\boldsymbol{Y}(0 \dots k_h)]) > 0\} \qquad (9.2)$$

of fault candidates.

9.2 Quantised systems

9.2.1 Continuous-variable system

The core of a quantised system is the continuous-variable discrete-time system

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{g}(\boldsymbol{x}(k), \boldsymbol{u}(k)), \qquad \boldsymbol{x}(0) \in \mathcal{X}_0 &\quad (9.3)\\ \boldsymbol{y}(k) &= \boldsymbol{h}(\boldsymbol{x}(k), \boldsymbol{u}(k)) &\quad (9.4) \end{aligned}$$

with input vector $\boldsymbol{u} \in \mathbb{R}^m$, output vector $\boldsymbol{y} \in \mathbb{R}^r$ and the vector of the internal state $\boldsymbol{x} \in \mathbb{R}^n$. $\mathcal{X}_0 \subseteq \mathbb{R}^n$ is the set of initial states that the system can assume. If $\boldsymbol{x}(0)$ is known, this set is a singleton. However, as the state $\boldsymbol{x}$ is not measurable, $\mathcal{X}_0$ is usually a subset of the state space $\mathbb{R}^n$.

It is assumed that for any initial state $\boldsymbol{x}(0) \in \mathcal{X}_0$ and input sequence

$$\boldsymbol{U}(0 \dots k_h) = (\boldsymbol{u}(0), \boldsymbol{u}(1), \dots, \boldsymbol{u}(k_h))$$

Eqs. (9.3), (9.4) generate a unique state and a unique output sequence

$$\begin{aligned} \boldsymbol{X}(0 \dots k_h) &= (\boldsymbol{x}(0), \boldsymbol{x}(1), \dots, \boldsymbol{x}(k_h)) \\ \boldsymbol{Y}(0 \dots k_h) &= (\boldsymbol{y}(0), \boldsymbol{y}(1), \dots, \boldsymbol{y}(k_h)), \end{aligned}$$

which are considered over the time interval $[0, k_h]$.

If the system is linear, Eqs. (9.3), (9.4) have the form

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k), \qquad \boldsymbol{x}(0) \in \mathcal{X}_0 &\quad (9.5)\\ \boldsymbol{y}(k) &= \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}\boldsymbol{u}(k) &\quad (9.6) \end{aligned}$$

with matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ of appropriate dimensions. Then, explicit solutions are known:

$$\begin{aligned}\boldsymbol{x}(k) &= \boldsymbol{A}^k\boldsymbol{x}(0) + \sum_{i=0}^{k-1}\boldsymbol{A}^{k-1-i}\boldsymbol{B}\boldsymbol{u}(i)\\ \boldsymbol{y}(k) &= \boldsymbol{C}\boldsymbol{A}^k\boldsymbol{x}(0) + \sum_{i=0}^{k-1}\boldsymbol{C}\boldsymbol{A}^{k-1-i}\boldsymbol{B}\boldsymbol{u}(i) + \boldsymbol{D}\boldsymbol{u}(k).\end{aligned}$$

Faults occurring in the system are described by an additional input signal $\boldsymbol{e}(k) \in \mathbb{R}^p$, which for the time horizon k_h is given by the sequence

$$\boldsymbol{E}(0 \dots k_h) = (\boldsymbol{e}(0), \boldsymbol{e}(1), \dots, \boldsymbol{e}(k_h)).$$

Accordingly, the state-space model has to be extended to become

$$\boldsymbol{x}(k+1) = \boldsymbol{g}(\boldsymbol{x}(k), \boldsymbol{u}(k), \boldsymbol{e}(k)), \quad \boldsymbol{x}(0) \in \mathcal{X}_0 \tag{9.7}$$

$$\boldsymbol{y}(k) = \boldsymbol{h}(\boldsymbol{x}(k), \boldsymbol{u}(k), \boldsymbol{e}(k)). \tag{9.8}$$

For any given input and fault sequences $\boldsymbol{U}$ and $\boldsymbol{E}$ this system is assumed to generate unique state and output sequences $\boldsymbol{X}$ and $\boldsymbol{Y}$.

9.2.2 Quantisation of the signal spaces

The continuous-variable system is considered in quantised signal spaces. The *quantisers* introduce partitions of the signal spaces $\mathbb{R}^m$ and $\mathbb{R}^r$ into a finite number of disjoint sets $\mathcal{Q}_u(\nu)$ $(\nu \in \mathcal{N}_u = \{0, 1, \dots, M\})$ and $\mathcal{Q}_y(\omega)$ $(\omega \in \mathcal{N}_y = \{0, 1, \dots, R\})$, where $\mathcal{Q}_u(\nu)$ or $\mathcal{Q}_y(\omega)$ denote the set of input values $\boldsymbol{u}$ or output values $\boldsymbol{y}$ with the same quantised values ν or ω. The mapping invoked by the quantiser is symbolised by $[\cdot]$:

$$[\boldsymbol{u}] = \nu \iff \boldsymbol{u} \in \mathcal{Q}_u(\nu) \tag{9.9}$$

$$[\boldsymbol{y}] = \omega \iff \boldsymbol{y} \in \mathcal{Q}_y(\omega). \tag{9.10}$$

The numbers ν or ω are called the quantised values or the *qualitative values* of the input or output, respectively, and $[\boldsymbol{u}]$ or $[\boldsymbol{y}]$ the *qualitative input* or *qualitative output*.

The sets $\mathcal{Q}_u(\nu)$ $(\nu \neq 0)$ and $\mathcal{Q}_y(\omega)$ $(\omega \neq 0)$ are assumed to be bounded while $\mathcal{Q}_u(0)$ and $\mathcal{Q}_y(0)$ are the unbounded "remaining" subsets of $\mathbb{R}^m$ or $\mathbb{R}^r$, respectively.

For discrete input or output, the quantisation is equivalent to the enumeration of the discrete signal values. For the tank system introduced in Section 2.1 the input $Pos(V_1)$ has two values, which are denoted by 1 and 0: $[Pos(V_1)] \in \{0, 1\}$.

In order to get a concise model of the quantised system, it is reasonable to introduce a quantisation of the state space $\mathbb{R}^n$ into the partitions $\mathcal{Q}_x(\zeta)$ $(\zeta \in \mathcal{N}_x = \{0, 1, \dots, N\})$ with

$$[\boldsymbol{x}] = \zeta \iff \boldsymbol{x} \in \mathcal{Q}_x(\zeta).$$

The number ζ of the partition $\mathcal{Q}_x(\zeta)$ to which the current state $\boldsymbol{x}$ belongs is called the *qualitative state* (although it is, more precisely, the qualitative value of the state).

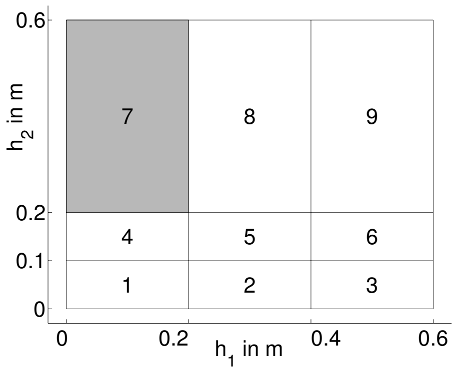


Fig. 9.4. Partition of a two-dimensional state space of the tank system

Figure 9.4 shows an example for the partition of the two-dimensional state space of the tank system. The grey region includes all states $\boldsymbol{x} = (h_1, h_2)'$ with the same qualitative value $[\boldsymbol{x}] = 7$. The two-dimensional state space outside the nine partitions constitutes the unbounded partition $Q_x(0)$.

A consequence of the introduction of the quantiser is the fact that no distinction can be made between different input, state and output values that belong to the same region $\mathcal{Q}_u(\nu)$, $\mathcal{Q}_x(\zeta)$ or $\mathcal{Q}_y(\omega)$. In an application, the regions have to be chosen in such a way that it does not matter for the solution of the process supervision task which real-valued input, state or output of a given region really occurs. The input or output quantisation may be given by practical circumstances, for example, by the sensor locations. The state quantisation can usually be arbitrarily chosen, which gives the possibility to adapt this quantisation to the dynamical phenomena that occur in the system.

Fault injector. The injector shown in Fig. 9.1 transforms the qualitative fault value $[\boldsymbol{e}]$ into a real-valued signal $\boldsymbol{e}$. This transformation can be considered as the inverse operation of quantisation. Every qualitative value $f = [\boldsymbol{e}]$ is associated with a partition $\mathcal{Q}_e$ of the signal space $\mathbb{R}^p$ of the fault signal $\boldsymbol{e}$. As only the qualitative fault value f is assumed to be known, the fault signal $\boldsymbol{e}$ is only known to belong to the partition $\mathcal{Q}_e(f)$:

$$[\boldsymbol{e}] = f \quad \Longleftrightarrow \quad \boldsymbol{e} \in \mathcal{Q}_e(f).$$

In the example considered in the following section the leakage in a tank is described by the flow constant c_2 of a hole which is partitioned into two intervals. The first inverval represents a very small leakage that is considered as negligible (faultless) and the second interval corresponds to the fault.

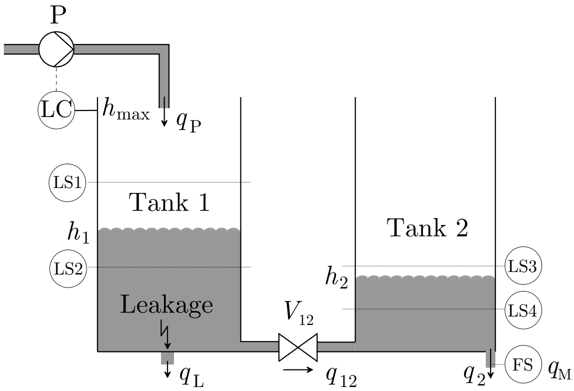


Fig. 9.5. Two-tank system with quantised level measurement

Example 9.1 *Quantised two-tank system*

Consider the tank example introduced in Section 2.1 and assume that the sensors give only quantised information (Fig. 9.5). That is, the sensors merely signal whether the liquid level is above or below their position. The result is a quantised measurement information for the tank levels h_1 and h_2 and the outflow q_M of Tank 2. The quantisation intervals are summarised in Table 9.1.

Table 9.1 Signal quantisation

$[h_1] = 1$	if	$0\,\text{m} \le h_1 < 0.2\,\text{m}$	$\Leftrightarrow$ LS2 $= 0$
$[h_1] = 2$	if	$0.2\,\text{m} \le h_1 < 0.4\,\text{m}$	$\Leftrightarrow$ LS2 $= 1 \wedge$ LS1 $= 0$
$[h_1] = 3$	if	$0.4\,\text{m} \le h_1 < 0.6\,\text{m}$	$\Leftrightarrow$ LS1 $= 1$
$[h_2] = 1$	if	$0\,\text{m} \le h_2 < 0.1\,\text{m}$	$\Leftrightarrow$ LS4 $= 0$
$[h_2] = 2$	if	$0.1\,\text{m} \le h_2 < 0.2\,\text{m}$	$\Leftrightarrow$ LS4 $= 1 \wedge$ LS3 $= 0$
$[h_2] = 3$	if	$0.2\,\text{m} \le h_2 < 0.6\,\text{m}$	$\Leftrightarrow$ LS3 $= 1$
$[q_M] = 1$	if	$0\,\text{l/min} \le q_M < 3\,\text{l/min}$	
$[q_M] = 2$	if	$3\,\text{l/min} \le q_M < 6\,\text{l/min}$	
$[q_M] = 3$	if	$6\,\text{l/min} \le q_M < 10\,\text{l/min}$	

With this quantisation, the input and output of the tank system have the signal values summarised in Table 9.2. The two quantised level measurements $[h_1]$ and $[h_2]$ yield the partition of the output space. Every region $[(h_1, h_2)] \in \{1, 2, ..., 9\}$ corresponds to one combination of quantised values for $[h_1]$ and $[h_2]$. In the following either the quantised level measurement $[(h_1,\ h_2)]$ or the quantised outflow $[q_M]$ is used as quantised output. □

Table 9.2 Quantised input, output and faults of the tank system

Symbol	Value set	Meaning
		Inputs
$[V_{12}]$	$\{1,2\}$	Connecting valve closed for $[V_{12}]=1$, open for $[V_{12}]=2$
$[u_P]$	$\{1,2\}$	Pump off for $[u_P]=1$, nominal velocity for $[u_P]=2$
		Outputs
$[h_1,h_2]$	$\{1,2,...,9\}$	Quantised level of the tanks
$[q_M]$	$\{1,2,3\}$	Quantised outflow of Tank 2
		Fault
$[c_L]$	$\{1,2\}$	No leakage for $[c_L]=1$, leakage for $[c_L]=2$

9.2.3 Behaviour of quantised systems

In the succeeding investigations the main ideas of modelling quantised systems and observing the qualitative state of such systems will be explained without reference to possible faults. Therefore, the faultless system (9.3), (9.4) will be considered.

The behaviour of the quantised system is the set of all I/O pairs

$$([\boldsymbol{U}(0\,...\,k_h)],[\boldsymbol{Y}(0\,...\,k_h)])$$

that are consistent with the system dynamics and the signal quantisation. As the qualitative input and output are considered, the behaviour is also referred to as the *qualitative behaviour* of the system (9.3), (9.4):

$$\mathcal{B}_{\text{qual}}(k_h)=\{([\boldsymbol{U}(0\,...\,k_h)],[\boldsymbol{Y}(0\,...\,k_h)])\ :\ \text{Eqs. (9.3), (9.4) hold}\}. \qquad (9.11)$$

As all measurements are qualitative, the initial state of the system is considered on the qualitative level of abstraction as well. This is why all the following investigations concern the set of those qualitative I/O pairs that the quantised system can generate for a *qualitatively* given initial state. If the qualitative initial state $\zeta(0)=[\boldsymbol{x}(0)]$ is precisely known, $\mathcal{X}_0=\mathcal{Q}_x(\zeta(0))$ holds. If the qualitative initial state is unknown, the set $\mathcal{X}_0$ is the union of several state partitions $\mathcal{Q}_x(\zeta)$ or $\mathcal{X}_0=\mathrm{I\!R}^n$ holds.

In order to get a better imagination of the qualitative behaviour $\mathcal{B}_{\text{qual}}$ note that for the elements of the I/O sequences the relations

$$[\boldsymbol{u}(k)]\in\mathcal{N}_u \quad = \quad \{0,1,\,...,\,M\} \qquad (9.12)$$

$$[\boldsymbol{y}(k)]\in\mathcal{N}_y \quad = \quad \{0,1,\,...,\,R\} \qquad (9.13)$$

and, hence,

$$[\boldsymbol{U}(0\,...\,k_h)]\in\mathcal{N}_u^{k_h+1} \quad = \quad \mathcal{N}_u\times\mathcal{N}_u\times\,...\,\times\mathcal{N}_u \qquad (9.14)$$

$$[\boldsymbol{Y}(0\,...\,k_h)]\in\mathcal{N}_y^{k_h+1} \quad = \quad \mathcal{N}_y\times\mathcal{N}_y\times\,...\,\times\mathcal{N}_y \qquad (9.15)$$

hold, where the Cartesian products on the right-hand side include the given sets k_h+1 times. Consequently, the qualitative behaviour is a subset of the Cartesian product of $\mathcal{N}_u^{k_h+1}$ and $\mathcal{N}_y^{k_h+1}$:

$$\mathcal{B}_{\text{qual}}(k_h) \subseteq \mathcal{N}_u^{k_h+1} \times \mathcal{N}_y^{k_h+1}. \tag{9.16}$$

The number of elements of this product is $((M+1)\cdot(R+1))^{k_h+1}$. If, for example, only 3 qualitative input and 4 qualitative output signals are considered and the time horizon is $k_h = 4$, this number is $160,000$. The behaviour $\mathcal{B}_{\text{qual}}(k_h)$ selects for a given time horizon k_h those elements out of this large set, which are consistent with the system. How large this number of elements is depends on the system properties. If the system were deterministic (which it is generally not as described below), it would generate a unique output sequence $[\boldsymbol{Y}(0\ldots k_h)]$ for every given input sequence $[\boldsymbol{U}(0\ldots k_h)]$ and initial state. In the given example, 243 different input sequences $[\boldsymbol{U}(0...4)]$ exist, which lead to the same number of elements of $\mathcal{B}_{\text{qual}}(4)$ for every initial state.

If later a faulty system is concerned, the behaviour is represented by all triples $([\boldsymbol{U}],[\boldsymbol{E}],[\boldsymbol{Y}])$ that are consistent with the system (9.7), (9.8), the quantisers and the injector. It is given by

$$\mathcal{B}_{\text{qual}}(k_h) = \{([\boldsymbol{U}(0\ldots k_h)],[\boldsymbol{E}(0\ldots k_h)],[\boldsymbol{Y}(0\ldots k_h)]) \;:\; \text{Eq. (9.7), (9.8) hold}\}. \tag{9.17}$$

Non-determinism of the qualitative behaviour. An important issue is the fact that it is impossible to predict the qualitative output sequence of a quantised system unambiguously for given qualitative initial state and qualitative input. The set $\mathcal{B}_{\text{qual}}(k_h)$ includes, in general, more than one element $([\boldsymbol{U}],[\boldsymbol{Y}])$ with the same qualitative input sequence $[\boldsymbol{U}]$. The reason for this is given by the fact that the system (9.3), (9.4) may start from any initial state $\boldsymbol{x}(0)$ with the given qualitative value $\zeta(0) = [\boldsymbol{x}(0)]$ and may obtain any input sequence $\boldsymbol{U}$ which is only described by the qualitative sequence $[\boldsymbol{U}]$. For these sets of initial states and input sequences, the resulting output sequences $\boldsymbol{Y}$ yield, in general, different qualitative sequences $[\boldsymbol{Y}]$. This phenomenon is referred to as the *non-determinism of the qualitative behaviour*.

