10

Optimal Design of the Suspension System of Railway Vehicles

In the present chapter the dynamic response of rail vehicles due to random excitation is dealt with by deriving very simple *analytical formulae*. The optimisation of railway vehicle suspension system is proposed by using multi-objective programming. Basic hints are given on how to select the railway vehicle suspension design variables to obtain the best trade-off between standard deviation of vertical acceleration at the body and standard deviation of secondary suspension stroke.

In the literature the authors have not found papers dealing with the problem of deriving simple analytical formulae for the estimation of the dynamic response of railway vehicles to random excitations generated by vertical track irregularity. With reference to vehicle–bogie vibrations and to vehicle–track interaction, a number of authors have dealt with the problem of deriving basic concepts useful for rail vehicle design [60, 76, 99, 199]. They usually have resorted to numerical simulations even when dealing with simple models.

Successful applications of multi-objective programming in the field of railway vehicle design are reported in [51, 147, 151].

In the first section of the chapter analytical formulae are derived [151] to describe the dynamic behaviour of a railway vehicle. These formulae are then used in the second section of the chapter to derive the best trade-off between the standard deviations of body acceleration and secondary suspension stroke [151].

10.1 System Model

10.1.1 Equations of Motion and Responses to Stochastic Excitation

The adopted system model is shown in Fig. 10.1. The system has two degrees of freedom. The mass m_1 represents one-fourth of the mass of the bogie

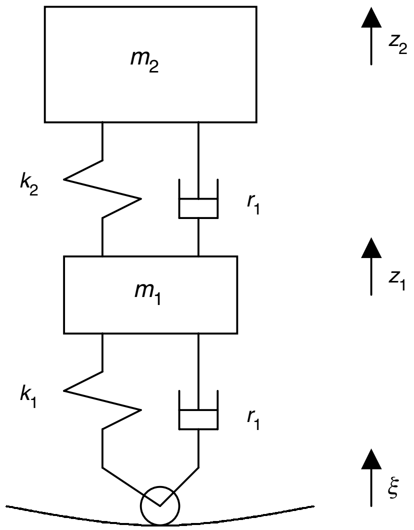

Fig. 10.1. Railway vehicle system model

and the mass m_2 represents one-eighth of the mass of the body. The excitation comes from the displacement (ξ) which represents the motion of the axle-box, the track being regarded as an uneven and infinitely stiff structure. This hypothesis had to be forcedly introduced to preserve both the analytical formulation and the analytical solution to the problem. In Sect. 9.2.3 the accuracy of the response of the model will be discussed.

The equations of motion may be written in matrix form as

$$\mathbf{M}\,\ddot{\mathbf{z}} + \mathbf{R}\,\dot{\mathbf{z}} + \mathbf{K}\,\mathbf{z} = \mathbf{F} \tag{10.1}$$

M, **R**, **K** are the mass, damping and stiffness matrixes, respectively

$$\mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \tag{10.2}$$

$$\mathbf{R} = \begin{pmatrix} r_1 + r_2 & -r_2 \\ -r_2 & r_2 \end{pmatrix} \tag{10.3}$$

$$\mathbf{K} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \tag{10.4}$$

F is the vector of external forces related to excitation from the uneven track

$$\mathbf{f} = \begin{pmatrix} r_1\dot{\xi} + k_1\xi \\ 0 \end{pmatrix} \tag{10.5}$$

z is the vector of the independent variables

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{10.6}$$

The displacement ξ (track irregularity) is assumed to be a random variable defined by a stationary and ergodic stochastic process. This assumption is consistent with the results of the studies, performed by many authors and organisations (see, e.g. [108,188]), on the stochastic properties of track (vertical) irregularity. A number of analytical formulae have been adopted (see [76,199]) to interpolate the measured data referring to the power spectral density (PSD) of the stochastic process defining ξ. In the present chapter two of those analytical formulae have been considered

$$S_\xi(\omega) = \frac{A_b\,(2\pi\ v)^3}{\omega^4} \tag{10.7}$$

$$S_\xi(\omega) = \frac{A_v \omega_c^2 v}{\omega^2\left(\omega^2 + \omega_c^2\right)} \tag{10.8}$$

In a log–log scaled plot (abscissa ω), the spectrum in Eq. (10.7) (reported in [199]) takes the shape of a line sloped at rate 4. In the following, it will be indicated as *one-slope* power spectral density (1S-PSD). The PSD in Eq. (10.8) has been reported in [76], and is widely used in railway vehicle dynamics simulations. In the log–log scaled plot shown in Fig. 10.2, Eq. (10.8) takes approximately the shape of a *two-slope* curve, thus reference to it will be made by the acronym 2S-PSD.

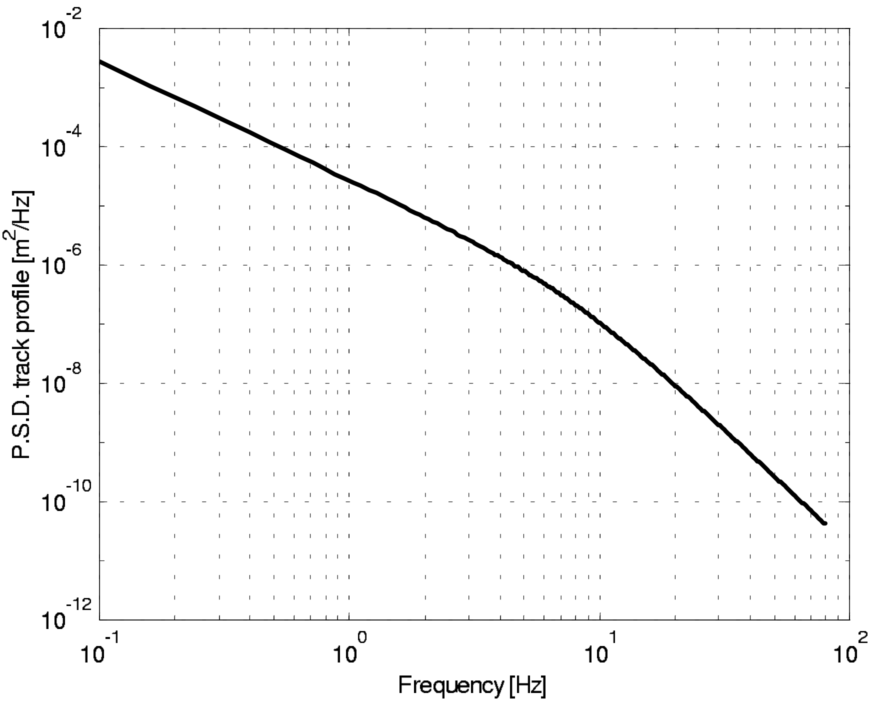

Fig. 10.2. Power spectral density (PSD) of the irregularity of the track in the vertical plane. Two slope PSD (2S-PSD, Eq. 10.8), at 177 km/h, (adapted from Ref. [76]).

The system (10.1) can be rewritten in a more convenient form

$$\begin{aligned} \dot{\underline{\mathbf{z}}} &= \mathbf{A}\,\underline{\mathbf{z}} + \mathbf{B}\,u \\ \mathbf{y} &= \mathbf{C}\,\underline{\mathbf{z}} + \mathbf{D}\,u \end{aligned} \tag{10.9}$$

$\underline{\mathbf{z}}$ is the vector of state variables

$$\underline{\mathbf{z}} = \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ z_1 \\ z_2 \end{pmatrix} \tag{10.10}$$

A and **B** are the state matrixes

$$\mathbf{A} = \begin{pmatrix} -\frac{r_1+r_2}{m_1} & \frac{r_2}{m_1} & -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{r_2}{m_2} & -\frac{r_2}{m_2} & \frac{k_2}{m_2} & -\frac{k_2}{m_2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1/m_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{10.11}$$

u is the input variable, related with track irregularity ξ

$$u = \left(r_1\dot{\xi} + k_1\xi \right) \tag{10.12}$$

$\mathbf{y}$ is the vector of output variables

$$\mathbf{y} = \begin{pmatrix} F_z \\ \ddot{z}_2 \\ z_2 - z_1 \end{pmatrix} \tag{10.13}$$

The output variables represent, respectively, the force on the axle-box (F_z), the vertical acceleration of the body ($\ddot{z}_2$), the (vertical) relative displacement body–bogie (stroke of the secondary suspension) ($z_2 - z_1$).

