A Universal Tool for Determining the Time Delay and the Frequency Shift of Light: Synge's World Function

Pierre Teyssandier[1], Christophe Le Poncin-Lafitte[1], and Bernard Linet[2]

[1] Département Systèmes de Référence Temps et Espace, CNRS/UMR 8630, Observatoire de Paris, 61 avenue de l'Observatoire, F-75014 Paris, France
[2] Laboratoire de Mathématiques et Physique Théorique, CNRS/UMR 6083, Université François Rabelais, F-37200 Tours, France

Summary. In almost all of the studies devoted to the time delay and the frequency shift of light, the calculations are based on the integration of the null geodesic equations. However, the above-mentioned effects can be calculated without integrating the geodesic equations if one is able to determine the bifunction $\Omega(x_A, x_B)$ giving half the squared geodesic distance between two points x_A and x_B (this bifunction may be called *Synge's world function*). In this chapter, $\Omega(x_A, x_B)$ is determined up to the order $1/c^3$ within the framework of the PPN formalism. The case of a stationary gravitational field generated by an isolated, slowly rotating axisymmetric body is studied in detail. The calculation of the time delay and the frequency shift is carried out up to the order $1/c^4$. Explicit formulae are obtained for the contributions of the mass, of the quadrupole moment, and of the internal angular momentum when the only post-Newtonian parameters different from zero are β and γ. It is shown that the relative frequency shift induced by the mass quadrupole moment of the Earth at the order $1/c^3$ will be bounded by 10^{-16} in space experiments like ESA's Atomic Clock Ensemble in Space (ACES) mission. Other contributions are briefly discussed.

1 Introduction

A lot of fundamental tests of gravitational theories rest on highly precise measurements of the travel time and/or the frequency shift of electromagnetic signals propagating through the gravitational field of the solar system. In practically all of the previous studies, the explicit expressions of such travel times and frequency shifts as predicted by various metric theories of gravity are derived from an integration of the null geodesic differential equations. This method works quite well within the first post-Minkowskian approximation, as it is shown by the results obtained, e.g., in [1–5]. Of course, it works also within the post-Newtonian approximation, especially in the case of a static, spherically symmetric space–time treated up to order $1/c^3$ [6, 7]. However, the solution of the geodesic equations requires heavy calculations when one has to take into account the presence of mass multipoles in the field or the tidal effects due to the planetary motions, and the calculations become quite complicated in the post-post-Minkowskian approximation [8], especially in the dynamical case [9].

The aim of this chapter is to present a quite different procedure recently developed by two of us. Based on Synge's world function [10], this procedure avoids the integration of the null geodesic equations and is particularly convenient for determining the light rays which connect an emitter and a receiver having specified spatial locations at a finite distance. Thus, we are able to extend the previous calculations of the time delay and of the frequency shift up to the order $1/c^4$. As a consequence, it is now possible to predict the time/frequency transfers in the vicinity of the Earth at a level of accuracy which amounts to 10^{-18} in fractional frequency. This level of accuracy is expected to be reached in the foreseeable future with optical atomic clocks [11].

The plan of the chapter is as follows. First, in Sect. 2, the definition of the time transfer function is given and the invariant expression of the frequency shift is recalled. It is shown that explicit expressions of the frequency shift can be derived when the time transfer functions are known. In Sect. 3, the relevant properties of Synge's world function are recalled. In Sect. 4, the general expressions of the world function and of the time transfer function are obtained within the Nordtvedt–Will parametrized post-Newtonian (PPN) formalism. In Sect. 5, the case of a stationary field generated by an isolated, slowly rotating axisymmetric body is analyzed in detail. It is shown that the contributions of the mass and spin multipoles can be obtained by straightforward derivations of a single function. Retaining only the terms due to the mass M, to the quadrupole moment J_2, and to the intrinsic angular momentum $\boldsymbol{S}$ of the rotating body, explicit expansions of the world function and of the time transfer function are derived up to the order $1/c^3$ and $1/c^4$, respectively. The same formalism yields the vectors tangent to the light ray at the emitter and at the receiver up to the order $1/c^3$. In Sect. 6, the frequency shift is developed up to the order $1/c^4$ on the assumption that β and γ are the only nonvanishing post-Newtonian parameters. Explicit expressions are obtained for the contributions of J_2 and $\boldsymbol{S}$. Numerical estimates are given for ESA's Atomic Clock Ensemble in Space (ACES) mission [12, 13]. Concluding remarks are given in Sect. 7.

Equivalent results formulated with slightly different notations may be found in [14] and an extension of the method to the general post-Minkowskian approximation is given in [15].

Notations

In this work, G is the Newtonian gravitational constant and c is the speed of light in a vacuum. The Lorentzian metric of space–time is denoted by g. The signature adopted for g is $(+---)$. We suppose that the space–time is covered by one global coordinate system $(x^\mu) = (x^0, \boldsymbol{x})$, where $x^0 = ct$, t being a time coordinate, and $\boldsymbol{x} = (x^i)$, the x^i being quasi-Cartesian coordinates. We choose coordinates x^i so that the curves of equations $x^i = $const are timelike. This choice means that $g_{00} > 0$ everywhere. We employ the vector notation $\boldsymbol{a}$ to denote either $\{a^1, a^2, a^3\} = \{a^i\}$ or $\{a_1, a_2, a_3\} = \{a_i\}$. Considering two such quantities $\boldsymbol{a}$ and $\boldsymbol{b}$ with for instance $\boldsymbol{a} = \{a^i\}$, we use $\boldsymbol{a}\cdot\boldsymbol{b}$ to denote $a^i b^i$ if $\boldsymbol{b} = \{b^i\}$ or $a^i b_i$ if $\boldsymbol{b} = \{b_i\}$ (the Einstein convention on the repeated indices is used). The quantity $|\,\boldsymbol{a}\,|$ stands for the ordinary Euclidean norm of $\boldsymbol{a}$.

2 Time Transfer Functions, Time Delay, and Frequency Shift

We consider here electromagnetic signals propagating through a vacuum between an emitter A and a receiver B. We suppose that these signals may be assimilated to light rays traveling along null geodesics of the metric (geometric optics approximation). We call x_A the point of emission by A and x_B the point of reception by B. We put $x_A = (ct_A, \boldsymbol{x}_A)$ and $x_B = (ct_B, \boldsymbol{x}_B)$. We assume that there do not exist two distinct null geodesics starting from x_A and intersecting the world line of B. These assumptions are clearly satisfied in all experiments currently envisaged in the solar system.

2.1 Time Transfer Functions and Time Delay

The quantity $t_B - t_A$ is the (coordinate) travel time of the signal. Upon the above-mentioned assumptions, $t_B - t_A$ may be considered either as a function of the instant of emission t_A and of $\boldsymbol{x}_A$, $\boldsymbol{x}_B$, or as a function of the instant of reception t_B and of $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$. So, we can in general define two distinct (coordinate) time transfer functions, $\mathcal{T}_e$ and $\mathcal{T}_r$ by putting:

$$t_B - t_A = \mathcal{T}_e(t_A, \boldsymbol{x}_A, \boldsymbol{x}_B) = \mathcal{T}_r(t_B, \boldsymbol{x}_A, \boldsymbol{x}_B)\,. \tag{1}$$

We call $\mathcal{T}_e$ the emission time transfer function and $\mathcal{T}_r$ the reception time transfer function. As we shall see below, the main problem will consist in determining explicitly these functions when the metric is given. Of course, it is, in principle, sufficient to determine one of these functions.

We shall put

$$R_{AB} = |\,\boldsymbol{x}_B - \boldsymbol{x}_A\,| \tag{2}$$

throughout this work. The time delay is then defined as $t_B - t_A - R_{AB}/c$. It is well known that this quantity is > 0 in Schwarzschild space–time, which explains its designation [16].

2.2 Frequency Shift

Denote by u^α_A and u^α_B the unit 4-velocity vectors of the emitter at x_A and of the receiver at x_B, respectively. Let Γ_{AB} be the null geodesic path connecting

x_A and x_B, described by parametric equations $x^\alpha = x^\alpha(\zeta)$, ζ being an affine parameter. Denote by l^μ the vector tangent to Γ_{AB} defined as

$$l^\mu = \frac{dx^\mu}{d\zeta}\,. \tag{3}$$

Let ν_A be the frequency of the signal emitted at x_A as measured by a clock comoving with A, and ν_B be the frequency of the same signal received at x_B as measured by a clock comoving with B. The ratio ν_A/ν_B is given by the well-known formula [10]

$$\frac{\nu_A}{\nu_B} = \frac{u_A^\mu (l_\mu)_A}{u_B^\mu (l_\mu)_B}\,. \tag{4}$$

Since it is assumed that the emission and reception points are connected by a single null geodesic, it is clear that $(l_\mu)_A$ and $(l_\mu)_B$ may be considered either as functions of the instant of emission t_A and of $\boldsymbol{x}_A$, $\boldsymbol{x}_B$, or as functions of the instant of reception t_B and of $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$. Therefore, we may write

$$\frac{\nu_A}{\nu_B} = \mathcal{N}_e(u_A, u_B; t_A, \boldsymbol{x}_A, \boldsymbol{x}_B) = \mathcal{N}_r(u_A, u_B; t_B, \boldsymbol{x}_A, \boldsymbol{x}_B)\,. \tag{5}$$

Denote by $\boldsymbol{v}_A = (d\boldsymbol{x}/dt)_A$ and $\boldsymbol{v}_B = (d\boldsymbol{x}/dt)_B$ the coordinate velocities of the observers at x_A and x_B, respectively:

$$\boldsymbol{v}_A = \left(\frac{d\boldsymbol{x}}{dt}\right)_A\,, \quad \boldsymbol{v}_B = \left(\frac{d\boldsymbol{x}}{dt}\right)_B\,. \tag{6}$$

It is easy to see that the formula (4) may be written as

$$\frac{\nu_A}{\nu_B} = \frac{u_A^0}{u_B^0}\,\frac{(l_0)_A}{(l_0)_B}\,\frac{q_A}{q_B}\,, \quad q_A = 1 + \frac{1}{c}\widehat{\boldsymbol{l}}_A \cdot \boldsymbol{v}_A\,, \quad q_B = 1 + \frac{1}{c}\widehat{\boldsymbol{l}}_B \cdot \boldsymbol{v}_B\,, \tag{7}$$

where $\widehat{\boldsymbol{l}}_A$ and $\widehat{\boldsymbol{l}}_B$ are the quantities defined as

$$\widehat{\boldsymbol{l}}_A = \left\{\left(\frac{l_i}{l_0}\right)_A\right\}\,, \qquad \widehat{\boldsymbol{l}}_B = \left\{\left(\frac{l_i}{l_0}\right)_B\right\}\,. \tag{8}$$

It is immediately deduced from (7) that an explicit expression of $\mathcal{N}_e$ (resp., $\mathcal{N}_r$) can be derived when the time transfer function $\mathcal{T}_e$ (resp., $\mathcal{T}_r$) is known. Indeed, one has Theorem 1 [15].

Theorem 1. *Consider a signal emitted at point $x_A = (ct_A, \boldsymbol{x}_A)$ and received at point $x_B = (ct_B, \boldsymbol{x}_B)$. Denote by l^μ the vector $dx^\mu/d\zeta$ tangent to the null geodesic at point $x(\zeta)$, ζ being any affine parameter, and put*

$$\widehat{l}_i = \left(\frac{l_i}{l_0}\right)\,. \tag{9}$$

Then, one has relations as follow at x_A and at x_B

$$\left(\widehat{l_i}\right)_A = c\,\frac{\partial \mathcal{T}_e}{\partial x_A^i}\left[1 + \frac{\partial \mathcal{T}_e}{\partial t_A}\right]^{-1} = c\,\frac{\partial \mathcal{T}_r}{\partial x_A^i}\,, \tag{10}$$

$$\left(\widehat{l_i}\right)_B = -c\,\frac{\partial \mathcal{T}_e}{\partial x_B^i} = -c\,\frac{\partial \mathcal{T}_r}{\partial x_B^i}\left[1 - \frac{\partial \mathcal{T}_r}{\partial t_B}\right]^{-1}\,, \tag{11}$$

$$\frac{(l_0)_A}{(l_0)_B} = 1 + \frac{\partial \mathcal{T}_e}{\partial t_A} = \left[1 - \frac{\partial \mathcal{T}_r}{\partial t_B}\right]^{-1}\,, \tag{12}$$

where $\mathcal{T}_e$ and $\mathcal{T}_r$ are taken at $(t_A, \boldsymbol{x}_A, \boldsymbol{x}_B)$ and $(t_B, \boldsymbol{x}_A, \boldsymbol{x}_B)$, respectively.

