6 Heat Transfer Fundamentals

In this chapter, the basics of heat conduction and transfer are discussed. The chapter contains 29 exercises, which illustrate Fourier Law, the solving of heat transfer coefficients for multi-layered flat and cylindrical partitions, the determination of a quasi-steady-state temperature field and the computation of a radiant tube temperature in boilers. Critical thickness of thermal insulation on the surface of the cylindrical tube is first determined analytically, then calculated. The methods for solving selected inverse steady-state heat conduction problems, which occur during heat flux measurement carried out by means of different types of sensors, are presented here. A great deal of attention is paid to the determination of temperature distribution and the efficiency of simple, circular, rectangular and hexagonal fins. The calculation results of efficiency in complex-shape fins, determined by means of an equivalent circular fin method and segment method, are compared with the results obtained from FEM. Examples that illustrate the computation of a heat transfer coefficient in pipes finned longitudinally and crosswise are presented here as well. Three exercises deal with the way steady-state temperature distribution is determined using control volume method. These exercises present the methods for solving problems and the computational programs used. In the last exercise of this chapter, temperature distribution and circular fin efficiency is determined under the assumption that thermal conductivity of the fin's material is temperature dependent. The problem is reduced to a two-point boundary problem for the system of two ordinary differential equations.

Exercise 6.1 Fourier Law in a Cylindrical Coordinate System

A radiant tube with an outer diameter d_z = 32 mm, wall thickness g = 5 mm and length L = 20 m is made of a steel with a thermal conductivity λ = 47 W/(m·K) (Fig. 6.1). Water-vapour mixture, heated by combustions gases that surround the tube on the outside, flows inside the tube. Inner surface temperature is T_w = 200°C, while the outer surface temperature is T_z = 250°C. The aim is to compute the heat flow transferred from the

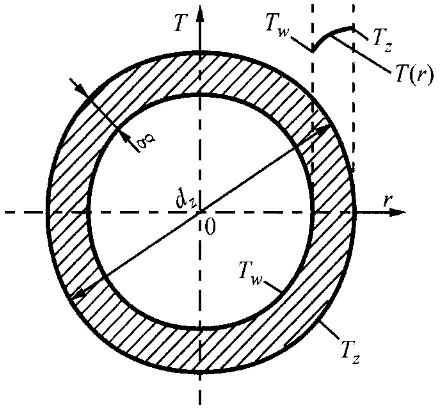

Fig. 6.1. Tube cross-section

combustion gases to the water-vapour mixture and the heat flux on the inner and outer surface.

Solution

Heat flux across the tube's wall thickness is formulated using Fourier Law

$$\dot{q}(r) = \lambda \frac{dT}{dr} . \tag{1}$$

Heat flow conducted through a flat wall can be written in the form

$$\dot{Q} = A(r)\dot{q} = A(r)\lambda \frac{dT}{dr} , \tag{2}$$

where $A(r) = 2\pi rL$.

The separation of variables in (2), gives

$$dT = \frac{\dot{Q}}{2\pi L\lambda} \frac{dr}{r} . \tag{3}$$

On the basis of known inner and outer surface temperature, one can write the boundary conditions as

$$T(r_z) = T_z , \qquad T(r_w) = T_w . \tag{4}$$

Once (3) is integrated, temperature distribution across the wall thickness of the tube is obtained

$$T = \frac{\dot{Q}}{2\pi L\lambda} \ln|r| + C, \tag{5}$$

where $\dot{Q}$ and C are constants computed from boundary conditions (4). Unknown value $\dot{Q}$ is equal to

$$\dot{Q} = \frac{(T_z - T_w) \cdot 2\pi L\lambda}{\ln(r_z / r_w)}.$$

Substituting of the data gives

$$\dot{Q} = \frac{2\pi \cdot 20 \cdot 47 \cdot (250 - 200)}{\ln\left(\frac{0.016}{0.011}\right)} = 788100\,\text{W}.$$

Inner surface heat flux is

$$\dot{q}_w = \frac{\dot{Q}}{2\pi r_w L} = \frac{788100}{2\pi \cdot 0.011 \cdot 20} = 5.702 \cdot 10^5\ \text{W/m}^2.$$

Outer surface heat flux is

$$\dot{q}_z = \frac{\dot{Q}}{2\pi r_z L} = \frac{788100}{2\pi \cdot 0.016 \cdot 20} = 3.92 \cdot 10^5\ \text{W/m}^2.$$

Exercise 6.2 The Equivalent Heat Transfer Coefficient Accounting for Heat Exchange by Convection and Radiation