C and **D** matrixes read respectively

$$\mathbf{C} = \begin{pmatrix} -r_1 & 0 & -k_1 & 0 \\ \frac{r_2}{m_2} & -\frac{r_2}{m_2} & \frac{k_2}{m_2} & -\frac{k_2}{m_2} \\ 0 & 0 & -1 & 1 \end{pmatrix} \tag{10.14}$$

$$\mathbf{D} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{10.15}$$

The frequency response of the linear dynamic system (10.9) is

$$\mathbf{H}_{\mathbf{y},u}(s) = \mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D} \tag{10.16}$$

where $\mathbf{I}$ is the identity matrix. The input is represented by u, the output is represented by vector $\mathbf{y}$.

Given the PSD, P_{SDu} of the excitation input, the P_{SDj} of the jth element of vector $\mathbf{y}$, can be computed as (see, e.g. Sect. 5.1.1)

$$P_{SDj}(s) = H_j(s)\ H_j(-s)\ P_{SDu}(s) \tag{10.17}$$

$$\mathbf{H}(s) = \begin{pmatrix} H_1(s) \\ H_2(s) \\ H_3(s) \end{pmatrix} = \begin{pmatrix} H_{F_z,u}(s) \\ G_{\ddot{z}_2,u}(s) \\ G_{z_2-z_1,u}(s) \end{pmatrix} \tag{10.18}$$

$H_j(s)$, being a frequency response function of a mechanical system, is obviously a ratio of two polynomials (of variable s), i.e. $H_j\ (s)=N(s)/D(s)$. P_{SDu} and $P_{SD\xi}$ are linked by the following expression:

$$P_{SDu}(s) = L_{u,\xi}(s)\ L_{u,\xi}(-s)\ P_{SD\xi}(s) \tag{10.19}$$

where $L_{u,\xi}(s) = k_1 + s\,r_1$.

In order to write $P_{SDj}(s)$ (Eq. (10.17)) in a form matching with that of following Eq. (10.24) (the reason for this need will be explained in the next section), $P_{SD\xi}(s)$ has to be written as

$$P_{SD\xi}(s) = W_{qS}\ L_{qS}(s)\ L_{qS}(-s)\,, q = 1, 2 \tag{10.20}$$

where for $q = 1$ reference is made to the 1S-PSD, and it follows from Eq. (10.7) that $s = j\omega$ ($j = \sqrt{-1}$), $L_{1S}(s) = 1/s^2$ and $W_{1S} = A_b\,(2\pi v)^3$; for $q = 2$ reference is made to the 2S-PSD (Eq. (10.8)), so $L_{2S}(s) = 1/\{s\,(s+\omega_c)\}$ and $W_{2S} = A_v\omega_c^2 v$.

The P_{SDj} of the jth element of vector y can be finally written as

$$P_{SDj}(s) = W_{qS}\ L_{qS}(s)\,L_{u,\xi}(s)\ H_j(s)\ H_j(-s)\,L_{u,\xi}(-s)\,L_{qS}(-s) \tag{10.21}$$

This form is convenient for computations that will be presented in the next section (Eqs. (10.23) and (10.24)).

The (vertical) relative displacement bogie–axle box (stroke of the primary suspension) $(z_1 - \xi)$ and the force on the axle-box are related by the following expression:

$$L_{z_1-\xi,F_z}(s) = -\frac{1}{k_1 + sr_1} = -\frac{1}{L_{u,\xi}(s)}$$

So, the PSD of $(z_1 - \xi)$ reads

$$P_{SD\,z_1-\xi}(s) = W_{qS}L_{qS}(s)\ H_{F_z,u}(s)\,H_{F_z,u}(-s)\,L_{qS}(-s) \tag{10.22}$$

10.1.2 Derivation of Standard Deviations in Analytical Form

By definition (see Sect. 5.1.1) the variance of a random variable described by a stationary and ergodic stochastic process is

$$\sigma_j^2 = \frac{1}{2} \int_{-\infty}^{+\infty} P_{SDj}(s)\ ds \tag{10.23}$$

In [180] it is shown that an analytical solution exists for σ_j^2 if P_{SDj} can be written as

$$P_{SDj} = \frac{N_{k-1}(s)\ N_{k-1}(-s)}{D_k(s)\ D_k(-s)} \tag{10.24}$$

where D_k is a polynomial of degree k, and N_{k-1} is a polynomial of maximum degree $k-1$ ($k \geqslant 1$). This is actually the case, in fact it can be verified that P_{SDj} may be written as in Eq. (10.24) both by inspection of Eqs. (10.21) and (10.22) and by analysing the expressions of H_j and $H_{F_z,u}$.

For example, considering the vertical acceleration of the vehicle body ($\ddot{z}_2$), the following expression for $N_{k-1}(s)/D_k(s)$ can be obtained:

- for the 1S-PSD ($q = 1$, Eqs. (10.20), (10.7)), $k = 4$

  $$\frac{N_3(s)}{D_4(s)} = \sqrt{W_{1s}} L_{1S}(s)\ L_{u,\xi}(s)\ \ H_{\ddot{z}_2,u}(s)$$

  where (setting $r_1 = 0$) assuming

  $$W_{1S} = A_b (2\pi v)^3, \quad L_{1S}(s) = \frac{1}{s^2}, \quad L_{u,\xi}(s) = k_1,$$

  $$H_{\ddot{z}_2,u}(s) = (k_2 + r_2 s)\, s^2 / \big(k_1 k_2 + k_1 r_2 s + k_2 m_1 s^2 + k_1 m_2 s^2 + k_2 m_2 s^2 + m_1 r_2 s^3 + m_2 r_2 s^3 + m_1 m_2 s^4\big)$$

  we obtain

  $$N_3(s) = (A_b)^{1/2} (2\pi v)^{3/2} (k_1 k_2 + k_1 r_2 s)$$

  $$D_4(s) = (k_1 k_2 + k_1 r_2 s + k_2 m_1 s^2 + k_1 m_2 s^2 + k_2 m_2 s^2 + m_1 r_2 s^3 + m_2 r_2 s^3 + m_1 m_2 s^4)$$

- for the 2S-PSD ($q = 2$, Eqs. (10.20), (10.8)), $k = 5$

  $$\frac{N_4(s)}{D_5(s)} = \sqrt{W_{2s}} L_{2S}(s)\ L_{u,\xi}(s)\ \ H_{\ddot{z}_2,u}(s)$$

  where (setting $r_1 = 0$) assuming

  $$W_{2S} = A_v \omega_c^2 v, \quad L_{2S}(s) = \frac{1}{s(s + \omega_c)}$$

  we obtain

$$N_4(s) = (A_v \omega_c^2 v)^{1/2}(k_1 k_2 s + k_1 r_2 s^2)$$

$$D_5(s) = (k_1 k_2 + k_1 r_2 s + k_2 m_1 s^2 + k_1 m_2 s^2 + k_2 m_2 s^2 + m_1 r_2 s^3 + m_2 r_2 s^3 + m_1 m_2 s^4)(s + \omega_c)$$

The analytical formulae presented in the following subsections have been derived by means of the analytical solutions of the integral (10.22) reported in Sect. 5.6.