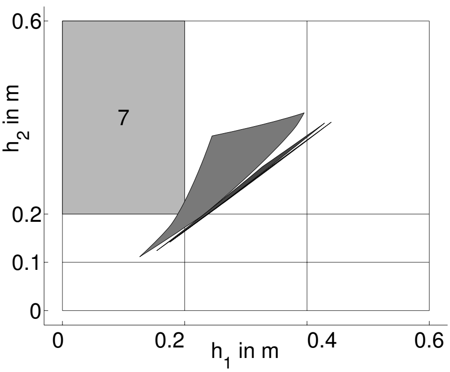


Fig. 9.6. Sets of states reached by the tank system for qualitative initial state $\boldsymbol{x}(0) \in \mathcal{Q}_x(7)$

Example 9.2 *Non-determinism of the qualitative behaviour of the tank system*

The reason why the qualitative behaviour is non-deterministic is illustrated by Fig. 9.6, which shows the set of state trajectories $\boldsymbol{X}$ of the tank system for the initial quantised set of states $[(h_1, h_2)'] = 7$. At time $k = 0$ the tank system may assume any state in the region 7 of the partitioned state space, because the initial state is only qualitatively known. For time $k = 1, 2, 3$, the system may be in any state of the succeeding regions, which were determined for constant input $[V_{12}] = 2$ and $[u_P] = 2$. The important point is that these regions overlap with more than one state partition. Hence, the system generates different qualitative state trajectories

$$[\boldsymbol{X}] = ([\boldsymbol{x}(0)], [\boldsymbol{x}(1)], [\boldsymbol{x}(2)], [\boldsymbol{x}(3)], ...),$$

for example

$$\begin{aligned}
[\boldsymbol{X}(0...3)] &= (7, 8, 8, 8) \\
[\boldsymbol{X}(0...3)] &= (7, 4, 4, 4) \\
[\boldsymbol{X}(0...3)] &= (7, 5, 8, 8) \\
[\boldsymbol{X}(0...3)] &= (7, 8, 9, 9)
\end{aligned}$$

and, hence, different output trajectories. If, for example, the output is identical to the second level h_2, the quantised output sequences are

$$\begin{aligned}
[\boldsymbol{Y}(0...3)] &= (3, 3, 3, 3) \\
[\boldsymbol{Y}(0...3)] &= (3, 2, 2, 2) \\
[\boldsymbol{Y}(0...3)] &= (3, 2, 3, 3),
\end{aligned}$$

where the output $y = h_2$ with three qualitative values has been used. Figure 9.7 shows a graphical representation of the set of qualitative state sequences.

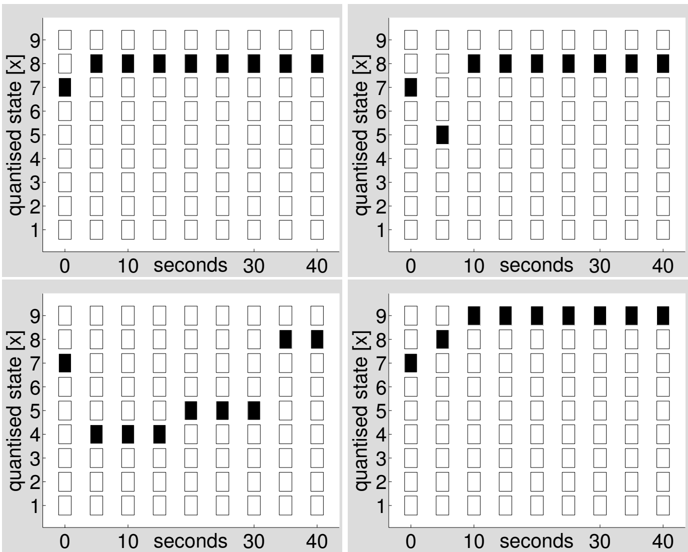
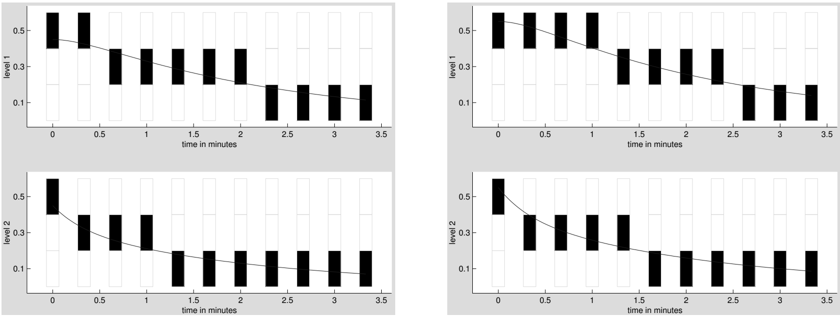


Fig. 9.7. Four quantised trajectories of the tank system for the same quantised initial state $[\boldsymbol{x}_0] = 7$ and constant input $[V_{12}] = 2$, $[u_P] = 2$

This example shows that the non-determinism occurs because a bundle of trajectories has to be considered rather than a unique trajectory. This bundle starts in the common qualitative state $[\boldsymbol{x}_0] = 7$ and has a common qualitative input sequence. The trajectories of this bundle generate different qualitative sequences $[\boldsymbol{Y}]$ and, hence, the qualitative output sequence cannot be predicted unambiguously for given qualitative input sequence and qualitative initial state.



(a) (b)

Fig. 9.8. Two different quantised trajectories of the tank system starting in the same quantised initial state $[\boldsymbol{x}_0]$

Figure 9.8 shows how this non-determinism becomes obvious in experiments with the tank system. In contrast to Fig. 9.7, Fig. 9.8 shows the two quantised tank levels rather than the enumerated state. The two parts of the figure concern two experiments, in which the initial liquid levels are qualitatively the same, but differ quantitatively. The thin lines show the quantitative tank behaviour. Obviously, the resulting qualitative trajectories are different.

This phenomenon has a direct consequence concerning the assertions that an experienced human operator can make about the tank system. If asked about the qualitative tank levels at time $k = 1$ under the assumption that the qualitative initial state is 7, the operator can only predict a set of qualitative levels rather than a unique one. This is, because the operator has merely the qualitative knowledge that Tank 1 has initially a low level h_1 and Tank 2 a high level h_2. The uncertainty of the knowledge about the future tank level is not based on insufficient knowledge about the dynamics of the system under consideration but on the uncertainty of the initial levels in both tanks resulting from the quantised information available. □

The example makes it obvious that the quantised system behaves like a stochastic process. The initial state $\boldsymbol{x}(0)$ can be assumed to be chosen randomly among all initial states with the given qualitative value $\zeta(0) = [\boldsymbol{x}(0)]$. In a more general experiment, also the input sequence $\boldsymbol{U}$ can be chosen randomly among all input sequences with a fixed qualitative value $[\boldsymbol{U}]$. For each of these initial states and input sequences the system generates a unique qualitative output sequence $[\boldsymbol{Y}]$, but since the initial state and the input vary from experiment to experiment so do the qualitative output sequences. No unambiguous prediction of $[\boldsymbol{Y}]$ can be made, but some probabilistic prediction is possible. Such a prediction will be considered in Section 9.4, where a stochastic automaton will be set up that generates a probability distribution for all the qualitative output values generated by the quantised system.

9.2.4 Stochastic properties of quantised systems

In the following, the set $\mathcal{X}_0$ of initial states $\boldsymbol{x}_0$ considered is assumed to be qualitatively given. That is, it is the union of one or more state partitions $Q_x(\zeta)$. The initial state is assumed to be randomly distributed over $\mathcal{X}_0$. Under these circumstances, consider the probability

$$\text{Prob}([\boldsymbol{Y}(0 \ldots k_h)] \,|\, [\boldsymbol{U}(0 \ldots k_h)], \mathcal{X}_0)$$

with which the qualitative output sequence $[\boldsymbol{Y}]$ occurs for a given qualitative input sequence $[\boldsymbol{U}]$ and qualitatively given initial state set $\mathcal{X}_0$. This probability characterises how often the I/O pair $([\boldsymbol{U}], [\boldsymbol{Y}])$ occurs if the probability of the occurrence of $[\boldsymbol{U}]$ and, moreover, the probability of the initial state $\boldsymbol{x}(0) \in \mathcal{X}_0$ are known. Note that the dependency of $\text{Prob}([\boldsymbol{Y}(0 \ldots k_h)] \,|\, [\boldsymbol{U}(0 \ldots k_h)], \mathcal{X}_0)$ upon $\boldsymbol{x}(0)$ is given implicitly by Eqs. (9.3) and (9.4).

To better understand the meaning of this probability, assume that the initial state $\boldsymbol{x}(0)$ is known to belong to one state partion $Q_x(\zeta_0)$ for given ζ_0. Then $\text{Prob}([\boldsymbol{Y}(0 \ldots k_h)] \,|\, [\boldsymbol{U}(0 \ldots k_h)], \mathcal{X}_0)$ says how often $[\boldsymbol{Y}]$ occurs for a given input sequence $[\boldsymbol{U}]$ if many experiments are made with the system starting in the same initial state $\boldsymbol{x}_0 \in \mathcal{X}_0$. If a specific I/O pair $([\boldsymbol{U}], [\boldsymbol{Y}])$ never occurs, because the system cannot follow the qualitative output sequence $[\boldsymbol{Y}]$ for the qualitative input sequence $[\boldsymbol{U}]$ and for some initial state $\boldsymbol{x}(0) \in \mathcal{X}_0$, then

$$\text{Prob}([\boldsymbol{Y}(0 \ldots k_h)] \,|\, [\boldsymbol{U}(0 \ldots k_h)], \mathcal{X}_0) = 0$$

holds. Other I/O pairs may occur with different frequencies, which lead to positive probability values.

If this probability should be determined, the "experiments" considered just now have to be investigated in more detail. Since $[\boldsymbol{U}]$ is the input to the quantised system whereas $[\boldsymbol{Y}]$ describes the effect of this input together with the initial state, the input $\boldsymbol{U}$ and the initial state $\boldsymbol{x}(0)$ have to be varied from experiment to experiment within the given sets of qualitatively equivalent input sequences and initial states. As the frequency, with which a certain output sequence $\boldsymbol{Y}$ occurs, also depends on how often a certain input sequence $\boldsymbol{U}$ and initial state $\boldsymbol{x}(0)$ is chosen, the probability distribution of these variables have to be fixed. This can be done, in principle, in an arbitrary way. To understand the following investigations, it can be assumed that *uniform* probability distributions are used. Then every value $\boldsymbol{x}(0)$ that belongs to the set $\mathcal{X}_0$ occurs with the same probability. The same holds for the input sequences. However, the following investigations are valid for arbitrary probability distributions.

The experiments to be made have to bring the system in the chosen initial state $\boldsymbol{x}(0)$ and to determine the qualitative system trajectory $[\boldsymbol{Y}(0 \ldots k_h)]$ for the chosen input sequence $\boldsymbol{U}(0 \ldots k_h)$. After all experiments have been made, the relative frequencies of the occurrence of the different qualitative output sequences $[\boldsymbol{Y}]$ give the (approximate) value of the probability $\text{Prob}([\boldsymbol{Y}(0 \ldots k_h)] \,|\, [\boldsymbol{U}(0 \ldots k_h)], \mathcal{X}_0)$ to be found.

With this probability, the qualitative behaviour of the quantised system defined in Eq. (9.11) can be expressed in the form

$$\begin{aligned} &\mathcal{B}_{\mathrm{qual}}(k_h) &(9.18)\\ &= \{[\boldsymbol{U}(0 \dots k_h)], [\boldsymbol{Y}(0 \dots k_h)] \;:\; \mathrm{Prob}([\boldsymbol{Y}(0 \dots k_h)] \,|\, [\boldsymbol{U}(0 \dots k_h)], \mathcal{X}_0) > 0\}. \end{aligned}$$

As the number of elements of $\mathcal{B}_{\mathrm{qual}}(k_h)$ is very large (cf. Section 9.2.3), it is not possible to graphically illustrate the probability of the I/O pairs. However, it is often more interesting to know with which probability a specific qualitative output value $[\boldsymbol{y}(k_h)]$ occurs. This probability can be obtained as boundary probability

$$\begin{aligned} &\mathrm{Prob}([\boldsymbol{y}(k_h)] \,|\, [\boldsymbol{U}(0 \dots k_h)], \mathcal{X}_0) &(9.19)\\ &= \sum_{[\boldsymbol{Y}(0 \dots k_h - 1)]} \mathrm{Prob}([\boldsymbol{Y}(0 \dots k_h)] \,|\, [\boldsymbol{U}(0 \dots k_h)], \mathcal{X}_0)\\ &= \sum_{[\boldsymbol{y}(0)], \dots, [\boldsymbol{y}(k_h - 1)]} \mathrm{Prob}([\boldsymbol{y}(0)], \dots, [\boldsymbol{y}(k_h)] \,|\, [\boldsymbol{U}(0 \dots k_h)], \mathcal{X}_0), \end{aligned}$$

where the summation is made over all elements $[\boldsymbol{y}(k)]$ of the output sequence with the exception of the last element $[\boldsymbol{y}(k_h)]$. This probability is depicted in Fig. 9.9 for the tank system with $\boldsymbol{y} = \boldsymbol{x} = (h_1, h_2)'$. This figure summarises the information given by the four state sequences depicted in Fig. 9.7 and associates these sequences with the probability of their occurrence. The darker the rectangles are, the higher is the probability. White rectangles show that the corresponding qualitative output value cannot occur at the corresponding time instant.

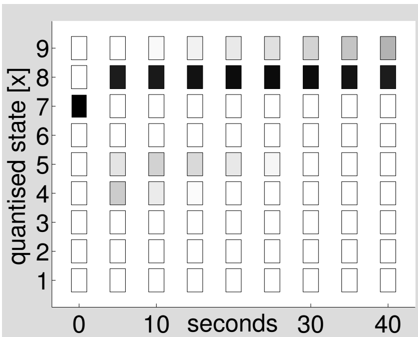


Fig. 9.9. Statistical properties of the tank system

Markov property. The probability considered so far concerns sequences of qualitative input and output values of arbitrary time horizon k_h:

$$\mathrm{Prob}([\boldsymbol{y}(0)], [\boldsymbol{y}(1)], \dots, [\boldsymbol{y}(k_h)] \mid [\boldsymbol{u}(0)], [\boldsymbol{u}(1)], \dots, [\boldsymbol{u}(k_h)], \mathcal{X}_0).$$

As these sequences may be arbitrarily long, it is impossible to store all the probability distributions,

$$\mathrm{Prob}([\boldsymbol{y}(0)] \mid [\boldsymbol{u}(0)], \mathcal{X}_0)$$

$$\mathrm{Prob}([\boldsymbol{y}(0)],[\boldsymbol{y}(1)] \mid [\boldsymbol{u}(0)],[\boldsymbol{u}(1)],\mathcal{X}_0)$$
$$\mathrm{Prob}([\boldsymbol{y}(0)],[\boldsymbol{y}(1)],[\boldsymbol{y}(2)] \mid [\boldsymbol{u}(0)],[\boldsymbol{u}(1)],[\boldsymbol{u}(2)],\mathcal{X}_0)$$
$$\vdots$$
$$\mathrm{Prob}([\boldsymbol{y}(0)],[\boldsymbol{y}(1)],\ ...,\ [\boldsymbol{y}(k_h)] \mid [\boldsymbol{u}(0)],[\boldsymbol{u}(1)],\ ...,\ [\boldsymbol{u}(k_h)],\mathcal{X}_0)$$

for all arguments. Therefore, it is interesting to known whether it is possible to determine these probability distributions recursively, where the probability distribution obtained for the time horizon k_h is determined from the probability distribution for the time horizon $k_h - 1$. Similarly, the probability considered in Eq. (9.19) should be determined from the probability of the output $[\boldsymbol{y}(k-1)]$.

Such recursive representations are possible, if the system possesses the Markov property. Then a transition probability p_{tr} exists with which the probability for time k_h can be obtained from the probability of time $k - 1$.

In general, dynamical systems possess the Markov property with respect to the state $\boldsymbol{x}$ although this property does not hold with respect to the output $\boldsymbol{y}$. This becomes obvious from Eq. (9.3),

$$\boldsymbol{x}(k+1) = \boldsymbol{g}(\boldsymbol{x}(k), \boldsymbol{u}(k)), \qquad \boldsymbol{x}(0) \in \mathcal{X}_0$$

where the state transition is represented by the function $\boldsymbol{g}$. This equation says that the state at time $k+1$ can be unambiguously determined from state $\boldsymbol{x}(k)$ without the knowledge of the states $\boldsymbol{x}(k-1)$, $\boldsymbol{x}(k-2)$ etc. occurring in the further history of the system. In a probabilistic setting, the probability of the state transition is one for the pair of states $(\boldsymbol{x}(k+1), \boldsymbol{x}(k))$ that occur in this equation together with the input $\boldsymbol{u}(k)$ and it is zero for all other pairs.

For the quantised system, the state $\boldsymbol{x}$ is replaced by the qualitative state $[\boldsymbol{x}]$. The Markov property would make it possible to represent the probability $\mathrm{Prob}([\boldsymbol{x}(k+1)])$ in dependence upon $\mathrm{Prob}([\boldsymbol{x}(k)])$. Then the relation

$$\mathrm{Prob}([\boldsymbol{x}(k+1)]) = \tilde{\boldsymbol{g}}(\mathrm{Prob}([\boldsymbol{x}(k)]), \mathrm{Prob}([\boldsymbol{u}(k)])) \tag{9.20}$$

would hold for some function $\tilde{\boldsymbol{g}}$. However, in general, a quantised system does *not* possess the Markov property and, hence, such a recursive representation does not exist. This has severe consequences for the solution of the modelling task which will be investigated in the next section:

> As the quantised system does not possess the Markov property with respect to the quantised input, state and output signals, every model that possesses the Markov property, can only be an approximate representation of the quantised system.

Example 9.3 *Violation of the Markov property by the tank system*

In order to explain why generally quantised systems do not possess the Markov property, Fig. 9.10 shows the movement of the initial set of states $\mathcal{X}_0$ of the tank system in the quantised state space for the constant input $[V_{12}] = 2$ and $[u_P] = 1$. Since the qualitative initial state is assumed to be $[\boldsymbol{x}(0)] = 3$,

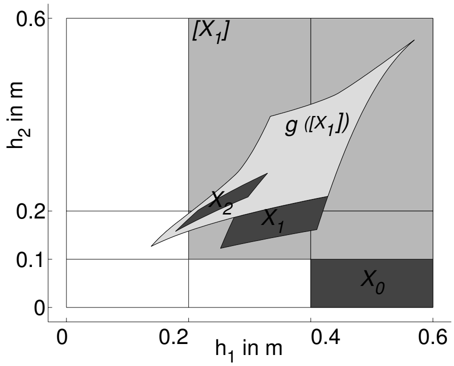


Fig. 9.10. Discussion of the Markov property of the quantised system

$$\mathcal{X}_0 = \mathcal{Q}_x(3),$$

the initial state $\boldsymbol{x}(0)$ lies in the partition denoted by X_0. The sets X_1 and X_2 are the sets of successor states $\boldsymbol{x}(1)$ and $\boldsymbol{x}(2)$ that can be obtained by the model given in Section 2.1. Obviously, $[\boldsymbol{x}(1)] \in [X_1] = \{5, 6, 8, 9\}$ and $[\boldsymbol{x}(2)] \in [X_2] = \{4, 5, 8\}$ hold. These results are obtained from a quantisation of the trajectory bundle of the continuous-variable system (9.3) that starts in the partition $\mathcal{Q}_x(3)$.

