Unified Formula for Comparison of Clock Rates and Its Applications

C. Xu[1], X. Wu[1], and E. Brüning[2]

[1] Purple Mountain Observatory, Nanjing 210008, China
Department of Physics, Nanjing Normal University, Nanjing 210097, China
[2] School of Mathematical Sciences, University of KwaZulu-Natal,
Durban 4000, South Africa

Summary. In this chapter, we deduce a unified formula which allows to discuss the comparison of clock rates at two different space–time points. In the case of a perturbed Robertson–Walker metric, our formula returns to an equation for the comparison of clock rates at different cosmic space–time points, which includes the Hubble redshift, the Doppler effect, the gravitational redshift, and the Rees–Sciama effects. In the case of the solar system, when the 2PN metric is substituted into the unified formula, the comparison of the clock rates both on the earth and a space station could be made. It might be useful for the discussion on the precise measurement on future ACES and ASTROD.

1 Introduction

One of the most basic experiments in physics is the measurement of times. Recently, atomic clocks with a time-keeping accuracy of the order of 10^{-18} in fractional frequency have been considered [1, 2]. Also a spatial experiment named Atomic Clock Ensemble in Space (ACES) mission [3, 4] is scheduled to be launched in 2006 by European Space Agency (ESA). The purpose of ACES is to obtain an accuracy of order 10^{-16} in fractional frequency. In such a situation (10^{-16}–10^{-18} level), 2PN (second post-Newtonian) approximate framework has to be carried out before hand. Also, Astrodynamical Space Test of Relativity using Optical Devices (ASTROD) [5, 6] is planned. The accuracy of measuring γ (about 10^{-9}) and other parameters will depend on the stability of the lasers or clocks. This plan also needs a 2PN level on the comparison of clock rates and equations of motion for planets. The precision of 2PN level on the comparison of clock rates (or time transfer) has been discussed in [7,8] by means of world function. But as we know, the calculation of the world function is not easy. Therefore, we deduce a unified formula in a different way. Our unified formula can also be applied to cosmos and easily extended to an even higher order (higher than 2PN level).

Many formulae have been suggested for the comparison of clock rates at different positions, based on certain simplifying assumptions about which effects are dominant. The change of the clock rates can be related to the relativistic Doppler effect, the gravitational redshift, the Hubble redshift, the Rees–Sciama effect, and so on. The physical conditions causing all these effects may prevail at the same time. In early 1990s, the Hubble redshift, the gravitational redshift, the Doppler effect, and the Rees-Sciama effect have been combined into one equation (to see (6) in [9]) in first-order approximation

$$1+z=\frac{R(\tau_o)}{R(\tau_e)}\left\{1+\frac{5}{3}(\phi_e-\phi_o)+2\int_{\tau_e}^{\tau_o}d\tau\mathbf{l}\cdot\nabla\phi+\mathbf{n}\cdot(\mathbf{v}_e-\mathbf{v}_o)\right\}\,, \tag{1}$$

where z is redshift, ϕ is gravitational potential, the subscript $e(o)$ denote the emitting (observer) point, $\mathbf{l}=\mathbf{k}/k_o$ (k^α is the tangent vector to the null geodesic connecting the emitting point and observer), and $\mathbf{v}$ is 3-velocity. Since all of terms are the level of the first-order approximation, the coupling terms do not exist. Also they do not deduce (1) through an exact method, it is difficult for us to extend the formula to higher-order precision.

Accordingly, a comprehensive approach, starting from first principles, is needed in which the physical conditions for all these effects are taken into account at the same time. Such an approach should lead us to a synthetic formula which reflects all these effects in a compact way and which should provide additional information, due to possible interactions which could not be incorporated in the isolated approaches for the individual effects.

In general, a comparison of the clock rates between $\Delta\tau_A$ and $\Delta\tau_B$ by means of differential coordinate time Δt_A and Δt_B in global coordinate can be achieved. The relation of the coordinate time between A and B is established by null geodesic line (light ray) [10]

$$cdt=\frac{-g_{0i}dx^i\pm\sqrt{(g_{0i}g_{oj}-g_{00}g_{ij})dx^idx^j}}{g_{00}}\,. \tag{2}$$

The minus and plus sign are taken in I and III quadrants (in $x-t$ coordinates) and II and IV quadrants, respectively. Normally we take the minus sign. Using these ideal we first time deduce a unified formula for the comparison of clock rates by means of "calculus of differences." Substituting the simplest perturbed Robertson–Walker metric into the unified formula, we obtain a formula for the comparison of clock rates at different cosmic space–time points, which includes the Hubble redshift, the Doppler effect, the gravitational redshift, and the Rees–Sciama effects. By using the 2PN metric in multiple coordinates [11], the 2PN comparison of clock rates both on the earth and a space station in the solar system is made, it may be useful for the precise measurement of ACES and ASTROD in future.