10.1.3 Complete Formulae Using the 1S-PSD (Eq. (10.7))

The 1S-PSD (Eq. (10.7)) has been considered because it allows a full and compact analytical solution of the problem. For the standard deviations of interest, by solving analytically Eq. (10.23) one gets

$$\sigma_{F_z} = 2\pi^2 v^{3/2}\sqrt{A_b}\sqrt{\frac{n_1}{d_1}} \tag{10.25}$$

$$\begin{aligned} n_1 = {} & r_1^2 m_1 m_2 \left(r_1 r_2^2 k_1 + r_1^2 r_2 k_2 + r_1 k_2^2 m_1 + r_2 k_1^2 m_2 + r_1 k_2^2 m_2\right) + \\ & (m_1 + m_2)(r_2 m_1 + r_1 m_2 + r_2 m_2)\left[(m_1 + m_2)\left(r_2^2 k_1^2 + r_1^2 k_2^2\right) - \right. \\ & \left. 2k_1^2 k_2 m_1 m_2\right] + k_1 k_2 (m_1 + m_2)^2 \left[-m_1 m_2 (r_2 k_1 + r_1 k_2) + \right. \\ & \left. (r_2 m_1 + r_1 m_2 + r_2 m_2)(r_1 r_2 + k_2 m_1 + k_1 m_2 + k_2 m_2)\right] + \\ & (r_2 k_1 + r_1 k_2)\Big[-2r_1 (r_2 k_1 + r_1 k_2) m_1 m_2 (m_1 + m_2) + \\ & (r_1 r_2 m_1 + r_1 r_2 m_2 + k_1 m_1 m_2)^2\Big] \\ d_1 = {} & \Big[- m_1 m_2 (r_2 k_1 + r_1 k_2)^2 - k_1 k_2 (r_2 m_1 + r_1 m_2 + r_2 m_2)^2 + \\ & (r_2 k_1 + r_1 k_2)(r_2 m_1 + r_1 m_2 + r_2 m_2)(r_1 r_2 + k_2 m_1 + k_1 m_2 + k_2 m_2)\Big] \end{aligned}$$

$$\sigma_{\ddot{z}_2} = 2\pi^2 v^{3/2}\sqrt{A_b}\sqrt{\frac{n_2}{d_2}} \tag{10.26}$$

$$\begin{aligned} n_2 = {} & r_1^2 r_2^2 (r_2 k_1 + r_1 k_2) + \left(r_2^2 k_1^2 + r_1^2 k_2^2\right)(r_2 m_1 + r_1 m_2 + r_2 m_2) + \\ & k_1 k_2 \left[-m_1 m_2 (r_2 k_1 + r_1 k_2) + (r_2 m_1 + r_1 m_2 + r_2 m_2)(r_1 r_2 + \right. \\ & \left. k_2 m_1 + k_1 m_2 + k_2 m_2)\right] \\ d_2 = {} & \Big[-m_1 m_2 (r_2 k_1 + r_1 k_2)^2 - k_1 k_2 (r_2 m_1 + r_1 m_2 + r_2 m_2)^2 + \\ & (r_2 k_1 + r_1 k_2)(r_2 m_1 + r_1 m_2 + r_2 m_2)(r_1 r_2 + k_2 m_1 + k_1 m_2 + k_2 m_2)\Big] \end{aligned}$$

$$
\begin{aligned}
\sigma_{z_2-z_1} &= 2\pi^2 v^{3/2}\sqrt{A_b}\sqrt{m_2^2\frac{n_3}{d_3}} \qquad (10.27)\\
n_3 &= r_1^2 k_2\left(r_2 m_1 + r_1 m_2 + r_2 m_2\right) + k_1\left[-m_1 m_2\left(r_2 k_1 + r_1 k_2\right) + \right.\\
&\quad \left.\left(r_2 m_1 + r_1 m_2 + r_2 m_2\right)\left(r_1 r_2 + k_2 m_1 + k_1 m_2 + k_2 m_2\right)\right]\\
d_3 &= k_2\left[-m_1 m_2\left(r_2 k_1 + r_1 k_2\right)^2 - k_1 k_2\left(r_2 m_1 + r_1 m_2 + r_2 m_2\right)^2 + \right.\\
&\quad \left.\left(r_2 k_1 + r_1 k_2\right)\left(r_2 m_1 + r_1 m_2 + r_2 m_2\right)\left(r_1 r_2 + k_2 m_1 + k_1 m_2 + k_2 m_2\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{z_1-\xi} &= 2\pi^2 v^{3/2}\sqrt{A_b}\sqrt{\frac{n_4}{d_4}} \qquad (10.28)\\
n_4 &= k_1\left(r_2 k_1 + r_1 k_2\right) m_1^2 m_2^2 + k_1\left(m_2 + m_1\right)\left(r_2 m_1 + r_1 m_2 + \right.\\
&\quad \left. r_2 m_2\right)\left(r_2^2 m_1 + r_2^2 m_2 - 2k_2 m_1 m_2\right) + \\
&\quad + k_2\left(m_1 + m_2\right)^2\left[-\left(r_2 k_1 + r_1 k_2\right) m_1 m_2 + \right.\\
&\quad \left.\left(r_2 m_1 + r_1 m_2 + r_2 m_2\right)\left(r_1 r_2 + k_2 m_1 + k_1 m_2 + k_2 m_2\right)\right]\\
d_4 &= k_1\left[-m_1 m_2\left(r_2 k_1 + r_1 k_2\right)^2 - k_1 k_2\left(r_2 m_1 + r_1 m_2 + r_2 m_2\right)^2 + \right.\\
&\quad \left.\left(r_2 k_1 + r_1 k_2\right)\left(r_2 m_1 + r_1 m_2 + r_2 m_2\right)\left(r_1 r_2 + k_2 m_1 + k_1 m_2 + k_2 m_2\right)\right]
\end{aligned}
$$

The above-mentioned standard deviations ($\sigma_{\ddot{z}_2}$, $\sigma_{z_2-z_1}$, σ_{F_z}) depend on the cube of the square root of the power of vehicle speed v, on the square root of track irregularity coefficient A_b, and on analytical functions of the model's parameters. These analytical expressions are rather complex. As the square roots do not depend on v, according to this formulation, the optimal settings of suspension design variables *do not* depend on vehicle speed.

10.1.4 Formulae for Vanishing Primary Damping Using the 1S-PSD (Eq. (10.7))

Primary dampers are commonly not fitted into the bogies of urban and suburban railway vehicles. This is due to the fact that, the primary stiffness (k_1) being relatively high (due to the high ratio payload/tare mass), the pitching of the bogie is already limited by the primary stiffness itself and dampers are not necessarily needed to limit the dynamic pitching oscillations. However, primary dampers can be adopted for reducing track wear.

Setting the primary damping to zero, i.e. $r_1 = 0$, the standard deviations (10.25), (10.26), (10.27), (10.28) assume relatively simple expressions

– Force on the axle-box

$$\sigma_{F_z} = 2\pi^2 v^{3/2}\sqrt{A_b}\cdot \sqrt{\frac{k_1^2 m_1^2 m_2^2 + k_1(m_1+m_2)^2\left[r_2^2(m_1+m_2) - 2k_2 m_1 m_2\right] + k_2(m_1+m_2)^2\left[k_2(m_1+m_2)^2 + k_1 m_2^2\right]}{r_2 k_1 m_2^2}} \tag{10.29}$$

– Body acceleration

$$\sigma_{\ddot{z}_2} = 2\pi^2 v^{3/2}\sqrt{A_b}\sqrt{\frac{r_2^2 k_1\left(m_1+m_2\right) + k_2^2\left(m_1+m_2\right)^2 + k_1 k_2 m_2^2}{r_2 k_1 m_2^2}} \tag{10.30}$$

– Secondary stroke

$$\sigma_{z_2 - z_1} = 2\pi^2 v^{3/2}\sqrt{A_b}\sqrt{\frac{k_2\left(m_1+m_2\right)^2 + k_1 m_2^2}{r_2 k_1 k_2}} \tag{10.31}$$

The primary stroke is obviously $\sigma_{z_1-\xi} = \sigma_{F_z}/k_1$.