A non-insulated tube (Fig. 6.2) with a nominal diameter d_n = 38 mm (1½″), and the following measurements: d_z = 38 mm, wall thickness g = 2.6 mm, length L = 5; the tube is kept in a room whose temperature is T_{ot} = 20°C. Water with temperature 80°C flows inside the tube. The tube's outer surface emissivity is ε = 0.8. Lets assume that outer surface temperature is identical to the temperature of a flowing medium inside and that heat transfer coefficient by means of convection is formulated as

$$\alpha_k = 5.0\sqrt[4]{\frac{T_z - T_{ot}}{T_{ot} d_z}} \text{ or } \alpha_k = 1.21\sqrt[4]{\frac{\Delta T}{d_z}},$$

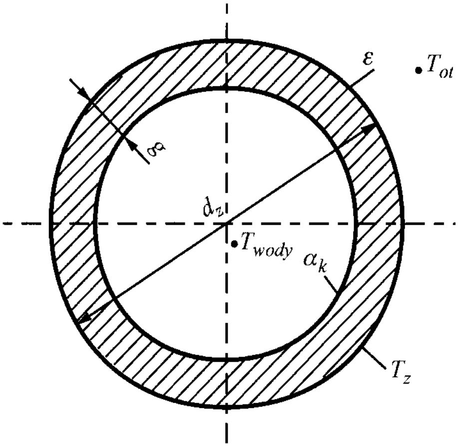

Fig. 6.2. Tube cross-section

where, d_z is the tube's outer diameter in meters, while T_{ot} is the temperature of surroundings in Kelvin [9]. The aim is to calculate heat loss $\dot{Q}$, which is related to heat transfer from hot water to surroundings by convection and radiation and to determine the equivalent heat transfer coefficient accounting for to convection and radiation.

Solution

Heat loss $\dot{Q}$ is the sum of losses from convection and radiation heat exchange

$$\dot{Q} = \dot{Q}_k + \dot{Q}_r .$$

Heat flow transferred by convection is

$$\dot{Q}_k = A_z \alpha_k \left(T_z - T_{ot}\right),$$

where $A_z = \pi d_z L$ is an outer surface of the tube.

By substituting the data, one obtains

$$\dot{Q}_k = \pi \cdot 0.038 \cdot 5 \cdot 7.627 \cdot \left(353.15 - 293.15\right) = 273.2 \text{ W} .$$

Heat flow transferred by radiation is given by

$$\dot{Q}_r = A_z \varepsilon \sigma \left(T_z^4 - T_{ot}^4\right),$$

where $\sigma = 5.67 \cdot 10^{-8}$ W/(m^2·K^4) is the Stefan-Boltzmann constant.

Thus

$$\dot{Q}_r = \pi \cdot 0{,}038 \cdot 5 \cdot 0.8 \cdot 5.67 \cdot 10^{-8} \cdot \left(353.15^4 - 293.15^4\right) = 221.17 \text{ W}.$$

Total heat loss amounts to

$$\dot{Q} = A_z \left(T_z - T_{ot}\right)\left[\alpha_k + \varepsilon\sigma\left(T_z^2 + T_{ot}^2\right)\left(T_z + T_{ot}\right)\right].$$

Total heat coefficient can be expressed as

$$\alpha_s = \alpha_k + \varepsilon\sigma\left(T_z^2 + T_{ot}^2\right)\left(T_z + T_{ot}\right).$$

Substitution of the numerical values yields

$$\alpha_s = 7.63 + 0.8 \cdot 5.67 \cdot 10^{-8}\left(353.15^2 + 293.15^2\right)\left(353.15 + 293.15\right) =$$

$$= 13.803 \text{ W/(m}^2 \cdot \text{K)}.$$

Exercise 6.3 Heat Transfer Through a Flat Single-Layered and Double-Layered Wall

A flat wall (Fig. 6.3) with a thickness of g = 0.4 m and surface area A = 15.6 m^2 is made of a material whose thermal conductivity equals λ = 1 W/(m·K). Air temperature in front of the wall is T_1 = 20°C, behind the wall T_2 = –20°C. Heat transfer coefficients for both wall surfaces are, correspondingly, α_1 = 5 W/(m^2·K) and α_2 = 15 W/(m^2·K). The aim is to calculate heat transfer coefficient, heat flux and heat flow transferred through the wall and the surface temperature of the wall. The question is how the heat flow transferred by the wall will be changed, if the wall is thermally insulated on its outer side by a layer of foamed polystyrene, which is 10 cm thick (g_{iz} = 10 cm) and whose thermal conductivity is λ_{iz} = 0.04 W/(m·K)? The second aim is to calculate surface temperature of the wall and the foamed polystyrene.