2 General Formula

In a global coordinates (ct, x^i), a source A moves with a velocity v_A^i and a receiver B with a velocity v_B^i. The clock rates in A and B are directly related with their own proper time $\Delta\tau_A$ and $\Delta\tau_B$. To compare them, we need to know the relation between the time interval Δt_A and Δt_B, because

$$\frac{\Delta\tau_A}{\Delta\tau_B} = \frac{\Delta\tau_A}{\Delta t_A}\frac{\Delta t_A}{\Delta t_B}\frac{\Delta t_B}{\Delta\tau_B}\,. \tag{3}$$

Since $-c^2 d\tau^2 = ds^2$, therefore if the velocity of a standard clock (A or B) in the global coordinates is $\mathbf{v}$, we have

$$\Delta t = \frac{\Delta\tau}{\sqrt{-[g_{00} + 2g_{0i}v^i/c + g_{ij}v^i v^j/c^2]}}\,, \tag{4}$$

where $g_{\mu\nu}$ (g_{00}, g_{0i}, and g_{ij}) are the global metric. As abbreviation we introduce

$$G_A = -(g_{00}(A) + 2g_{0i}(A)v_A^i/c + g_{ij}(A)v_A^i v_A^j/c^2)\,,$$
$$G_B = -(g_{00}(B) + 2g_{0i}(B)v_B^i/c + g_{ij}(B)v_B^i v_B^j/c^2)\,.$$

One of the main purpose of our chapter is to calculate the relation between Δt_A and Δt_B by means of a "calculus of differences." Assuming that, at t_{A_1} (coordinate time) a source A emits a first pulse at position $A_1(x^i_{A_1})$, then a receiver B received the first pulse at position $B_1(x^i_{B_1})$ at time t_{B_1}. A second pulse is emitted from A at position $A_2(x^i_{A_2})$ at t_{A_2}, which is received by B at position $B_2(x^i_{B_2})$ at time t_{B_2}. Then the relation between the emission time and reception time can be rewritten as

$$t_B = t_A + \frac{1}{c}\int_A^B \frac{-g_{0i}\frac{dx^i}{dx} - \sqrt{(g_{0i}g_{0j} - g_{00}g_{ij})\frac{dx^i}{dx}\frac{dx^j}{dx}}}{g_{00}}dx\,, \tag{5}$$

where we define $dx^2 \equiv \delta_{ij}dx^i dx^j$, the geometric meaning of dx is the spatial differential length of the line in the flat space.

In a weak field, $g_{0i}g_{0j}$ is a small quantities ($\sim O(6)$) and the spatial conformal isotropic condition [11, 12] is

$$g_{00}g_{ij} = -\delta_{ij} - \frac{q_{ij}}{c^4} + O(6)\,, \tag{6}$$

where q_{ij} is a spatial anisotropic contribution in the second order, $O(6)$ is the abbreviation symbol for $O(c^{-6})$ as well as $O(n)$ for $O(c^{-n})$. Then (5) simplifies to

$$t_B = t_A - \frac{1}{c}\int_A^B \frac{1}{g_{00}}\left[1 + g_{0i}\frac{dx^i}{dx} + \frac{q_{ij}}{2c^4}\frac{dx^i}{dx}\frac{dx^j}{dx}\right]dx\,. \tag{7}$$

Briefly, we define

$$F(t,x^i) \equiv \frac{-g_{0i}\frac{dx^i}{dx} - \sqrt{(g_{0i}g_{0j} - g_{00}g_{ij})\frac{dx^i}{dx}\frac{dx^j}{dx}}}{cg_{00}}\,, \tag{8}$$

Equation (5) becomes $t_B = t_A + \int_A^B F(t,x^i)dx$. According to the "calculus of differences," we have

$$\Delta t_B = \Delta t_A + \Delta\left[\int_A^B F(t,x^i)dx\right]. \tag{9}$$

Note that in this formula we are dealing with finite differences, not with infinitesimals ones as in the calculus of variations. In particular in (6) one would have to use $x(A)$ and $x(B)$, respectively, as integration boundaries.

If we only consider a linear approximation in "calculus of differences," the difference of the integral in (6) can be divided into three parts

$$\Delta\int_A^B Fdx = \int_{x(B)}^{x(B+\Delta B)} Fdx - \int_{x(A)}^{x(A+\Delta A)} Fdx + \int_{x(A)}^{x(B)} \Delta F dx\,, \tag{10}$$

where $A+\Delta A$ and $B+\Delta B$ are corresponding to A_2 and B_2. $\Delta x(A)$ and $\Delta x(B)$ are given by

$$\Delta x(A) \equiv x(A+\Delta A) - x(A) = \frac{\mathbf{k}_A}{|k_A|}\cdot\left.\frac{d\mathbf{x}}{dx}\right|_A \Delta x = \frac{\mathbf{k}_A}{|k_A|}\cdot\mathbf{v}_A\Delta t_A\,, \tag{11}$$

$$\Delta x(B) \equiv x(B+\Delta B) - x(B) = \frac{\mathbf{k}_B}{|k_B|}\cdot\left.\frac{d\mathbf{x}}{dx}\right|_B \Delta x = \frac{\mathbf{k}_B}{|k_B|}\cdot\mathbf{v}_B\Delta t_B\,. \tag{12}$$