Now consider what a qualitative model of the form (9.20) can do. As the input is constant, the system can be considered as autonomous with the simpler model

$$\mathrm{Prob}([\boldsymbol{x}(k+1)]) = \tilde{\boldsymbol{g}}(\mathrm{Prob}([\boldsymbol{x}(k)])) \,.$$

For $k = 0$ and $\mathrm{Prob}([\boldsymbol{x}(0)]=3) = 1$ the function $\tilde{\boldsymbol{g}}$ yields

$$\begin{aligned} &\mathrm{Prob}([\boldsymbol{x}(1)]=8) = 0.11, &&\mathrm{Prob}([\boldsymbol{x}(1)]=9) = 0.06\\ &\mathrm{Prob}([\boldsymbol{x}(1)]=5) = 0.78, &&\mathrm{Prob}([\boldsymbol{x}(1)]=6) = 0.05 \end{aligned}$$

which is related to the relative overlap of the set X_1 with the partitions $\mathcal{Q}_x(5)$, $\mathcal{Q}_x(6)$, $\mathcal{Q}_x(8)$ and $\mathcal{Q}_x(9)$ (for a detailed analysis cf. Section 9.4). For $k = 1$ the function $\tilde{\boldsymbol{g}}$ has to determine $\mathrm{Prob}([\boldsymbol{x}(k+1)])$ by using this result only. As this function does not know that the system has started at $k = 0$ in the set $\mathcal{Q}_x(3)$, it assumes that the state can lie anywhere in the sets $\mathcal{Q}_x(5)$, $\mathcal{Q}_x(6)$, $\mathcal{Q}_x(8)$ and $\mathcal{Q}_x(9)$ whose union

$$[X_1] = \mathcal{Q}_x(5) \cup \mathcal{Q}_x(6) \cup \mathcal{Q}_x(8) \cup \mathcal{Q}_x(9)$$

is depicted in medium grey in Fig. 9.10. Therefore, the function $\tilde{\boldsymbol{g}}$ concerns the mapping of this set, which results in the light grey set in Fig. 9.10 which conservatively overapproximates the true set $X_2 = \boldsymbol{g}(\boldsymbol{g}(X_0))$ denoted by $\boldsymbol{g}([X_1])$. The light grey set intersects with the regions 4, 5, 8 and 9 of the partitioned state space. Consequently, $\tilde{\boldsymbol{g}}$ yields a positive probability $\mathrm{Prob}([\boldsymbol{x}(2)] = 9) > 0$ although the set X_2 does not intersect with $\mathcal{Q}_x(9)$ and, hence, the correct result is $\mathrm{Prob}([\boldsymbol{x}(2)]=9) = 0$.

The reason for the "error" in the calculation with the model 9.20 is the missing Markov property. The example demonstrates that with the additional knowledge about $[\boldsymbol{x}(0)]$ the determination of $[\boldsymbol{x}(2)]$ is better compared to the determination based on the information about $[\boldsymbol{x}(1)]$ only. In contrast to this, the Markov property requires that this additional knowledge has no effect. □

9.3 A behavioural view on the process supervision problems for quantised systems

As the Fig. 9.2 and 9.3 show, the observation and the diagnostic methods have access to the I/O pair $[\boldsymbol{U}(0 \ldots k_h)]$ and $[\boldsymbol{Y}(0 \ldots k_h)]$ of the quantised system. The observation problem is solved by looking for qualitative states $[\boldsymbol{x}(k_h)]$ for which the given I/O pair may occur. In general, the solution will not be unique but represents a set $\mathcal{Z}(k_h \mid k_h)$ of such qualitative states. The diagnostic problem is solved by searching for all faults $f(k_h)$ for which the given I/O pair may occur. It yields the set $\mathcal{F}(k_h)$ of fault candidates.

Note that in both problems, the distinction between input and output is of no importance. Both sequences included in the I/O pair together provide the information about the current movement of the quantised system used when solving the observation or diagnostic task. It has only to be known which I/O pairs may occur and which I/O pairs may not occur. It is just the information provided by the behaviour $\mathcal{B}_{\text{qual}}$ of the quantised system. Any model of the quantised system used when solving the observation or diagnostic task should provide this information.

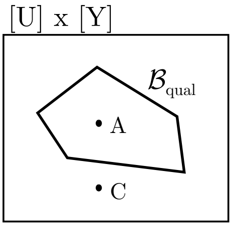


Fig. 9.11. Behaviour $\mathcal{B}_{\text{qual}}$

The behaviour $\mathcal{B}_{\text{qual}}(k_h)$ for given time horizont k_h has a nice graphical interpretation, because it is a subset of the Cartesian product of the sets of sequences $[\boldsymbol{U}(0 \ldots k_h)]$ and $[\boldsymbol{Y}(0 \ldots k_h)]$ (cf. Eq. (9.16) and Fig. 9.11). Qualitative I/O pairs $([\boldsymbol{U}(0 \ldots k_h)], [\boldsymbol{Y}(0 \ldots k_h)])$ for which the relation

$$([\boldsymbol{U}(0 \ldots k_h)], [\boldsymbol{Y}(0 \ldots k_h)]) \in \mathcal{B}_{\text{qual}}(k_h)$$

is valid are called *consistent* with the quantised system. For example, the I/O pair symbolised by the point A is consistent with the quantised system whereas the I/O pair C is inconsistent.

State observation of quantised systems. From the behavioural viewpoint, the observation problem consists in determining those qualitative state sequences

$$[\boldsymbol{X}(0 \ldots k_h)] = ([\boldsymbol{x}(0)], [\boldsymbol{x}(1)], \ldots, [\boldsymbol{x}(k_h)])$$

that may occur for the measured input sequence $[\boldsymbol{U}(0 \ldots k_h)]$ and may produce the measured output sequence $[\boldsymbol{Y}(0 \ldots k_h)]$. Then the I/O pair is called consistent with

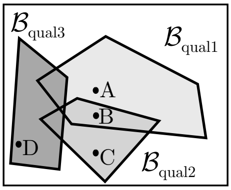


Fig. 9.12. Consistency-based observation and diagnosis

the behaviour that occurs for the initial state $\zeta_0 = [\boldsymbol{x}(0)]$ and ζ_0 is a possible qualitative initial state and $[\boldsymbol{x}(k_h)]$ a possible qualitative state at time k_h.

In more detail, assume that, as in Fig. 9.12, three different behaviours are given, which represent the quantised system for three different qualitative initial states $\mathcal{X}_0 = \mathcal{Q}_x(\zeta_{01})$, $\mathcal{X}_0 = \mathcal{Q}_x(\zeta_{02})$, and $\mathcal{X}_0 = \mathcal{Q}_x(\zeta_{03})$. If the measured I/O pair corresponds to point A, it is consistent with $\mathcal{B}_{\rm qual1}$, which implies that the qualitative initial state is found to be ζ_{01}. If C or D are measured, the qualitative initial state ζ_{02} or ζ_{03} are uniquely determined. The point B illustrates that the observation problem may not have a unique solution. The I/O pair represented by B is consistent with two behaviours and, hence, the quantised system may have one of the two qualitative initial states ζ_{01} and ζ_{02}.

Fault diagnosis of quantised systems. The same way of solution is used for the diagnostic problem with the only difference that the quantised system is now subject to the fault sequence $\boldsymbol{E}$. Then, the behaviours $\mathcal{B}_{\rm qual1}$, $\mathcal{B}_{\rm qual2}$ and $\mathcal{B}_{\rm qual3}$ used in Fig. 9.12 show the sets of triples

$$([\boldsymbol{U}(0 \dots k_h)], [\boldsymbol{E}(0 \dots k_h)], [\boldsymbol{Y}(0 \dots k_h)])$$

that are consistent with the quantised system for three different qualitative initial states. In the graphical representation, the behaviour is now a subset of the set of such triples.

The measured I/O pair $([\boldsymbol{U}], [\boldsymbol{Y}])$ includes the first and the third element of the triple $([\boldsymbol{U}], [\boldsymbol{E}], [\boldsymbol{Y}])$ describing the behaviour of the faulty system. To solve the diagnostic problem it has to be checked whether the measured pair is consistent with some triple in the sense that the measurements are identical with the first and the third element of the triple. Then a possible fault sequence is given by the second element $[\boldsymbol{E}]$ of this triple for the initial state $\zeta(0)$ to which the behaviour belongs. That is, the diagnostic task is solved by testing the consistency of the measured I/O pair with the behaviour of the quantised system. This illustrates *consistency-based diagnosis* of quantised systems.

Way of solution for both problems. The development of a solution to the observation and the diagnostic problems consists of two major steps. First, a suitable representation of the quantised system has to be found. Section 9.4 will show that a stochastic automaton is such a suitable representation and that the automaton can be determined from the description of the quantised system by means of Eqs. (9.3), (9.4) together with the quantisers. Second, methods that use this model and the observed input and output sequences must be elaborated in order to solve the state observation or the diagnostic problems. As the model used is a stochastic automaton, the methods developed in Chapter 8 can be applied for these purposes. As the results obtained for the stochastic automaton should be used for the quantised system, the automaton has to satisfy some completeness requirements, which will be developed in the sequel.

Requirement on the model. The behavioural view on the supervision problems to be solved shows that the model of the quantised system has to provide the behaviour $\mathcal{B}_{\mathrm{qual}}(k_h)$ of the quantised system for the relevant time horizon k_h. Therefore, any model that describes the qualitative behaviour $\mathcal{B}_{\mathrm{qual}}$ can be used. The model of the hybrid system given by Eqs. (2.1) – (2.3) together with the description of the quantisers given in Table 9.1 provides such a representation of the qualitative behaviour $\mathcal{B}_{\mathrm{qual}}$.

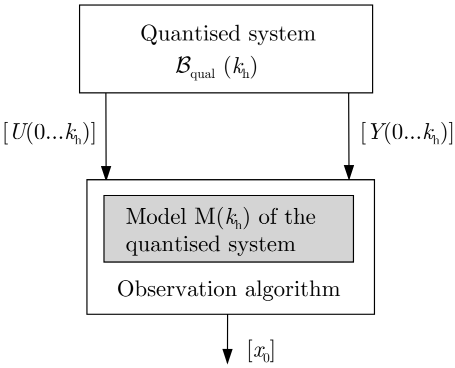


Fig. 9.13. A behavioural view on the observation problem

However, this hybrid representation of the quantised system leads to very complex observation and diagnostic algorithms because it combines differential equations with inequalities or logical formulae that describe the quantisers. Therefore, a more compact model will be introduced in Section 9.4 which has the form of a stochastic automaton whose state, input and output correspond directly with the quantised state, quantised input and quantised output of the given system. Instead of the behaviour $\mathcal{B}_{\mathrm{qual}}$, the supervision problems will be solved by means of the behaviour $\mathcal{M}$ of the stochastic automaton, which likewise depends on the qualitative initial state and the fault occurring in the quantised system (cf. Fig. 9.13). This model directly refers to the qualitative versions of the input, state and output and is called *qualitative model*.

As both the observation and the diagnostic methods are based on a consistency check for a given I/O pair and the model, the model has to represent *all* I/O pairs that may occur for the given quantised system, i.e. it has to be *complete* according to the following definition.

Definition 9.1 (Completeness)
A model with the behaviour $\mathcal{M}$ *that satisfies the relation*

$$\mathcal{M}(k_h) \supseteq \mathcal{B}_{\text{qual}}(k_h) \tag{9.21}$$

for all k_h *is called* complete.

Complete models include all I/O pairs for a given time horizon k_h that are consistent with the quantised system.

From Eq. 9.21 it follows that there may exist pairs

$$\begin{aligned}(V(0 \ldots k_h), W(0 \ldots k_h)) &\in \mathcal{M}(k_h)\\ (V(0 \ldots k_h), W(0 \ldots k_h)) &\notin \mathcal{B}_{\text{qual}}(k_h)\end{aligned}$$

which are consistent with the model but not with the quantised system. These pairs are called *spurious*. Their existence is a typical phenomenon encountered in qualitative modelling. The reason for the existence of spurious solutions is given by the fact that the qualitative model should be less complex than the precise model. Hence, it has to ignore some information about the properties of the quantised system. In particular, the qualitative model has the Markov property to provide a recursive representation of the behaviour $\mathcal{M}$ whereas the quantised system does not possess the Markov property.

The importance of the completeness of the model used for state observation and fault diagnosis is given by the following corollary:

> A model is suitable for solving the state observation problem or the diagnostic problem if and only if it is complete.

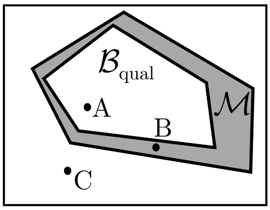


Fig. 9.14. Diagnosis with complete models

For a complete model with behaviour $\mathcal{M}$ an I/O pair like C in Fig. 9.14, which is not consistent with $\mathcal{M}$, is also not consistent with $\mathcal{B}_{\text{qual}}$:

$$([\boldsymbol{U}], [\boldsymbol{Y}]) \notin \mathcal{M}(k_h) \;\Rightarrow\; ([\boldsymbol{U}], [\boldsymbol{Y}]) \notin \mathcal{B}_{\text{qual}}(k_h). \tag{9.22}$$

This has a direct consequence for the observation problem. Assume that $\mathcal{B}_{\text{qual}}$ is the behaviour for $\mathcal{X}_0 = \mathcal{Q}_x(\zeta_0)$ and that the measured I/O pair corresponds to the point

C. Then the inconsistency of the I/O pair C with $\mathcal{M}$ implies the inconsistency of C with $\mathcal{B}_{\text{qual}}$ and it is known that the initial state cannot have been in $\mathcal{X}_0$: $\boldsymbol{x}_0 \notin \mathcal{X}_0$. On the other hand, an I/O pair like A, which is consistent with $\mathcal{B}_{\text{qual}}$ is also consistent with $\mathcal{M}$ and again the result obtained for $\mathcal{M}$ corresponds to the result that would be obtained for $\mathcal{B}_{\text{qual}}$.

Different results are obtained for I/O pairs that lie within $\mathcal{M}$ but not in $\mathcal{B}_{\text{qual}}$ like the pair B (spurious I/O pairs). Obviously, B is consistent with $\mathcal{M}$ but inconsistent with the quantised system. As a consequence, the sets of qualitative initial states or fault candidates that are obtained by a supervision algorithm that uses the model $\mathcal{M}$ are supersets of the solution sets that would be obtained if $\mathcal{B}_{\text{qual}}$ were used as representation of the quantised system. The fact that a superset of the solution is obtained rather than the precise solution, is the "price" for using the simple model $\mathcal{M}$ rather than the more complex representation of the quantised system by the behaviour $\mathcal{B}_{\text{qual}}$. The more spurious solutions exist, the more qualitative initial states or fault candidates occur in the solution set that should not occur there. Hence, the precision of the solution decreases. It is, therefore, the aim of modelling to find a model that satisfies the completeness requirement (9.21) with the smallest possible set of spurious solutions.

9.4 Discrete-event models of quantised systems

9.4.1 Modelling problem

From the point of view of the diagnostic algorithm that only has access to the qualitative I/O sequences, the input and output of the quantised system switch from one discrete value to another when time proceeds. Hence, the quantised system behaves like a discrete-event system. In more detail, Section 9.2.4 has shown that the quantised system is non-deterministic and can be considered as a stochastic process. The model used for diagnostic purposes has to describe this stochastic process.

> As the qualitative behaviour is non-deterministic, the model has to be non-deterministic.

This section shows how a model can be obtained that satisfies the completeness requirement (9.21).

To state the modelling problem more formally, denote the model input by $\nu(k)$, the model output by $\omega(k)$ and the behaviour of the model by $\mathcal{M}$. The solution to the modelling problem will be explained for the faultless system (9.3), (9.4) and later extended to faulty systems. For given initial state $\zeta(0)$ and given input sequence

$$\mathcal{V}(0 \ldots k_h) = (\nu(0), \nu(1), \nu(2), \ldots, \nu(k_h))$$

the model generates the output sequence

$$\Omega(0 \ldots k_h) = (\omega(0), \omega(1), \omega(2), \ldots, \omega(k_h)).$$

The model behaviour $\mathcal{M}(k_h) = \{(\mathcal{V}(0 \ldots k_h), \Omega(0 \ldots k_h))\}$ includes all I/O pairs with time horizon k_h that the model may produce. Since the model should be a representation of the quantised system, its input and output can assume the same values as the qualitative input and qualitative output signals of the quantised system: $\nu(k) \in \mathcal{N}_u$, $\omega(k) \in \mathcal{N}_y$.

Moreover, the model state ζ is interpreted as the qualitative state $[\boldsymbol{x}]$ of the system and, hence, $\zeta \in \mathcal{N}_x$ holds. This choice of the model state is very important because it makes it possible to associate with each model state a qualitative system state. In particular, for the initial state the relation

$$\zeta(0) = [\boldsymbol{x}(0)] \tag{9.23}$$

is valid.

As the considerations of Section 9.3 have shown, the completeness of the model is a fundamental requirement. The following sections show how stochastic automata can be found that provide a complete description of the quantised system.

9.4.2 Representation of autonomous quantised systems by stochastic automata

This and the next sections investigate how a quantised system can be described by a stochastic automaton (8.12)

$$\mathcal{S} = (\mathcal{N}_x, \mathcal{N}_u, \mathcal{N}_y, L, \mathrm{Prob}(z(0))) \tag{9.24}$$

whose state, input or output sets are identical to the sets of qualitative states, qualitative input values or qualitative output values, respectively. The modelling problem is to find the behavioural relation L such that the automaton $\mathcal{S}$ is a complete model of the quantised system. Then, the automaton is also called an *abstraction* of the system (9.3), (9.4).

Qualitative modelling of autonomous quantised systems. First, an autonomous quantised system ($\boldsymbol{u} = 0$) is considered whose qualitative state can be measured ($[\boldsymbol{y}(k)] = [\boldsymbol{x}(k)]$). The continuous-variable part of this system is given by

$$\boldsymbol{x}(k+1) = \boldsymbol{g}(\boldsymbol{x}(k)), \qquad \boldsymbol{x}(0) \in \mathcal{X}_0 \tag{9.25}$$

$$\boldsymbol{y}(k) = \boldsymbol{x}(k). \tag{9.26}$$

The quantised system should be described by the stochastic automaton

$$\mathcal{S}_a = (\mathcal{N}_x, G, \mathrm{Prob}(z(0)))$$

where G denotes the state transition probability (cf. Eq. (8.18) for $\omega = 0$).