This theorem may be straightforwardly deduced from a fundamental property of the world function that we introduce in Sect. 3.

Case of a stationary space–time. In a stationary space–time, we can choose coordinates (x^μ) such that the metric does not depend on x^0. Then, the travel time of the signal only depends on $\boldsymbol{x}_A, \boldsymbol{x}_B$. This means that (1) reduces to a single relation of the form

$$t_B - t_A = \mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)\,. \tag{13}$$

It immediately follows from (10) and (11) that

$$(\widehat{l_i})_A = c\frac{\partial}{\partial x_A^i}\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)\,, \tag{14}$$

$$(\widehat{l_i})_B = -c\frac{\partial}{\partial x_B^i}\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)\,, \tag{15}$$

$$\frac{(l_0)_A}{(l_0)_B} = 1\,. \tag{16}$$

As a consequence, the formula (7) reduces now to

$$\frac{\nu_A}{\nu_B} = \frac{u_A^0}{u_B^0}\,\frac{1 + \boldsymbol{v}_A \cdot \boldsymbol{\nabla}_{\boldsymbol{x}_A}\mathcal{T}}{1 - \boldsymbol{v}_B \cdot \boldsymbol{\nabla}_{\boldsymbol{x}_B}\mathcal{T}}\,, \tag{17}$$

where $\boldsymbol{\nabla}_{\boldsymbol{x}} f$ denotes the usual gradient operator acting on $f(\boldsymbol{x})$.

It is worthy of note that $(1, \{(\widehat{l_i})_A\})$ and $(1, \{(\widehat{l_i})_B\})$ constitute a set of covariant components of the vector tangent to the light ray at $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$, respectively. This tangent vector corresponds to the affine parameter chosen so that $(l_0)_A = (l_0)_B = 1$.

3 The World Function and Its Post-Newtonian Limit

3.1 Definition and Fundamental Properties

For a moment, consider x_A and x_B as arbitrary points. We assume that there exists one and only one geodesic path, say Γ_{AB}, which links these two points.

This assumption means that point x_B belongs to the normal convex neighborhood [17] of point x_A (and conversely that x_A belongs to the normal convex neighborhood of point x_B). The world function is the two-point function $\Omega(x_A, x_B)$ defined by

$$\Omega(x_A, x_B) = \frac{1}{2}\epsilon_{AB}[s_{AB}]^2 , \tag{18}$$

where s_{AB} is the geodesic distance between x_A and x_B, namely

$$s_{AB} = \int_{\Gamma_{AB}} \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \tag{19}$$

and $\epsilon_{AB} = 1, 0, -1$ according as Γ_{AB} is a timelike, a null, or a spacelike geodesic. An elementary calculation shows that $\Omega(x_A, x_B)$ may be written in any case as [10]

$$\Omega(x_A, x_B) = \frac{1}{2}\int_0^1 g_{\mu\nu}(x^\alpha(\lambda))\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}d\lambda , \tag{20}$$

where the integral is taken along Γ_{AB}, λ denoting the unique affine parameter along Γ_{AB} which fulfills the boundary conditions $\lambda_A = 0$ and $\lambda_B = 1$.

It follows from (16) or (18) that the world function $\Omega(x_A, x_B)$ is unchanged if we perform any admissible coordinate transformation.

The utility of the world function for our purpose comes from the following properties [10, 15]:

1. The vectors $(dx^\alpha/d\lambda)_A$ and $(dx^\alpha/d\lambda)_B$ tangent to the geodesic Γ_{AB}, respectively, at x_A and x_B are given by

$$\left(g_{\alpha\beta}\frac{dx^\beta}{d\lambda}\right)_A = -\frac{\partial\Omega}{\partial x_A^\alpha} , \quad \left(g_{\alpha\beta}\frac{dx^\beta}{d\lambda}\right)_B = \frac{\partial\Omega}{\partial x_B^\alpha} . \tag{21}$$

   As a consequence, if $\Omega(x_A, x_B)$ is explicitly known, the determination of these vectors does not require the integration of the differential equations of the geodesic.
2. Two points x_A and x_B are linked by a null geodesic if and only if the condition

$$\Omega(x_A, x_B) = 0 \tag{22}$$

   is fulfilled. Thus, $\Omega(x_A, x) = 0$ is the equation of the null cone $\mathcal{C}(x_A)$ at x_A.

Consequently, if the bifunction $\Omega(x_A, x_B)$ is explicitly known, it is, in principle, possible to determine the emission time transfer function $\mathcal{T}_e$ by solving the equation

$$\Omega(ct_A, \boldsymbol{x}_A, ct_B, \boldsymbol{x}_B) = 0 \tag{23}$$

for t_B. It must be pointed out, however, that solving (23) for t_B yields two distinct solutions t_B^+ and t_B^- since the timelike curve $x^i = x_B^i$ cuts the light cone

$\mathcal{C}(x_A)$ at two points x_B^+ and x_B^-; x_B^+ being in the future of x_B^-. Since we regard x_A as the point of emission of the signal and x_B as the point of reception, we shall exclusively focus our attention on the determination of $t_B^+ - t_A$ (clearly, the determination of $t_B^- - t_A$ comes within the same methodology). For the sake of brevity, we shall henceforth write t_B instead of t_B^+.

Of course, solving (23) for t_A yields the reception time transfer function $\mathcal{T}_r$.

Generally, extracting the time transfer functions from (23), next using (10) or (11) will be more straightforward than deriving the vectors tangent at x_A and x_B from (21), next imposing the constraint (22).

To finish, note that Theorem 1 is easily deduced from the identities

$$\Omega(ct_A, \boldsymbol{x}_A, ct_A + c\mathcal{T}_e(t_A, \boldsymbol{x}_A, \boldsymbol{x}_B), \boldsymbol{x}_B) \equiv 0$$

and

$$\Omega(ct_B - c\mathcal{T}_r(t_B, \boldsymbol{x}_A, \boldsymbol{x}_B), \boldsymbol{x}_A, ct_B, \boldsymbol{x}_B) \equiv 0\,.$$

3.2 General Expression of the World Function in the Post-Newtonian Limit

We assume that the metric may be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{24}$$

throughout space–time, with $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. Let $\Gamma_{AB}^{(0)}$ be the straight line defined by the parametric equations $x^\alpha = x_{(0)}^\alpha(\lambda)$, with

$$x_{(0)}^\alpha(\lambda) = (x_B^\alpha - x_A^\alpha)\lambda + x_A^\alpha\,, \quad 0 \le \lambda \le 1\,. \tag{25}$$

With this definition, the parametric equations of the geodesic Γ_{AB} connecting x_A and x_B may be written in the form

$$x^\alpha(\lambda) = x_{(0)}^\alpha(\lambda) + X^\alpha(\lambda)\,, \quad 0 \le \lambda \le 1\,, \tag{26}$$

where the quantities $X^\alpha(\lambda)$ satisfy the boundary conditions

$$X^\alpha(0) = 0\,, \quad X^\alpha(1) = 0\,. \tag{27}$$

Inserting (24) and $dx^\mu(\lambda)/d\lambda = x_B^\mu - x_A^\mu + dX^\mu(\lambda)/d\lambda$ in (16), then developing and noting that

$$\int_0^1 \eta_{\mu\nu}(x_B^\mu - x_A^\mu)\frac{dX^\nu}{d\lambda}d\lambda = 0$$

by virtue of (27), we find the rigorous formula

$$\begin{aligned}\Omega(x_A, x_B) = \Omega^{(0)}(x_A, x_B) &+ \frac{1}{2}(x_B^\mu - x_A^\mu)(x_B^\nu - x_A^\nu)\int_0^1 h_{\mu\nu}(x^\alpha(\lambda))d\lambda \\ &+ \frac{1}{2}\int_0^1\left[g_{\mu\nu}(x^\alpha(\lambda))\frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda} + 2(x_B^\mu - x_A^\mu)h_{\mu\nu}(x^\alpha(\lambda))\frac{dX^\nu}{d\lambda}\right]d\lambda,\end{aligned} \tag{28}$$

where the integrals are taken over $\mathit{\Gamma}_{AB}$ and $\mathit{\Omega}^{(0)}(x_A, x_B)$ is the world function in Minkowski space–time

$$\mathit{\Omega}^{(0)}(x_A, x_B) = \frac{1}{2}\eta_{\mu\nu}(x^\mu_B - x^\mu_A)(x^\nu_B - x^\nu_A)\,. \tag{29}$$

Henceforth, we shall consider only weak gravitational fields generated by self-gravitating extended bodies within the slow-motion, post-Newtonian approximation. So, we assume that the potentials $h_{\mu\nu}$ may be expanded as follows

$$\begin{aligned} h_{00} &= \frac{1}{c^2}h^{(2)}_{00} + \frac{1}{c^4}h^{(4)}_{00} + O(6)\,,\\ h_{0i} &= \frac{1}{c^3}h^{(3)}_{0i} + O(5)\,,\\ h_{ij} &= \frac{1}{c^2}h^{(2)}_{ij} + O(4)\,. \end{aligned} \tag{30}$$

From these expansions and from the Euler–Lagrange equations satisfied by any geodesic curve, namely

$$\frac{d}{d\lambda}\left(g_{\alpha\beta}\frac{dx^\beta}{d\lambda}\right) = \frac{1}{2}\partial_\alpha h_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\,, \tag{31}$$

it results that $X^\mu(\lambda) = O(2)$ and that $dx^\mu/d\lambda = x^\mu_B - x^\mu_A + O(2)$. As a consequence, $h_{\mu\nu}(x^\alpha(\lambda)) = h_{\mu\nu}(x^\alpha_{(0)}(\lambda)) + O(4)$ and the third and fourth terms in the RHS of (28) are of order $1/c^4$. These features result in an expression for $\mathit{\Omega}(x_A, x_B)$ as follows

$$\mathit{\Omega}(x_A, x_B) = \mathit{\Omega}^{(0)}(x_A, x_B) + \mathit{\Omega}^{(PN)}(x_A, x_B) + O(4)\,, \tag{32}$$

where

$$\begin{aligned} \mathit{\Omega}^{(PN)}(x_A, x_B) = &\frac{1}{2c^2}(x^0_B - x^0_A)^2\int_0^1 h^{(2)}_{00}(x^\alpha_{(0)}(\lambda))d\lambda\\ &+\frac{1}{2c^2}(x^i_B - x^i_A)(x^j_B - x^j_A)\int_0^1 h^{(2)}_{ij}(x^\alpha_{(0)}(\lambda))d\lambda\\ &+\frac{1}{c^3}(x^0_B - x^0_A)(x^i_B - x^i_A)\int_0^1 h^{(3)}_{0i}(x^\alpha_{(0)}(\lambda))d\lambda, \end{aligned} \tag{33}$$

the integral being now taken over the line $\mathit{\Gamma}^{(0)}_{AB}$ defined by (25).

The formulae (32) and (33) yield the general expression of the world function up to the order $1/c^3$ within the framework of the 1 PN approximation. We shall see in Sect. 3.3 that this approximation is sufficient to determine the time transfer functions up to the order $1/c^4$. It is worthy of note that the method used above would as well lead to the expression of the world function in the linearized weak-field limit previously found by Synge [10].

3.3 Time Transfer Functions at the Order $1/c^4$

Suppose that x_B is the point of reception of a signal emitted at x_A. Taking (32) into account, (22) may be written in the form

$$\Omega^{(0)}(x_A, x_B) + \Omega^{(PN)}(x_A, x_B) = O(4)\,, \tag{34}$$

which implies the relation

$$t_B - t_A = \frac{1}{c}R_{AB} - \frac{\Omega^{(PN)}(ct_A, \boldsymbol{x}_A, ct_B, \boldsymbol{x}_B)}{cR_{AB}} + O(4)\,. \tag{35}$$

Using iteratively this relation, we find for the emission time transfer function

$$\mathcal{T}_e(t_A, \boldsymbol{x}_A, \boldsymbol{x}_B) = \frac{1}{c}R_{AB} - \frac{\Omega^{(PN)}(ct_A, \boldsymbol{x}_A, ct_A + R_{AB}, \boldsymbol{x}_B)}{cR_{AB}} + O(5) \tag{36}$$

and for the reception time transfer function

$$\mathcal{T}_r(t_B, \boldsymbol{x}_A, \boldsymbol{x}_B) = \frac{1}{c}R_{AB} - \frac{\Omega^{(PN)}(ct_B - R_{AB}, \boldsymbol{x}_A, ct_B, \boldsymbol{x}_B)}{cR_{AB}} + O(5)\,. \tag{37}$$

These last formulae show that the time transfer functions can be explicitly calculated up to the order $1/c^4$ when $\Omega^{(PN)}(x_A, x_B)$ is known. This fundamental result will be exploited in the following sections.