Here $\mathbf{k}_A$ is the wave vector at point A of the light signal emitted from A and received at B. $\mathbf{k}_B$ is the value of this wave vector at the point B. In (11) and (12) we can replace $\frac{dx^i}{dx}\Delta x$ by $\frac{dx^i}{dt}\Delta t$. The first and second terms in (10) can be written as

$$\int_{x(B)}^{x(B)+x(\Delta B)} Fdx = F(B)\frac{\mathbf{k}_B\cdot\mathbf{v}_B}{|k_B|}\Delta t_B\,, \tag{13}$$

$$-\int_{x(A)}^{x(A)+x(\Delta A)} Fdx = -F(A)\frac{\mathbf{k}_A\cdot\mathbf{v}_A}{|k_A|}\Delta t_A\,. \tag{14}$$

The last term of (10) is the difference of the integral between line 2 and line 1 when $\mathbf{v}_A = \mathbf{v}_B = 0$ (i.e., the boundary of integral is fixed), which can be expanded as

$$\int_{x(A)}^{x(B)}\left(F(t+\Delta t, x^i+\Delta x^i) - F(t,x^i)\right)dx = \int_{x(A)}^{x(B)}\left(\frac{\partial F}{\partial t}\Delta t + \frac{\partial F}{\partial x^i}\Delta x^i\right)dx\,. \tag{15}$$

As we know, if F is independent of time, then $\int_{x(A)}^{x(B)}\Delta F dx = \int_{x(A)}^{x(B)}\frac{\partial F}{\partial x^i}\Delta x^i dx = 0$, since for fixed boundaries the light ray is unique (no deviation). When F is dependent on time, there are two curves. The second term $\int_{x(A)}^{x(B)} \frac{\partial F}{\partial x^i} \Delta x^i dx$ caused by time-dependent metric is a higher-order term compared with $\int_{x(A)}^{x(B)} \frac{\partial F}{\partial t} \Delta t dx$, i.e.,

$$\int_{x(A)}^{x(B)} \frac{\partial F}{\partial x^i} \Delta x^i dx \ll \int_{x(A)}^{x(B)} \frac{\partial F}{\partial t} \Delta t dx \,. \tag{16}$$

Finally we substitute (13–15) into (6), and use (4). The unified formula is obtained as

$$\begin{aligned} \frac{\Delta\tau_B}{\Delta\tau_A} &= \sqrt{\frac{G_B}{G_A}} \left(\frac{1 - F(A)\frac{\mathbf{k}_A \cdot \mathbf{v}_A}{|k_A|}}{1 - F(B)\frac{\mathbf{k}_B \cdot \mathbf{v}_B}{|k_B|}} \right) \\ &\quad + \frac{\sqrt{G_B}}{\Delta\tau_A \left(1 - F(B)\frac{\mathbf{k}_B \cdot \mathbf{v}_B}{|k_B|}\right)} \int_{x(A)}^{x(B)} \left(\frac{\partial F}{\partial t} \Delta t + \frac{\partial F}{\partial x^i} \Delta x^i \right) dx \,. \end{aligned} \tag{17}$$

As an example, we consider the Doppler effect of a moving source in Minkowski metric

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & \delta_{ij} \end{pmatrix} , \tag{18}$$

since $\mathbf{v}_B = 0$, $G_B = 1$, $G_A = 1 - \frac{v_A^2}{c^2}$, and $\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x^i} = 0$, $F(A) = \frac{1}{c}$, so that

$$\frac{\Delta\tau_B}{\Delta\tau_A} = \frac{1 - \frac{\mathbf{v}}{c} \cdot \frac{\mathbf{k}_A}{|k_A|}}{\sqrt{1 - \frac{v_A^2}{c^2}}} \,. \tag{19}$$

This is just the formula of the Doppler effect in the special relativity.

The other simple example is the gravitational redshift. Considering a static gravitational field (e.g., Schwarzschild metric), in which both source and receiver without moving ($\mathbf{v}_A = \mathbf{v}_B = 0$), the unified form then becomes

$$\frac{\Delta\tau_B}{\Delta\tau_A} = \sqrt{\frac{-g_{00}(B)}{-g_{00}(A)}} \simeq 1 - \frac{w(B)}{c^2} + \frac{w(A)}{c^2} \,, \tag{20}$$

where the last step of above equation is the Newtonian limitation. Equation (20) is just the formula of gravitational redshift in ordinary textbooks of gravity.