Solution

a) Non-insulated wall (Fig. 6.4)
Heat transfer coefficient through the flat wall :

$$k = \frac{1}{\dfrac{1}{\alpha_1} + \dfrac{g}{\lambda} + \dfrac{1}{\alpha_2}}.$$

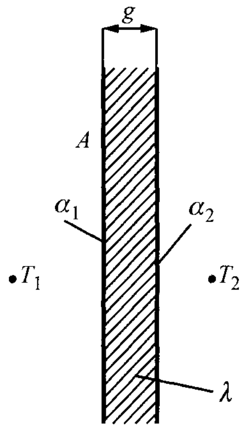

Fig. 6.3. Flat wall

Thus

$$k = \frac{1}{\frac{1}{5} + \frac{0.4}{1} + \frac{1}{15}} = 1.5\ \text{W/(m}^2\cdot\text{K)}.$$

Heat flux transferred by the flat wall can be determined from the following formula:

$$\dot{q} = k\left(T_1 - T_2\right),$$

$$\dot{q} = 1.5 \cdot \left(20 + 20\right) = 60\ \text{W/m}^2$$

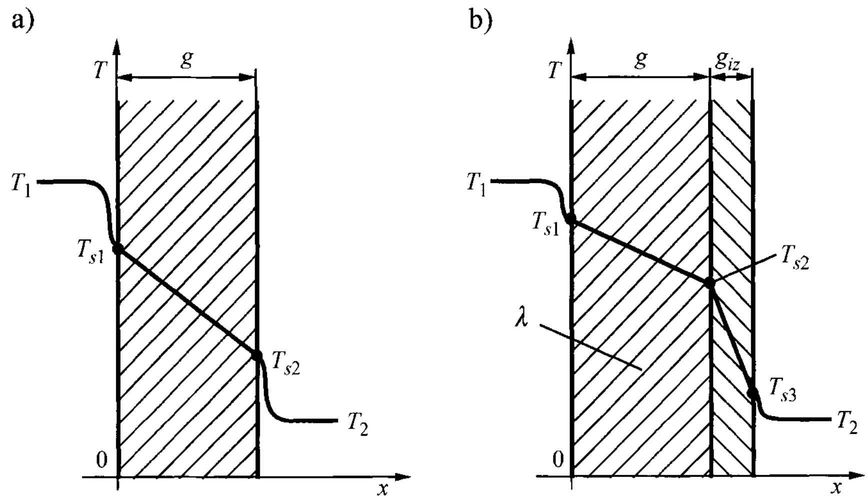

Fig. 6.4. Temperature distribution: (**a**) non-insulated wall, (**b**) insulated wall

Heat flow conducted by the flat wall can be determined as follows:

$$\dot{Q} = A \cdot k(T_1 - T_2) = 15.6 \cdot 1.5 \cdot (20 + 20) = 936 \text{ W}.$$

Temperature T_{s1} is (Fig. 6.4)

$$T_{s1} = T_1 - \frac{\dot{q}}{\alpha_1} = 20 - \frac{60}{5} = 20 - 12 = 8^\circ\text{C}.$$

Temperature T_{s2} is calculated using similar expression

$$T_{s2} = T_2 + \frac{\dot{q}}{\alpha_2} = -20 + \frac{60}{15} = -16^\circ\text{C}.$$

b) Thermally insulated wall
Heat transfer coefficient

$$k = \frac{1}{\frac{1}{\alpha_1} + \frac{g}{\lambda} + \frac{g_{iz}}{\lambda_{iz}} + \frac{1}{\alpha_2}} = \frac{1}{\frac{1}{5} + \frac{0.4}{1} + \frac{0.1}{0.04} + \frac{1}{15}} = 0.3158 \text{ W/(m}^2\text{·K)}.$$

Heat flux transferred by the wall is computed in the following way:

$$\dot{q} = k(T_1 - T_2) = 0.3158 \left[20 - (-20)\right] = 12.632 \text{ W/m}^2.$$

Heat flow transferred by an insulated wall is:

$$\dot{Q} = A\dot{q} = Ak(T_1 - T_2) = 15.6 \cdot 12.632 = 197.06 \text{ W}.$$

Surface temperature T_{s1} is:

$$T_{s1} = T_1 - \frac{\dot{q}}{\alpha_1} = 20 - \frac{12.632}{5} = 17.47\,^\circ\text{C}.$$

Temperature T_{s2} is calculated by subtracting a temperature drop across the wall thickness from temperature T_{s1}

$$T_{s2} = T_{s1} - \frac{\dot{q}g}{\lambda} = 17.47 - \frac{12.632 \cdot 0.4}{1} = 12.42^\circ\text{C}.$$