As the set of automaton states coincides with the set $\mathcal{N}_x$ of qualitative states of the system 9.25, in the graphical representation each partition of the state space is associated with an automaton state. This is illustrated by Fig. 9.15 for the quantised tank system with 9 state partitions.

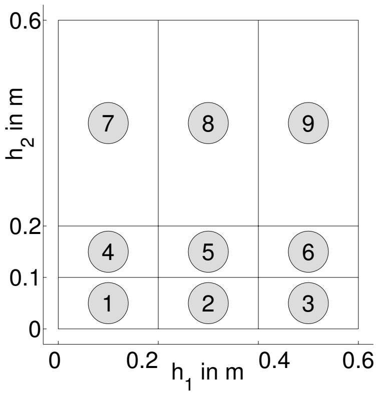


Fig. 9.15. Definition of the automata states for the quantised tank system

Completeness of the qualitative model. In order to illustrate the meaning of the completeness of the obtained model, use the model for a prediction of the qualitative behaviour of the quantised system for the time horizon $[0, k_h]$ by determining the state trajectory of the automaton. Assume that the qualitative initial state $[\boldsymbol{x}(0)]$ of the quantised system has been measured. Then the initial probability distribution of the automaton is given by

$$\text{Prob}(z(0) = z) \begin{cases} > 0 & \text{if} \quad z = [\boldsymbol{x}_0] \text{ for some } \boldsymbol{x}_0 \in \mathcal{X}_0 \\ = 0 & \text{else.} \end{cases}$$

The future state trajectory can be determined by Eq. (8.24), which for the autonomous automaton reads as

$$\text{Prob}(Z(0 \ldots k_h)) = G(z(k_h) \,|\, z(k_h - 1)) \cdot \; G(z(k_h - 1) \,|\, z(k_h - 2)) \cdot G(z(1) \,|\, z(0)) \cdot \text{Prob}(z(0)). \tag{9.27}$$

The behaviour of the automaton includes all state trajectories that occur with non-vanishing probability:

$$\mathcal{M}(k_h) = \{Z(0 \ldots k_h) \,:\, \text{Prob}(Z(0 \ldots k_h)) > 0\} \tag{9.28}$$

$$= \{(z(0), z(1), \ldots, z(k_h)) \,: \text{Prob}(z(0)) > 0 \;\; G(z(k+1) \mid z(k)) > 0 \text{ for } k = 0, 1, \ldots k_h - 1\}. \tag{9.29}$$

Due to the one-to-one relation between the qualitative states of the quantised system and the automaton states, the state trajectories of the automaton can be interpreted directly as qualitative state trajectories of the quantised system:

$$Z(0 \ldots k_h) = [\boldsymbol{X}(0 \ldots k_h)]\,. \tag{9.30}$$

The completeness requirement of the model claims that if the qualitative state trajectory

$$[\boldsymbol{X}(0\ldots k_h)] = ([\boldsymbol{x}(0)]{=}z(0),\ [\boldsymbol{x}(1)]{=}z(1),\ \ldots, [\boldsymbol{x}(k_h)]{=}z(k_h))$$

can be generated by the quantised system, the trajectory

$$Z(0\ldots k_h) = (z(0), z(1),\ \ldots,\ z(k_h))$$

has to belong to $\mathcal{M}(k_h)$.

Unfortunately, Eq. (9.30) is not valid for all $Z(0\ldots k_h)$. The completeness relation (9.21) implies that the quantised system may not generate all qualitative state trajectory that belong to the set $\mathcal{M}(k_h)$, but there may be some state trajectories Z that the quantised system cannot follow. These are the spurious state sequences of the model.

Abstraction of the stochastic automaton. The initial state probability distribution $\text{Prob}(z(0))$ and the state transition relation G of the automaton have to be chosen so that the model is complete. The question how to find such an automaton for a given quantised system is answered in the following lemma:

Lemma 9.1 (Complete model of the autonomous quantised system)
A stochastic automaton $\mathcal{S}_a = (\mathcal{N}_z, G, \text{Prob}(z(0)))$ *is a complete model of the autonomous quantised system if and only if the following conditions are satisfied:*

$$G(z' \mid z) \quad > \quad 0 \iff \text{Prob}([\boldsymbol{x}(1)]{=}z' \mid [\boldsymbol{x}(0)]{=}z) > 0 \qquad (9.31)$$

$$\text{Prob}(z(0) = z) \quad > \quad 0 \ \text{ for all } \ z = [\boldsymbol{x}_0],\ \ \boldsymbol{x}_0 \in \boldsymbol{\mathcal{X}}_0. \qquad (9.32)$$

A stochastic automaton that satisfies these requirements is called an *abstraction* of the autonomous quantised system.

Equation (9.32) ensures that the automaton states, which correspond to possible qualitative initial states, have a non-vanishing probability. If $[\boldsymbol{x}(0)]$ is known, $\text{Prob}(z(0))$ is chosen according to

$$\text{Prob}(z(0)) = \begin{cases} 1 & \text{for } \ z(0) = [\boldsymbol{x}(0)] \\ 0 & \qquad \text{otherwise.} \end{cases} \qquad (9.33)$$

If $[\boldsymbol{x}(0)]$ is not known, the condition (9.32) can be satisfied by associating a positive probability with all possible qualitative initial states. That is, if a set $\mathcal{Z}(0) \subseteq \mathcal{N}_x$ is available which is known to include the true value of $[\boldsymbol{x}(0)]$, the relation

$$\text{Prob}(z(0) = z) > 0 \text{ for all } z \in \mathcal{Z}(0)$$

has to be satisfied.

In order to determine the function $G(z' \mid z)$, the value of

$$\text{Prob}([\boldsymbol{x}(1)]{=}z' \mid [\boldsymbol{x}(0)]{=}z)$$

is determined for all possible combinations of z and z' by means of the state-space model (9.25) of the continuous-variable system and the definitions of the quantiser. To do this, assume that the initial state $\boldsymbol{x}(0)$ is uniformly distributed over the sets

$\mathcal{Q}_x(z)$ and determine the probability that $[\boldsymbol{x}(1)] = z'$ holds for given z and z'. For linear systems $\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k)$ with rectangular partitions this can be done by mapping the corner points of the set $\mathcal{Q}_x(z)$ by applying the matrix $\boldsymbol{A}$ and by determining the intersection of the resulting set with $\mathcal{Q}_x(z')$. For nonlinear systems the conditional probability (9.31) can be approximately determined by mapping a grid of initial states and "counting" those points $\boldsymbol{x}(1)$ that fall into the set $\mathcal{Q}_x(z')$.

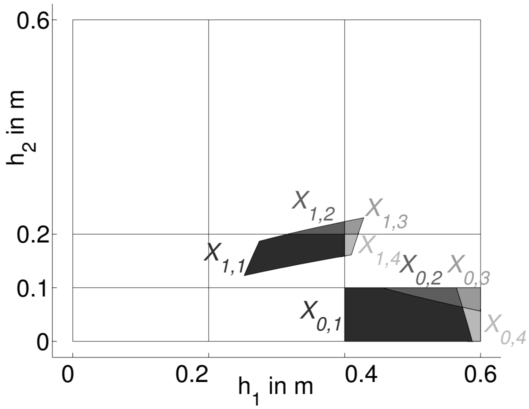


Fig. 9.16. Abstraction of a stochastic automaton describing the autonomous tank system

Example 9.4 *Abstraction of the autonomous tank system*

Consider the tank system for the same configuration as in Fig. 9.10. Figure 9.16 shows the map of the set $X_0 = \mathcal{Q}_x(3)$ of initial states towards the set $X_1 = \boldsymbol{g}(X_0)$ which includes all successor states $\boldsymbol{x}(1)$. In order to determine the transition probability, the set $\mathcal{X}_1$ has to be decomposed into the sets $X_{1,1}$, $X_{1,2}$, $X_{1,3}$ and $X_{1,4}$ where the relations

$$X_{1,2} = X_1 \cap \mathcal{Q}_x(8), \qquad X_{1,3} = X_1 \cap \mathcal{Q}_x(9)$$
$$X_{1,1} = X_1 \cap \mathcal{Q}_x(5), \qquad X_{1,4} = X_1 \cap \mathcal{Q}_x(6)$$

hold. Then it is determined which initial states $\boldsymbol{x}(0)$ lead to a state $\boldsymbol{x}(1) \in X_{1,1}$. The set of such states is denoted by $X_{0,1}$. The same is done for $\boldsymbol{x}(0)$ that yield successor states in $X_{1,2}$, $X_{1,3}$ or $X_{1,4}$, which are summarised into the sets $X_{0,2}$, $X_{0,3}$ or $X_{0,4}$, respectively. That is, the trajectories are followed in reverse direction leading to the decomposition of the set $\mathcal{Q}_x(3)$ into the disjoint sets $X_{0,1}$ to $X_{0,4}$. Then the transition probability can be determined as the quotient of the areas of these sets, which are given by the measure $\lambda(\cdot)$ of these sets:

$$\text{Prob}([\boldsymbol{x}(1)]=5 \mid [\boldsymbol{x}(0)]=3) \quad = \quad \frac{\lambda(X_{0,1})}{\lambda(X_0)}$$

$$\text{Prob}([\boldsymbol{x}(1)]=6 \mid [\boldsymbol{x}(0)]=3) \quad = \quad \frac{\lambda(X_{0,4})}{\lambda(X_0)}$$

$$\text{Prob}([\boldsymbol{x}(1)]=8 \mid [\boldsymbol{x}(0)]=3) \quad = \quad \frac{\lambda(X_{0,2})}{\lambda(X_0)}$$

$$\text{Prob}([\boldsymbol{x}(1)]=9 \mid [\boldsymbol{x}(0)]=3) \quad = \quad \frac{\lambda(X_{0,3})}{\lambda(X_0)}$$

More precisely, these areas are the Lebesgue measures of the sets.

These steps have to be performed for all partitions $\mathcal{Q}_x(z)$, $z \in \mathcal{N}_x$. Note that this investigation includes only the behaviour of the continuous-variable system for one time step (i.e. from $k = 0$ towards $k = 1$), although the automaton can be used later to generate state trajectories $\mathcal{Z}(0 \ldots k_h)$ of arbitrary length k_h. □

Reasonable choice of the state transition probability. Equation (9.31) claims that whenever the quantised system can perform a state transition from the qualitative state z towards the qualitative state z' then the automaton has to be able to perform the same state transition. As the automaton should not be used only to generate a complete behaviour but also to predict the probability with which the I/O pairs occur, the transition probability is chosen as follows:

$$G(z' \mid z) = \text{Prob}([\boldsymbol{x}(1)]=z' \mid [\boldsymbol{x}(0)]=z). \tag{9.34}$$

Then, Eq. (9.27) gives an estimate of the probability with which a given state trajectory Z is generated if the quantised systems is at $k = 0$ in a given qualitative initial state $[\boldsymbol{x}(0)] = z_0$. This probability is only an *estimate* and not the true probability, because the relation (9.21) is, in general, not satisfied with the equality sign and the spurious trajectories are predicted with non-vanishing probability. Hence, some non-spurious trajectories must be associated with imprecise probability measures. [1]

Often, instead of the probability of the whole state trajectory Z the probability of a certain qualitative state $[\boldsymbol{x}(k)]$ is to be predicted. Equation (8.32), which simplifies for the autonomous case to the recursive relation

$$\text{Prob}(z(k+1)) = \sum_{z(k) \in \mathcal{N}_z} G(z(k+1) \mid z(k)) \cdot \text{Prob}(z(k)),$$

can be used to determine $\text{Prob}(z(k+1))$ for the given time horizon $k = 0, \ldots, k_h - 1$. The result gives an estimate of the probability with which the qualitative state $[\boldsymbol{x}(k)] = z(k)$ is assumed on any state trajectory that the quantised system may generate for the given qualitative initial state. The completeness property of the qualitative model implies that only qualitative states of the set

$$\mathcal{Z}(k) = \{[\boldsymbol{x}(k)] \; : \; \text{Prob}(z(k)=[\boldsymbol{x}(k)]) > 0\}$$

can be assumed or, vice versa, that the quantised system is known not to assume any qualitative state $[\boldsymbol{x}(k)]$ for which the relation

$$\text{Prob}(z=[\boldsymbol{x}(k)]) = 0$$

holds.

[1] Even if, under very restrictive conditions, the relation (9.21) were satisfied with the equality sign, the stochastic automaton would yield merely an estimate rather than the true probability distributions.

Example 9.5 *Prediction of the qualitative state of the tank system*

Figure 9.17 shows the qualitative state trajectory of the tank system that has been generated by means of the qualitative model. The grey rectangles show with which probability the given qualitative state is assumed at the corresponding time instants. White rectangles say that it is impossible that the quantised tank system assumes the corresponding qualitative state for the time instant k considered. The completeness of the model implies that the qualitative states assumed by the tank system during the trajectories depicted in Fig. 9.7 are associated with a non-vanishing probability (grey box) in Fig. 9.17. The probabilities obtained by means of the qualitative model differ from the true values depicted in Fig. 9.9 because the stochastic automaton is only an approximate model of the quantised system. □

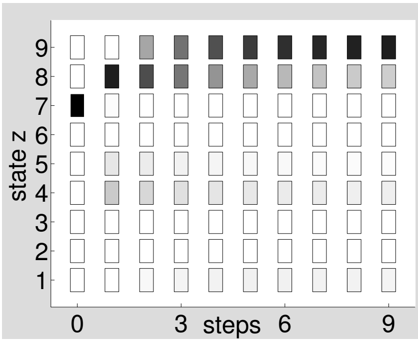


Fig. 9.17. Qualitative state trajectory of the tank system predicted by means of the qualitative model

Abstraction algorithms. The determination of the right-hand side of Eq. (9.34) is a numerical problem of its own. The main difficulty lies in the determination of the set $\mathcal{X}_1$ of successor states $\boldsymbol{x}(1)$ for a given set $\mathcal{X}_0 = \mathcal{Q}_x(z)$ of initial state $\boldsymbol{x}(0)$. Several methods have been elaborated.

The simplest method uses a grid of $\mathcal{N}$ points distributed uniformly over the set $\mathcal{Q}_x(z)$ and determines the set of successor states $\boldsymbol{x}(1)$. Then the number $\mathcal{N}'$ of states with the same qualitative value $[\boldsymbol{x}(1)] = z'$ is obtained and the probability approximated by the relative frequency as follows:

$$\text{Prob}\left([\boldsymbol{x}(1)] = z' \mid [\boldsymbol{x}(0)] = z\right) \approx \frac{\mathcal{N}'}{\mathcal{N}}.$$

The larger the number $\mathcal{N}$ the better is this approximation. However, even for a low dynamical order $\nu = \dim(\boldsymbol{x})$ a very large number of grid points have to be used to get a reasonable approximation.

The main problem of applying this method is the fact that the completeness of the model cannot be ensured. The reason for this is given by the fact that even for a large number of grid points not all state transitions $z \mapsto z'$ are found. Hence, the algorithm yields

$$\text{Prob}\left([\boldsymbol{x}(1)] = z' \mid [\boldsymbol{x}(0)] = z\right) = 0$$

even if the state transition $z \mapsto z'$ is possible for the quantised system.

A complete model can be obtained by an abstraction method that uses a Lipschitz condition for the function $\boldsymbol{g}$. Accordingly, a constant L is determined such that

$$||\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x}')|| \ \leq L\, ||\boldsymbol{x} - \boldsymbol{x}'||$$

holds for all $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{Q}_x(z)$. The main idea of this abstraction method is to map a subset $\mathcal{Q} \subset \mathcal{Q}_x(z)$ which results in the set

$$\mathcal{Q}' = \{\boldsymbol{x}(1) \mid \boldsymbol{x}(1) = \boldsymbol{g}(\boldsymbol{x}(0)), \boldsymbol{x}(0) \in \mathcal{Q}\}$$

of successor states. Instead of determining $\mathcal{Q}'$, a superset $\bar{\mathcal{Q}}'$ can be found by mapping the center point $\boldsymbol{c}$ of $\mathcal{Q}$ into the point $\boldsymbol{c}' = \boldsymbol{g}(\boldsymbol{c})$ and by using the Lipschitz constant L for determining an upper bound of the distance that any point $\boldsymbol{x}(1)$ may have from the center $\boldsymbol{c}'$. If $\mathcal{Q}_x(z)$ is partitioned into sufficiently small sets $\mathcal{Q}$, the union of the sets $\bar{\mathcal{Q}}'$ found in this way is a reasonable approximation of $\mathcal{X}_1$. The advantage of this method with respect to the point-mapping method described above results from the fact that a superset of $\mathcal{X}_1$ is found and, hence, no qualitative state transition is missed. The model obtained in this way is complete.

If the function $\boldsymbol{g}$ has specific properties, simpler algorithms can be used. For example, if $\boldsymbol{g}$ is linear

$$\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x},$$

and the quantiser includes a rectangular partitioning of the state space, it is sufficient to map the corner points of $\mathcal{Q}_x(z)$ by multiplying them by the matrix $\boldsymbol{A}$. The points obtained are the corner points of $\mathcal{X}_1$ (cf. Fig. 9.16).

9.4.3 Extensions to quantised systems with input and output

The abstraction method explained in the preceding section for autonomous systems should be extended now for systems with input and output. A stochastic automaton

$$\mathcal{S} = (\mathcal{N}_x, \mathcal{N}_u, \mathcal{N}_y, L, \text{Prob}(z(0)))$$

is used whose state, input and output sets are identical to the sets of qualitative states, qualitative input and qualitative output values of the quantised system. The behavioural relation L and the initial state probability have to be chosen according to the following theorem.

Theorem 9.1 (Complete model of the quantised system)
A stochastic automaton is a complete model of the quantised system if and only if the following conditions are satisfied:

$$L(z', w \mid z, v) > 0$$
$$\Longleftrightarrow \text{Prob}([\boldsymbol{x}(1)] = z', [\boldsymbol{y}(0)] = w \mid [\boldsymbol{x}(0)] = z, [\boldsymbol{u}(0)] = v) > 0 \quad (9.35)$$
$$\text{Prob}(z(0) = z) > 0 \quad \text{for all} \quad z = [\boldsymbol{x}_0] \;\; \boldsymbol{x}_0 \in \mathcal{X}_0 \quad (9.36)$$

Like in the autonomous case (cf. Eq. (9.34)), L is best chosen according to

$$L(z', w \mid z, v) = \text{Prob}([\boldsymbol{x}(1)] = z', [\boldsymbol{y}(0)] = w \mid [\boldsymbol{x}(0)] = z, [\boldsymbol{u}(0)] = v). \quad (9.37)$$

Note that the right-hand side of Eq. (9.35) can be determined by means of the state-space model (9.3), (9.4) of the continuous-variable system and the definitions of the quantisers. The initial state $\boldsymbol{x}(0)$ and the input $\boldsymbol{u}(0)$ are distributed over the sets $\mathcal{Q}_x(z)$ or $\mathcal{Q}_u(v)$ and the task is to determine the probability that $[\boldsymbol{x}(1)] = z'$ and $[\boldsymbol{y}(0)] = w$ holds for given z, v, z' and w. An approximation of L can be obtained by mapping a grid of initial states for selected input values and "counting" those points $\boldsymbol{x}(1)$ or $\boldsymbol{y}(0)$ that fall into the sets $\mathcal{Q}_x(z')$ or $\mathcal{Q}_y(w)$, respectively.