3 Application in Cosmos with Perturbed R–W Metric

First we recall the unperturbed Robertson–Walker metric

$$ds^2 = -c^2 dt^2 + \frac{R(t)^2 \delta_{ij} dx^i dx^j}{\left(1 + \frac{k}{4} r^2\right)^2} \,, \tag{21}$$

where $R(t)$ is the cosmic scalar factor, $k = -1, 0, +1$ is corresponding to the open, flat, and closed universe, respectively. $R(t)$ has the dimension of length and dx^i is dimensionless. As we already know, $R(t)$ is model dependent. Since usually we do not consider the local gravitational redshift and the Doppler effect in the problem of cosmological expansion, we then have $\Delta t_A = \Delta\tau_A$ and $\Delta t_B = \Delta\tau_B$, and thus the formula for the Hubble redshift is

$$\frac{\Delta\tau_A}{\Delta\tau_B} = \frac{\Delta t_A}{\Delta t_B} = \frac{R(t_A)}{R(t_B)}\,. \tag{22}$$

The results of (22) can easily be deduced directly from the unified formula (17), if we take $v_A = v_B = w_A = w_B = 0$.

Next, we consider a linearly simplest perturbed Robertson–Walker metric of the form

$$ds^2 = -c^2\left(1 - \frac{2w}{c^2}\right)dt^2 + \left(1 + \frac{2w}{c^2}\right)\frac{R^2\delta_{ij}dx^i dx^j}{\left(1 + \frac{k}{4}r^2\right)^2}\,, \tag{23}$$

where the gravitational potential $w = w(t, x^i)$ is assumed to be a small quantity. Later we only consider the Doppler effect caused by the motion of the source, then $\mathbf{v}_B = 0$ (also possible $\mathbf{v}_A = 0$, then $\mathbf{v}_B \neq 0$). The velocity of the source A is

$$v_A^i = R(t_A)\frac{dx_A^i}{dt}\,. \tag{24}$$

$F(A)$, G_A, and G_B can be calculated as follows

$$F(A) = \frac{\left(1 + \frac{2w}{c^2}\right)\sqrt{R^2\delta_{ij}\frac{dx^i}{dx}\frac{dx^j}{dx}}}{c\left(1 + \frac{kr^2}{4}\right)} = \frac{R(t_A)}{c} + O(3)\,, \tag{25}$$

where we have neglected all of higher-order terms and consider $\delta_{ij}n^in^j = 1$ and kr^2 as higher-order term also. Then

$$\sqrt{\frac{G_B}{G_A}} = 1 + \frac{1}{c^2}w(t_A, x_A^i) - \frac{1}{c^2}w(t_B, x_B^i) + \frac{v_A^2}{c^2}\,. \tag{26}$$

Now we calculate the last term of (17). Consider the integral in (17)

$$I \equiv \int_{x(A)}^{x(B)} \frac{\partial F}{\partial t}\Delta t dx\,, \tag{27}$$

where we have omitted the term of $\int_{x(A)}^{x(B)} \frac{\partial F}{\partial x^i}\Delta x^i dx$ because of (16). Therefore we have

$$I = \int_{x(A)}^{x(B)} \frac{\partial}{\partial t}\left[R(t)\left(1 + \frac{2w}{c^2}\right)\right]\frac{\Delta t\sqrt{\delta_{ij}dx^idx^j}}{c\left(1 + \frac{k}{4}r^2\right)}\,. \tag{28}$$

From (22) we have $\Delta t = \Delta t_A R(t)/R(t_A)$, then (28) becomes

$$I = \int_{x(A)}^{x(B)} \frac{R(t)\Delta t_A}{R(t_A)} \left(\dot{R}(t) + \dot{R}(t)\frac{2w}{c^2} + R(t)\frac{\partial(2w/c^2)}{\partial t} \right) \frac{\sqrt{\delta_{ij}dx^i dx^j}}{c\left(1+\frac{k}{4}r^2\right)} . \quad (29)$$

Considering null geodetic line, (23) yields

$$\left(1 - \frac{w}{c^2}\right) dt = \pm \left(1 + \frac{w}{c^2}\right) \frac{R(t)\sqrt{\delta_{ij}dx^i dx^j}}{c\left(1+\frac{k}{4}r^2\right)} , \quad (30)$$

which we use to evaluate the integral I and get

$$\begin{aligned} I &= \frac{\Delta t_A}{R(t_A)} \int_{x(A)}^{x(B)} \left(\dot{R}(t) + 2\frac{R(t)}{c^2}\frac{\partial w}{\partial t} \right) dt \\ &= \Delta t_A \left[\left(\frac{R(t_B) - R(t_A)}{R(t_A)} \right) + \frac{2}{c^2 R(t_A)} \int_{x(A)}^{x(B)} R(t)\frac{\partial w}{\partial t} dt \right] . \end{aligned} \quad (31)$$

The second term of (17) then becomes

$$\sqrt{\frac{G_B}{G_A}} \left(\frac{R(t_B) - R(t_A)}{R(t_A)} + \frac{2}{c^2 R(t_A)} \int_{x(A)}^{x(B)} R(t)\frac{\partial w}{\partial t} dt \right) . \quad (32)$$