10.1.5 Simplified Formulae Using the 1S-PSD (Eq. (10.7))

Equations (10.29)–(10.31) can be simplified by neglecting those terms which vanish when parameter values refer to actual railway vehicles[1]:

$$\sigma_{F_z} = B\sqrt{\frac{k_1 m_1^2}{r_2}\left(1+\beta\mu^2+\beta^2\mu^2\right) + m_1 r_2(3+\mu)} \tag{10.32}$$

$$\sigma_{\ddot{z}_2} = B\sqrt{\frac{r_2}{m_2} + \frac{k_2}{r_2}} \tag{10.33}$$

$$\sigma_{z_2-z_1} = B\sqrt{\frac{m_2^2}{r_2 k_2}} \tag{10.34}$$

A comparison has been made between the standard deviations $\sigma_{\ddot{z}_2}, \sigma_{z_2-z_1}, \sigma_{F_z}$ computed by Eqs. (10.29)–(10.31) and the corresponding standard deviations $\sigma_{\ddot{z}_2}$, $\sigma_{z_2-z_1}$, σ_{F_z} computed by Eqs. (10.32)–(10.34) (vehicle parameters in Table 10.1.). The absolute value of the error was 0.49% for the force on the axle-box (Eq. (10.32)), 1.7% for the body vertical acceleration (Eq. (10.33)), 0.37% for the secondary stroke (Eq. (10.34)).

[1]For actual railway vehicles usually one finds

$$5 < \mu < 10, \quad 0.01 < \beta < 0.2$$

with $\mu = m_2/m_1$ and $\beta = k_2/k_1$. Assuming that $(1+\mu)/\mu \approx 1$ and $\sqrt{\beta+1} \approx 1$ Eqs. (10.29)–(10.31) and Eqs. (10.39)–(10.41) can be simplified

Table 10.1. Data of the reference railway vehicle taken into consideration (data from [76])

m_{1r}	773 kg
m_{2r}	5,217 kg
k_{1r}	28,240,000 N/m
k_{2r}	162,000 N/m
r_{1r}	21,890 Ns/m
r_{2r}	14,600 Ns/m

10.1.6 Complete Formulae Using the 2S-PSD (Eq. (10.8))

The analytical expressions of the standard deviations of the force on the axle-box (F_z), of the vertical acceleration of the body ($\ddot{z}_2$), of the (vertical) relative displacement body–bogie have been derived by solving analytically Eq. (10.23). They are not reported here because of their extreme complexity. In this case it is more convenient to solve Eq. (10.23) numerically. It has to be noticed that, due to the employed spectrum, contrary to what happens in Eqs. (10.25)–(10.28), the speed v is mixed among the parameters A_v, ω_c, m_i, r_i, k_i. According to this formulation the optimal suspension design variables do depend on vehicle speed v.

$$\sigma_{F_z} = l_{1DS}\left(v, A_v, \omega_c, m_1, m_2, r_1, k_1, r_2, k_2\right) \tag{10.35}$$

$$\sigma_{\ddot{z}_2} = l_{2DS}\left(v, A_v, \omega_c, m_1, m_2, r_1, k_1, r_2, k_2\right) \tag{10.36}$$

$$\sigma_{z_2-z_1} = l_{3DS}\left(v, A_v, \omega_c, m_1, m_2, r_1, k_1, r_2, k_2\right) \tag{10.37}$$

$$\sigma_{z_1-\xi} = l_{4DS}\left(v, A_v, \omega_c, m_1, m_2, r_1, k_1, r_2, k_2\right) \tag{10.38}$$

10.1.7 Formulae for Vanishing Primary Damping Using the 2S-PSD (Eq. (10.8))

If the primary damping vanishes (i.e. $r_1 = 0$) the standard deviations in Eqs. (10.35)–(10.38) assume relatively simple expressions

– Force on the axle-box

$$\sigma_{F_z} = \omega_c \sqrt{\pi\ v\ A_v} \sqrt{\frac{n_{v1}}{d_{v1}}} \tag{10.39}$$

$$\begin{aligned} n_{v1} = {} & k_1^2 m_1^2 m_2^2 \left(k_2 + r_2\omega_c + m_2\omega_c^2\right) + k_1 \left(m_1 + m_2\right) \left(-2k_2 m_1 m_2 + \right. \\ & \left. m_1 r_2^2 + m_2 r_2^2\right) \left(k_2 m_1 + k_2 m_2 + m_1 r_2 \omega_c + m_2 r_2 \omega_c + m_1 m_2 \omega_c^2\right) + \\ & + k_2^2 \left(m_1 + m_2\right)^2 \left(k_2 m_1^2 + 2k_2 m_1 m_2 + k_1 m_2^2 + k_2 m_2^2 + \right. \\ & \left. m_1^2 r_2 \omega_c + 2m_1 m_2 r_2 \omega_c + m_2^2 r_2 \omega_c + m_1^2 m_2 \omega_c^2 + m_1 m_2^2 \omega_c^2\right) \\ d_{v1} = {} & 2m_2^2 r_2 \left(k_1 k_2 + k_1 r_2 \omega_c + k_2 m_1 \omega_c^2 + k_1 m_2 \omega_c^2 + k_2 m_2 \omega_c^2 + \right. \\ & \left. m_1 r_2 \omega_c^3 + m_2 r_2 \omega_c^3 + m_1 m_2 \omega_c^4\right) \end{aligned}$$

– Body acceleration

$$\sigma_{\ddot{z}_2} = \omega_c \sqrt{\pi\ v\ A_v} \sqrt{\frac{n_{v2}}{d_{v2}}} \tag{10.40}$$

$$\begin{aligned}
n_{v2} = & \left[k_1 r_2^2 \left(k_2 m_1 + k_2 m_2 + m_1 r_2 \omega_c + r_2 m_2 \omega_c + \right.\right. \\
& \left. m_1 m_2 \omega_c^2\right) + k_2^2 \left(k_2 m_1^2 + 2k_2 m_1 m_2 + k_1 m_2^2 + k_2 m_2^2 + \right. \\
& \left.\left. m_1^2 r_2 \omega_c + 2m_1 m_2 r_2 \omega_c + m_2^2 r_2 \omega_c + m_1^2 m_2 \omega_c^2 + m_2^2 m_1 \omega_c^2\right)\right] \\
d_{v2} = & \left[2m_2^2 r_2 \left(k_1 k_2 + k_1 r_2 \omega_c + k_2 m_1 \omega_c^2 + k_1 m_2 \omega_c^2 + k_2 m_2 \omega_c^2 + \right.\right. \\
& \left.\left. m_1 r_2 \omega_c^3 + m_2 r_2 \omega_c^3 + m_1 m_2 \omega_c^4\right)\right]
\end{aligned}$$

– Secondary stroke

$$\sigma_{z_2 - z_1} = \omega_c \sqrt{\pi\ v\ A_v} \sqrt{\frac{n_{v3}}{d_{v3}}} \tag{10.41}$$

$$\begin{aligned}
n_{v3} = & \ k_2 m_1^2 + 2k_2 m_1 m_2 + k_1 m_2^2 + k_2 m_2^2 + m_1^2 r_2 \omega_c + \\
& 2m_1 m_2 r_2 \omega_c + m_2^2 r_2 \omega_c + m_1^2 m_2 \omega_c^2 + m_1 m_2^2 \omega_c^2 \\
d_{v3} = & \ 2r_2 \left(k_1 k_2 + k_1 r_2 \omega_c + k_2 m_1 \omega_c^2 + k_1 m_2 \omega_c^2 + k_2 m_2 \omega_c^2 + \right. \\
& \left. m_1 r_2 \omega_c^3 + m_2 r_2 \omega_c^3 + m_1 m_2 \omega_c^4\right)
\end{aligned}$$

The primary stroke is obviously $\sigma_{z_1 - \xi} = \sigma_{F_z}/k_1$.

10.1.8 Simplified Formulae Using the 2S-PSD (Eq. (10.8))

Equations (10.39)–(10.41) can be simplified by neglecting those terms which vanish when parameter values refer to actual railway vehicles (see footnote 1).