It is worthy of note that a comparison of (36) and (37) immediately gives the following relations:

$$\mathcal{T}_r(t_B, \boldsymbol{x}_A, \boldsymbol{x}_B) = \mathcal{T}_e\left(t_B - \frac{R_{AB}}{c}, \boldsymbol{x}_A, \boldsymbol{x}_B\right) + O(5) \tag{38}$$

and conversely

$$\mathcal{T}_e(t_A, \boldsymbol{x}_A, \boldsymbol{x}_B) = \mathcal{T}_r\left(t_A + \frac{R_{AB}}{c}, \boldsymbol{x}_A, \boldsymbol{x}_B\right) + O(5)\,. \tag{39}$$

The quantity $\Omega^{(PN)}(ct_A, \boldsymbol{x}_A, ct_A + R_{AB}, \boldsymbol{x}_B)$ in (36) may be written in an integral form by using (33), in which R_{AB} and $R_{AB}\lambda + ct_A$ are substituted for $x_B^0 - x_A^0$ and for $x_{(0)}^0(\lambda)$, respectively. As a consequence

$$\begin{aligned}\mathcal{T}_e(t_A, \boldsymbol{x}_A, \boldsymbol{x}_B) = \frac{1}{c}R_{AB}\Bigg\{1 - \frac{1}{2c^2}\int_0^1 \Big[&h_{00}^{(2)}(z_+^\alpha(\lambda)) \\ &+ h_{ij}^{(2)}(z_+^\alpha(\lambda))N^iN^j + \frac{2}{c}h_{0i}^{(3)}(z_+^\alpha(\lambda))N^i\Big]d\lambda\Bigg\} + O(5),\end{aligned} \tag{40}$$

the integral being taken over curve $\mathit{\Gamma}_{AB}^{(0)+}$ defined by the parametric equations $x^\alpha = z_+^\alpha(\lambda)$, where

$$z_+^0(\lambda) = R_{AB}\lambda + ct_A\,, \quad z_+^i(\lambda) = R_{AB}N^i\lambda + x_A^i\,, \quad 0 \le \lambda \le 1\,, \tag{41}$$

with

$$R_{AB} = \mid \boldsymbol{R}_{AB} \mid, \qquad N^i = \frac{x_B^i - x_A^i}{R_{AB}}\,. \tag{42}$$

We note that $\mathit{\Gamma}_{AB}^{(0)+}$ is a null geodesic path of Minkowski metric from x_A, having the above-defined quantities N^i as direction cosines.

A similar reasoning leads to an expression as follows for $\mathcal{T}_r$

$$\mathcal{T}_r(t_B, \boldsymbol{x}_A, \boldsymbol{x}_B) = \frac{1}{c}R_{AB}\Bigg\{1 - \frac{1}{2c^2}\int_0^1 \Big[h_{00}^{(2)}(z_-^\alpha(\lambda)) + h_{ij}^{(2)}(z_-^\alpha(\lambda))N^iN^j + \frac{2}{c}h_{0i}^{(3)}(z_-^\alpha(\lambda))N^i\Big]d\lambda\Bigg\} + O(5), \tag{43}$$

the integral being now taken over curve $\mathit{\Gamma}_{AB}^{(0)-}$ defined by the parametric equations $x^\alpha = z_-^\alpha(\lambda)$, where

$$z_-^0(\lambda) = -R_{AB}\lambda + ct_B\,, \quad z_-^i(\lambda) = -R_{AB}N^i\lambda + x_B^i\,, \quad 0 \le \lambda \le 1\,. \tag{44}$$

Curve $\mathit{\Gamma}_{AB}^{(0)-}$ is a null geodesic path of Minkowski metric arriving at x_B and having N^i as direction cosines.

4 World Function and Time Transfer Functions Within the Nordtvedt–Will PPN Formalism

4.1 Metric in the 1 PN Approximation

In this section, we use the Nordtvedt–Will post-Newtonian formalism involving ten parameters β, γ, ξ, α_1, …, ζ_4 [18]. We introduce slightly modified notations to be closed of the formalism recently proposed by Klioner and Soffel [20] as an extension of the post-Newtonian framework elaborated by Damour et al. [21] for general relativity. In particular, we denote by $\boldsymbol{v}_r$ the velocity of the center of mass O relative to the universe rest frame.[1]

Although our method is not confined to any particular assumption on the matter, we suppose here that each source of the field is described by the energy–momentum tensor of a perfect fluid

$$T^{\mu\nu} = \rho c^2\left[1 + \frac{1}{c^2}\left(\mathit{\Pi} + \frac{p}{\rho}\right)\right]u^\mu u^\nu - pg^{\mu\nu}\,, \tag{45}$$

[1] This velocity is noted $\boldsymbol{w}$ in [18].

where ρ is the rest mass density, Π is the specific energy density (ratio of internal energy density to rest mass density), p is the pressure, and u^μ is the unit 4-velocity of the fluid. In this section and in the following one, $\boldsymbol{v}$ is the coordinate velocity $d\boldsymbol{x}/dt$ of an element of the fluid. We introduce the conserved mass density ρ^* given by

$$\rho^* = \rho\sqrt{-g}u^0 = \rho\left[1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 + 3\gamma U\right) + O(4)\right], \tag{46}$$

where $g = \det(g_{\mu\nu})$ and U is the Newtonian-like potential

$$U(x^0, \boldsymbol{x}) = G\int \frac{\rho^*(x^0, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3\boldsymbol{x}'\,. \tag{47}$$

To obtain a more simple form than the usual one for the potentials h_{0i}, we suppose that the chosen (x^μ) are related to a standard post-Newtonian gauge $(\overline{x}^\mu)$ by the transformation

$$x^0 = \overline{x}^0 + \frac{1}{c^3}\left[(1 + 2\xi + \alpha_2 - \zeta_1)\partial_t\chi - 2\alpha_2\boldsymbol{v}_r\cdot\boldsymbol{\nabla}\chi\right], \quad x^i = \overline{x}^i\,, \tag{48}$$

where χ is the superpotential defined by

$$\chi(x^0, \boldsymbol{x}) = \frac{1}{2}G\int \rho^*(x^0, \boldsymbol{x}')\,|\boldsymbol{x} - \boldsymbol{x}'|\,d^3\boldsymbol{x}'\,. \tag{49}$$

Moreover, we define $\widehat{\rho}$ by

$$\begin{aligned}\widehat{\rho} = \rho^*\Bigg[1 &+ \frac{1}{2}(2\gamma + 1 - 2\xi + \alpha_3 + \zeta_1)\frac{v^2}{c^2} + (1 - 2\beta + \xi + \zeta_2)\frac{U}{c^2} + (1 + \zeta_3)\frac{\Pi}{c^2}\\ &+(3\gamma - 2\xi + 3\zeta_4)\frac{p}{\rho^* c^2} - \frac{1}{2}(\alpha_1 - \alpha_3)\frac{v_r^2}{c^2} - \frac{1}{2}(\alpha_1 - 2\alpha_3)\frac{\boldsymbol{v}_r\cdot\boldsymbol{v}}{c^2} + O(4)\Bigg].\end{aligned} \tag{50}$$

Then, the post-Newtonian potentials read

$$h_{00} = -\frac{2}{c^2}w + \frac{2\beta}{c^4}w^2 + \frac{2\xi}{c^4}\phi_W + \frac{1}{c^4}(\zeta_1 - 2\xi)\phi_v - \frac{2\alpha_2}{c^4}v_r^i v_r^j\partial_{ij}\chi + O(6), \tag{51}$$

$$\boldsymbol{h} \equiv \{h_{0i}\} = \frac{2}{c^3}\left[\left(\gamma + 1 + \frac{1}{4}\alpha_1\right)\boldsymbol{w} + \frac{1}{4}\alpha_1 w\,\boldsymbol{v}_r\right] + O(5), \tag{52}$$

$$h_{ij} = -\frac{2\gamma}{c^2}w\delta_{ij} + O(4)\,, \tag{53}$$

where

$$\begin{aligned}w(x^0, \boldsymbol{x}) = G\int &\frac{\widehat{\rho}(x^0, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}d^3\boldsymbol{x}'\\ &+\frac{1}{c^2}\left[(1 + 2\xi + \alpha_2 - \zeta_1)\partial_{tt}\chi - 2\alpha_2\boldsymbol{v}_r\cdot\boldsymbol{\nabla}(\partial_t\chi)\right],\end{aligned} \tag{54}$$

$$\phi_W(x^0,\boldsymbol{x}) = G^2 \int \frac{\rho^*(x^0,\boldsymbol{x}')\rho^*(x^0,\boldsymbol{x}'')(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3} \times \left(\frac{\boldsymbol{x}'-\boldsymbol{x}''}{|\boldsymbol{x}-\boldsymbol{x}''|} - \frac{\boldsymbol{x}-\boldsymbol{x}''}{|\boldsymbol{x}'-\boldsymbol{x}''|}\right) d^3\boldsymbol{x}' d^3\boldsymbol{x}'' , \tag{55}$$

$$\phi_v(x^0,\boldsymbol{x}) = G \int \frac{\rho^*(x^0,\boldsymbol{x}')[\boldsymbol{v}(x^0,\boldsymbol{x}')\cdot(\boldsymbol{x}-\boldsymbol{x}')]^2}{|\boldsymbol{x}-\boldsymbol{x}'|^3} d^3\boldsymbol{x}' , \tag{56}$$

$$\boldsymbol{w}(x^0,\boldsymbol{x}) = G \int \frac{\rho^*(x^0,\boldsymbol{x}')\boldsymbol{v}(x^0,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3\boldsymbol{x}' . \tag{57}$$

4.2 Determination of the World Function and of the Time Transfer Functions

For the post-Newtonian metric given by (51–57), it follows from (33) that $\Omega(x_A, x_B)$ may be written up to the order $1/c^3$ in the form given by (32) with

$$\Omega^{(PN)}(x_A,x_B) = \Omega_w^{(PN)}(x_A,x_B) + \Omega_{\boldsymbol{w}}^{(PN)}(x_A,x_B) + \Omega_{\boldsymbol{v}_r}^{(PN)}(x_A,x_B) , \tag{58}$$

where

$$\Omega_w^{(PN)}(x_A,x_B) = -\frac{1}{c^2}\left[(x_B^0-x_A^0)^2 + \gamma R_{AB}^2\right] \int_0^1 w(x_{(0)}^\alpha(\lambda))d\lambda , \tag{59}$$

$$\Omega_{\boldsymbol{w}}^{(PN)}(x_A,x_B) = \frac{2}{c^3}\left(\gamma+1+\frac{1}{4}\alpha_1\right)(x_B^0-x_A^0) \times \boldsymbol{R}_{AB}\cdot \int_0^1 \boldsymbol{w}(x_{(0)}^\alpha(\lambda))d\lambda , \tag{60}$$

$$\Omega_{\boldsymbol{v}_r}^{(PN)}(x_A,x_B) = \frac{1}{2c^3}\alpha_1(x_B^0-x_A^0)(\boldsymbol{R}_{AB}\cdot\boldsymbol{v}_r) \int_0^1 w(x_{(0)}^\alpha(\lambda))d\lambda , \tag{61}$$

the integrals being calculated along the curve defined by (25).

The emission time transfer function is easily obtained by using (36) or (40). We get

$$\mathcal{T}_e(t_A,\boldsymbol{x}_A,\boldsymbol{x}_B) = \frac{1}{c}R_{AB} + \frac{1}{c^3}(\gamma+1)R_{AB}\int_0^1 w(z_+^\alpha(\lambda))d\lambda - \frac{2}{c^4}\boldsymbol{R}_{AB}\cdot\left[(\gamma+1+\frac{1}{4}\alpha_1)\int_0^1 \boldsymbol{w}(z_+^\alpha(\lambda))d\lambda + \frac{1}{4}\alpha_1\boldsymbol{v}_r\int_0^1 w(z_+^\alpha(\lambda))d\lambda\right] + O(5) , \tag{62}$$

the integral being evaluated along the curve $\Gamma_{AB}^{(0)+}$ defined by (41).