Substituting (25), (26), and (32) into (17) (pay attention to that, in cosmology the term $\frac{\mathbf{k}}{|k|} \cdot \mathbf{v}$ should be replaced by $\frac{\mathbf{k}}{|k|} \cdot \frac{\mathbf{v}}{R}$ in (11), (12), and (17)), we finally obtain

$$\begin{aligned} \frac{\Delta\tau_B}{\Delta\tau_A} &= \left[1 + \frac{1}{c^2}\left(w(t_A, x_A^i) - w(t_B, x_B^i) \right) + \frac{v_A^2}{c^2} \right] \\ &\quad \times \left\{ \frac{R(t_B)}{R(t_A)} - \frac{\mathbf{k}_A \cdot \mathbf{v}_A}{c|k_A|} + \frac{2}{c^2 R(t_A)} \int_{x(A)}^{x(B)} R(t)\frac{\partial w}{\partial t} dt \right\} , \end{aligned} \quad (33)$$

where $\frac{1}{c^2}\left(w(t_A, x_A^i) - w(t_B, x_B^i) \right)$ is the contribution from the normal gravitational redshift; $\frac{\mathbf{k}_A \cdot \mathbf{v}_A}{c|k_A|}$ and $\frac{v_A^2}{c^2}$ are the Doppler effect and transverse Doppler effect (or relativistic Doppler effect), respectively; $R(t_B)/R(t_A)$ just contributes to Hubble redshift; and the last term is related to Rees–Sciama effect [13–16]. If we put $\mathbf{v}_A = 0$ and $R(t) = R(T_A) = R(t_B)$, then

$$\frac{\Delta\tau_B}{\Delta\tau_A} = 1 + \frac{1}{c^2}\left(w(t_A, x_A^i) - w(t_b, x_B^i) \right) + \frac{2}{c^2} \int_{x(A)}^{x(B)} \frac{\partial w}{\partial t} dt , \quad (34)$$

where w is the same as u in [17] where $c = 1$ units is taken, and their results totally agree with ours. We thought that our unified formula allows to derive the Birkinshaw–Gull effect [18,19] too, if we use suitable perturbation functions for $w(t, x^i)$ and $w_j(t, x^i)$. This will be discussed in another chapter.

4 Application in Solar System with DSX Metric

In near future, high-precision measurement will be done up to 2PN level as we mentioned before, thus allowing the coupling term (i.e., the term connecting the gravitational redshift, the Doppler redshift, and so on) to be measured. Our scheme (the unified form (17)) offers the possibility for this if an appropriate assumptions about the metric are used. Accordingly, in this section we start from DSX formalism [12,20,21] and its extension [11] and evaluate formula (17) for this metric. This extended DSX metric is described by

$$g_{00} = -\exp\left(-\frac{2w}{c^2}\right) + O(6)\,, \tag{35}$$

$$g_{0i} = -\frac{4w_i}{c^3} + O(5)\,, \tag{36}$$

$$g_{ij} = \delta_{ij}\exp\left(\frac{2w}{c^2}\right) + \frac{q_{ij}}{c^4} + O(6)\,, \tag{37}$$

$$g_{ij}g_{00} = -\delta_{ij} - \frac{q_{ij}}{c^4} + O(6)\,. \tag{38}$$

In fact, in the following calculation, q_{ij} appears only in the function $F(x^i, t)$, but in the final 2PN formula of clock rates (see (53)), q_{ij} does not exist which agree with the result in [7]. Substituting (35–38) into (17), calculating all of components, we can get a unified formula for the comparison of clock rates at 2PN level.

We begin by evaluating the terms G_A and G_B for the extended DSX metric:

$$\begin{aligned} G_A &= -g_{00}(A) - 2g_{0i}(A)\frac{v_A^i}{c} - g_{ij}(A)\frac{v_A^i v_A^j}{c^2} \\ &= 1 - \frac{2w(A)}{c^2} + \frac{2w^2(A)}{c^4} + \frac{8w_i(A)v_A^i}{c^4} - \frac{v_A^2}{c^2} - \frac{2w(A)v_A^2}{c^4} + O(6)\,, \end{aligned} \tag{39}$$

$$G_B = 1 - \frac{2w(B)}{c^2} + \frac{2w^2(B)}{c^4} + \frac{8w_i(B)v_B^i}{c^4} - \frac{v_B^2}{c^2} - \frac{2w(B)v_B^2}{c^4} + O(6)\,. \tag{40}$$