This abstraction step can be explained by again using Fig. 9.16. As now different output values belong to different values of L, the decomposition of the sets X_0 and X_1 into the sets $X_{0,1} \ldots X_{0,4}$ or $X_{1,1} \ldots X_{1,4}$, respectively, has to be refined. All these sets have to be further decomposed according to the output $[\boldsymbol{y}]$ that the system produces when it moves from the set $X_{0,i}$ towards the set $X_{1,i}$. The resulting partitions are "numbered" by the output w, which leads to the notations $X_{0,i}(w)$ or $X_{1,i}(w)$ and the relations

$$\begin{aligned} X_{0,i} &= \cup_{w \in \mathcal{N}_w} X_{0,i}(w) \\ X_{1,i} &= \cup_{w \in \mathcal{N}_w} X_{1,i}(w) \end{aligned}$$

holds. Furthermore, the state transitions have to be considered for all the possible input values v. Therefore, the "departure" sets $X_{0,i}$ and the "arrival" sets $X_{1,i}$ depend on the input v, which is symbolised by the additional argument: $X_{0,i}(v, w)$ and $X_{1,i}(v, w)$. Then the behavioural relation L can be determined for the state transition starting in the qualitative state 3 and ending in the qualitative states $5, 6, 8$ or 9 as follows:

$$\begin{aligned} L(w, 5 \,|\, 3, v) &= \text{Prob}([\boldsymbol{y}(0)] = w, [\boldsymbol{x}(1)] = 5 \mid [\boldsymbol{x}(0)] = 3, [\boldsymbol{u}(0)] = v) \\ &= \frac{\lambda(X_{0,1}(v, w))}{\lambda(\mathcal{Q}_x(3))} \end{aligned}$$

$$\begin{aligned} L(w,6\,|\,3,v) &= \text{Prob}([\boldsymbol{y}(0)]=w,[\boldsymbol{x}(1)]=6 \mid [\boldsymbol{x}(0)]=3,[\boldsymbol{u}(0)]=v) \\ &= \frac{\lambda(X_{0,4}(v,w))}{\lambda(\mathcal{Q}_x(3))} \\ &\;\;\vdots \end{aligned}$$

9.4.4 Representation of faulty quantised systems

The abstraction method developed so far can be used to find a complete model for the faulty quantised system. The description of the stochastic automaton has to be extended so as to refer to the fault as an additional input (cf. Fig. 9.1):

$$\mathcal{S} = (\mathcal{N}_z, \mathcal{N}_v, \mathcal{N}_w, \mathcal{N}_f, L, \text{Prob}(z(0))). \tag{9.38}$$

The behavioural relation $L(z', w \mid z, v, f)$ has to satisfy the condition (9.35) for given fault f:

$$\begin{aligned} &L(z', w \mid z, v, f) \\ &\quad = \text{Prob}([\boldsymbol{x}(1)]=z', [\boldsymbol{y}(0)]=w \mid [\boldsymbol{x}(0)]=z, [\boldsymbol{u}(0)]=v, [\boldsymbol{e}]=f). \end{aligned} \tag{9.39}$$

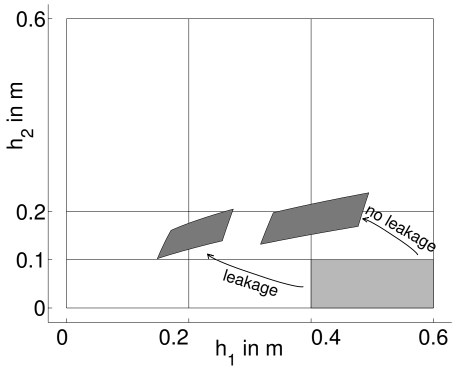


Fig. 9.18. Set of states of the tank system reached with and without fault

Example 9.6 *Model of the faulty tank system*

For the tank system, a stochastic automaton has been obtained by means of the abstraction method explained in this section. Figure 9.18 shows the set of states reached by the faultless and by the faulty system for $k = 1$ when starting in $\boldsymbol{x}(0) \in \mathcal{Q}_x(3)$ for the input signals $[V_{12}] = 2$ and $[u_P] = 2$. In the faultless case the region overlaps with the regions $5, 6, 8$ and 9 of the partitioned state space.

In this example, the quantised outflow $[q_M]$ of Tank 2 is considered as output. The automaton graph for the faultless case and for the above input is shown in Fig. 9.19. The vertices

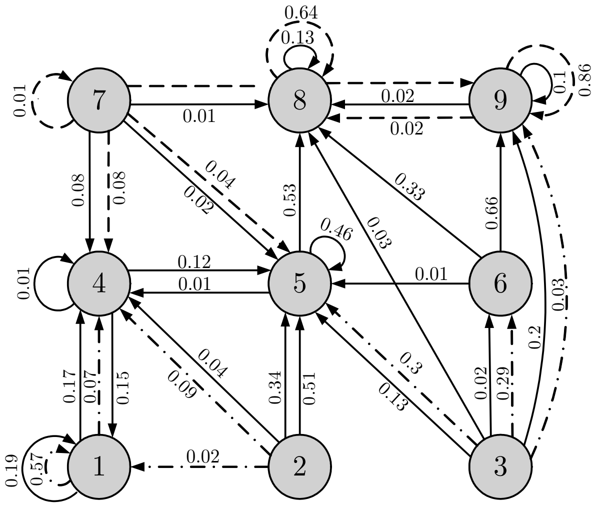


Fig. 9.19. Automaton graph of faultless tank system ($[c_L] = 1$) for the input $[V_{12}] = 2$ and $[u_P] = 2$

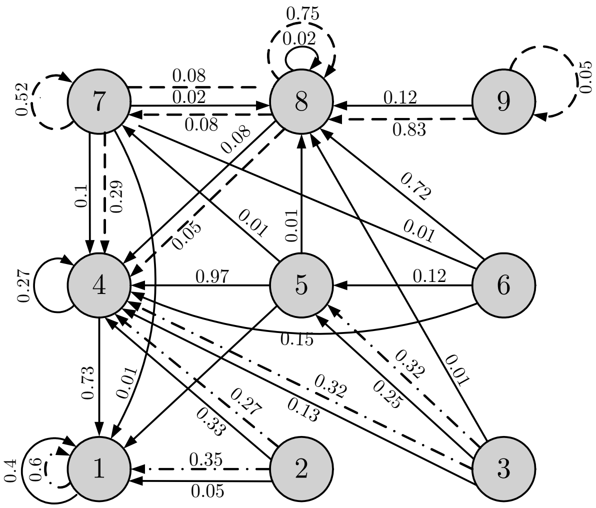


Fig. 9.20. Automaton graph of faulty tank system ($[c_L] = 2$) for the input $[V_{12}] = 2$ and $[u_P] = 2$

correspond to the quantised states shown in Fig. 9.15. The colour and thickness of the edges denote the different quantised output values, from thin black for $[q_M]=1$ to thick light grey for $[q_M]=3$. The edge labels refer to the probability of the transition.

For a leakage $[c_L]=2$ the qualitative behaviour of the tank system changes considerably, which can be seen from the difference between the automaton graphs for the faultless and the faulty systems in Figs. 9.19 and 9.20. □

The fault $f = [\boldsymbol{e}(k)]$ is used here as an additional (unknown) input to the quantised system. In general, some information about the frequency of the change of this signal is known and should be used during the diagnosis. Like for the stochastic automaton (cf. Fig. 8.9 on page 388) a fault model is used to represent this information (Fig. 9.21).

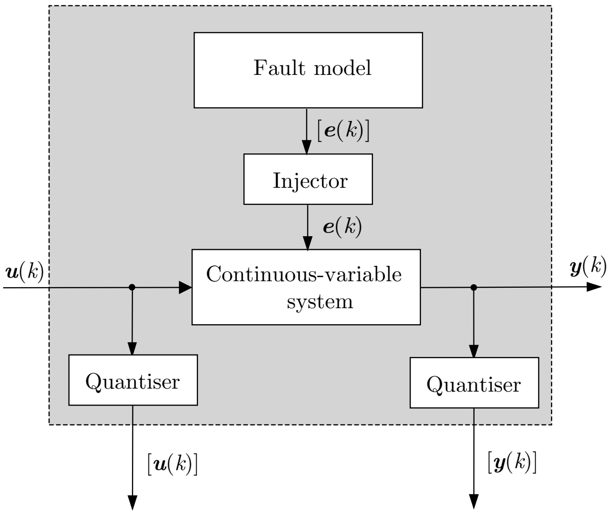


Fig. 9.21. Quantised system with fault model

The fault is assumed to be the output of the stochastic automaton

$$\mathcal{S}_f = (\mathcal{N}_f, G_f, \mathrm{Prob}(f(0))) \tag{9.40}$$

which is called the *fault model*. The transition relation G_f describes the fault-state transition probability

$$\begin{aligned} &G_f : \mathcal{N}_f \times \mathcal{N}_f \longrightarrow [0,1] \\ &G_f(f' \mid f) = \mathrm{Prob}([\boldsymbol{e}(1)]=f' \mid [\boldsymbol{e}(0)]=f) \ , \end{aligned} \tag{9.41}$$

which is the conditional probability that the fault changes from $[\boldsymbol{e}(0)] = f$ towards $[\boldsymbol{e}(1)]=f'$ within one time step. The a-priori probability distribution over the initial fault set is given by $\mathrm{Prob}(f(0))$.

The combination of the stochastic automaton, which represents the quantised system, with the fault model is depicted in Fig. 9.22. It is a stochastic automaton

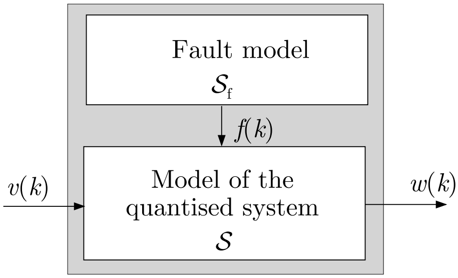


Fig. 9.22. Model of the quantised system combined with the fault model

$$\tilde{\mathcal{S}} = (\mathcal{N}_{\tilde{z}}, \mathcal{N}_v, \mathcal{N}_w, \tilde{L}, \mathrm{Prob}(\tilde{z}(0))) \tag{9.42}$$

whose state set is given by the Cartesian product

$$\mathcal{N}_{\tilde{z}} = \mathcal{N}_z \times \mathcal{N}_f \tag{9.43}$$

and whose behavioural relation $\tilde{L}$ can be obtained from L and G_f according to the relation

$$\tilde{L}(z', f', w \mid z, f, v) = L(z', w \mid z, f, v) \cdot G_f(f' \mid f) \tag{9.44}$$

with $z, z' \in \mathcal{N}_z$, $v \in \mathcal{N}_v$, $w \in \mathcal{N}_w$ and $f, f' \in \mathcal{N}_f$ (cf. Eqs. (8.42) – (8.44)). This model will be used later for solving diagnostic tasks.

9.5 State observation of quantised systems

9.5.1 Observation method

This section deals with the state observation problem for quantised systems given in Section 9.1.2. The task is to find the current qualitative state $[\boldsymbol{x}(k_h)]$ for the measured sequences of input values

$$[\boldsymbol{U}(0 \dots k_h)] = ([\boldsymbol{u}(0)], \dots, [\boldsymbol{u}(k_h)])$$

and output values

$$[\boldsymbol{Y}(0 \dots k_h)] = ([\boldsymbol{y}(0)], \dots, [\boldsymbol{y}(k_h)]).$$

The model $\mathcal{S}$ used for solving this problem is the stochastic automaton

$$\mathcal{S} = (\mathcal{N}_x, \mathcal{N}_u, \mathcal{N}_y, L, \mathrm{Prob}(z(0)))$$

introduced in Eq. (9.24). Therefore, the observation problem can be solved by directly applying the observation method developed in Section 8.3 where

$$\begin{array}{lll} V(0 \dots k_h) & \text{is replaced by} & [\boldsymbol{U}(0 \dots k_h)] \\ W(0 \dots k_h) & & [\boldsymbol{Y}(0 \dots k_h)]\,. \end{array} \tag{9.45}$$

The automaton state $z(k)$ describes the qualitative value $[\boldsymbol{x}(k)]$ of the system state $\boldsymbol{x}$. Equations (8.64) – (8.66) yield

$$\text{Prob}([\boldsymbol{x}(k_h)] \mid k_h) = \frac{\sum\limits_{[\boldsymbol{x}(k_h+1)]} L(k_h) \cdot \text{Prob}([\boldsymbol{x}(k_h)] \mid k_h-1)}{\sum\limits_{[\boldsymbol{x}(k_h)],\,[\boldsymbol{x}(k_h+1)]} L(k_h) \cdot \text{Prob}([\boldsymbol{x}(k_h)] \mid k_h-1)} \qquad (k_h = 0, 1, ...) \tag{9.46}$$

with

$$\text{Prob}([\boldsymbol{x}(k_h)] \mid k_h-1) = \frac{\sum\limits_{[\boldsymbol{x}(k_h-1)]} L(k_h-1) \cdot \text{Prob}([\boldsymbol{x}(k_h!-!1)] \mid k_h-2)}{\sum\limits_{[\boldsymbol{x}(k_h)],\,[\boldsymbol{x}(k_h-1)]} L(k_h!-!1) \cdot \text{Prob}([\boldsymbol{x}(k_h-1)] \mid k_h-2)} \qquad (k_h = 1, 2, ...) \tag{9.47}$$

$$\text{Prob}([\boldsymbol{x}(0)] \mid -1) := \text{Prob}([\boldsymbol{x}(0)]), \tag{9.48}$$

where the abbreviation

$$L(k_h) = L([\boldsymbol{x}(k_h+1)], [\boldsymbol{y}(k_h)] \mid [\boldsymbol{x}(k_h)], [\boldsymbol{u}(k_h)])$$

has been used. The set

$$\mathcal{Z}(k_h \mid k_h) = \{[\boldsymbol{x}(k_h)] \;:\; \text{Prob}([\boldsymbol{x}(k_h)] \mid k_h) > 0\} \tag{9.49}$$

describes all current qualitative states $[\boldsymbol{x}(k_h)]$ that the observation method provides.

9.5.2 Discussion of the result

The recursion relations (9.46) – (9.48) have been obtained by using the relations (8.64) – (8.66) developed for the stochastic automaton, where the abbreviation (8.50)

$$\text{Prob}(z(k_h) \,|\, k_h) := \text{Prob}(z(k_h) \mid W(0 \dots k_h), V(0 \dots k_h))$$

had been used. Therefore, it is reasonable to expect that $\text{Prob}([\boldsymbol{x}(k_h)] \mid k_h)$ which occurs in Eqs. (9.46) – (9.48) is likewise an abbreviation of the probability

$$\text{Prob}([\boldsymbol{x}(k_h)] \,|\, [\boldsymbol{U}(0 \dots k_h)], [\boldsymbol{Y}(0 \dots k_h)])$$

with which the qualitative state $[\boldsymbol{x}]$ can be assumed by the quantised system provided that the quantised system has generated the I/O pair $([\boldsymbol{U}(0 \dots k_h)], [\boldsymbol{Y}(0 \dots k_h)])$. However, this conjecture is not true, because the stochastic automaton used to solve the observation problem is only a complete but not a precise model of the quantised system. Therefore, the result obtained by the observation method can only be an approximation of the probability to be found:

$$\text{Prob}([\boldsymbol{x}(k_h)] \mid k_h) \approx \text{Prob}([\boldsymbol{x}(k_h)] \,|\, [\boldsymbol{U}(0 \dots k_h)], [\boldsymbol{Y}(0 \dots k_h)])\,. \tag{9.50}$$

As a consequence, the set $\mathcal{Z}(k_h \,|\, k_h)$ obtained from Eq. (9.49) need not coincide with the solution set of the observation problem defined in Eq. (9.1):

$$\mathcal{X}(k_h \mid k_h) = \{[\boldsymbol{x}(k_h)] \;:\; \mathrm{Prob}([\boldsymbol{x}(k_h)] \mid [\boldsymbol{U}(0 \dots k_h)], [\boldsymbol{Y}(0 \dots k_h)]) > 0\}.$$

However, due to the completeness of the model the set $\mathcal{Z}$ is know to be a superset of $\mathcal{X}$:

$$\mathcal{Z}(k_h \mid k_h) \supseteq \mathcal{X}(k_h \mid k_h)\,. \qquad (9.51)$$

Theorem 9.2 (Observation of quantised systems)
If the stochastic automaton $\mathcal{S}$ is a complete model of the quantised system, the qualitative state $[\boldsymbol{x}(k_h)]$ of the quantised system belongs to the set $\mathcal{Z}(k_h \mid k_h)$ described by Eq. (9.49), where $\mathrm{Prob}([\boldsymbol{x}(k_h)] \mid k_h)$ is given by Eqs. (9.46) – (9.48).

For the application, this theorem yields the following conclusions:

Corollary 9.1 (Observation result for the quantised system)

- *The quantised system is known to be in some qualitative state*

  $$[\boldsymbol{x}(k_h)] \in \mathcal{Z}(k_h \mid k_h)$$

  and not to be in any state $[\boldsymbol{x}(k_h)] \notin \mathcal{Z}(k_h \mid k_h)$.
- $\mathrm{Prob}([\boldsymbol{x}(k_h)] \mid k_h)$ *is an estimate of the probability with which the quantised system assumes the state $[\boldsymbol{x}(k_h)]$.*

A-priori knowledge about the initial state. An a-priori probability distribution $\mathrm{Prob}([\boldsymbol{x}(0)])$ has to be known to initialise the observation method. As for the stochastic automaton, it is most important to ensure that

$$\mathrm{Prob}([\boldsymbol{x}(0)]) > 0$$

holds for the true qualitative initial state $[\boldsymbol{x}(0)]$, which is unknown. If nothing is known about $[\boldsymbol{x}(0)]$, a good choice of the a-priori probability distribution is the uniform distribution over the set $\mathcal{N}_x$ of all qualitative states

$$\mathrm{Prob}([\boldsymbol{x}(0)]) = 1/(N+1) \text{ for all } [\boldsymbol{x}(0)] \in \mathcal{N}_x\,,$$

where $N+1$ is the number of qualitative states defined by the state quantiser.