– Force on the axle-box

$$\sigma_{F_z} = \omega_c \sqrt{\pi\ v\ A_v} \sqrt{\frac{n_{a1}}{d_{a1}}} \tag{10.42}$$

$$\begin{aligned}
n_{a1} = & \ k_1^2 m_1^2 \left(\beta\ k_1 + r_2 \omega_c + m_1 \mu\ \omega_c^2 + \frac{1}{k_1 m_1} \left(-2\beta\ k_1 m_1 + r_2^2\right) \cdot \right. \\
& \left. \left(\beta\ k_1 \mu + r_2 \omega_c \mu + m_1 \mu\ \omega_c^2\right)\right) \\
d_{a1} = & \ 2r_2 \left(\beta\ k_1^2 + k_1 r_2 \omega_c + m_1 k_1 \omega_c^2 \mu + r_2 m_1 \omega_c^3 \mu + m_1^2 \mu\ \omega_c^4\right)
\end{aligned}$$

– Body acceleration

$$\sigma_{\ddot{z}_2} = \omega_c \sqrt{\pi\ v\ A_v} \cdot \sqrt{\frac{\frac{k_1 r_2}{m_1 \mu} \left(\beta\ k_1 + r_2 \omega_c + m_1 \omega_c^2\right) + \beta^2 \frac{k_1^2}{r_2} \left(k_1 + r_2 \omega_c + m_1 \omega_c^2\right)}{2\left(\beta\ k_1^2 + k_1 r_2 \omega_c + m_1 k_1 \omega_c^2 \mu + r_2 m_1 \omega_c^3 \mu + m_1^2 \mu\ \omega_c^4\right)}} \tag{10.43}$$

– Secondary stroke

$$\sigma_{z_2-z_1} = \omega_c \sqrt{\pi \ v \ A_v} \cdot \sqrt{\frac{m_1^2 \mu^2 \left(k_1 + r_2 \omega_c + m_1 \omega_c^2\right)}{2 r_2 \left(\beta \ k_1^2 + k_1 r_2 \omega_c + m_1 k_1 \omega_c^2 \mu + r_2 m_1 \omega_c^3 \mu + m_1^2 \mu \ \omega_c^4\right)}} \tag{10.44}$$

The primary stroke is obviously $\sigma_{z_1-\xi} = \sigma_{F_z}/k_1$.

A comparison has been made between the $\sigma_{\ddot{z}_2}$, $\sigma_{z_2-z_1}$, σ_{F_z} computed by Eqs. (10.39)–(10.41) and the $\sigma_{\ddot{z}_2}$, $\sigma_{z_2-z_1}$, σ_{F_z} computed by Eqs. (10.42)–(10.44) (vehicle parameters in Table 10.1.). In the vehicle speed range 20–100 m/s, the error varies from −1.5 to 0.3% for the force on the axle-box (Eq. (10.42)), from −2.2 to 1.2% for the body vertical acceleration (Eq. (10.43)), from −0.2 to −1.4 for the secondary stroke (Eq. (10.44)).

10.2 Validation

In order to validate the simple model described in Sect. 10.1, a comparison with the data presented in [76] is performed. In [76] a model for the study of the vertical dynamics of railway vehicles has been proposed. The model is rather complex as it accounts for the heave, pitch and roll of body and bogies. Vehicle body bending and torsional modes of vibration have been also included. From [76] it is argued that the output model responses were validated experimentally with satisfactory results. The data of the vehicle studied in [76] are reported in Table 10.1.

In [76] the adopted PSD of the stochastic process defining track irregularity in the vertical plane (2S-PSD, see Eq. (10.8)) is

$$P_{SD\xi}\left(\omega\right) = \frac{A_v \omega_c^2 v}{\omega^2 \left(\omega^2 + \omega_c^2\right)} = \frac{A_v \Omega_c^2 v^3}{\omega^2 \left(\omega^2 + v^2 \Omega_c^2\right)} \tag{10.45}$$

where A_v=0.035 cm^2 rad/m is the track quality coefficient and $\Omega_C = 0.99$ rad/m is the break wave number. $P_{SD\xi}\ (\omega)$ is plotted in Fig. 10.2.

In [76] an in-depth sensitivity analysis was performed referring to the standard deviations of body acceleration and of secondary stroke. Unfortunately, no data are available on primary stroke and force on the axle-box.

The following validation has been performed by considering Eqs. (10.36) and (10.37) which refer to the model shown in Fig. 10.1 and described in Sect. 10.1. The adopted PSD of the track irregularity is given by Eq. (10.45). A steady speed v equal to 177 km/h has been considered.

10.2.1 Primary Stiffness

In [76] the primary suspension stiffness k_1 was at first increased by a factor 4, then decreased by the same factor relatively to the reference value

Table 10.2. Comparison between computed results and data referring to an actual vehicle (adapted from [76])

k_1	$\sigma_{\ddot{z}_2}$(m/s^2), from [76]	$\sigma_{\ddot{z}_2}$(m/s^2), Eq. (10.36)	$\sigma_{z_2-z_1}$(mm), from [76]	$\sigma_{z_2-z_1}$(mm), Eq. (10.37)
$k_{1r}/4$	0.56	0.62	9.6	9.8
k_{1r}	0.72	0.68	9.5	9.6
$4k_{1r}$	0.67	0.69	9.5	9.5

Variation of the primary stiffness k_1 (k_{1r} is reported in Table 10.1)

(Table 10.1). In Table 10.2 the values of standard deviations $\sigma_{\ddot{z}_2}$ and $\sigma_{z_2-z_1}$ computed by means of Eqs. (10.36) and (10.37) are compared with the corresponding values reported in [76]. Body acceleration is substantially unaffected by primary stiffness variation: The error given by the model with respect to the corresponding data in [76] is always less than 10%. For the secondary stroke the error is always less than 2%.

10.2.2 Natural Frequency

The natural frequencies of the body rigid modes (heave, pitch and roll) were varied in [76] without changing their damping ratios. This was achieved by suitable variations in the secondary suspension stiffness, k_2 and damping r_2. The effect of halving and doubling the reference natural frequencies was investigated. The results are reported in Table 10.3. The effect is relevant both on body acceleration and on secondary suspension stroke. Again the simple model is able to give the responses of the reference vehicle with a limited error.

10.2.3 Damping Ratio

The effect of the secondary damping on body acceleration and secondary stroke was studied by varying the coefficient r_2, such that body's heave damping ratio $\varsigma = r_2/\left(2\sqrt{k_2 m_2}\right)$ was increased from the reference value (0.25)

Table 10.3. Effect of the vehicle body natural frequency

Natural frequency	f_h (Hz)	k_2 (N/m)	r_2 (Ns/m)	$\sigma_{\ddot{z}_2}$(m/s^2), from [76]	$\sigma_{\ddot{z}_2}$(m/s^2), Eq. (10.36)	$\sigma_{z_2-z_1}$(mm), from [76]	$\sigma_{z_2-z_1}$(mm), Eq. (10.37)
Halved	0.445	162,000/4	14,600/2	0.39	0.41	15.1	13.8
Base	0.885	162,000	14,600	0.72	0.68	9.5	9.6
Double	1.769	162,000·4	14,600·2	1.17	1.27	5.3	6.6

Comparison between computed results and data referring to an actual vehicle (adapted from [76])

Table 10.4. Comparison between computed results and data referring to an actual vehicle (adapted from [76])

Damping ratio	ς	r_2 (Ns/m)	$\sigma_{\ddot{z}_2}$ (m/s^2), from [76]	$\sigma_{\ddot{z}_2}$ (m/s^2), Eq. (10.33)	$\sigma_{z_2-z_1}$ (mm), from [76]	$\sigma_{z_2-z_1}$ (mm), Eq. (10.34)
r_2	0.250	14,600	0.72	0.68	9.5	9.6
$r_2 \cdot 1.5$	0.375	14,600 · 1.5	0.85	0.81	7.5	7.7
$r_2 \cdot 2$	0.50	14,600 · 2	1.02	0.96	6.4	6.6
$r_2 \cdot 2\sqrt{2}$	0.707	14,600 · $2\sqrt{2}$	1.20	1.19	5.4	5.5

Variation of the secondary suspension damping r_2 (r_{2r} is reported in Table 10.1)

to 0.375, 0.50 and 0.707 (Table 10.4) (ς is defined as if k_1 were infinity). Referring to the vertical body acceleration, the error given by the model is always less than 6%. For the secondary stroke the error is always less than 3%.