The reception time transfer function is given by

$$\begin{aligned}\mathcal{T}_r(t_B,\boldsymbol{x}_A,\boldsymbol{x}_B) = {} & \frac{1}{c}R_{AB} + \frac{1}{c^3}(\gamma+1)R_{AB}\int_0^1 w(z_-^\alpha(\lambda))d\lambda \\ & -\frac{2}{c^4}\boldsymbol{R}_{AB}\cdot\left[(\gamma+1+\frac{1}{4}\alpha_1)\int_0^1 \boldsymbol{w}(z_-^\alpha(\lambda))d\lambda\right. \\ & \left. +\frac{1}{4}\alpha_1\boldsymbol{v}_r\int_0^1 w(z_-^\alpha(\lambda))d\lambda\right] + O(5)\,, \quad (63)\end{aligned}$$

the integral being evaluated along the curve $\Gamma_{AB}^{(0)-}$ defined by (44).

Let us emphasize that, since $w = U + O(2)$, w may be replaced by the Newtonian-like potential U in (59–62).

4.3 Case of a Stationary Source

In what follows, we suppose that the gravitational field is generated by a single stationary source. Then, $\partial_t\chi = 0$ and the potentials w and $\boldsymbol{w}$ do not depend on time. In this case, the integration involved in (59–61) can be performed by a method due to Buchdahl [19]. Introducing the auxiliary variables $\boldsymbol{y}_A = \boldsymbol{x}_A - \boldsymbol{x}'$ and $\boldsymbol{y}_B = \boldsymbol{x}_B - \boldsymbol{x}'$, and replacing in (25) the parameter λ by $u = \lambda - 1/2$, a straightforward calculation yields

$$\int_0^1 w(\boldsymbol{x}_{(0)}(\lambda))d\lambda = G\int \widehat{\rho}(\boldsymbol{x}')F(\boldsymbol{x}',\boldsymbol{x}_A,\boldsymbol{x}_B)d^3\boldsymbol{x}', \quad (64)$$

$$\int_0^1 \boldsymbol{w}(\boldsymbol{x}_{(0)}(\lambda))d\lambda = G\int \rho^*(\boldsymbol{x}')\boldsymbol{v}(\boldsymbol{x}')F(\boldsymbol{x}',\boldsymbol{x}_A,\boldsymbol{x}_B)d^3\boldsymbol{x}'\,, \quad (65)$$

where the kernel function $F(\boldsymbol{x}',\boldsymbol{x}_A,\boldsymbol{x}_B)$ has the expression

$$F(\boldsymbol{x}',\boldsymbol{x}_A,\boldsymbol{x}_B) = \int_{-1/2}^{1/2}\frac{du}{\mid(\boldsymbol{y}_B-\boldsymbol{y}_A)u + \frac{1}{2}(\boldsymbol{y}_B+\boldsymbol{y}_A)\mid}\,. \quad (66)$$

Noting that $\boldsymbol{y}_B - \boldsymbol{y}_A = \boldsymbol{R}_{AB}$, which implies that $\mid\boldsymbol{y}_B - \boldsymbol{y}_A\mid = R_{AB}$, we find

$$F(\boldsymbol{x},\boldsymbol{x}_A,\boldsymbol{x}_B) = \frac{1}{R_{AB}}\ln\left(\frac{\mid\boldsymbol{x}-\boldsymbol{x}_A\mid + \mid\boldsymbol{x}-\boldsymbol{x}_B\mid + R_{AB}}{\mid\boldsymbol{x}-\boldsymbol{x}_A\mid + \mid\boldsymbol{x}-\boldsymbol{x}_B\mid - R_{AB}}\right)\,. \quad (67)$$

Inserting (64), (65), and (67) in (59–61) and (62) will enable one to obtain quite elegant expressions for $\Omega^{(PN)}(x_A,x_B)$ and for $\mathcal{T}(\boldsymbol{x}_A,\boldsymbol{x}_B)$.

5 Isolated, Slowly Rotating Axisymmetric Body

Henceforth, we suppose that the light is propagating in the gravitational field of an isolated, slowly rotating axisymmetric body. The gravitational field is assumed to be stationary. The main purpose of this section is to determine the influence of the mass and spin multipole moments of the rotating body on the coordinate time transfer and on the direction of light rays. From these results, it will be possible to obtain a relativistic modeling of the one-way time transfers and frequency shifts up to the order $1/c^4$ in a geocentric nonrotating frame.

Since we treat the case of a body located very far from the other bodies of the universe, the global coordinate system (x^μ) used until now can be considered as a local (i.e., geocentric) one. So, in agreement with the UAI/UGG Resolution B1 (2000) [22], we shall henceforth denote by W and $\boldsymbol{W}$ the quantities w and $\boldsymbol{w}$, respectively, defined by (54) and (57) and we shall denote by $G_{\mu\nu}$ the components of the metric. However, we shall continue here with using lower case letters for the geocentric coordinates to avoid too heavy notations.

The center of mass O of the rotating body is taken as the origin of the quasi-Cartesian coordinates $(\boldsymbol{x})$; we choose the axis of symmetry as the x^3-axis. We assume that the body is rotating about $\mathrm{O}x^3$ with a constant angular velocity $\boldsymbol{\omega}$, so that

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{\omega} \times \boldsymbol{x} \,. \tag{68}$$

In what follows, we put $r = \mid \boldsymbol{x} \mid$, $r_A = \mid \boldsymbol{x}_A \mid$, and $r_B = \mid \boldsymbol{x}_A \mid$. We call θ the angle between $\boldsymbol{x}$ and $\mathrm{O}x^3$. We consider only the case where all points of the segment joining $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ are outside the body. We denote by r_e the radius of the smallest sphere centered on O and containing the body (for celestial bodies, r_e is the equatorial radius). In this section, we assume the convergence of the multipole expansions formally derived below at any point outside the body, even if $r < r_e$.

5.1 Multipole Expansions of the Potentials

According to (54), (57), and (68), the gravitational potentials W and $\boldsymbol{W}$ obey the equations

$$\boldsymbol{\nabla}^2 W = -4\pi G \widehat{\rho}\,, \quad \boldsymbol{\nabla}^2 \boldsymbol{W} = -4\pi G \rho^* \boldsymbol{\omega} \times \boldsymbol{x} \,. \tag{69}$$

It follows from (69) that the potential W is a harmonic function outside the rotating body. As a consequence, W may be expanded in a multipole series of the form

$$W(\boldsymbol{x}) = \frac{GM}{r}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{r_e}{r}\right)^n P_n(\cos\theta)\right] . \tag{70}$$

In this expansion, P_n is the Legendre polynomial of degree n and the quantities M, $J_2, \ldots, J_n, \ldots$ correspond to the generalized Blanchet–Damour mass multipole moments in general relativity [23].

For the sake of simplicity, put

$$z = x^3 \,. \tag{71}$$

Taking into account the identity

$$\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right) = \frac{(-1)^n n!}{r^{1+n}} P_n(z/r)\,, \quad z = x^3\,, \tag{72}$$

it may be seen that

$$W(\boldsymbol{x}) = GM\left[\frac{1}{r} - \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} J_n r_e^n \frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right)\right]. \tag{73}$$

Substituting for W from (73) into (69) yields an expansion for $\widehat{\rho}$ as follows

$$\widehat{\rho}(\boldsymbol{x}) = M\left[\delta^{(3)}(\boldsymbol{x}) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} J_n r_e^n \frac{\partial^n}{\partial z^n}\delta^{(3)}(\boldsymbol{x})\right], \tag{74}$$

where $\delta^{(3)}(\boldsymbol{x})$ is the Dirac distribution supported by the origin O. This expansion of $\widehat{\rho}$ in a multipole series will be exploited in Sect. 5.2.

Now, substituting (68) into (57) yields for the vector potential $\boldsymbol{W}$

$$\boldsymbol{W}(\boldsymbol{x}) = G\int \frac{\rho^*(\boldsymbol{x}')\boldsymbol{\omega}\times\boldsymbol{x}'}{|\,\boldsymbol{x}-\boldsymbol{x}'\,|} d^3\boldsymbol{x}'\,. \tag{75}$$

It is possible to show that this vector may be written as

$$\boldsymbol{W} = -\frac{1}{2}\boldsymbol{\omega}\times\boldsymbol{\nabla}\mathcal{V}\,, \tag{76}$$

where $\mathcal{V}$ is an axisymmetric function satisfying the Laplace equation $\boldsymbol{\nabla}^2\mathcal{V} = 0$ outside the body. Consequently, we can expand $\mathcal{V}$ in a series of the form

$$\mathcal{V}(\boldsymbol{x}) = \frac{GI}{r}\left[1 - \sum_{n=1}^{\infty} K_n \left(\frac{r_e}{r}\right)^n P_n(\cos\theta)\right], \tag{77}$$

where I and each K_n are constants. Substituting for $\mathcal{V}$ from (77) into (76), and then using the identity

$$(n+1)P_n(z/r) + (z/r)P_n'(z/r) = P_{n+1}'(z/r)\,, \tag{78}$$

we find an expansion for $\boldsymbol{W}$ as follows

$$\boldsymbol{W}(\boldsymbol{x}) = \frac{GI\boldsymbol{\omega}\times\boldsymbol{x}}{2r^3}\left[1 - \sum_{n=1}^{\infty} K_n \left(\frac{r_e}{r}\right)^n P_{n+1}'(\cos\theta)\right], \tag{79}$$

which coincides with a result previously obtained by one of us [24]. This coincidence shows that I is the moment of inertia of the body about the z-axis. Thus, the quantity $\boldsymbol{S} = I\boldsymbol{\omega}$ is the intrinsic angular momentum of the rotating body. The coefficients K_n are completely determined by the density distribution ρ^*

and by the shape of the body [24, 25]. The quantities $I, K_1, K_2, \ldots K_n, \ldots$ correspond to the Blanchet–Damour spin multipoles in the special case of a stationary axisymmetric gravitational field.

Equation (79) may also be written as

$$\boldsymbol{W}(\boldsymbol{x}) = -\frac{1}{2} G\boldsymbol{S} \times \boldsymbol{\nabla} \left[\frac{1}{r} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} K_n r_e^n \frac{\partial^n}{\partial z^n} \left(\frac{1}{r} \right) \right] . \tag{80}$$

Consequently, the density of mass current can be expanded in the multipole series

$$\rho^*(\boldsymbol{x})(\boldsymbol{\omega} \times \boldsymbol{x}) = -\frac{1}{2} \boldsymbol{S} \times \boldsymbol{\nabla} \left[\delta^{(3)}(\boldsymbol{x}) - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} K_n r_e^n \frac{\partial^n}{\partial z^n} \delta^{(3)}(\boldsymbol{x}) \right] . \tag{81}$$

This expansion may be compared with the expansion of $\widehat{\rho}$ given by (74).

5.2 Multipole Structure of the World Function

The function $\Omega^{(PN)}(x_A, x_B)$ is determined by (58–61) where w and $\boldsymbol{w}$ are, respectively, replaced by W and $\boldsymbol{W}$. The integrals involved in the RHS of (58–61) are given by (64) and (65). Substituting (74) into (64) and using the properties of the Dirac distribution, we obtain

$$\int_0^1 W\left(\boldsymbol{x}_{(0)}(\lambda)\right) d\lambda = GM \left[1 - \sum_{n=2}^{\infty} \frac{1}{n!} J_n r_e^n \frac{\partial^n}{\partial z^n} \right] F(\boldsymbol{x}, \boldsymbol{x}_A, \boldsymbol{x}_B) \Big|_{\boldsymbol{x}=0} . \tag{82}$$

Similarly, substituting (81) into (65), we get[2]

$$\int_0^1 \boldsymbol{W}\left(\boldsymbol{x}_{(0)}(\lambda)\right) d\lambda = \frac{1}{2} G\boldsymbol{S} \times \boldsymbol{\nabla} \left[1 - \sum_{n=1}^{\infty} \frac{1}{n!} K_n r_e^n \frac{\partial^n}{\partial z^n} \right] F(\boldsymbol{x}, \boldsymbol{x}_A, \boldsymbol{x}_B) \Big|_{\boldsymbol{x}=0} . \tag{83}$$

These formulae show that the multipole expansion of $\Omega^{(PN)}(x_A, x_B)$ can be thoroughly calculated by straightforward differentiations of the kernel function $F(\boldsymbol{x}, \boldsymbol{x}_A, \boldsymbol{x}_B)$ given by (67). They constitute an essential result, since they give an algorithmic procedure for determining the multipole expansions of the time transfer function and of the frequency shift in a stationary axisymmetric field (see also [2]).