Since $\mathbf{k}_A \cdot \mathbf{v}_A/|k_A|$ and $\mathbf{k}_B \cdot \mathbf{v}_B/|k_B|$ are first order already, so $F(A)$ and $F(B)$ need to be calculated up to c^{-5} level. From (8), by using (35–38), we get

$$\begin{aligned} F(A) = \frac{1}{c}\Bigg\{ & 1 + \frac{2w(A)}{c^2} - \frac{4w_i(A)}{c^3}\left.\frac{dx^i}{dx}\right|_A + \frac{2w^2(A)}{c^4} \\ & + \frac{1}{2c^4}q_{ij}(A)\left.\left(\frac{dx^i}{dx}\frac{dx^j}{dx}\right)\right|_A \Bigg\} + O(6)\,, \end{aligned} \tag{41}$$

where we have neglected $g_{0i}g_{0j}(\sim O(6))$. Similarly

$$\begin{aligned} F(B) = \frac{1}{c}\Bigg\{ & 1 + \frac{2w(B)}{c^2} - \frac{4w_i(B)}{c^3}\left.\frac{dx^i}{dx}\right|_B + \frac{2w^2(B)}{c^4} \\ & + \frac{1}{2c^4}q_{ij}(B)\left.\left(\frac{dx^i}{dx}\frac{dx^j}{dx}\right)\right|_B \Bigg\} + O(6)\,. \end{aligned} \tag{42}$$

At last, we consider the integral (the second term) in (17)

$$\int_A^B \left(\frac{\partial F}{\partial t}\Delta t + \frac{\partial F}{\partial x^i}\Delta x^i\right) dx\,. \tag{43}$$

Because of (16) we will omit the second term in it. We only consider $\int_A^B \frac{\partial F}{\partial t}\Delta t dx$. As we know Δt_A at A and Δt_B at B, we could calculate Δt at an arbitrary point between A and B approximately. Then the definite integral can be evaluated by the median method, i.e.,

$$\int_A^B \frac{\partial F}{\partial t}\Delta t dx = \Delta\bar{t}\int_A^B \frac{\partial F}{\partial t}dx\,, \tag{44}$$

where $\Delta\bar{t}$ is the median value, for which we introduce a parameter η:

$$\Delta\bar{t} = \eta\Delta t_A\,. \tag{45}$$

η is a value closed to 1. For the term $\frac{\partial F}{\partial t}$ one finds

$$\frac{\partial F}{\partial t} = \frac{1}{c}\left(\frac{2}{c^2}\frac{\partial w}{\partial t} - \frac{4}{c^3}\frac{\partial w_i}{\partial t}\frac{dx^i}{dx} + \frac{4w}{c^4}\frac{\partial w}{\partial t} + \frac{1}{2c^4}\frac{\partial q_{ij}}{\partial t}\frac{dx^i}{dx}\frac{dx^j}{dx}\right) + O(6)\,. \tag{46}$$

In the solar system the change of the potential (the other metric much smaller) with time is very small (less then $O(2)$ level), only the leading term is considered. Therefore the term $\frac{2}{c^2}\int_A^B \frac{\partial w}{\partial t}dt$ is already on $O(4)$ level, but not $O(2)$. Furthermore, we have $dx = dl + O(2) = cdt + O(2)$. Then (46) simplifies to

$$\int_A^B \frac{\partial F}{\partial t}dx = \frac{2}{c^2}\int_A^B \frac{\partial w}{\partial t}dt + O(6)\,. \tag{47}$$

If we consider quick variable field (e.g., field in pulsar) we have to take (46) to substitute into (47).

The second integral term of (17) then becomes

$$\frac{\sqrt{G_B}}{\sqrt{G_A}}\frac{2\eta}{\left(1 - F(B)\frac{\mathbf{k}_B\cdot\mathbf{v}_B}{|k_B|}\right)c^2}\int_A^B \frac{\partial w}{\partial t}dt\,. \tag{48}$$

Gathering all evaluations done thus far in this section, we arrive at following general formula for the solar system:

$$\frac{\Delta\tau_B}{\Delta\tau_A} = \sqrt{\frac{G_B}{G_A}}\frac{1}{\left(1 - F(B)\frac{\mathbf{k}_B\cdot\mathbf{v}_B}{|k_B|}\right)}\left(1 - F(A)\frac{\mathbf{k}_A\cdot\mathbf{v}_A}{|k_A|} + \frac{2\eta}{c^2}\int_A^B \frac{\partial w}{\partial t}dt\right)\,. \tag{49}$$

In (49) we only consider the leading term, namely the scalar potential changing with the time, but in our scheme all of the second post-Newtonian terms (see

(46)) can be included in. Maybe in a system of binary pulsars, the higher-order terms in (46) are important. Formula (49) in static metric and in 1PN level agrees with the known formula [22].