10.3 Parameter Sensitivity Analysis

The dynamic response of the railway vehicle system model in Fig. 10.1 is analysed on the basis of Eqs. (10.29)–(10.31) and Eqs. (10.39)–(10.41). The same analysis (not reported here for sake of space) has been performed by means of Eqs. (10.32)–(10.34) and Eqs. (10.42)–(10.44). The discrepancies were negligible.

A typical railway passenger vehicle for intercity service is taken into consideration (Table 10.1). The results of the parameter sensitivity analysis are shown in Figs. 10.3–10.5. The parameters are varied within wide ranges. The data are presented in non-dimensional form, i.e. the standard deviation of interest σ is divided by the corresponding one (σ_r) computed by considering the parameters at their reference values (see Table 10.1)

$$\sigma_{F_z r} = \sigma_{F_z}(m_{1r}, m_{2r}, r_{1r}, r_{2r}, k_{1r}, k_{2r}) \tag{10.46}$$

$$\sigma_{\ddot{z}_2 r} = \sigma_{\ddot{z}_2}(m_{1r}, m_{2r}, r_{1r}, r_{2r}, k_{1r}, k_{2r}) \tag{10.47}$$

$$\sigma_{z_2 - z_1 r} = \sigma_{z_2 - z_1}(m_{1r}, m_{2r}, r_{1r}, r_{2r}, k_{1r}, k_{2r}) \tag{10.48}$$

The non-dimensional standard deviations derived from Eqs. (10.29)–(10.31) do not depend on vehicle speed. The opposite occurs for the non-dimensional standard deviations derived from Eqs. (10.39)–(10.41) (referring to 2S-PSD, Eq. (10.8)). For this reason the last non-dimensional standard deviations are analysed at two different vehicle speeds, low speed (10 m/s) and high speed (100 m/s).

10.3.1 Standard Deviation of Force on Axle-box

Figure 10.3 shows that

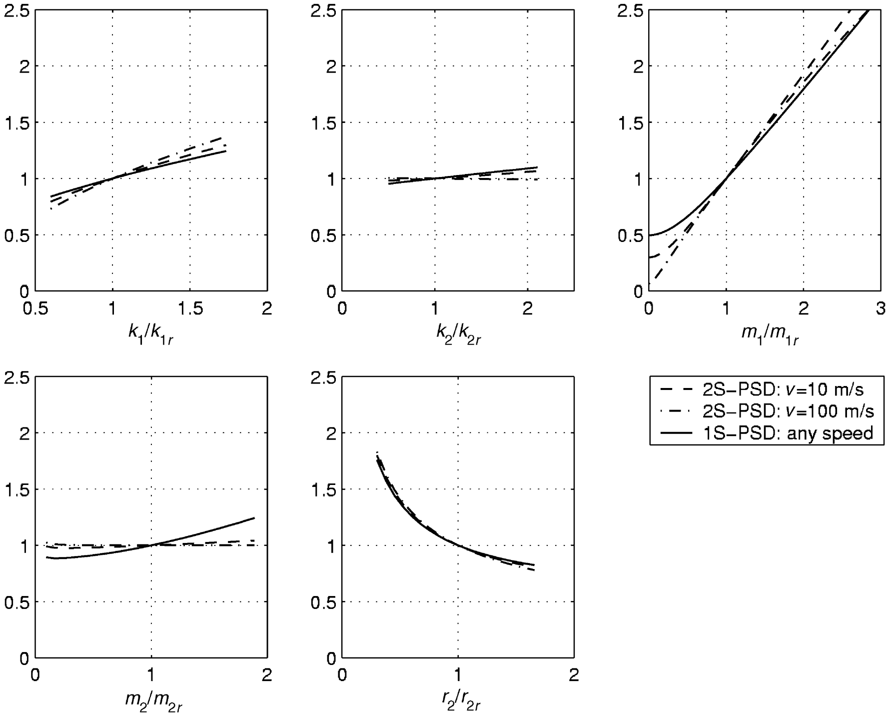

Fig. 10.3. $\sigma_{F_z}/\sigma_{F_z r}$, non-dimensional standard deviation of force on the axle-box as function of model parameters. Data of the reference vehicle in Table 10.1. Each diagram has been obtained by varying one single parameter, the other ones being constant and equal to those of the reference vehicle

- σ_{F_z} depends almost linearly on the primary suspension stiffness k_1,
- σ_{F_z} does not depend significantly on the secondary suspension stiffness k_2,
- σ_{F_z} depends almost linearly on m_1 (non-linear at start),
- σ_{F_z} does not depend significantly on m_2 if the excitation is given by Eq. (10.8) (2S-PSD),
- the secondary suspension damping r_2 has an important influence on the standard deviation σ_{F_z}.

Some of the above considerations can be derived by a simple inspection of Eqs. (10.32) and (10.42).

10.3.2 Standard Deviation of Body Acceleration

By inspection of Fig. 10.4 one may notice that

- k_1 does not influence significantly $\sigma_{\ddot{z}_2}$,
- m_1 does not influence significantly $\sigma_{\ddot{z}_2}$ if the excitation is given by Eq. (10.7) (1S-PSD),

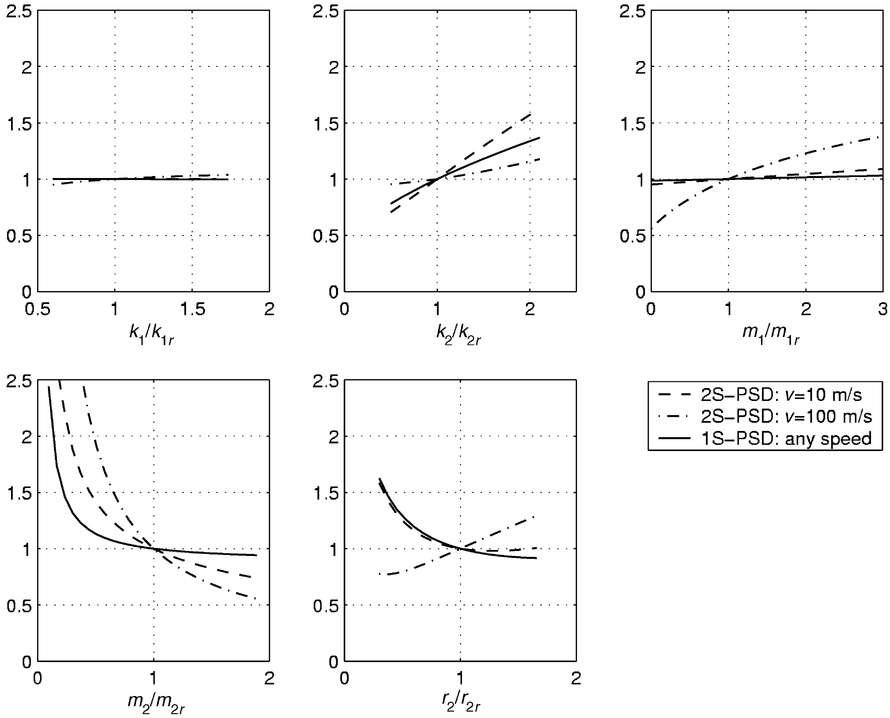

Fig. 10.4. $\sigma_{\ddot{z}_2}/\sigma_{\ddot{z}_{2r}}$, non-dimensional standard deviation of body acceleration as function of model parameters. Data of the reference vehicle in Table 10.1. Each diagram has been obtained by varying one single parameter, the other ones being constant and equal to those of the reference vehicle

- $\sigma_{\ddot{z}_2}$ may decrease strongly as m_2 increases,
- the parameters of secondary suspension (k_2, r_2) have a remarkable influence on the $\sigma_{\ddot{z}_2}$. For a 2S-PSD excitation the effect of an increase of the secondary suspension damping r_2 is positive at low speed but negative at high speed. For a 1S-PSD excitation the effect of an increase of the secondary suspension damping r_2 is positive at any speed.

Some of the above considerations can be derived by a simple inspection of Eqs. (10.33) and (10.43).