To obtain explicit formulae, we shall only retain the contributions due to M, J_2, and $\boldsymbol{S}$ in the expansion yielding $\Omega_W^{(PN)}$ and $\Omega_{\boldsymbol{W}}^{(PN)}$. Then, denoting the unit vector along the z-axis by $\boldsymbol{k}$ and noting that $\boldsymbol{S} = S\boldsymbol{k}$, we get for $\Omega_W^{(1)}(x_A, x_B)$

[2] Note that the sign of (55) in [14] is erroneous.

$$
\begin{aligned}
&\Omega_{W}^{(PN)}(x_A, x_B) \\
&\quad = -\frac{GM}{c^2}\frac{(x_B^0 - x_A^0)^2 + \gamma R_{AB}^2}{R_{AB}} \ln\left(\frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}}\right) \\
&\qquad + \frac{2GM}{c^2} J_2 r_e^2 \frac{(x_B^0 - x_A^0)^2 + \gamma R_{AB}^2}{[(r_A + r_B)^2 - R_{AB}^2]^2}(r_A + r_B)\left(\frac{\boldsymbol{k}\cdot\boldsymbol{x}_A}{r_A} + \frac{\boldsymbol{k}\cdot\boldsymbol{x}_B}{r_B}\right)^2 \\
&\qquad - \frac{GM}{c^2} J_2 r_e^2 \frac{(x_B^0 - x_A^0)^2 + \gamma R_{AB}^2}{(r_A + r_B)^2 - R_{AB}^2}\left[\frac{(\boldsymbol{k}\times\boldsymbol{x}_A)^2}{r_A^3} + \frac{(\boldsymbol{k}\times\boldsymbol{x}_B)^2}{r_B^3}\right] + \ldots
\end{aligned} \tag{84}
$$

and for $\Omega_{\boldsymbol{W}}^{(PN)}(x_A, x_B)$

$$
\begin{aligned}
\Omega_{\boldsymbol{W}}^{(PN)}(x_A, x_B) = &\left(\gamma + 1 + \frac{1}{4}\alpha_1\right)\frac{2GS}{c^3} \\
&\times (x_B^0 - x_A^0)\frac{r_A + r_B}{r_A r_B}\frac{\boldsymbol{k}\cdot(\boldsymbol{x}_A \times \boldsymbol{x}_B)}{(r_A + r_B)^2 - R_{AB}^2} + \ldots .
\end{aligned} \tag{85}
$$

Finally, owing to the limit $|\alpha_1| < 4\times 10^{-4}$ furnished in [18], we shall henceforth neglect all the multipole contributions in $\Omega_{\boldsymbol{v}_r}^{(PN)}(x_A, x_B)$. Thus, we get

$$
\Omega_{\boldsymbol{v}_r}^{(PN)}(x_A, x_B) = \alpha_1 \frac{GM}{2c^3}(x_B^0 - x_A^0)\frac{\boldsymbol{R}_{AB}\cdot\boldsymbol{v}_r}{R_{AB}} \ln\left(\frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}}\right) + \cdots . \tag{86}
$$

In this section and in the following one, the symbol $+\ldots$ stands for the contributions of higher multipole moments which are neglected. For the sake of brevity, when $+\cdots$ is used, we systematically omit to mention the symbol $O(n)$ which stands for the neglected post-Newtonian terms.

5.3 Time Transfer Function up to the Order $1/c^4$

In what follows, we put

$$
\boldsymbol{n}_A = \frac{\boldsymbol{x}_A}{r_A}, \qquad \boldsymbol{n}_B = \frac{\boldsymbol{x}_B}{r_B}, \tag{87}
$$

and

$$
\boldsymbol{N}_{AB} = \{N^i\} = \frac{\boldsymbol{x}_B - \boldsymbol{x}_A}{R_{AB}} . \tag{88}
$$

Furthermore, we use systematically the identity

$$
(r_A + r_B)^2 - R_{AB}^2 = 2 r_A r_B (1 + \boldsymbol{n}_A\cdot\boldsymbol{n}_B) . \tag{89}
$$

By substituting R_{AB} for $x_B^0 - x_A^0$ into (84–86) and inserting the corresponding expression of $\Omega^{(PN)}$ into (36), we get an expression for the time transfer function as follows

$$
\begin{aligned}
\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B) = &\frac{1}{c}R_{AB} + \mathcal{T}_M(\boldsymbol{x}_A, \boldsymbol{x}_B) \\
&+ \mathcal{T}_{J_2}(\boldsymbol{x}_A, \boldsymbol{x}_B) + \mathcal{T}_{\boldsymbol{S}}(\boldsymbol{x}_A, \boldsymbol{x}_B) + \mathcal{T}_{\boldsymbol{v}_r}(\boldsymbol{x}_A, \boldsymbol{x}_B) + \ldots ,
\end{aligned} \tag{90}
$$

where

$$\mathcal{T}_M(\boldsymbol{x}_A, \boldsymbol{x}_B) = (\gamma + 1)\frac{GM}{c^3} \ln\left(\frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}}\right), \tag{91}$$

$$\begin{aligned}\mathcal{T}_{J_2}(\boldsymbol{x}_A, \boldsymbol{x}_B) = &-\frac{\gamma+1}{2}\frac{GM}{c^3} J_2 \frac{r_e^2}{r_A r_B}\frac{R_{AB}}{1 + \boldsymbol{n}_A \cdot \boldsymbol{n}_B} \\ &\times \left[\left(\frac{1}{r_A} + \frac{1}{r_B}\right)\frac{(\boldsymbol{k}\cdot\boldsymbol{n}_A + \boldsymbol{k}\cdot\boldsymbol{n}_B)^2}{1 + \boldsymbol{n}_A \cdot \boldsymbol{n}_B}\right. \\ &\left. -\frac{1 - (\boldsymbol{k}\cdot\boldsymbol{n}_A)^2}{r_A} - \frac{1 - (\boldsymbol{k}\cdot\boldsymbol{n}_B)^2}{r_B}\right],\end{aligned} \tag{92}$$

$$\mathcal{T}_{\boldsymbol{S}}(\boldsymbol{x}_A, \boldsymbol{x}_B) = -\left(\gamma + 1 + \frac{1}{4}\alpha_1\right)\frac{GS}{c^4}\left(\frac{1}{r_A} + \frac{1}{r_B}\right)\frac{\boldsymbol{k}\cdot(\boldsymbol{n}_A \times \boldsymbol{n}_B)}{1 + \boldsymbol{n}_A \cdot \boldsymbol{n}_B}, \tag{93}$$

$$\mathcal{T}_{\boldsymbol{v}_r}(\boldsymbol{x}_A, \boldsymbol{x}_B) = -\alpha_1 \frac{GM}{2c^4}(\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_r) \ln\left(\frac{r_A + r_B + R_{AB}}{r_A + r_B - R_{AB}}\right). \tag{94}$$

The time transfer is thus explicitly determined up to the order $1/c^4$. The term of order $1/c^3$ given by (91) is the well-known Shapiro time delay [16]. Equations (92) and (93) extend results previously found for $\gamma = 1$ and $\alpha_1 = 0$ [1]. However, our derivation is more straightforward and yields formulae which are more convenient to calculate the frequency shifts. As a final remark, it is worthy of note that $\mathcal{T}_M$ and $\mathcal{T}_{J_2}$ are symmetric in $(\boldsymbol{x}_A, \boldsymbol{x}_B)$, while $\mathcal{T}_{\boldsymbol{S}}$ and $\mathcal{T}_{\boldsymbol{v}_r}$ are antisymmetric in $(\boldsymbol{x}_A, \boldsymbol{x}_B)$.

5.4 Directions of Light Rays at $\boldsymbol{x_A}$ and $\boldsymbol{x_B}$ up to the Order $1/c^3$

To determine the vectors tangent to the ray path at x_A and x_B, we use (14) and (15) where $\mathcal{T}$ is replaced by the expression given by (90–94). It is clear that $\widehat{\boldsymbol{l}}_A$ and $\widehat{\boldsymbol{l}}_B$ may be written as

$$\widehat{\boldsymbol{l}}_A = -\boldsymbol{N}_{AB} + \boldsymbol{\lambda}_e(\boldsymbol{x}_A, \boldsymbol{x}_B), \tag{95}$$

$$\widehat{\boldsymbol{l}}_B = -\boldsymbol{N}_{AB} + \boldsymbol{\lambda}_r(\boldsymbol{x}_A, \boldsymbol{x}_B), \tag{96}$$

where $\boldsymbol{\lambda}_e$ and $\boldsymbol{\lambda}_r$ are perturbation terms due to $\mathcal{T}_M$, $\mathcal{T}_{J_n}$, $\mathcal{T}_{\boldsymbol{S}}$, $\mathcal{T}_{K_n}$, ... For the expansion of $\mathcal{T}$ given by (90–94), we find

$$\boldsymbol{\lambda}_e(\boldsymbol{x}_A, \boldsymbol{x}_B) = -\boldsymbol{\lambda}_M(\boldsymbol{x}_B, \boldsymbol{x}_A) - \boldsymbol{\lambda}_{J_2}(\boldsymbol{x}_B, \boldsymbol{x}_A) + \boldsymbol{\lambda}_{\boldsymbol{S}}(\boldsymbol{x}_B, \boldsymbol{x}_A) + \boldsymbol{\lambda}_{\boldsymbol{v}_r}(\boldsymbol{x}_B, \boldsymbol{x}_A) + \cdots, \tag{97}$$

$$\boldsymbol{\lambda}_r(\boldsymbol{x}_A, \boldsymbol{x}_B) = \boldsymbol{\lambda}_M(\boldsymbol{x}_A, \boldsymbol{x}_B) + \boldsymbol{\lambda}_{J_2}(\boldsymbol{x}_A, \boldsymbol{x}_B) + \boldsymbol{\lambda}_{\boldsymbol{S}}(\boldsymbol{x}_A, \boldsymbol{x}_B) + \boldsymbol{\lambda}_{\boldsymbol{v}_r}(\boldsymbol{x}_A, \boldsymbol{x}_B) + \cdots, \tag{98}$$

where $\boldsymbol{\lambda}_M$, $\boldsymbol{\lambda}_{J_2}$, $\boldsymbol{\lambda}_{\boldsymbol{S}}$, and $\boldsymbol{\lambda}_{\boldsymbol{v}_r}$ stand for the contributions of $\mathcal{T}_M$, $\mathcal{T}_{J_2}$, $\mathcal{T}_{\boldsymbol{S}}$, and $\mathcal{T}_{\boldsymbol{v}_r}$, respectively. We get from (91)

$$\boldsymbol{\lambda}_M(\boldsymbol{x}_A, \boldsymbol{x}_B) = -(\gamma{+}1)\frac{GM}{c^2}\left(\frac{1}{r_A}+\frac{1}{r_B}\right)\frac{1}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}\left(\boldsymbol{N}_{AB} - \frac{R_{AB}}{r_A+r_B}\boldsymbol{n}_B\right). \tag{99}$$