We proceed by evaluating the remaining terms in (49). From (39) and (40), we find

$$\sqrt{G_B} = 1 - \frac{w(B)}{c^2} - \frac{v_B^2}{2c^2} + \frac{w^2(B)}{2c^4} + \frac{4w_i(B)v_B^i}{c^4} - \frac{3w(B)v_B^2}{2c^4} - \frac{v_B^4}{8c^4}\,, \quad (50)$$

$$\frac{1}{\sqrt{G_A}} = 1 + \frac{w(A)}{c^2} + \frac{v_A^2}{2c^2} + \frac{w^2(A)}{2c^4} - \frac{4w_i(A)v_A^i}{c^4} + \frac{5w(A)v_A^2}{2c^4} + \frac{3v_A^4}{8c^4}\,. \quad (51)$$

We also have

$$\begin{aligned}\left(1 - F(B)\frac{\mathbf{k}_B\cdot\mathbf{v}_B}{|k_B|}\right)^{-1} = {} & 1 + \frac{\mathbf{k}_B\cdot\mathbf{v}_B}{c|k_B|} + \frac{1}{c^2}\left(\frac{\mathbf{k}_B\cdot\mathbf{v}_B}{|k_B|}\right)^2 + \frac{2w(B)}{c^3}\frac{\mathbf{k}_B\cdot\mathbf{v}_B}{|k_B|} \\ & + \frac{1}{c^3}\left(\frac{\mathbf{k}_B\cdot\mathbf{v}_B}{|k_B|}\right)^3 - \frac{4w_i(B)}{c^4}\left.\frac{dx^i}{dx}\right|_B\frac{\mathbf{k}_B\cdot\mathbf{v}_B}{|k_B|} \\ & + \frac{4w(B)}{c^4}\left(\frac{\mathbf{k}_B\cdot\mathbf{v}_B}{|k_B|}\right)^2 + \frac{1}{c^4}\left(\frac{\mathbf{k}_B\cdot\mathbf{v}_B}{|k_B|}\right)^4 . \end{aligned} \quad (52)$$

Substituting (41), (42), and (50–52) into (49), we finally have a unified formula for the comparison of clock rates in the solar system, on the 2PN level of precision

$$\begin{aligned}\frac{\Delta\tau_B}{\Delta\tau_A} = 1 & + \left\{\frac{1}{c^2}\Big(w(A) - w(B)\Big) + \frac{1}{2c^2}\left(v_A^2 - v_B^2\right) - \left(\frac{\mathbf{k}_A\cdot\mathbf{v}_A}{c|k_A|} - \frac{\mathbf{k}_B\cdot\mathbf{v}_B}{c|k_B|}\right)\right. \\ & \left. -\frac{1}{c^2}\left(\frac{\mathbf{k}_B\cdot\mathbf{v}_B}{|k_B|}\right)\left(\frac{\mathbf{k}_A\cdot\mathbf{v}_A}{|k_A|} - \frac{\mathbf{k}_B\cdot\mathbf{v}_B}{|k_B|}\right)\right\} \\ & + \frac{1}{c^3}\left\{\Big(w(B) - w(A)\Big)\left(\frac{\mathbf{k}_B\cdot\mathbf{v}_B}{c|k_B|} + \frac{\mathbf{k}_A\cdot\mathbf{v}_A}{c|k_A|}\right)\right. \\ & + 2w(A)\left(\frac{\mathbf{k}_B\cdot\mathbf{v}_B}{c|k_B|} - \frac{\mathbf{k}_A\cdot\mathbf{v}_A}{c|k_A|}\right) \\ & - \frac{1}{2}\Big(v^2(B) - v^2(A)\Big)\left(\frac{\mathbf{k}_B\cdot\mathbf{v}_B}{c|k_B|} - \frac{\mathbf{k}_A\cdot\mathbf{v}_A}{c|k_A|}\right) \\ & \left. + \left(\frac{\mathbf{k}_B\cdot\mathbf{v}_B}{c|k_B|}\right)^2\left(\frac{\mathbf{k}_B\cdot\mathbf{v}_B}{c|k_B|} - \frac{\mathbf{k}_A\cdot\mathbf{v}_A}{c|k_A|}\right)\right\} \\ & + \frac{1}{c^4}\left\{\frac{1}{2}\left(w(B) - w(A)\right)^2 + \frac{1}{2}\left(10w(A) - w(B)\right)v_A^2\right. \\ & - \frac{1}{2}\left(w(A) + 6w(B)\right)v_B^2 + \frac{1}{8}\left(3v_A^4 - 2v_A^2 v_B^2 - v_B^4\right) \\ & + 4\left(w_i(B)v_B^i - w_i(A)v_B^i - w_i(B)\frac{k_B^i}{|k_B|}\frac{\mathbf{k}_B\cdot\mathbf{v}_B}{c|k_B|} + w_i(A)\frac{k_A^i}{|k_A|}\frac{\mathbf{k}_A\cdot\mathbf{v}_A}{c|k_A|}\right)\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{\mathbf{k}_B\cdot\mathbf{v}_B}{c|k_B|}\right)^2\left(3w(B)+w(A)+\frac{1}{2}\left(v_A^2-v_B^2\right)\right)\\
&-2\left(\frac{\mathbf{k}_B\cdot\mathbf{v}_B}{c|k_B|}\right)\left(\frac{\mathbf{k}_A\cdot\mathbf{v}_A}{c|k_A|}\right)\left(w(A)+w(B)\right)\\
&\left.+\left(\frac{\mathbf{k}_B\cdot\mathbf{v}_B}{c|k_B|}\right)^3\left(\frac{\mathbf{k}_B\cdot\mathbf{v}_B}{c|k_B|}-\frac{\mathbf{k}_A\cdot\mathbf{v}_A}{c|k_A|}\right)\right\}\\
&+\frac{2\eta}{c^2}\int_A^B\frac{\partial w}{\partial t}dt+O(5)\,. \qquad (53)
\end{aligned}
$$