10.3.3 Standard Deviation of Secondary Stroke

By inspection of Fig. 10.5, one may notice that

- $\sigma_{z_2-z_1}$ is not influenced by primary suspension stiffness k_1,
- the stiffness of the secondary suspension k_2 has a remarkable influence on the $\sigma_{z_2-z_1}$. For a 2S-PSD excitation the effect of a variation of k_2 is less relevant increasing the vehicle speed,

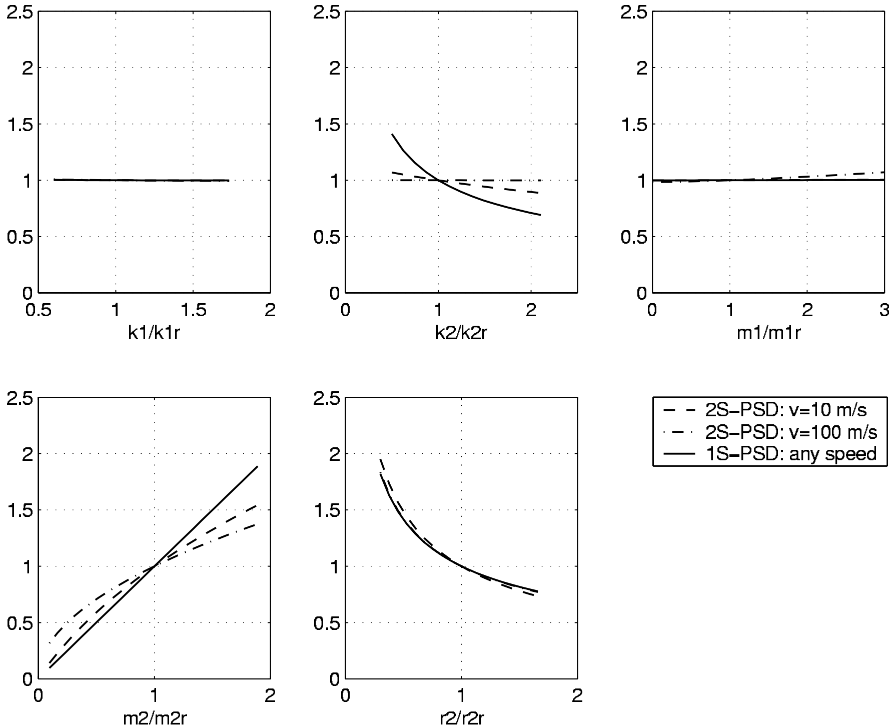

Fig. 10.5. $\sigma_{z_2-z_1}/\sigma_{z_2-z_1r}$, non–dimensional standard deviation of secondary stroke as function of model parameters. Data of the reference vehicle in Table 10.1. Each diagram has been obtained by varying one single parameter, the other ones being constant and equal to those of the reference vehicle.

– $\sigma_{z_2-z_1}$ is not influenced significantly by m_1,
– $\sigma_{z_2-z_1}$ depends almost linearly on m_2 considering a 1S-PSD excitation. The relationship is non-linear considering the 2S-PSD excitation,
– $\sigma_{z_2-z_1}$ is influenced remarkably by secondary suspension damping r_2.

Some of the above considerations can be derived by a simple inspection of Eqs. (10.34) and (10.44).

10.3.4 Optimal Secondary Suspension Design Variables

The constraints method introduced in Sect. 3.7.1 has been used to optimise the design variables of the secondary suspension of a railway vehicle described by the simple system model in Fig. 10.1. The design variables to be optimised were the stiffness k_2 and the damping r_2 of the secondary suspension, the objective functions were $\sigma_{\ddot{z}_2}$and $\sigma_{z_2-z_1}$.

Derivation of Optimal $\sigma_{\ddot{z}_2}$, $\sigma_{z_2-z_1}$ and Optimal k_2, r_2 Using 1S-PSD

The objective functions are defined by Eqs. (10.33) and (10.34). The optimisation procedure based on constraints method and described in Sect. 3.7.1 is applied as follows:

a) from Eq. (10.34) the expression of r_2 as function of $\sigma_{z_2-z_1}$ and k_2 is derived

$$r_2 = \frac{B^2 m_2^2}{k_2 \cdot \sigma_{z_2-z_1}^2} \tag{10.49}$$

b) the expression of $\sigma_{\ddot{z}_2}$ as function of r_2 and $\sigma_{z_2-z_1}$ by substituting the expression of r_2 (Eq. (10.49)) in Eq. (10.33)

$$\sigma_{\ddot{z}_2} = B\sqrt{\frac{B^2 m_2}{k_2 \sigma_{z_2-z_1}^2} + \frac{k_2^2 \sigma_{z_2-z_1}^2}{B^2 m_2^2}} = B\sqrt{(\cdot)} \tag{10.50}$$

c) the following derivative equal to zero gives the stationary solution

$$\frac{d\sigma_{\ddot{z}_2}}{dk_2} = B\frac{d\sqrt{(\cdot)}}{dk_2} = B\frac{1}{2\sqrt{(\cdot)}}\frac{d(\cdot)}{dk_2} = \quad 0 \tag{10.51}$$

the term $\frac{1}{2\sqrt{(\cdot)}}$ is always greater than zero, so solving with respect to k_2

$$-\frac{B^2 m_2}{\sigma_{z_2-z_1}^2 k_2^2} + 2\frac{k_2}{B^2 m_2^2}\sigma_{z_2-z_1}^2 = 0 \tag{10.52}$$

thus

$$k_2 = \sqrt[3]{\frac{B^4 m_2^3}{2\,\sigma_{z_2-z_1}^4}} \tag{10.53}$$

d) finally, by substituting Eq. (10.53) in Eq. (10.49) and Eq. (10.33), the expression of $\sigma_{\ddot{z}_2}$ as function of $\sigma_{z_2-z_1}$ is obtained

$$\sigma_{\ddot{z}_2} = \sqrt[6]{\frac{27}{4}\frac{B^8}{\sigma_{z_2-z_1}^2}} \tag{10.54}$$

this equation defines the relationship between the standard deviation of the acceleration of the body and the standard deviation of the secondary suspension stroke when both the standard deviations are minimised.

e) The equation which defines the optimal design variable set is

$$r_2 = r_{2ott} = \sqrt{2k_2 m_2} \tag{10.55}$$

For the system composed by the mass m_2 , the damper r_2, and the spring k_2 (mass m_1 fixed), the critical damping may be defined as

$$r_{2crit} = \sqrt{4k_2m_2} \tag{10.56}$$

it follows

$$r_{2ott} = \frac{1}{\sqrt{2}} \cdot r_{2crit} \tag{10.57}$$

By setting the stiffness and the damping of the secondary suspension as indicated above, the best compromise between the standard deviation of the body acceleration and the standard deviation of the secondary stroke is obtained.

Derivation of Optimal $\sigma_{\ddot{z}_2}$, $\sigma_{z_2-z_1}$ and Optimal k_2, r_2 Using 2S-PS

On the basis of Eqs. (10.40) and (10.41), a numerical search has been undertaken to find both the optimal set $\sigma_{\ddot{z}_2}$,$\sigma_{z_2-z_1}$ and the optimal set k_2, r_2. The corresponding plots are reported in Figs. 10.6 and 10.7, respectively.