Consistent I/O pairs. It has been discussed in detail in Section 8.3.3 that state observation problems can be solved for stochastic automata only if the measured

I/O pair is consistent with the automaton. The same holds true here for the quantised system. However, this does not pose any problems, because the model is set up to be a complete abstraction of the quantised system. Therefore, any I/O pair that the quantised system may generate is consistent with the stochastic automaton used in the observation method. Consequently, the denominators occurring in Eqs. (9.46) and (9.47) are positive in each recursion step. If they become zero, either the a-priori knowledge about the initial state was wrong, the measurements has been perturbed by some disturbances and, hence, became inconsistent with the model or, for some reason, the qualitative model is not complete. If the qualitative model has been obtained by the abstraction method given in Section 9.4 the latter situation can only occur if the continuous-variable state-space model (9.3), (9.4) is wrong.

9.5.3 Observation algorithm

In the observation algorithm the following abbreviations are used:

$$\begin{aligned} & h([\boldsymbol{x}(k_h)]) \\ & \quad = \sum_{[\boldsymbol{x}(k_h+1)]} L(k_h) \cdot \text{Prob}([\boldsymbol{x}(k_h)] \mid [\boldsymbol{U}(0 \ldots k_h - 1)], [\boldsymbol{Y}(0 \ldots k_h - 1)]) \end{aligned}$$

$$p_k([\boldsymbol{x}(k_h)]) = \text{Prob}([\boldsymbol{x}(k_h)] \mid [\boldsymbol{U}(0 \ldots k_h)], [\boldsymbol{Y}(0 \ldots k_h)])$$

$$p_r([\boldsymbol{x}(k_h+1)]) = \text{Prob}([\boldsymbol{x}(k_h+1)] \mid [\boldsymbol{U}(0 \ldots k_h)], [\boldsymbol{Y}(0 \ldots k_h)])$$

Algorithm 9.1 *Observation algorithm for quantised systems*

Given: Complete model $\mathcal{S}$ of the quantised system.
A-priori initial state probability $\mathrm{Prob}([\boldsymbol{x}(0)])$.

Initialise: $p_r([\boldsymbol{x}]) = \mathrm{Prob}([\boldsymbol{x}(0)])$
$k_h = 0.$

Do:

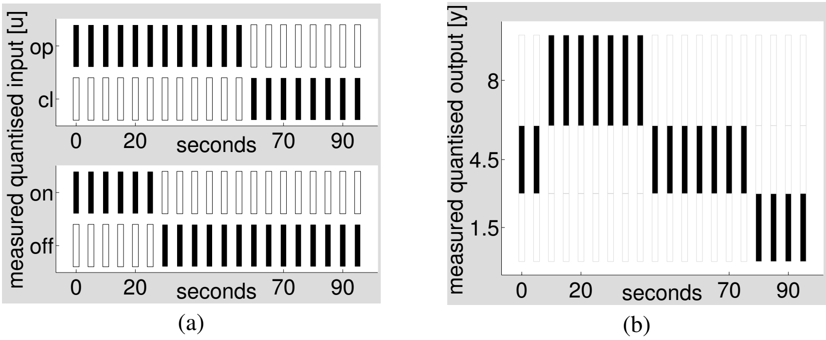
1. Measure $[\boldsymbol{u}(k_h)],\ [\boldsymbol{y}(k_h)]$.
2. For all $[\boldsymbol{x}] \in \mathcal{N}_x$ determine
$h([\boldsymbol{x}]) = \sum_{[\bar{\boldsymbol{x}}]} L([\bar{\boldsymbol{x}}], [\boldsymbol{y}(k_h)] \,|\, [\boldsymbol{x}], [\boldsymbol{u}(k_h)]) \cdot p_r([\boldsymbol{x}])$.
3. If $\sum_{[\boldsymbol{x}]} h([\boldsymbol{x}]) = 0$ holds, stop the algorithm (inconsistent I/O pair or wrong initial state distribution).
4. For all $[\boldsymbol{x}] \in \mathcal{N}_x$ determine $p_k([\boldsymbol{x}]) = \dfrac{h([\boldsymbol{x}])}{\sum\limits_{[\boldsymbol{x}]} h([\boldsymbol{x}])}$.
5. For all $[\boldsymbol{x}] \in \mathcal{N}_x$ determine
$$p_r([\boldsymbol{x}]) = \frac{\sum\limits_{[\bar{\boldsymbol{x}}]} L([\boldsymbol{x}], [\boldsymbol{y}(k_h)] \,|\, [\bar{\boldsymbol{x}}],\ [\boldsymbol{u}(k_h)])\, p_r([\bar{\boldsymbol{x}}])}{\sum\limits_{[\boldsymbol{x}]} h([\boldsymbol{x}])}.$$
6. Determine $\mathcal{Z}(k_h \,|\, k_h) = \{[\boldsymbol{x}] \ :\ p_k([\boldsymbol{x}]) \neq 0\}$.
7. $k_h := k_h + 1$
Continue with Step 1.

Result: $p_k([\boldsymbol{x}(k_h)]) \approx \mathrm{Prob}([\boldsymbol{x}(k_h)] \,|\, [\boldsymbol{U}(0 \ldots k_h)], [\boldsymbol{Y}(0 \ldots k_h)])$ and $\mathcal{Z}(k_h \,|\, k_h)$ for increasing time horizon k_h .

Example 9.7 *State observation of a tank system*

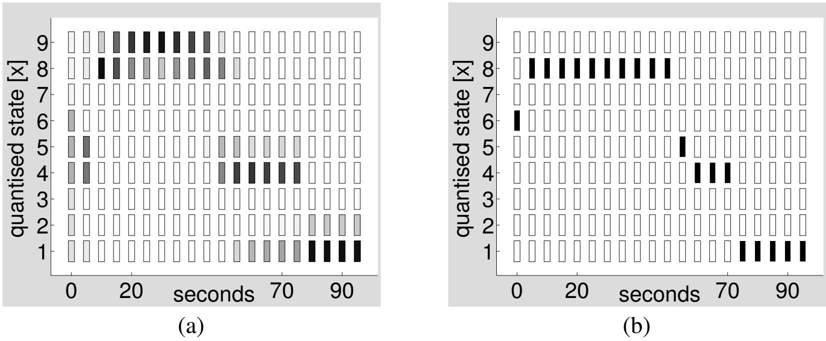
The observation algorithm is now applied to the two-tank system described in Section 2.1 in the quantised configuration given in Section 9.4. On the left-hand side of Fig. 9.23 the sequence $[\boldsymbol{U}(0 \ldots k_h)]$ of measured quantised input signals is shown. The upper sequence corresponds to the valve position $[V_{12}]$ (open or closed) and the lower to the quantised pump input $[u_P]$. The right-hand side of this figure shows the measured output sequence $[\boldsymbol{Y}(0 \ldots k_h)]$, where the output corresponds to the quantised outflow $[q_M]$ of Tank 2. The task is to determine from these sequences the quantised states $[\boldsymbol{x}(k_h)]$ at each time instant $k_h = 0, 1, \ldots$ for unknown initial state set $\mathcal{X}_0$.

On the left-hand side of Fig. 9.24 the observation result obtained by the above algorithm is shown. The grey boxes depict the probability distributions $\mathrm{Prob}([\boldsymbol{x}(k_h)] \mid k_h)$ for



(a) (b)

Fig. 9.23. Quantised input and output sequences used to solve the state observation problem



(a) (b)

Fig. 9.24. Observation result (left) and state sequence of the considered experiment (right)

$k_h = 0, 1, ...$ for the 9 different quantised states. For comparison, the state sequence of the experiment for which the output sequence was obtained is shown on the right-hand side of the figure. It can be seen, that the observation result is nonzero in the states of the "real" sequence at all times. Note that the state sequence obtained in this experiment is not the only possible one yielding the measurements of Fig. 9.23. However, the observation result is guaranteed to cover all possible sequences. □

9.6 Diagnosis of quantised systems

9.6.1 Diagnostic method

This section deals with the diagnosis of quantised system. To solve the problem given in Section 9.1.2, the fault $[e(k_h)]$ has to be determined for the measured sequences of qualitative input values

$$[\boldsymbol{U}(0 \ldots k_h)] = ([\boldsymbol{u}(0)], \ldots, [\boldsymbol{u}(k_h)])$$

and qualitative output values

$$[\boldsymbol{Y}(0 \ldots k_h)] = ([\boldsymbol{y}(0)],\ \ldots,\ [\boldsymbol{y}(k_h)]).$$

The model of the faulty quantised system used here is given by the automaton

$$\tilde{\mathcal{S}} = (\mathcal{N}_{\tilde{z}}, \mathcal{N}_v, \mathcal{N}_w, \tilde{L}, \mathrm{Prob}(\tilde{z}(0)))$$

defined in Section 9.4.4. Therefore, the diagnostic problem can be solved by directly applying the diagnostic method developed for stochastic automata in Section 8.4 after

$$\begin{array}{lll} V(0 \ldots k_h) & \text{is replaced by} & [\boldsymbol{U}(0 \ldots k_h)] \\ W(0 \ldots k_h) & & [\boldsymbol{Y}(0 \ldots k_h)] \,. \end{array}$$

With these substitutions, Eqs. (8.91) – (8.93) yield the following result:

$$\mathrm{Prob}([\boldsymbol{e}(k_h)] \mid k_h) = \frac{\sum\limits_{\substack{[\boldsymbol{e}(k_h+1)] \\ [\boldsymbol{x}(k_h+1)],[\boldsymbol{x}(k_h)]}} L(k_h) \cdot G_f(k_h) \cdot \mathrm{Prob}([\boldsymbol{e}(k_h)], [\boldsymbol{x}(k_h)] \mid k_h - 1)}{\sum\limits_{\substack{[\boldsymbol{e}(k_h)],[\boldsymbol{e}(k_h+1)] \\ [\boldsymbol{x}(k_h)],[\boldsymbol{x}(k_h+1)]}} L(k_h) \cdot G_f(k_h) \cdot \mathrm{Prob}([\boldsymbol{e}(k_h)], [\boldsymbol{x}(k_h)] \mid k_h - 1)} \qquad (k_h = 0, 1, \ldots) \tag{9.52}$$

with

$$\mathrm{Prob}([\boldsymbol{e}(k_h)], [\boldsymbol{x}(k_h)] \mid k_h - 1) = \frac{\sum\limits_{[\boldsymbol{e}(k_h-1)],[\boldsymbol{x}(k_h-1)]} L(k_h-1) \cdot G_f(k_h-1) \cdot \mathrm{Prob}([\boldsymbol{e}(k_h-1)], [\boldsymbol{x}(k_h-1)] \mid k_h - 2)}{\sum\limits_{\substack{[\boldsymbol{e}(k_h)],[\boldsymbol{e}(k_h-1)] \\ [\boldsymbol{x}(k_h)],[\boldsymbol{x}(k_h-1)]}} L(k_h-1) \cdot G_f(k_h-1) \cdot \mathrm{Prob}([\boldsymbol{e}(k_h-1)], [\boldsymbol{x}(k_h-1)] \mid k_h - 2)} \qquad (k_h = 1, 2, \ldots) \tag{9.53}$$

$$\mathrm{Prob}([\boldsymbol{x}(0)], [\boldsymbol{e}(0)] \mid -1) := \mathrm{Prob}([\boldsymbol{e}(0)]) \cdot \mathrm{Prob}([\boldsymbol{x}(0)]) \tag{9.54}$$

and the abbreviation

$$L(k_h) := L([\boldsymbol{x}(k_h+1)], [\boldsymbol{y}(k_h)] \mid [\boldsymbol{x}(k_h)], [\boldsymbol{u}(k_h)], [\boldsymbol{e}(k_h)]) \,. \tag{9.55}$$

9.6.2 Discussion of the result

Like in the observation algorithm, $\mathrm{Prob}([\boldsymbol{e}(k_h)] \mid k_h)$ determined by these equations is not identical to the probability distribution of the current fault $[\boldsymbol{e}(k_h)]$, but it is an approximation:

$$\mathrm{Prob}([\boldsymbol{e}(k_h)] \mid k_h) \approx \mathrm{Prob}([\boldsymbol{e}(k_h)] \mid [\boldsymbol{U}(0 \ldots k_h)], [\boldsymbol{Y}(0 \ldots k_h)]).$$

The set of faults for which $\mathrm{Prob}([\boldsymbol{e}(k_h)] \mid k_h) > 0$ holds is a superset of the set of fault candidates that would be obtained if the hybrid model of the quantised system were used to solve the diagnostic problem that may really occur in the quantised system.

Theorem 9.3 (Diagnosis of quantised systems)
If the stochastic automaton $\mathcal{S}$ is a complete model of the quantised system, the fault that occurs in the quantised systems belongs to the set

$$\mathcal{F}(k_h) = \{[\boldsymbol{e}(k_h)] \,:\, \text{Prob}([\boldsymbol{e}(k_h)] \,|\, k_h) > 0\}, \tag{9.56}$$

where $\text{Prob}([\boldsymbol{e}(k_h)] \,|\, k_h)$ *is described by Eqs. (9.52) – (9.54).*

Note that the algorithm does not include a simulation of the system behaviour. The main idea is to determine the probability with which the system has made the state transitions that are necessary to produce the currently measured output for the currently measured input.

Corollary 9.2 (Diagnostic results for the quantised system)

- *The quantised system is known to be subjected to some fault* $f \in \mathcal{F}(k_h)$.
- **Fault detection:** *If* $f_0 \notin \mathcal{F}(k_h)$ *holds, the quantised system is known to be subjected to some fault (where* f_0 *denotes the faultless system).*
- **Fault identification:** *If* $\mathcal{F}(k_h) = \{f_i\}$ *is a singleton, the system is known to be subjected to fault* f_i *provided that the occurring fault belongs to the set* $\mathcal{F}$.

As long as $\mathcal{F}(k_h)$ has more than one element, the probability $\text{Prob}(f \,|\, k_h)$ describes an approximation of the probability that the fault f occurs.

Diagnosability of quantised systems. The results on diagnosability of stochastic automata apply directly to the quantised system if the automaton represents a *complete* model. The quantised system is diagnosable whenever the automaton is diagnosable. That is, if the fault is diagnosable from the less precise model it can also be diagnosed from the precise representation given by the continuous-variable model together with the quantisers and the injector.

9.6.3 Diagnostic algorithm

These results lead to the following diagnostic algorithm in which $[\boldsymbol{U}]$ and $[\boldsymbol{Y}]$ denote the sequences $[\boldsymbol{U}(0 \dots k_h)]$ and $[\boldsymbol{Y}(0 \dots k_h)]$ of length k_h+1. The following abbreviations are used:

$$h([\boldsymbol{e}(k_h)], [\boldsymbol{x}(k_h)]) \\ = \sum_{[\boldsymbol{e}(k_h+1)]} \sum_{[\boldsymbol{x}(k_h+1)]} L(k_h) \cdot G_f(k_h) \cdot \text{Prob}([\boldsymbol{x}(k_h)] \mid [U(0 \dots k_h - 1)], [Y(0 \dots k_h - 1)])$$

$$p_k([\boldsymbol{e}(k_h)], [\boldsymbol{x}(k_h)]) = \text{Prob}([\boldsymbol{e}(k_h)], [\boldsymbol{x}(k_h)] \mid [U(0 \dots k_h)], [Y(0 \dots k_h)])$$

$$p_r([\boldsymbol{e}(k_h+1)], [\boldsymbol{x}(k_h+1)]) \\ = \text{Prob}([\boldsymbol{e}(k_h+1)], [\boldsymbol{x}(k_h+1)] \mid [U(0 \dots k_h)], [Y(0 \dots k_h)])$$

Algorithm 9.2 *Diagnosis of quantised systems*

Given: Complete model $\mathcal{S}$ of the quantised system.
Fault model $\mathcal{S}_f$.
Initial state probability distribution $\text{Prob}([\boldsymbol{x}(0)])$.
Initial fault probability distribution $\text{Prob}([\boldsymbol{e}(0)])$.

Initialisation: $p_r([\boldsymbol{e}], [\boldsymbol{x}]) = \text{Prob}([\boldsymbol{e}(0)] = f) \cdot \cdot \text{Prob}([\boldsymbol{x}(0)] = z)$ for all $f \in \mathcal{N}_f$ and $z \in \mathcal{N}_z$
$k_h = 0$.

1. Measure the current input $[\boldsymbol{u}(k_h)] = v$ and output $[\boldsymbol{y}(k_h)] = w$.

2. For all $[\boldsymbol{e}] \in \mathcal{N}_f$ and $[\boldsymbol{x}] \in \mathcal{N}_x$ determine
$$h([\boldsymbol{e}], [\boldsymbol{x}]) = \sum_{[\bar{\boldsymbol{e}}],[\bar{\boldsymbol{x}}]} L([\bar{\boldsymbol{x}}], [\boldsymbol{y}(k_h)] \mid [\boldsymbol{x}], [\boldsymbol{e}], [\boldsymbol{u}(k_h)]) \cdot \\ \cdot G_f([\bar{\boldsymbol{e}}] \mid [\boldsymbol{e}]) \cdot p_r([\boldsymbol{e}], [\boldsymbol{x}]).$$

3. If $\sum_{[\boldsymbol{e}],[\boldsymbol{x}]} h([\boldsymbol{e}], [\boldsymbol{x}]) = 0$ holds, stop the algorithm (inconsistent I/O pair or wrong initial distributions).

4. For all $[\boldsymbol{e}] \in \mathcal{N}_f$ and $[\boldsymbol{x}] \in \mathcal{N}_x$ determine
$$p_k([\boldsymbol{e}], [\boldsymbol{x}]) = \frac{h([\boldsymbol{e}], [\boldsymbol{x}])}{\sum_{[\boldsymbol{e}],[\boldsymbol{x}]} h([\boldsymbol{e}], [\boldsymbol{x}])}.$$

5. For all $[e] \in \mathcal{N}_f$ and $[x] \in \mathcal{N}_x$ determine

$$p_r([e],[x]) = \frac{\sum_{[\bar{e}],[\bar{x}]} L([x], w \mid [\bar{x}], [u(k_h)], [\bar{e}]) \cdot G_f([e] \mid [\bar{e}]) \cdot p_r([\bar{e}], [\bar{x}])}{\sum_{[e],[x]} h([e],[x])}.$$

6. Determine $\text{Prob}([e(k_h)] \mid k_h) = \sum_{[x]} p_k([e],[x])$.

7. Determine $\mathcal{F}(k_h)$ according to Eq. (9.56).

8. $k_h := k_h + 1$

   Continue with Step 1.