From (92), we get

$$\begin{aligned}
&\boldsymbol{\lambda}_{J_2}(\boldsymbol{x}_A, \boldsymbol{x}_B) \\
&\quad = (\gamma+1)\frac{GM}{c^2}\left(\frac{1}{r_A}+\frac{1}{r_B}\right) J_2\,\frac{r_e^2}{r_A r_B}\frac{1}{(1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)^2} \\
&\qquad \times\left\{\boldsymbol{N}_{AB}\left[\frac{(\boldsymbol{k}\cdot\boldsymbol{n}_A+\boldsymbol{k}\cdot\boldsymbol{n}_B)^2}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}\left(\frac{r_A}{r_B}+\frac{r_B}{r_A}+\frac{1}{2}-\frac{3}{2}\boldsymbol{n}_A\cdot\boldsymbol{n}_B\right)\right.\right. \\
&\qquad\quad \left. -\frac{1}{2}\frac{r_A r_B}{r_A+r_B}\left(\frac{1-(\boldsymbol{k}\cdot\boldsymbol{n}_A)^2}{r_A}+\frac{1-(\boldsymbol{k}\cdot\boldsymbol{n}_B)^2}{r_B}\right)\left(\frac{r_A}{r_B}+\frac{r_B}{r_A}+1-\boldsymbol{n}_A\cdot\boldsymbol{n}_B\right)\right] \\
&\qquad\quad -\boldsymbol{n}_B\frac{R_{AB}}{r_A+r_B}\left[\frac{(\boldsymbol{k}\cdot\boldsymbol{n}_A+\boldsymbol{k}\cdot\boldsymbol{n}_B)^2}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}\left(\frac{r_A}{r_B}+\frac{r_B}{r_A}+\frac{3}{2}-\frac{1}{2}\boldsymbol{n}_A\cdot\boldsymbol{n}_B\right)\right. \\
&\qquad\quad -\frac{1}{2}\left[1-3(\boldsymbol{k}\cdot\boldsymbol{n}_B)^2\right]\frac{r_A(2+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)+r_B}{r_B} \\
&\qquad\quad \left. -\frac{1}{2}(r_A+r_B)\left(\frac{1-(\boldsymbol{k}\cdot\boldsymbol{n}_A)^2}{r_A}-\frac{2(\boldsymbol{k}\cdot\boldsymbol{n}_A)(\boldsymbol{k}\cdot\boldsymbol{n}_B)}{r_B}\right)\right] \\
&\qquad\quad \left. +\boldsymbol{k}\,\frac{R_{AB}}{r_B}\left[(\boldsymbol{k}\cdot\boldsymbol{n}_A)+(\boldsymbol{k}\cdot\boldsymbol{n}_B)\frac{r_A(2+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)+r_B}{r_A+r_B}\right]\right\}.
\end{aligned} \tag{100}$$

From (93) and (94), we derive the other contributions that are not neglected here:

$$\begin{aligned}
\boldsymbol{\lambda}_{\boldsymbol{S}}(\boldsymbol{x}_A, \boldsymbol{x}_B) &= \left(\gamma+1+\frac{1}{4}\alpha_1\right)\frac{GS}{c^3 r_B}\left(\frac{1}{r_A}+\frac{1}{r_B}\right)\frac{1}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B} \\
&\quad \times\left\{\boldsymbol{k}\times\boldsymbol{n}_A - \frac{\boldsymbol{k}\cdot(\boldsymbol{n}_A\times\boldsymbol{n}_B)}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}\left[\boldsymbol{n}_A+\frac{r_A(2+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)+r_B}{r_A+r_B}\boldsymbol{n}_B\right]\right\},
\end{aligned} \tag{101}$$

$$\begin{aligned}
\boldsymbol{\lambda}_{\boldsymbol{v}_r}(\boldsymbol{x}_A, \boldsymbol{x}_B) &= \alpha_1\frac{GM}{2c^3}\left[\frac{\boldsymbol{v}_r-(\boldsymbol{v}_r\cdot\boldsymbol{N}_{AB})\boldsymbol{N}_{AB}}{R_{AB}}\ln\left(\frac{r_A+r_B+R_{AB}}{r_A+r_B-R_{AB}}\right)\right. \\
&\quad \left. +\frac{(\boldsymbol{v}_r\cdot\boldsymbol{N}_{AB})}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}\left(\frac{1}{r_A}+\frac{1}{r_B}\right)\left(\boldsymbol{N}_{AB}-\frac{R_{AB}}{r_A+r_B}\boldsymbol{n}_B\right)\right].
\end{aligned} \tag{102}$$

We note that the mass and the quadrupole moment yield contributions of order $1/c^2$, while the intrinsic angular momentum and the velocity relative to the universe rest frame yield contributions of order $1/c^3$.

5.5 Sagnac Terms in the Time Transfer Function

In experiments like ACES Mission, recording the time of emission t_A will be more practical than recording the time of reception t_B. So, it will be very convenient to form the expression of the time transfer $\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)$ from $\boldsymbol{x}_A(t_A)$ to $\boldsymbol{x}_B(t_B)$ in terms of the position of the receiver B at the time of emission t_A. For any quantity $Q_B(t)$ defined along the world line of the station B, let us put $\widetilde{Q}_B = Q(t_A)$. Thus we may write $\widetilde{\boldsymbol{x}}_B(t_A)$, $\widetilde{r}_B(t_A)$, $\widetilde{\boldsymbol{v}}_B(t_A)$, $\widetilde{v}_B =\mid \widetilde{\boldsymbol{v}}_B \mid$, etc.

Now, let us introduce the instantaneous coordinate distance $\boldsymbol{D}_{AB} = \widetilde{\boldsymbol{x}}_B - \boldsymbol{x}_A$ and its norm D_{AB}. Since we want to know $t_B - t_A$ up to the order $1/c^4$, we can use the Taylor expansion of $\boldsymbol{R}_{AB}$

$$\boldsymbol{R}_{AB} = \boldsymbol{D}_{AB} + (t_B - t_A)\widetilde{\boldsymbol{v}}_B + \frac{1}{2}(t_B - t_A)^2\, \widetilde{\boldsymbol{a}}_B + \frac{1}{6}(t_B - t_A)^3\, \widetilde{\boldsymbol{b}}_B + \cdots ,$$

where $\boldsymbol{a}_B$ is the acceleration of B and $\boldsymbol{b}_B = d\boldsymbol{a}_B/dt$. Using iteratively this expansion together with (90), we get

$$\begin{aligned}
\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B) = \mathcal{T}(\boldsymbol{x}_A, \widetilde{\boldsymbol{x}}_B) &+ \frac{1}{c^2}\boldsymbol{D}_{AB} \cdot \widetilde{\boldsymbol{v}}_B \\
&+ \frac{1}{2c^3} D_{AB} \left[\frac{(\boldsymbol{D}_{AB} \cdot \widetilde{\boldsymbol{v}}_B)^2}{D_{AB}^2} + \widetilde{v}_B^2 + \boldsymbol{D}_{AB} \cdot \widetilde{\boldsymbol{a}}_B \right] \\
&+ \frac{1}{c^4} \Bigg[(\boldsymbol{D}_{AB} \cdot \widetilde{\boldsymbol{v}}_B) \left(\widetilde{v}_B^2 + \boldsymbol{D}_{AB} \cdot \widetilde{\boldsymbol{a}}_B\right) \\
&+ \frac{1}{2} D_{AB}^2 \left(\widetilde{\boldsymbol{v}}_B \cdot \widetilde{\boldsymbol{a}}_B + \frac{1}{3}\boldsymbol{D}_{AB} \cdot \widetilde{\boldsymbol{b}}_B \right) \Bigg] \\
&+ \frac{1}{c}\frac{\boldsymbol{D}_{AB}}{D_{AB}} \cdot \widetilde{\boldsymbol{v}}_B \left[\mathcal{T}_M(\boldsymbol{x}_A, \widetilde{\boldsymbol{x}}_B) + \mathcal{T}_{J_2}(\boldsymbol{x}_A, \widetilde{\boldsymbol{x}}_B) \right] \\
&- \frac{1}{c^2} D_{AB} \widetilde{\boldsymbol{v}}_B \cdot \left[\boldsymbol{\lambda}_M(\boldsymbol{x}_A, \widetilde{\boldsymbol{x}}_B) + \boldsymbol{\lambda}_{J_2}(\boldsymbol{x}_A, \widetilde{\boldsymbol{x}}_B) \right] + \cdots , \quad (103)
\end{aligned}$$

where $\mathcal{T}(\boldsymbol{x}_A, \widetilde{\boldsymbol{x}}_B)$ is obtained by substituting $\widetilde{\boldsymbol{x}}_B$, $\widetilde{r}_B$, and $\boldsymbol{D}_{AB}$, respectively, for $\boldsymbol{x}_B$, r_B, and $\boldsymbol{R}_{AB}$ into the time transfer function defined by (90–94). This expression extends the previous formula [6] to the next order $1/c^4$. The second, the third, and the fourth terms in (103) represent pure Sagnac terms of order $1/c^2$, $1/c^3$, and $1/c^4$, respectively. The fifth and the sixth terms are contributions of the gravitational field mixed with the coordinate velocity of the receiving station. Since these last two terms are of order $1/c^4$, they might be calculated for the arguments $(\boldsymbol{x}_A, \boldsymbol{x}_B)$.

6 Frequency Shift in the Field of a Rotating Axisymmetric Body

6.1 General Formulae up to the Fourth Order

It is possible to derive the ratio q_A/q_B up to the order $1/c^4$ from our results in Sect. 4 since $\widehat{\boldsymbol{l}}_A$ and $\widehat{\boldsymbol{l}}_B$ are given up to the order $1/c^3$ by (95–98). Denoting by $\widehat{\boldsymbol{l}}^{(n)}/c^n$ the $O(n)$ terms in $\widehat{\boldsymbol{l}}$, q_A/q_B may be expanded as

$$\begin{aligned}\frac{q_A}{q_B} = 1 - \frac{1}{c}\frac{\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)}{1-\boldsymbol{N}_{AB}\cdot\dfrac{\boldsymbol{v}_B}{c}} + \frac{1}{c^3}\left[\widehat{\boldsymbol{l}}_A^{(2)}\cdot\boldsymbol{v}_A - \widehat{\boldsymbol{l}}_B^{(2)}\cdot\boldsymbol{v}_B\right] \\ +\frac{1}{c^4}\left[\widehat{\boldsymbol{l}}_A^{(3)}\cdot\boldsymbol{v}_A - \widehat{\boldsymbol{l}}_B^{(3)}\cdot\boldsymbol{v}_B\right] \\ +\frac{1}{c^4}\boldsymbol{N}_{AB}\cdot\left[\left(\widehat{\boldsymbol{l}}_B^{(2)}\cdot\boldsymbol{v}_B\right)(\boldsymbol{v}_A-2\boldsymbol{v}_B)+\left(\widehat{\boldsymbol{l}}_A^{(2)}\cdot\boldsymbol{v}_A\right)\boldsymbol{v}_B\right]+O(5)\,. \quad (104)\end{aligned}$$

To be consistent with this expansion, we have to perform the calculation of u_A^0/u_B^0 at the same level of approximation. For a clock delivering a proper time τ, $1/u^0$ is the ratio of the proper time $d\tau$ to the coordinate time dt. To reach the suitable accuracy, it is therefore necessary to take into account the terms of order $1/c^4$ in g_{00}. For the sake of simplicity, we shall henceforth confine ourselves to the fully conservative metric theories of gravity without preferred location effects, in which all the PPN parameters vanish except β and γ. Since the gravitational field is assumed to be stationary, the chosen coordinate system is then a standard post-Newtonian gauge and the metric reduces to its usual form

$$\begin{aligned} G_{00} &= 1-\frac{2}{c^2}W+\frac{2\beta}{c^4}W^2+O(6), \\ \{G_{0i}\} &= \frac{2(\gamma+1)}{c^3}\boldsymbol{W}+O(5), \quad (105) \\ G_{ij} &= -\left(1+\frac{2\gamma}{c^2}W\right)\delta_{ij}+O(4)\,, \quad (106)\end{aligned}$$

where W given by (54) reduces to

$$W(\boldsymbol{x}) = U(\boldsymbol{x})+\frac{G}{c^2}\int\frac{\rho^*(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\left[\left(\gamma+\frac{1}{2}\right)v^2+(1-2\beta)U+\Pi+3\gamma\frac{p}{\rho^*}\right]d^3\boldsymbol{x}'\,, \quad (107)$$

and $\boldsymbol{W}$ is given by (75). As a consequence, for a clock moving with the coordinate velocity $\boldsymbol{v}$, the quantity $1/u^0$ is given by the formula

$$\begin{aligned}\frac{1}{u^0}\equiv\frac{d\tau}{dt} = 1-\frac{1}{c^2}\left(W+\frac{1}{2}v^2\right)+\frac{1}{c^4}\left[\left(\beta-\frac{1}{2}\right)W^2-\left(\gamma+\frac{1}{2}\right)Wv^2\right. \\ \left.-\frac{1}{8}v^4+2(\gamma+1)\boldsymbol{W}\cdot\boldsymbol{v}\right]+O(6), \quad (108)\end{aligned}$$

from which it is easily deduced that

$$\frac{u_A^0}{u_B^0} = 1 + \frac{1}{c^2}\left(W_A - W_B + \frac{1}{2}v_A^2 - \frac{1}{2}v_B^2\right) + \frac{1}{c^4}\Big\{(\gamma+1)(W_A v_A^2 - W_B v_B^2)$$
$$+\frac{1}{2}(W_A - W_B)\left[W_A - W_B + 2(1-\beta)(W_A + W_B) + v_A^2 - v_B^2\right]$$
$$-2(\gamma+1)(\boldsymbol{W}_A\cdot\boldsymbol{v}_A - \boldsymbol{W}_B\cdot\boldsymbol{v}_B) + \frac{3}{8}v_A^4 - \frac{1}{4}v_A^2 v_B^2 - \frac{1}{8}v_B^4\Big\} + O(6)\,. \tag{109}$$