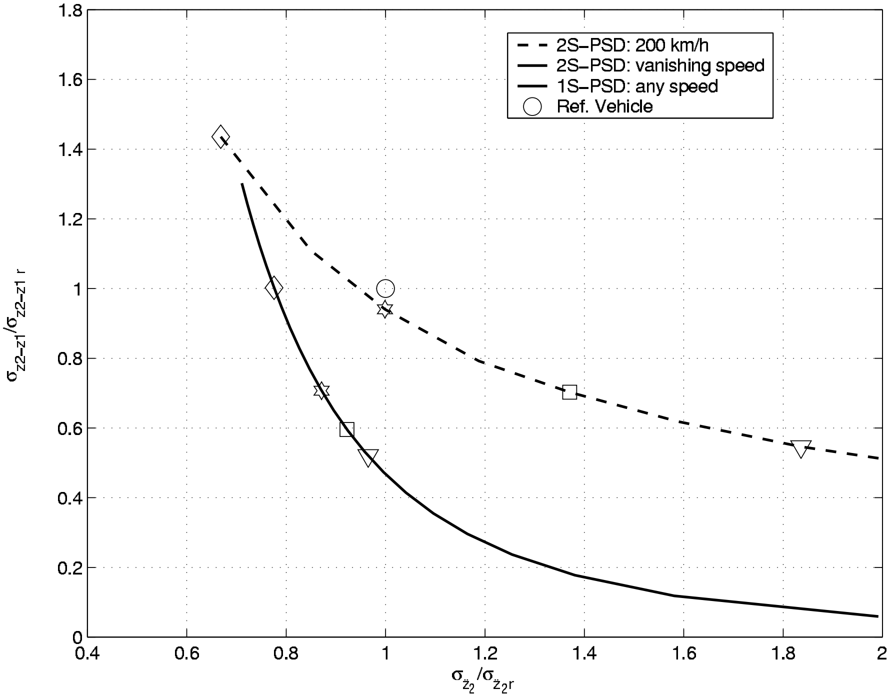

Fig. 10.6. Optimal $\sigma_{\ddot{z}_2}$ and optimal $\sigma_{z_2-z_1}$ plotted in non-dimensional form. The curves are obtained by varying k_2 and r_2, the points highlighted by using special symbols (triangle, square, ...) refer to the points in Fig. 10.7. Vehicle parameters in Table 10.1.

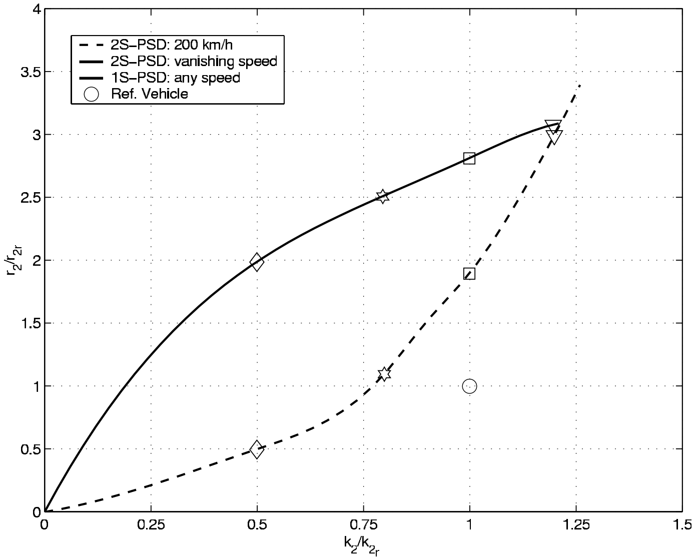

Fig. 10.7. Optimal k_2 and optimal r_2 plotted in non-dimensional form for minimising $\sigma_{\ddot{z}_2}$ and $\sigma_{z_2-z_1}$, the points highlighted by using special symbols (triangle, square, ...) refer to the points in Fig. 10.6. Vehicle parameters in Table 10.1.

Optimal $\sigma_{\ddot{z}_2}$, $\sigma_{z_2-z_1}$ and Optimal k_2, r_2

Both for the 1S-PSD (Eqs. (10.54), (10.55)) and for the 2S-PSD excitations, optimal $\sigma_{\ddot{z}_2}$, $\sigma_{z_2-z_1}$and optimal k_2, r_2 are plotted in non-dimensional form in Figs. 10.6 and 10.7, respectively. To obtain non-dimensional values, reference is made to a railway passenger vehicle whose relevant parameters are reported in Table 10.1. $\sigma_{\ddot{z}_2}$ increases when $\sigma_{z_2-z_1}$ decreases, i.e. these two standard deviations are conflicting. The designer should choose, on the basis of given technical specifications, the desired compromise between $\sigma_{\ddot{z}_2}$ and $\sigma_{z_2-z_1}$ by selecting one point lying on the curves plotted in Fig. 10.6, e.g. one of those marked with special symbols (triangle, square, ...). Having chosen the preferred compromise between $\sigma_{\ddot{z}_2}$and $\sigma_{z_2-z_1}$, the corresponding values of the design variables k_2, r_2 are uniquely defined. This correspondence between the points of the curves plotted in the $\sigma_{\ddot{z}_2}$, $\sigma_{z_2-z_1}$ plane (Fig. 10.6) and the points of the curves in the k_2, r_2 plane (Fig. 10.7) are highlighted by special symbols in Figs. 10.6 and 10.7. By inspection of Fig. 10.6 one may notice that for the 1S-PSD the non-dimensional $\sigma_{\ddot{z}_2}$ and $\sigma_{z_2-z_1}$ do not depend on vehicle speed. This is consequence of the fact that for this excitation spectrum, the speed parameter v is not mixed with those of the vehicle system k_2, r_2, ... (see Sect. 10.1). On the contrary, for the 2S-PSD, which is very frequently found in actual applications, the non-dimensional $\sigma_{\ddot{z}_2}$and $\sigma_{z_2-z_1}$ do depend on vehicle speed (see Sect. 10.1). This suggests that vehicle suspension design variables should vary with vehicle speed in order to keep the optimality conditions. This is technically easily achievable and hopefully in the future adaptive suspensions could be adopted for railway vehicles.

In Fig. 10.7 the relationship between optimal stiffness k_2 and optimal damping r_2 is highlighted. To keep the best compromise between $\sigma_{\ddot{z}_2}$ and $\sigma_{z_2-z_1}$, the damping r_2 has to increase with the stiffness k_2, both for the 1S-PSD and for the 2S-PSD excitation. The rate of change of r_2 with respect to k_2 does not depend on vehicle speed for the 1S-PSD and varies considerably with the vehicle speed for the 2S-PSD excitation.

10.4 Conclusion

Analytical formulae have been derived in order to estimate the response of railway vehicles to random excitations generated by the vertical track irregularity. The accuracy of the derived formulae has been assessed by comparison with data presented in the literature. The analytical formulae should estimate with reasonable accuracy the dynamic behaviour of an actual railway vehicle running on rigid track. Referring to the performed validation, the sensitivity of body acceleration and secondary stroke to vehicle suspension design variables (primary and secondary stiffness, secondary damping) is captured satisfactorily by the analytical formulae. It has been found that analytical formulae (in complete form) predicted the standard deviations of both body acceleration and secondary stroke with an error always less than 10%, and often less than 2%. On the basis of the validated analytical formulae, a theoretical parameter sensitivity analysis has been performed with reference to the standard deviations of force on the axle-box, of body acceleration, of secondary suspension stroke. All these standard deviations are influenced by secondary suspension design variables. Particularly, the secondary damping affects significantly the body acceleration. A general result (confirmed by common experience) is the strong influence of the type of track irregularity on all the considered standard deviations. The bogie mass and the primary stiffness do not seem to influence considerably the secondary stroke.

By using the derived analytical formulae in the second part of the chapter, the constraints method has been applied to find the best trade-off between conflicting objective functions such as $\sigma_{\ddot{z}_2}$ (standard deviation of the body acceleration) and $\sigma_{z_2-z_1}$(the standard deviation of the secondary stroke). The design variables of the secondary suspension (stiffness k_2 and damping r_2) of a railway vehicle have been optimised with the aim of minimising both $\sigma_{\ddot{z}_2}$ and $\sigma_{z_2-z_1}$. Simple analytical formulae have been derived for the optimal $\sigma_{\ddot{z}_2}$,$\sigma_{z_2-z_1}$ and correspondingly optimal k_2, r_2. Optimal $\sigma_{\ddot{z}_2}$ increases when both optimal k_2 and optimal r_2 increase, the opposite occurs for optimal $\sigma_{z_2-z_1}$. If the excitation is defined by the 1S-PSD the optimal secondary suspension settings do not depend on vehicle speed. The opposite occurs for the more realistic 2S-PSD excitation, thus it seems reasonable to recommend, for future research, comprehensive studies on the application of adaptive stiffness and damping elements to railway vehicle secondary suspension systems.