The Formula (53) "contains" the Doppler effect, transverse Doppler effect (relativistic Doppler effect), gravitational redshift, and their complete coupling effects to 2PN level in the solar system. In addition there is a term which is the integral of the rates of change of the scalar potential along the null geodetic line from source A to receiver B. This is probably the most interesting result in our chapter. Hopefully this integral term and the coupling effects can be tested in the future with a deep-space explorer and are confirmed.

5 Conclusion Remarks

We have synthesized all known effects for the comparison of clock rates in one formula (17). The synthesized formula contains additional coupling terms and a new integral terms and thus gives essential new but untested information. Therefore to get this synthesized formula is not an end in itself, but a starting point for the further test work. We hope that this work could contribute to the further comparison of clock rates, such as ACES mission planned in 2006. The general form may be taken as the basis for a starting point to compare clock rates at any two different space–time points. For example, the frequency shift caused by gravitomagnetic effect (or Lens–Thirring effect) can also be considered in our scheme.

The general form is valid not only for any metric gravitational theory, but also for general relativity. If we substitute the parametrized 2PN metric into the formula, (53) could include parameters. In Sect. 4, we have discussed the comparison of two clock rates both on the earth and a space station with 2PN precision. In fact, the calculation of the higher precision might be done in a similar way, if we know the metric to higher order.

Equation (17) or (49) says that the clock rates depend on the trajectory of the transmitted signal and on the metric (especially the scalar potential) varying with time. While (17) is a unified formula, (49) is valid only in the solar system; however if replace (47) by the integral of (46), a general 2PN formula results.

In the case of cosmology, the general form can be used for any linearly perturbed metric, in particular it allows to include the general Sachs–Wolfe effects.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant No. 10273008). In the workshop "Relativistic Astrophysics and High-Precision Astrodynamics" led by Prof. Wei-tou Ni, we had several very fruitful discussions, and especially we would like to thank Prof. Tianyi Huang for his useful discussion.

References

1. R. Holzwarth et al., Phys. Rev. Lett. **85**, 2264 (2000).
2. Th. Udem et al., Phys. Rev. Lett. **86**, 4996 (2001).
3. C. Salomon and C. Veillet, in Proceeding of the Symposium on Space Station Utilisation, ESA–SP, **385**, 295 (1996).
4. A. Spallicci et al., Class. Quantum Grav. **14**, 2971 (1997).
5. W.-T. Ni, Int. J. Mod. Phys. D **11**(7) 947 (2002).
6. W.-T. Ni, S. Shiomi and A.-C., Liao, Class. Quantum Grav. **21**, S641 (2004).
7. B. Linet and P. Teyssandier, Phys. Rev. D **66**, 024045 (2002).
8. C. Poncin-Lafitte, B. Linet and P. Teyssandier, Class. Quantum Grav. **21**, 4463 (2004).
9. E. Martinez-Gonzalez, J.L. Sanz and J. Silk, Astrophys. J. **355**, L5–L9 (1990).
10. S. Weinberg, Gravitation and Cosmology (Wiley) (1972).
11. C. Xu and X. Wu, Chin. Phys. Lett. **20**, 195 (2003).
12. T. Damour, M. Soffel and C. Xu, Phys. Rev. D **43**, 3273 (1991).
13. M. Rees and D.W. Sciama, Nature **217**, 511 (1968).
14. G.R. Blumenthal et al., Astrophys. J. **388**, 234 (1992).
15. R.B. Tully et al., Astrophys. J. **388**, 9 (1992).
16. A. Meszaros, Astrophys. J. **423**, 19 (1994).
17. R. Tuluie, P. Laguna and P. Anninos, Astrophys. J. **463**, 15 (1996).
18. M. Birkinshaw and S.F. Gull, Nature **302**, 315 (1983).
19. M. Birkinshaw, in Lecture Notes in Physics **330**, Gravitational Lenses (eds Moran JM, Hewitt JN and Lo KL, Springer, Berlin Heidelberg New York), p. 59 (1989).
20. T. Damour, M. Soffel and C. Xu, Phys. Rev. D **45**, 1017 (1992).
21. T. Damour, M. Soffel and C. Xu, Phys. Rev. D **47**, 3124 (1993).
22. R. Adler, M. Bazin and M. Schiffer, Introduction to General Relativity (McGraw-Hill Book) (1975).