It follows from (104) and (109) that the frequency shift $\delta\nu/\nu$ is given by

$$\frac{\delta\nu}{\nu} \equiv \frac{\nu_A}{\nu_B} - 1 = \left(\frac{\delta\nu}{\nu}\right)_c + \left(\frac{\delta\nu}{\nu}\right)_g\,, \tag{110}$$

where $(\delta\nu/\nu)_c$ is the special-relativistic Doppler effect

$$\left(\frac{\delta\nu}{\nu}\right)_c = -\frac{1}{c}\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A - \boldsymbol{v}_B)$$
$$+\frac{1}{c^2}\left[\frac{1}{2}v_A^2 - \frac{1}{2}v_B^2 - (\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A - \boldsymbol{v}_B))(\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_B)\right]$$
$$-\frac{1}{c^3}\left[(\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A - \boldsymbol{v}_B))\left(\frac{1}{2}v_A^2 - \frac{1}{2}v_B^2 + (\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_B)^2\right)\right]$$
$$+\frac{1}{c^4}\Big[\frac{3}{8}v_A^4 - \frac{1}{4}v_A^2 v_B^2 - \frac{1}{8}v_B^4$$
$$-(\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A - \boldsymbol{v}_B))(\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_B)\left(\frac{1}{2}v_A^2 - \frac{1}{2}v_B^2 + (\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_B)^2\right)\Big]$$
$$+O(5) \tag{111}$$

and $(\delta\nu)/\nu)_g$ contains all the contribution of the gravitational field, eventually mixed with kinetic terms

$$\left(\frac{\delta\nu}{\nu}\right)_g = \frac{1}{c^2}(W_A - W_B)$$
$$-\frac{1}{c^3}\left[(W_A - W_B)(\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A - \boldsymbol{v}_B)) - \widehat{\boldsymbol{l}}_A^{(2)}\cdot\boldsymbol{v}_A + \widehat{\boldsymbol{l}}_B^{(2)}\cdot\boldsymbol{v}_B\right]$$
$$+\frac{1}{c^4}\Big\{(\gamma+1)(W_A v_A^2 - W_B v_B^2)$$
$$+\frac{1}{2}(W_A - W_B)\Big[W_A - W_B + 2(1-\beta)(W_A + W_B) + v_A^2 - v_B^2$$
$$-2(\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A - \boldsymbol{v}_B))(\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_B)\Big]$$
$$+\boldsymbol{N}_{AB}\cdot\left[\left(\widehat{\boldsymbol{l}}_B^{(2)}\cdot\boldsymbol{v}_B\right)(\boldsymbol{v}_A - 2\boldsymbol{v}_B) + \left(\widehat{\boldsymbol{l}}_A^{(2)}\cdot\boldsymbol{v}_A\right)\boldsymbol{v}_B\right]$$
$$+\left(\widehat{\boldsymbol{l}}_A^{(3)} - 2(\gamma+1)\boldsymbol{W}_A\right)\cdot\boldsymbol{v}_A - \left(\widehat{\boldsymbol{l}}_B^{(3)} - 2(\gamma+1)\boldsymbol{W}_B\right)\cdot\boldsymbol{v}_B\Big\}$$
$$+O(5)\,. \tag{112}$$

It must be emphasized that the formulae (108) and (109) are valid within the PPN framework without adding special assumption, provided that β and γ are the only nonvanishing post-Newtonian parameters. On the other hand, (112) is valid only for stationary gravitational fields. In the case of an axisymmetric rotating body, we shall obtain an approximate expression of the frequency shift by inserting the following developments in (112), yielded by (97–102):

$$\frac{\widehat{\boldsymbol{l}}_A^{(2)}}{c^2} = -\boldsymbol{\lambda}_M(\boldsymbol{x}_B, \boldsymbol{x}_A) - \boldsymbol{\lambda}_{J_2}(\boldsymbol{x}_B, \boldsymbol{x}_A) + \dots\,, \qquad \frac{\widehat{\boldsymbol{l}}_A^{(3)}}{c^3} = \boldsymbol{\lambda}_{\boldsymbol{S}}(\boldsymbol{x}_B, \boldsymbol{x}_A) + \dots\,,$$

$$\frac{\widehat{\boldsymbol{l}}_B^{(2)}}{c^2} = \boldsymbol{\lambda}_M(\boldsymbol{x}_A, \boldsymbol{x}_B) + \boldsymbol{\lambda}_{J_2}(\boldsymbol{x}_A, \boldsymbol{x}_B) + \dots\,, \qquad \frac{\widehat{\boldsymbol{l}}_B^{(3)}}{c^3} = \boldsymbol{\lambda}_{\boldsymbol{S}}(\boldsymbol{x}_A, \boldsymbol{x}_B) + \dots\,,$$

the function $\boldsymbol{\lambda}_{\boldsymbol{S}}$ being now given by (101) written with $\alpha_1 = 0$. Let us recall that the symbol $+\cdots$ stands for the contributions of the higher multipole moments which are neglected.

6.2 Application in the Vicinity of the Earth

To perform numerical estimates of the frequency shifts in the vicinity of the Earth, we suppose now that A is onboard the International Space Station (ISS) orbiting at the altitude $H = 400\,\mathrm{km}$ and that B is a terrestrial station. It will be the case for the ACES mission. We use $r_B = 6.37 \times 10^6\,\mathrm{m}$ and $r_A - r_B = 400\,\mathrm{km}$. For the velocity of ISS, we take $v_A = 7.7 \times 10^3\,\mathrm{m\,s^{-1}}$ and for the terrestrial station, we have $v_B \leq 465\,\mathrm{m\,s^{-1}}$. The other useful parameters concerning the Earth are $GM = 3.986 \times 10^{14}\,\mathrm{m^3\,s^{-2}}$, $r_e = 6.378 \times 10^6\,\mathrm{m}$, $J_2 = 1.083 \times 10^{-3}$; for $n \geq 3$, the multipole moments J_n are in the order of 10^{-6}. With these values, we get $W_B/c^2 \approx GM/c^2 r_B = 6.95 \times 10^{-10}$ and $W_A/c^2 \approx GM/c^2 r_A = 6.54 \times 10^{-10}$. From these data, it is easy to deduce the following upper bounds: $\mid \boldsymbol{N}_{AB} \cdot \boldsymbol{v}_A/c \mid \leq 2.6 \times 10^{-5}$ for the satellite, $|\boldsymbol{N}_{AB} \cdot \boldsymbol{v}_B/c| \leq 1.6 \times 10^{-6}$ for the ground station, and $|\boldsymbol{N}_{AB} \cdot (\boldsymbol{v}_A - \boldsymbol{v}_B)/c| \leq 2.76 \times 10^{-5}$ for the first-order Doppler term.

Our purpose is to obtain correct estimates of the effects in (112) which are greater than or equal to 10^{-18} for an axisymmetric model of the Earth. At this level of approximation, it is not sufficient to take into account the J_2-terms in $(W_A - W_B)/c^2$. First, the higher-multipole moments J_3, J_4, ... yield contribution of order 10^{-15} in W_A/c^2. Second, owing to the irregularities in the distribution of masses, the expansion of the geopotential in a series of spherical harmonics is probably not convergent at the surface of the Earth. For these reasons, we do not expand $(W_A - W_B)/c^2$ in (112).

However, for the higher-order terms in (112), we can apply the explicit formulae obtained in Sect. 5. Indeed, since the difference between the geoid and the reference ellipsoid is less than 100 m, W_B/c^2 may be written as [26]

$$\frac{1}{c^2} W_B = \frac{GM}{c^2 r_B} + \frac{GM r_e^2 J_2}{2c^2 r_B^3}(1 - 3\cos^2\theta) + \frac{1}{c^2} \triangle W_B\,, \tag{113}$$

where the residual term $\triangle W_B/c^2$ is such that $|\triangle W_B/c^2| \leq 10^{-14}$. At a level of experimental uncertainty about 10^{-18}, this inequality allows to retain only the contributions due to M, J_2, and $\boldsymbol{S}$ in the terms of orders $1/c^3$ and $1/c^4$. As a consequence, the formula (112) reduces to

$$\left(\frac{\delta\nu}{\nu}\right)_g = \frac{1}{c^2}(W_A - W_B) + \frac{1}{c^3}\left(\frac{\delta\nu}{\nu}\right)^{(3)}_M + \frac{1}{c^3}\left(\frac{\delta\nu}{\nu}\right)^{(3)}_{J_2} + \ldots$$
$$+\frac{1}{c^4}\left(\frac{\delta\nu}{\nu}\right)^{(4)}_M + \frac{1}{c^4}\left(\frac{\delta\nu}{\nu}\right)^{(4)}_{\boldsymbol{S}} + \ldots\,, \tag{114}$$

where the different terms involved in the RHS are separately explicited and discussed in what follows.

By using (89), it is easy to see that $(\delta\nu/\nu)^{(3)}_M$ is given by

$$\left(\frac{\delta\nu}{\nu}\right)^{(3)}_M = -\frac{GM(r_A+r_B)}{r_A r_B}\left[\left(\frac{\gamma+1}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B} - \frac{r_A-r_B}{r_A+r_B}\right)[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)]\right.$$
$$\left.+(\gamma+1)\frac{R_{AB}}{r_A+r_B}\frac{\boldsymbol{n}_A\cdot\boldsymbol{v}_A+\boldsymbol{n}_B\cdot\boldsymbol{v}_B}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}\right]\,. \tag{115}$$

The contribution of this term is bounded by 5×10^{-14} for $\gamma=1$, in accordance with a previous analysis [6].

6.3 Influence of the Quadrupole Moment at the Order $1/c^3$

It follows from (100) and (112) that the term $(\delta\nu/\nu)^{(3)}_{J_2}$ in (114) is given by

$$\left(\frac{\delta\nu}{\nu}\right)^{(3)}_{J_2}$$
$$= \frac{GM}{2r_e}J_2\left(\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)\right)\left[\left(\frac{r_e}{r_A}\right)^3\left[3(\boldsymbol{k}\cdot\boldsymbol{n}_A)^2-1\right] - \left(\frac{r_e}{r_B}\right)^3\left[3(\boldsymbol{k}\cdot\boldsymbol{n}_B)^2-1\right]\right]$$
$$+(\gamma+1)GM\left(\frac{1}{r_A}+\frac{1}{r_B}\right)J_2\,\frac{r_e^2}{r_A r_B}\,\frac{1}{(1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)^2}$$
$$\times\left\{[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)]\left[\frac{(\boldsymbol{k}\cdot\boldsymbol{n}_A+\boldsymbol{k}\cdot\boldsymbol{n}_B)^2}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}\left(\frac{r_A}{r_B}+\frac{r_B}{r_A}+\frac{1}{2}-\frac{3}{2}\boldsymbol{n}_A\cdot\boldsymbol{n}_B\right)\right.\right.$$
$$-\frac{1}{2}\left(1-\frac{r_A(\boldsymbol{k}\cdot\boldsymbol{n}_B)^2+r_B(\boldsymbol{k}\cdot\boldsymbol{n}_A)^2}{r_A+r_B}\right)$$
$$\left.\times\left(\frac{r_A}{r_B}+\frac{r_B}{r_A}+1-\boldsymbol{n}_A\cdot\boldsymbol{n}_B\right)\right]$$
$$+\frac{R_{AB}}{r_A+r_B}(\boldsymbol{n}_A\cdot\boldsymbol{v}_A+\boldsymbol{n}_B\cdot\boldsymbol{v}_B)\frac{(\boldsymbol{k}\cdot\boldsymbol{n}_A+\boldsymbol{k}\cdot\boldsymbol{n}_B)^2}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}\left(\frac{r_A}{r_B}+\frac{r_B}{r_A}+\frac{3}{2}-\frac{1}{2}\boldsymbol{n}_A\cdot\boldsymbol{n}_B\right)$$
$$-\frac{1}{2}\frac{R_{AB}}{r_A}(\boldsymbol{n}_A\cdot\boldsymbol{v}_A)\left[1-3(\boldsymbol{k}\cdot\boldsymbol{n}_A)^2\right]\frac{r_A+r_B(2+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)}{r_A+r_B}$$