Result: $\text{Prob}([e(k_h)] \mid k_h)$ and $\mathcal{F}(k_h)$ for increasing time horizon k_h.

Example 9.8 *Diagnosis of a quantised tank system*

To demonstrate this diagnostic algorithm, the tank system of Section 2.1 is considered as a quantised system. On the left-hand side of Fig. 9.25 the sequence $[U(0 \ldots k_h)]$ of measured quantised input values is shown. The upper sequence corresponds to the valve position $[V_{12}]$ and the lower to the quantised pump input $[u_P]$. The right-hand side of this figure shows the measured output sequence $[Y(0 \ldots k_h)]$, which describes to the quantised outflow $[q_M]$ of Tank 2. The task is to determine from these sequences the unknown quantised fault $[e(k)] = [c_L]$ at each time instant $k_h = 0, 1, \ldots$, for unknown initial state set $\mathcal{X}_0$. The fault is assumed to be constant during the experiment, so that the fault model (8.95) can be used.

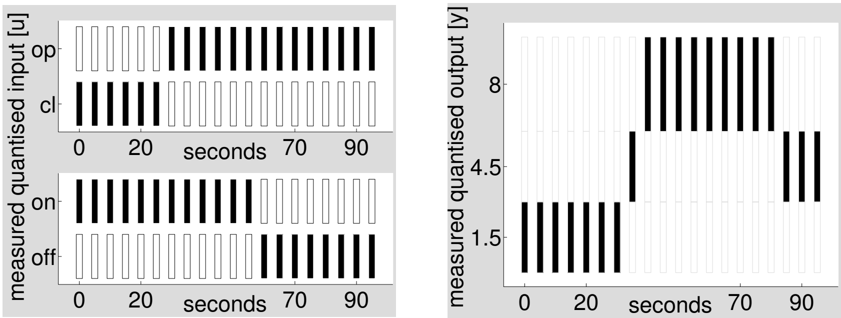
On the left-hand side of Fig. 9.26 the diagnostic result is depicted. It is shown how the probability distribution $\text{Prob}([e(k_h)] \mid k_h)$ changes for $k_h = 0, 1, \ldots$. Initially both faults have the same probability. The probabilities change with increasing time horizon k_h. After 8 steps, the measured sequences are only consistent with the model for the faultless case $[e] = 1$, showing that no leakage occurred in this experiment. The case $[e] = 2$ corresponding to a leakage in Tank 1 is excluded. □

9.6.4 Reconfiguration in case of sensor or actuator failures

If sensor or actuator failures are not included in the fault set $\mathcal{F}$, the diagnostic Algorithm 9.2 stops in Step 3 as soon as a faulty sensor or actuator causes that the denominator $\sum_{[e],[x]} h([e],[x])$ to become zero. In this case, the faulty sensor or actuator can be isolated by using the observer schemes introduced in Section 8.7. Furthermore, the diagnostic algorithm can be automatically reconfigured as described in Section 8.7.2. This is illustrated by the following example.

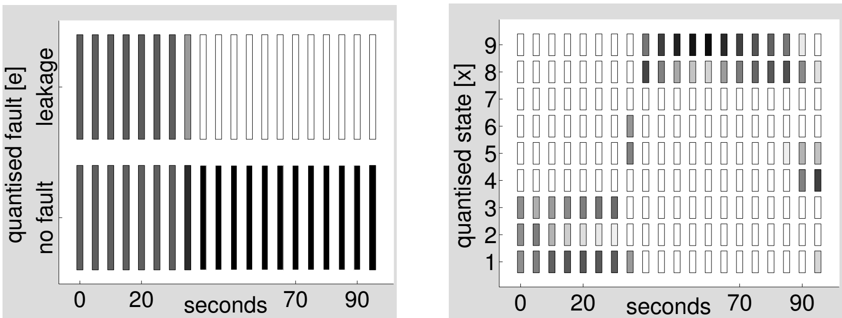
Example 9.9 *Automatic reconfiguration of diagnosis*

The automatic reconfiguration of diagnosis is shown using the example of the two-tank system of Section 2.1. Actuators are the valve V_{12} which can be either closed or opened and the pump P which can be switched on with $u_P = u_{P,nom}$ or switched off with $u_P = 0$. As plant fault, a



(a) (b)

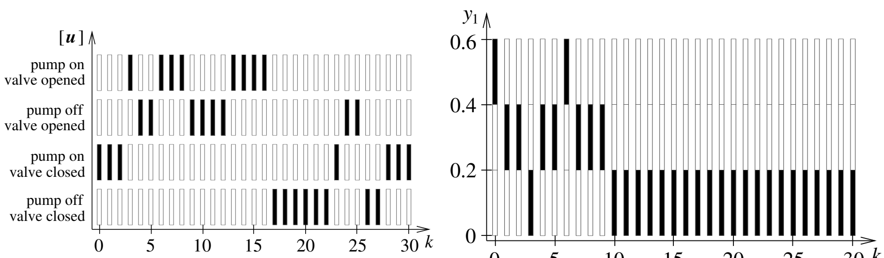
Fig. 9.25. Quantised input and output sequences available for fault diagnosis



(a) (b)

Fig. 9.26. Diagnostic result (left) and observed states (right)

leakage in the left tank is considered. The levels in both tanks are measured by means of the discrete level sensors at the positions leading to the state-space partition shown in Fig. 9.4. For simplification, the discrete level sensors at Tank 1 are treated as quantised level sensor 1 and those at Tank 2 as quantised level sensor 2. The sampling time is $T_s = 10\,\mathrm{s}$.



(a) (b)

Fig. 9.27. Quantised input sequence (left) and faulty interval measurements of level sensor 1 (right).

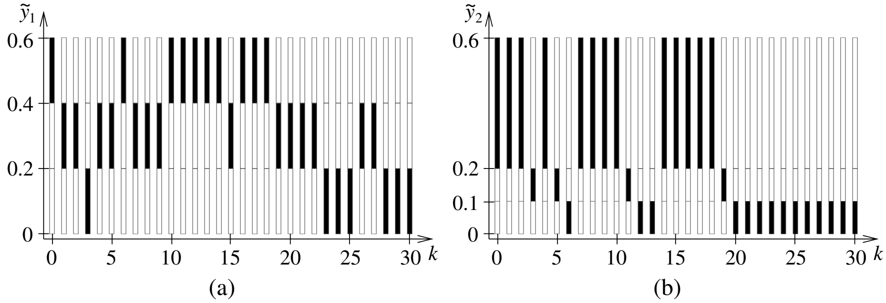


Fig. 9.28. Sequence of the true quantised liquid levels in the left and right tank, respectively.

In an experiment, the quantised input sequence shown on the left-hand side of Fig. 9.27 has been applied to the tank system. This yields the sequences of quantised tank levels shown in Fig. 9.28. Instead of the correct output shown on the left-hand side of the second figure, sensor 1 yields the sequence shown on the right-hand side of the first figure. It can be seen that from time $k = 10$ onwards, the sensor yields the measurement value $[0, 0.2)$ independently the actual level in the left tank.

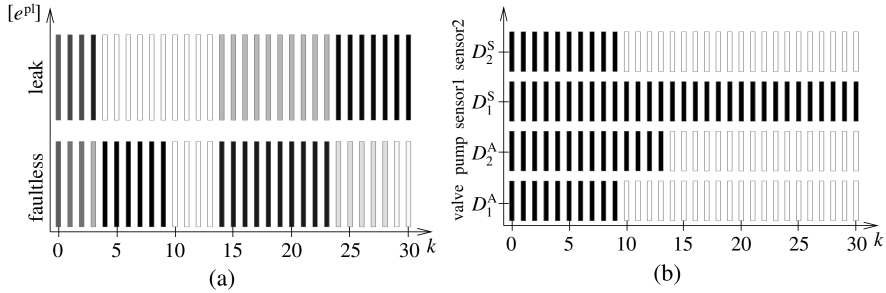


Fig. 9.29. Diagnostic result for the plant fault (left) and denominators of the blocks of the generalised observer scheme (right).

To the measured sequences, the generalised observer scheme shown in Fig. 8.23 is applied. Furthermore, two blocks for actuator supervision as shown in Fig. 8.27 are used. That is, besides the diagnostic result, the four signals $D_1^S(k)$, $D_2^S(k)$, $D_1^A(k)$ and $D_2^A(k)$ are determined, which correspond to the denominators of the diagnostic blocks without Sensor 1, Sensor 2, Actuator 1 (valve) or Actuator 2 (pump), respectively. The diagnostic result is shown on the left-hand side of Fig. 9.29. The values of the four denominators are shown on the right-hand side of the figure, where a black bar indicates a nonzero denominator.

From the left-hand side of Fig. 9.29, it can be seen that until $k = 9$ the diagnostic algorithm works correctly and that the faultless case is isolated for unknown initial plant fault after a few steps. At $k = 10$, when Sensor 1 breaks down, the denominator $D(s)$ of the main diagnostic block becomes zero. The diagnostic block yields no further diagnostic results which is indicated by the white bars from $k = 10$ onwards. The right-hand side of the figure shows that at $k = 10$, two other denominators, namely $D_2^S(k)$ and $D_1^A(k)$, also become zero indicating that neither a valve fault nor a failure of in Sensor 2 could have caused the inconsistency with the model of the main diagnostic block. After a few more steps at $k = 14$, the measured sequences also become inconsistent with the model that has no regard of the pump input. This indicates that Sensor 1 must be faulty because the only block which is still consistent with the measurement sequences is the block of the generalised observer scheme that does not use the information of Sensor 1.

Having identified the faulty component, the diagnostic system is reconfigured so that only Sensor 2 is used for diagnosis. Such a diagnostic block is already included in the generalised observer scheme. The probabilities of the plant faults determined by this block are shown on the left-hand side of Fig. 9.29 from time $k = 14$ onwards. Note that the new diagnostic block implicitly performs a state observation of the level in Tank 1.

At time $k = 18$, a leakage occurs in the left tank and is present until the end of the experiment. It can be seen that after some time, this plant fault is identified. However, the reconfigured diagnostic system has a lower performance because it obtains less information due to the loss of Sensor 1. If Sensor 1 was still operating, the leakage could be detected already at time $k = 19$ when the level in the left tank decreases for a closed connecting valve, cf. Fig. 9.27 (left) and Fig. 9.28 (left). The reconfigured scheme is slower but allows that the diagnosis can be continued though the original diagnostic block could no longer be used from time $k = 10$ onwards. □

9.6.5 Extensions and application examples

Diagnosis of transient faults. The diagnostic problem dealt with so far concerned the task to find the current fault $f(k_h)$ by using the measured I/O pair. The solution to this problem is not appropriate if the fault is apparent in the measurements only after some time delay. Transient faults, which vary quickly in time, represent such faults, where the moment of the fault occurrence lies several time instants before the effects of the fault become measurable.

To understand the difference with respect to the diagnostic problem tackled until now, remember that the fault is described by the sequence

$$F(0 \dots k_h) = (f(0), f(1), f(2), \dots, f(k_h))$$

and that until now only the last element of this sequence has to be found. If the fault becomes "visible" from the measurements only after some time delay, at time k_h the task to be solved is to find all possible sequences $F(0 \dots k_h)$ and to decide whether these sequences include faults $f(k_h - k) \neq f_0$ that occurred k time instants ago, where f_0 again denotes the faultless operation mode of the system.

As a consequence, the problem has to be posed in such a way that all possible fault sequences $F(0 \dots k_h)$ have to be found for which the measured I/O pair $([\boldsymbol{U}], [\boldsymbol{Y}])$ belongs to the system behaviour $\mathcal{B}_{\text{qual}}(k_h)$ of the quantised system. Then it is known that the sequence $F(0 \dots k_h)$ can have happened and, consequently, the faults occurring in this sequence may have occurred at the corresponding time instants.

The diagnostic method explained in this chapter can be extended to solve this problem. The main idea remains the same but the summation occurring in the formulas have to be made over all possible fault sequences $F(0 \dots k_h)$.

Diagnosis of discrete-event quantised systems. This chapter concerned discrete-time quantised systems, whose continuous-variable core can be described by the

Eqs. (9.3), (9.4) with given sampling time. However, another viewpoint can be adopted concerning the temporal quantisation. If a continuous-time system is considered, the quantiser may not only determine the current qualitative state, but it may also identify the time instants at which these qualitative values change. Then the continuous-time continuous-variable system "moves" whenever its output trajectory $\boldsymbol{y}(t)$ crosses a border between two adjacent output space partitions $\mathcal{Q}_y(i)$ and $\mathcal{Q}_y(j)$. A change of the qualitative value $[\boldsymbol{y}]$ is called an event.

The main ideas described in this chapter can be used to diagnose discrete-event quantised systems. Then, the stochastic automaton describes the sequence of discrete events generated by the quantised system. The diagnostic algorithm can be directly applied if the interface between the quantised system and the algorithm is modified such that the I/O sequences are composed of events rather than of qualitative input and output values obtained by sampling.

Although the main ideas are the same, the results may differ considerably. The reason for this is given by the fact that stochastic automata of discrete-event quantised systems have no information about the temporal distance of the events, but this temporal distance may be decisive for the diagnosability.

Therefore, timed discrete-event models as, for example, semi-Markov processes have to be used. Such models can be set up in a similar way as described in Section 9.4 if the probability of the occurrence of events are evaluated also with respect to their occurrence times. The diagnostic algorithm presented for stochastic automata can be extended to this kind of models.

Application examples. The methods described in this chapter have been applied to different laboratory experiments and practical problems.

- **Diagnosis of a batch process:** The example used to illustrate the methods throughout this chapter presents a hybrid system with switching dynamics.

- **Diagnosis and fault-tolerant control of a neutralisation process:** Many important signals that occur in the process industry are not precisely measurable or even immeasurable. A neutralisation process has been used to demonstrate the applicability of the observation and diagnostic methods. The fault-tolerant control of a part of this process is described in Section 10.2.

- **Diagnosis of an H_2 compressor:** For large industrial compressor systems, no quantitative model (9.3), (9.4) is available. By approximating the behavioural relation by the relative frequency with which the system under consideration has changed its qualitative state during its operation over several years, a stochastic automaton as qualitative model of the compressor system has been obtained and used for on-line diagnosis.

- **Diagnosis of the power stage of a diesel engine:** Automotive systems have quick dynamics and, hence, rather strong real-time constraints for diagnosis. The quantised system approach has the advantage of reducing the information available to that which is necessary to solve the diagnostic task. Dealt with as a discrete-event quantised system, the power stage of a diesel engine has successfully been equipped with a diagnostic module.

Example 9.10 *Diagnosis of the air path of a diesel engine*

Growing demands on automobiles in terms of reliability, economy, and safety necessiate severe improvements of automated on-board diagnosis. To guarantee low emission levels the diagnosis of the injection, compression, and combustion plays a key role. This example concerns the qualitative diagnosis of the air path of a diesel engine with a single turbo charger (Fig. 9.30).

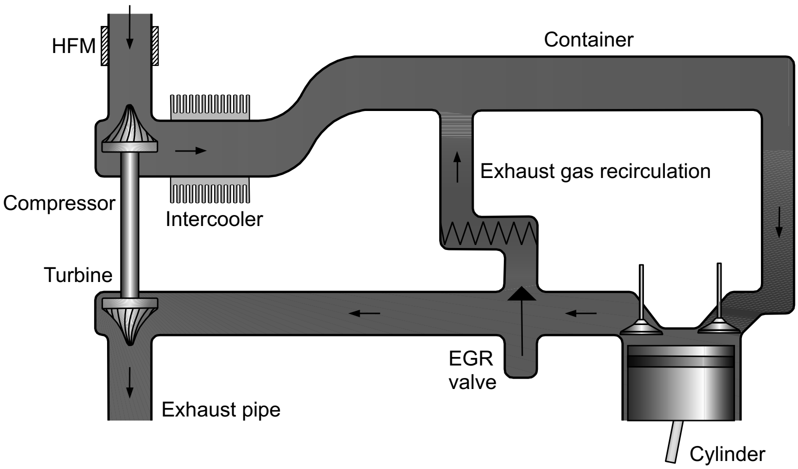


Fig. 9.30. Air path of a diesel engine

The air path consists of the following components (Fig. 9.30):

- Hot-wire air flow meter (HFM): Measurement of the incoming air flow.
- Compressor: Compression of the incoming air flow.
- Intercooler: Cooling of the compressed air to increase the density.
- Container: Blending of the compressed air with parts of the waste gas.
- Cylinder: Combustion of the fuel and airmix from the container.
- Exhaust gas recirculation valve (EGR valve): Control of the percentage of the recirculated exhaust gas.
- Turbine: Driving the compressor; equipped with a variable turbine geometry (VTG).
- Exhaust pipe: Channelling of the waste gas to the outside.

The air path is subject to the input signals n_E (engine speed), m_F (fuel flow per stroke), A_{EGR} (effective area of the EGR valve) and A_{VTG} (effective area of the turbine). The following output signals can be measured: q_1 (incoming air flow), p_2 (pressure in the container) and T_2 (temperature in the container). The continuous-variable state-space model (9.3), (9.4) has seven state variables, four input and three output signals (Fig. 9.31).

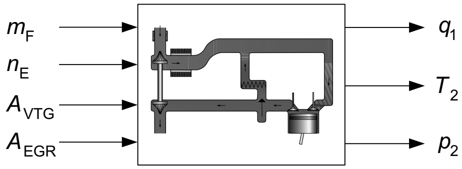


Fig. 9.31. Block diagram of the air path

The special structure of the air path results in a strong coupling among the different system states. The recirculated exhaust gas affects the airmix that streams into the engine by changing its temperature, pressure, and oxygen content. In turn, the airmix influences the exhaust gas that is leaving the engine. A second feedback is realised through the turbo charger. It couples the input air flow with the exhaust gas flow.

Faults that occur in the components of this system influence the emission, which is subject to legal restrictions. Therefore, the sensor and actuator faults listed in the table below have to be considered for diagnosis. The diagnostic task is to detect faults in the air system and to identify the faults f_1 throught f_9 by using measurement sequences of the sensors and input sequences to the actuators mentioned above.

Fault	Fault description
f_0	Faultless operation
f_1	HFM offset +0,02 kg/s
f_2	HFM breakdown 0,0001 kg/s
f_3	HFM drift -20
f_4	EGR blocked close
f_5	EGR blocked open
f_6	T_2-sensor drift +30
f_7	T_2-sensor drift -30
f_8	p_2-sensor drift +30
f_9	p_2-sensor drift -30

The airpath is considered as a quantised system. For all measured signals, a parition of the signal space is introduced. For example, for the air flow q_1 six regions starting from "very low" up to "maximum" are distinguished after the discretisation (Fig. 9.32). The dark or light rectangles indicate in which interval the signal q_1 lies at the respective time instant. As can be seen in the figure, the trajectories of the signal q_1 for the two fault cases f_0 and f_3 can be distinguished easily in spite of the rough discretisation.