$$
\begin{aligned}
&-\frac{1}{2}\frac{R_{AB}}{r_B}(\boldsymbol{n}_B\cdot\boldsymbol{v}_B)\left[1-3(\boldsymbol{k}\cdot\boldsymbol{n}_B)^2\right]\frac{r_A(2+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)+r_B}{r_A+r_B}\\
&+R_{AB}\left[\left(\frac{\boldsymbol{n}_A\cdot\boldsymbol{v}_A}{r_A}+\frac{\boldsymbol{n}_B\cdot\boldsymbol{v}_B}{r_B}\right)(\boldsymbol{k}\cdot\boldsymbol{n}_A)(\boldsymbol{k}\cdot\boldsymbol{n}_B)\right.\\
&\qquad\left.-\frac{1}{2}(\boldsymbol{n}_A\cdot\boldsymbol{v}_A)\frac{1-(\boldsymbol{k}\cdot\boldsymbol{n}_B)^2}{r_B}-\frac{1}{2}(\boldsymbol{n}_B\cdot\boldsymbol{v}_B)\frac{1-(\boldsymbol{k}\cdot\boldsymbol{n}_A)^2}{r_A}\right]\\
&-\frac{R_{AB}}{r_A}(\boldsymbol{k}\cdot\boldsymbol{v}_A)\left[\boldsymbol{k}\cdot\boldsymbol{n}_A\frac{r_A+r_B(2+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)}{r_A+r_B}+\boldsymbol{k}\cdot\boldsymbol{n}_B\right]\\
&-\frac{R_{AB}}{r_B}(\boldsymbol{k}\cdot\boldsymbol{v}_B)\left[\boldsymbol{k}\cdot\boldsymbol{n}_A+\boldsymbol{k}\cdot\boldsymbol{n}_B\frac{r_A(2+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)+r_B}{r_A+r_B}\right]\Bigg\}. \qquad (116)
\end{aligned}
$$

One has $|\boldsymbol{v}_A/c| = 2.6\times10^{-5}$, $|\boldsymbol{v}_B/c| \leq 1.6\times10^{-6}$, and $K_{AB} = 3.77\times10^{-3}$. A crude estimate can be obtained by neglecting in (116) the terms involving the scalar products $\boldsymbol{n}_B\cdot\boldsymbol{v}_B$ and $\boldsymbol{k}\cdot\boldsymbol{v}_B$. Since the orbit of ISS is almost circular, the scalar product $\boldsymbol{n}_A\cdot\boldsymbol{v}_A$ can also be neglected. On these assumptions, we find for $\gamma = 1$

$$
\left|\frac{1}{c^3}\left(\frac{\delta\nu}{\nu}\right)^{(3)}_{J_2}\right| \leq 1.3\times10^{-16}. \qquad (117)
$$

As a consequence, it will perhaps be necessary to take into account the $O(3)$ contributions of J_2 in the ACES mission. This conclusion is to be compared with the order of magnitude given in [6] without a detailed calculation. Of course, a better estimate might be found if the inclination $i = 51.6$ deg of the orbit with respect to the terrestrial equatorial plane and the latitude $\pi/2-\theta_B$ of the ground station was taken into account.

6.4 Frequency Shifts of Order $1/c^4$

The term $(\delta\nu/\nu)^{(4)}_M$ in (114) is given by

$$
\begin{aligned}
\left(\frac{\delta\nu}{\nu}\right)^{(4)}_M &= (\gamma+1)\left(\frac{GM}{r_A}v_A^2-\frac{GM}{r_B}v_B^2\right)-\frac{GM(r_A-r_B)}{2\,r_Ar_B}(v_A^2-v_B^2)\\
&\quad+\frac{1}{2}\left(\frac{GM}{r_Ar_B}\right)^2\left[(r_A-r_B)^2+2(\beta-1)(r_A^2-r_B^2)\right]-\frac{GM(r_A+r_B)}{r_Ar_B}\\
&\quad\times\left[\left(\frac{2(\gamma+1)}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}-\frac{r_A-r_B}{r_A+r_B}\right)[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-\boldsymbol{v}_B)]\,(\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_B)\right.\\
&\qquad+\frac{\gamma+1}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B}\frac{R_{AB}}{r_A+r_B}\Big\{(\boldsymbol{n}_A\cdot\boldsymbol{v}_A)\,(\boldsymbol{N}_{AB}\cdot\boldsymbol{v}_B)\\
&\qquad\left.-[\boldsymbol{N}_{AB}\cdot(\boldsymbol{v}_A-2\boldsymbol{v}_B)]\,(\boldsymbol{n}_B\cdot\boldsymbol{v}_B)\Big\}\right]. \qquad (118)
\end{aligned}
$$

The dominant term $(\gamma+1)GMv_A^2/r_A$ in (118) induces a correction to the frequency shift which amounts to 10^{-18}. So, it will certainly be necessary to take this correction into account in experiments performed in the foreseeable future.

The term $(\delta\nu/\nu)^{(4)}_{\boldsymbol{S}}$ is the contribution of the intrinsic angular momentum to the frequency shift. Substituting (79) and (101) into (112), it may be seen that

$$\left(\frac{\delta\nu}{\nu}\right)^{(4)}_{\boldsymbol{S}} = (\mathcal{F}_{\boldsymbol{S}})_A - (\mathcal{F}_{\boldsymbol{S}})_B\,, \tag{119}$$

where

$$\begin{aligned}(\mathcal{F}_{\boldsymbol{S}})_A = (\gamma+1)\frac{GS}{r_A^2}\left(1+\frac{r_A}{r_B}\right)\boldsymbol{v}_A\cdot\Bigg\{&\frac{\boldsymbol{k}\times\boldsymbol{n}_B}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B} - \frac{r_B}{r_A+r_B}\boldsymbol{k}\times\boldsymbol{n}_A \\ &+\frac{\boldsymbol{k}\cdot(\boldsymbol{n}_A\times\boldsymbol{n}_B)}{(1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)^2}\left[\frac{r_A+r_B(2+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)}{r_A+r_B}\boldsymbol{n}_A+\boldsymbol{n}_B\right]\Bigg\}\,,\end{aligned} \tag{120}$$

$$\begin{aligned}(\mathcal{F}_{\boldsymbol{S}})_B = (\gamma+1)\frac{GS}{r_B^2}\left(1+\frac{r_B}{r_A}\right)\boldsymbol{v}_B\cdot\Bigg\{&\frac{\boldsymbol{k}\times\boldsymbol{n}_A}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B} - \frac{r_A}{r_A+r_B}\boldsymbol{k}\times\boldsymbol{n}_B \\ &-\frac{\boldsymbol{k}\cdot(\boldsymbol{n}_A\times\boldsymbol{n}_B)}{(1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)^2}\left[\boldsymbol{n}_A+\frac{r_A(2+\boldsymbol{n}_A\cdot\boldsymbol{n}_B)+r_B}{r_A+r_B}\boldsymbol{n}_B\right]\Bigg\}\,.\end{aligned} \tag{121}$$

To make easier the discussion, it is useful to introduce the angle ψ between $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$, and the angle i_p between the plane of the photon path and the equatorial plane. These angles are defined by

$$\cos\psi = \boldsymbol{n}_A\cdot\boldsymbol{n}_B\,,\quad 0\le\psi<\pi\,,\quad \boldsymbol{k}\cdot(\boldsymbol{n}_A\times\boldsymbol{n}_B)=\sin\psi\cos i_p\,,\quad 0\le i_p<\pi\,.$$

With these definitions, it is easily seen that

$$\frac{\boldsymbol{k}\cdot(\boldsymbol{n}_A\times\boldsymbol{n}_B)}{1+\boldsymbol{n}_A\cdot\boldsymbol{n}_B} = \cos i_p\tan\frac{\psi}{2}\,.$$

Let us apply our formulas to ISS. Due to the inequality $v_B/v_A\le 6\times10^{-2}$, the term $(\mathcal{F}_{\boldsymbol{S}})_B$ in (119) may be neglected. From (120), it is easily deduced that

$$\mid(\mathcal{F}_{\boldsymbol{S}})_A\mid\le\,(\gamma+1)\frac{GS}{r_A^2}\left(1+\frac{r_A}{r_B}\right)\frac{2+3\,|\tan\psi/2|}{|1+\cos\psi|}\,v_A\,.$$

Assuming $0\le\psi\le\pi/2$, we have $(2+3\,|\tan\psi/2\,|)/\,|1+\cos\psi\,|\le 5$. Inserting this inequality in the previous one and taking for the Earth $GS/c^3r_A^2 = 3.15\times 10^{-16}$, we find

$$\left|\frac{1}{c^4}\left(\frac{\delta\nu}{\nu}\right)^{(4)}_{\boldsymbol{S}}\right|\le\,(\gamma+1)\times10^{-19}\,. \tag{122}$$

Thus, we get an upper bound which is slightly greater than the one estimated by retaining only the term $h_{0i}v^i/c$ in (109). However, our formula confirms that the intrinsic angular momentum of the Earth will not affect the ACES experiment.

7 Conclusion

It is clear that the world function $\Omega(x_A, x_B)$ constitutes a powerful tool for determining the time delay and the frequency shift of electromagnetic signals in a weak gravitational field. The analytical derivations given here are obtained within the Nordtvedt–Will PPN formalism. We have found the general expression of $\Omega(x_A, x_B)$ up to the order $1/c^3$. This result yields the expression of the time transfer functions $\mathcal{T}_e(t_A, \boldsymbol{x}_A, \boldsymbol{x}_B)$ and $\mathcal{T}_r(t_B, \boldsymbol{x}_A, \boldsymbol{x}_B)$ up to the order $1/c^4$. We point out that γ and α_1 are the only post-Newtonian parameters involved in the expressions of the world function and of the time transfer functions within the limit of the considered approximation.

We have treated in detail the case of an isolated, axisymmetric rotating body, assuming that the gravitational field is stationary and that the body is moving with a constant velocity $\boldsymbol{v}_r$ relative to the universe rest frame. We have given a systematic procedure for calculating the terms due to the multipole moments in the world function $\Omega(x_A, x_B)$ and in the single time transfer function $\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)$. These terms are obtained by straightforward differentiations of a kernel function. We have explicitly derived the contributions due to the mass M, to the quadrupole moment J_2, and to the intrinsic angular momentum $\boldsymbol{S}$ of the rotating body.

Assuming for the sake of simplicity that only β and γ are different from zero, we have determined the general expression of the frequency shift up to the order $1/c^4$. We have derived an explicit formula for the contributions of J_2 at the order $1/c^3$. Our method would give as well the quadrupole contribution at the order $1/c^4$ in case of necessity. Furthermore, we have obtained a thorough expression for the contribution of the mass monopole at the fourth order, as well as the contribution of the intrinsic angular momentum $\boldsymbol{S}$, which is also of order $1/c^4$. It must be pointed out that our calculations give also the vectors tangent to the light ray at the emission and reception points. So, our results could be used for determining the contributions of J_2 and $\boldsymbol{S}$ to the deflection of light.

On the assumption that the gravitational field is stationary, our formulae yield all the gravitational corrections to the frequency shifts up to 10^{-18} in the vicinity of the Earth. Numerically, the influence of the Earth quadrupole moment at the order $1/c^3$ is in the region of 10^{-16} for a clock installed onboard ISS and compared with a ground clock. As a consequence, this effect will probably be observable during the ACES mission. We also note that the leading term in the fourth-order frequency shift due to the mass monopole is equal to 10^{-18} for a clock installed onboard ISS and compared with a ground clock. As a consequence, this effect could be observable in the foreseeable future with atomic clocks using optical transitions.

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