For the evaluation of the diagnostic method, the input signals from test series with an experimental car have been used. In the tests the EGR valve was closed, which was adopted as a modelling assumption.

The qualitative model has been determined by the abstraction algorithm explained in Section 9.4.

Results. The results obtained by diagnosing the different faults are briefly presented in the following. It has turned out that the faults f_2, f_6 – f_9 are very easy to be diagnosed and therefore the attention is focused on the remaining faults. The diagnostic result describes the probability for the occurrence of each fault, which is marked by the intensity of the bars in the following diagrams.

Figure 9.33 shows the diagnostic result for the faultless case f_0. The faultless operation is unambigously recognised. After starting the diagnosis at time 0 the faultless case is the most

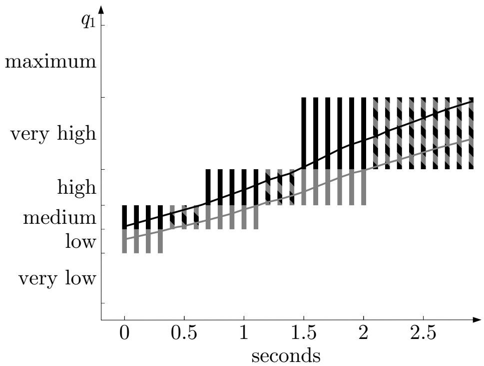


Fig. 9.32. Air flow q_1 during faultless operation f_0 (black) and during fault f_3 (grey)

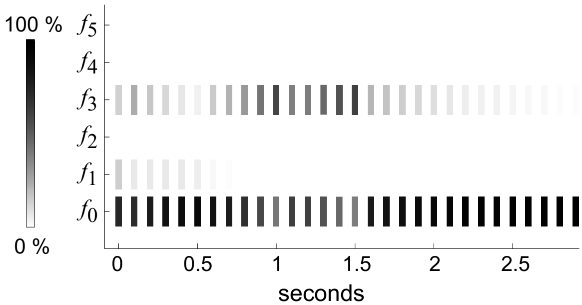


Fig. 9.33. Identification of the faultless case

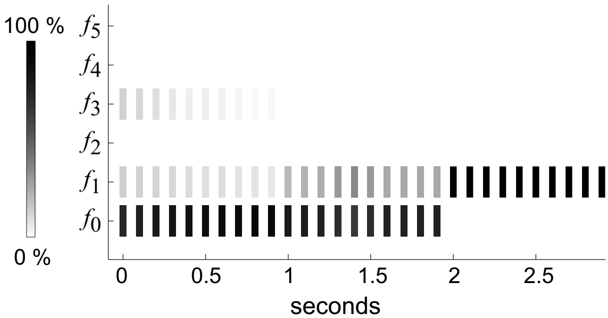


Fig. 9.34. Identification of the fault f_1

likely one but the faults f_1 and f_3 cannot be excluded. The higher initial probability of the faultless case is given to the diagnostic system because it is assumed that the system normally works without any faults. After a short time of diagnosing, the fault f_1 is excluded. During the time span from one to two seconds the signals from the air path coincide as well with the faultless model as with the model for the fault f_3. After two seconds the probability for the fault f_3 decreases continuously and the diagnostic system shows the correct result: no fault is present in this experiment.

The figure illustrates the fact that the diagnostic result is obtained by accumulating the information included in the measurement sequences. After about 2.5 seconds, the diagnostic algorithm has sufficient information for determining the faultless case unambiguously.

If the air path is affected by the fault f_1, the faults f_0, f_1 and f_3 are stated possible when the diagnosis starts (Fig. 9.34). The higher probability of f_0 is due to the higher initial probability explained above. During the diagnosis the probability of f_3 decreases continuously and the probability of f_1 increases accordingly. At 1.9 seconds a measurement has occurred that is impossible in the faultless case. Therefore, the faultless case is excluded and the diagnostic system returns the correct result.

Evaluation of the results. The presented qualitative diagnostic method allows diagnosing faults that, in general, cannot be diagnosed by signal-based methods. The qualitative approach means that the behaviour of the air path is modelled as coarse as possible so that many details can be ignored but the faulty behaviour can still be distinguished from the nominal behaviour. Thereby the unavoidable uncertainties of the model are taken into account and the air path can be modelled in a simple manner. □

These examples show that difficulties in the measurement of important output signals are not the only reason for using the quantised system approach to the diagnosis of continuous-variable systems. If the system has a hybrid dynamics, the method to use a discrete-event abstraction as the model for diagnosis has the advantages to deal with a unique discrete-event system and to use recursive algorithms that can be applied in real time.

9.7 Fault-tolerant control of quantised systems

9.7.1 Reconfiguration problem

As the actions to be made in fault-tolerant control refer to severe changes of the control input and possibly include switches to new sensors or actuators, the quantised system approach is particularly reasonable for these steps. The discrete-event model of the plant has direct reference to the discrete decision variables to be found. This fact will become obvious in this section. In a more elaborate form, this method is illustrated by its application to the neutralisation process described in Section 10.2.

The control problem considered here concerns the following situation. Assume that a fault has appeared in the plant and a diagnostic algorithm has found this fault. As the diagnosis takes some time, the system has moved from its nominal operation point to another point, because the nominal controller is applied to the faulty system. This situation occurs, in particular, if a sensor or an actuator fails, because then the control law is restricted to that part of the plant-controller interface that is still working. The control to be found requires the reconfiguration of the control loop, because the controller can have access only to the faultless sensors and faultless actuators.

In this situation a fault-tolerant controller has first to move the system state back into the nominal operation point and then to stabilise the system in this point. Therefore, the control algorithm consists of two steps:

1. A discrete controller has to be found that moves the operation point back into the nominal one.
2. A new continuous controller has to be found that stabilises the faulty system in the operation point.

The first controller has, usually, discrete input and output signals, because the operation point has to be moved through a large part of the state space, which does not necessitate precise numerical measurements. The stabilising controller is a feedback controller of the faulty system, which can be designed by well known controller design methods after the available actuators and sensors have been identified and the reconfigurability investigated (cf. Chapters 4 and 5). The following concentrates on the first part of the fault-tolerant control algorithm.

Main idea. As the movement of the operation point requires broad changes of the input, the quantised systems approach is reasonable for the solution of this task even if the plant has continuous inputs and outputs. Hence, quantisers are introduced deliberately for continuous-variable signals. This step facilitates a uniform discrete-event view on the system, which may have both discrete and continuous inputs and outputs.

A discrete-event model is abstracted by the methods described in Section 9.4. For the explanation of the main idea of the control design for this model, a nondeterministic automaton is sufficient, because the extension to stochastic automata is straightforward. The automaton can be graphically interpreted by the automaton graph. The directed edges of the graph show which state transitions among the quantised states the system can perform, where the edge label $([\boldsymbol{u}], f)$ denotes the quantised control input and the fault under which this transition occurs. By using this representation, the control problem can be formulated as a graph search problem. A path from the vertex representing the current qualitative state has to be found towards one of those vertices that represent the claimed operation point. This path may include only those edges that are valid for the current fault, which is assumed to be given by the diagnostic algorithm. The labels belonging to the edges in the path signify which control actions have to be used. The sequence of this control actions represents the solution to the control problem.

This method will be explained now in more detail. It is applied in Section 10.1.3 to reconfigure the controller of a three-tank system.

9.7.2 Graph-theoretic formulation of the control problem

Since a diagnostic algorithm can only find the fault after the system under consideration has sufficiently changed its behaviour and, thus, left its nominal state, the qualitative initial state $[\boldsymbol{x}(0)]$ is different from the required state and assumed in the following to be arbitrary, but known. The problem is to find a controller

$$[\boldsymbol{u}(k)] = k_q([\boldsymbol{x}(k)], f) \tag{9.57}$$

such that the operation point is moved from the current point $[\boldsymbol{x}(0)]$ into one operation point included in the set $\mathcal{Z}_{\mathrm{Aim}}$

$$\mathcal{Z}_{\mathrm{Aim}}(f) = \{[\boldsymbol{x}] \,|\, \text{Control specifications are satisfied}\} \subset \mathcal{N}_z. \tag{9.58}$$

These points are selected so as to satisfy the given specifications on the closed-loop system under the faulty conditions. Note that the controller (9.57) uses the quantised information about the current state $\boldsymbol{x}$ and represents a qualitative state feedback. More restrictive control laws like a qualitative output feedback

$$[\boldsymbol{u}(k)] = \tilde{k}_q([\boldsymbol{y}(k)], f)$$

can be found in a similar way.

A graph-theoretic interpretation of this problem can be obtained as follows. At time $k = 0$ the quantised system is in the qualitative initial state $z_0 = [\boldsymbol{x}(0)]$, which is represented by the vertex of the automaton graph with the same name. The controller (9.57) has to be chosen such that the system arrives at the set $\mathcal{Z}_{\mathrm{Aim}}(f)$ and remains there. That is, the quantised system together with the controller has to be *qualitatively stable*.

Graph-theoretic characterisation of qualitative stability. The qualitative stability can be tested by the automaton graph G_c of the closed-loop system. Assume that the controller (9.57) is already known. Then the non-deterministic automaton

$$N_c = (\mathcal{N}_z, \mathcal{N}_f, L_c, z_0)$$

of the closed-loop system has the state transition relation

$$L_c(z, f) = L_n(z, k_q(z, f), f)$$

where L_n is the state transition relation of the non-deterministic automaton of the plant subject to fault f. The graph G_c of this automaton can be obtained from the automaton graph of the plant by deleting all edges (z_i, z_j) whose labels $([\boldsymbol{u}], f)$ do not satisfy the control law (9.57) with $[\boldsymbol{x}(k)] = z_i$.

For the stability analysis the set of vertices $\mathcal{Z}_{\mathrm{Aim}}$ is replaced in the graph G_c by a new vertex z_{Aim}. All edges (z, z') starting in some vertex $z \notin \mathcal{Z}_{\mathrm{Aim}}$ and going to some $z' \in \mathcal{Z}_{\mathrm{Aim}}$ are replaced by an edge (z, z_{Aim}). In this way a new graph denoted by $G'_c(\mathcal{N}'_z, \mathcal{E}'_c)$ is obtained. The reconfiguration problem is solved if the conditions given in the following theorem are satisfied:

Theorem 9.4 (Qualitative stability)
A qualitative controller (9.57) solves the control problem, if the graph $G'_c(\mathcal{N}'_z, \mathcal{E}'_c)$ *satisfies the following three conditions:*

1. *The graph* G'_c *has no strongly connected vertices.*
2. *There are no self-circles around any vertex* $z \neq z_{\rm Aim}$.
3. *The vertex* $z_{\rm Aim}$ *is the only end vertex of* G'_c. *That is,* $z_{\rm Aim}$ *is the only vertex that has no outgoing edge.*

The conditions appearing in the theorem have been proved to be necessary and sufficient for the qualitative stability of the quantised closed-loop system provided that the qualitative model is complete. Hence, the reconfiguration problem can be formulated as follows:

> **Reconfiguration problem:** *Find a controller (9.57) such that the reduced automaton graph* $G'_c(\mathcal{N}'_z, \mathcal{E}'_c)$ *of the closed-loop system satisfies the three conditions given in Theorem 9.4.*

9.7.3 A reconfiguration method

First, some notions from graph theory have to be recalled. It is assumed that in the automaton graph there is at most one edge between a predecessor state and a successor state. This assumption can be satisfied by lumping parallel edged together, where the labels are combined accordingly.

A subgraph $T(\mathcal{N}_z, \mathcal{E}_t)$ of $G(\mathcal{N}_z, \mathcal{E})$ with $\mathcal{E}_t \subseteq \mathcal{E}$ is called a *spanning tree* if any pair of its vertices z_i and z_j are connected and if T has no cycle. Two vertices z_i and z_j are called *strongly connected* if there exists a path from z_i towards z_j and a path from z_j towards z_i. A subgraph $G_i(\mathcal{N}_{zi}, \mathcal{E}_i)$ of G is called a strongly connected component if all vertices of $\mathcal{N}_{zi}$ are strongly connected. The graph that results after replacing every strongly connected component by a new vertex is called the *condensed graph*.

Algorithm 9.3 *Solution of the reconfiguration problem by means of a qualitative model*

Given: Automaton graph $G(\mathcal{N}_z, \mathcal{E})$,
Goal set $\mathcal{Z}_{\rm Aim}$.

1. Determine the graph $G'(\mathcal{N}'_z, \mathcal{E}')$ by replacing the set of vertices $\mathcal{Z}_{\rm Aim}$ by the single vertex $z_{\rm Aim}$.
2. Determine the condensed graph $\hat{G}(\hat{\mathcal{N}}_z, \hat{\mathcal{E}})$ of $G'(\mathcal{N}'_z, \mathcal{E}')$ by first determining all strongly connected components G_i of G' and second replacing G_i by hyper vertices.
3. For all strongly connected components G_i determine a spanning tree T_i.
4. Using the condensed graph and the spanning trees, construct a new graph $G_{new}(\mathcal{N}'_z, \mathcal{E}_{new})$ with $G_{new} = \hat{G} \cup T_1 \cup T_2 \cup \cdots \cup T_s$ where s is the number of strongly connected components.
5. Determine the control law $k_q(z, f)$ as follows: For every vertex $z \in \mathcal{N}'_z$ and every fault f, select the qualitative output $[\boldsymbol{u}]$ associated with the edge (z, z') in the graph G_{new}.

Result: Qualitative controller (9.57).

The strongly connected components G_i can be constructed using a depth-first-search algorithm whose complexity is linear with respect to the number of vertices. For each vertex of the condensed graph there exists a spanning tree T_i. Such trees can be found by Tremaux and Tarjan's algorithm whose complexity is linear. Each vertex $z \neq z_i$ of graph G' will be transferred to the vertex z_i of G'_{z_i} using spanning tree T_i. Then, for each vertex z_i there exists a path to the desired vertex $\mathcal{Z}_{\rm Aim}$ in the condensed graph $\hat{G}$.

The result is the mapping k_q of the controller (9.57), which can be represented by a table and which can be easily implemented.

9.8 Exercises

Exercise 9.1 *Selection of reasonable state-space partitions*

Consider an autonomous first-order discrete-time system. Under what conditions on the state partitions does the system map one state set precisely into another state set with the consequence that the quantised system can be described by a *deterministic* automaton?

Can this determinism be retained if the system has an input by appropriately partitioning the input space? □

Exercise 9.2 *Determinstic discrete-event behaviour of an undamped oscillator*
Consider an undamped oscillator described by the state-space model

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}.$$

Show that the sampling time can be chosen such that the oscillator has a deterministic discrete-event behaviour in a state space where both state variables x_1 and x_2 are independently partitioned.

Does this result hold if the oszillator is damped (where the damping factor δ replaces the zeros in the system matrix)? □

Exercise 9.3 *Quantised tank system*
Consider a two-tank system in a state space, whose partitioning distinguishes merely between a full and an empty tank. At time $t = 0$ both tanks are filled. An outlet valve of the right tank is open. Set up a simple second-order state-space model and use the abstraction algorithm to get a non-deterministic or a stochastic automaton describing the quantised tank system.

Extend the model, if the left tank has a pump, which can be switched on and off and which produces an inflow into the left tank that is twice as large as the outflow from the right tank. How do you have to extend the automaton to cope with this extension of the quantised system? □

Exercise 9.4 *Spring-mass system*
Consider a damped spring-mass system. A fault reduces the mass be the factor two. Draw the movement of the system in the faultless and the faulty case and decide whether this fault can be diagnosed by only measuring quantised values of the mass position. □

9.9 Bibliographical notes

Results on the diagnosis of quantised systems have been obtained in two fields. The problem of abstracting discrete-event representations for quantised systems has been investigated as a step for the analysis of hybrid systems or for the verification of discrete controllers. The publications [138], [141] and [153] have shown that quantised systems have, in general, a non-deterministic qualitative behaviour and do not possess the Markov property. Hence, they cannot be represented precisely by any model that possesses the Markov property like stochastic automata.

Methods for abstracting discrete-event representations of discrete-time or discrete-event quantised systems have been proposed in [118], [138], [141], [153], [117], [201], [204], [246]all of which aim at finding complete models in the sense defined in Section 9.4. [204] showed that by using different definitions of the model state, a hierarchy of discrete abstractions can be obtained, which generate different numbers of spurious solutions.

In the field of computer-aided modelling the abstraction problem has been considered in [136] and [262] with the aim to find deterministic discrete-event representations, which, according to [138] is possible only for a very small class of quantised systems. There are only preliminary results concerning the question how to partition the signal spaces in order to obtain abstractions with a small number of spurious solutions, cf. [153], [147], [115]. A connection of the stochastic automaton as discrete-event description of quantised systems and the Frobenius-Perron operator is given in [158].

On the other hand, the diagnosis of quantised systems is based on methods for diagnosing discrete-event systems described by automata, which have been elaborated in [128], [157], [211] and [210]. The complete solution to the diagnostic problem for stochastic automata given in [157] is the basis for the results reviewed in Section 8.4. First results for quantised systems have been described in [125] and [156]. In [63] it is shown that discrete-event representations of quantised systems can be used for diagnosis if and only if they are complete. This reference also gives an example to demonstrate that different models can be used for the same quantised system all of which are complete but differ concerning the number of spurious solutions and, hence, yield diagnostic results of different precision.

The solution to the state observation problem for the quantised system is based on the solution to the observation problem for stochastic automata. This problem has been dealt with only by a few authors, for example in [157] and [191].

The extensions to transient faults is described in [216].

The diagnosis of discrete-event quantised systems has be developed in [63] and [143], where the latter reference presents a method that takes the temporal event distances into account. The corresponding abstraction methods, which transforms the continuous-variable description of the system into a discrete-event model, are developed in [60] and [142].

Theorem 9.4 has been proved in [140]. The problem how to partition the state space in order to avoid the non-determinsm on the quantised system level and to get a deterministic automaton as the precise qualitative model of the quantised system is still open [147].

Application examples of fault diagnosis of quantised systems are presented in [156] (neutralisation process), [124] (H_2 compressor), [61], [62] (diesel injection system) and [64], [174] (air path of a diesel motor), which is summarised here as Example 9.10.

The reconfiguration method described in Section 9.7 has been first published in Chapter 12 of [3].