Temperature T_{s3} is calculated as follows:

$$T_{s3} = T_2 + \frac{\dot{q}}{\alpha_2} = -20 + \frac{12.632}{15} = -19.16^\circ\text{C}.$$

One can observe that insulating a wall with a foamed polystyrene has a significant effect on the heat flow transferred by the wall and on the wall's temperature. In the case when there is a lack of insulation, heat flow $Q^{\cdot}$ is

$n = 936/197.06 = 4.75$ times larger than when the wall is insulated. When insulation is applied, the wall does not freeze, since its outer surface temperature increases from $T_{s2} = -16°C$ to $T_{s2} = 12.42°C$.

From the conducted analysis, one can deduce that buildings should be insulated on the outside surface, since the temperature of the walls remains then positive (is above zero).

Exercise 6.4 Overall Heat Transfer Coefficient and Heat Loss Through a Pipeline Wall

Pipeline (Fig. 6.5) with an outer diameter $d_z = 273$ mm, wall thickness $g = 16$ mm and length $L = 70$ m is made of a material whose thermal conductivity is $\lambda = 45$ W/(m·K). The pipeline is thermally insulated by a layer, which is 10 cm thick ($g_{iz} = 10$ cm) and made of a material with $\lambda_{iz} = 0.08$ W/(m·K). A medium with temperature $T_w = 400°C$ flows inside the pipeline, while on the inner surface a heat transfer coefficient is $\alpha_w = 500$ W/(m^2·K). Air temperature, which surrounds the pipeline on the outside, is $T_z = 20°C$, while a heat transfer coefficient on an outer surface is $\alpha_z = 10$ W/(m^2·K). The aim here is to compute:

1. overall heat transfer coefficient related to:
   a) outer insulation surface
   b) inner surface of the pipeline
   c) tube's length
2. heat loss

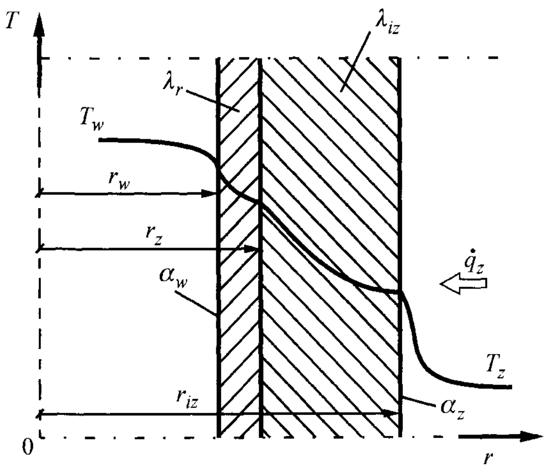

Fig. 6.5. Longitudinal cross-section of a pipeline

Solution

a) Total drop in temperature is equal to the sum of temperature decreases due to, correspondingly, convectional heat exchange on an outer surface of the insulation, the heat conduction in the insulation, the pipeline conduction and the convectional inner surface heat exchange

$$\frac{\dot{q}_z}{k_z} = \frac{\dot{q}_z}{\alpha_z} + \frac{\dot{q}_z r_{iz}}{\lambda_{iz}} \ln\frac{r_{iz}}{r_z} + \frac{\dot{q}_z r_{iz}}{\lambda_r} \ln\frac{r_z}{r_w} + \frac{\dot{q}_z}{\alpha_w}\frac{r_{iz}}{r_w}.$$

From the equation above, one can determine the overall heat transfer coefficient related to the outer surface of the tube

$$k_z = \frac{1}{\frac{1}{\alpha_z} + \frac{r_{iz}}{\lambda_{iz}} \ln\frac{r_{iz}}{r_z} + \frac{r_{iz}}{\lambda_r} \ln\frac{r_z}{r_w} + \frac{1}{\alpha_w}\frac{r_{iz}}{r_w}}.$$

Substitution of the numerical values gives

$$k_z = \frac{1}{\frac{1}{10} + \frac{0.2365}{0.08} \ln\frac{0.2365}{0.1365} + \frac{0.2365}{45} \ln\frac{0.1365}{0.1205} + \frac{1}{500}\frac{0.2365}{0.1205}} =$$

$$= 0.5782\ \text{W/(m}^2 \cdot \text{K)}.$$

Heat flow conducted by the pipeline and the insulation heat loss is given by:

$$\dot{Q} = A_{iz} k_z \left(T_w - T_z\right) = 2\pi r_{iz} L k_z \left(T_w - T_z\right).$$

Thus

$$\dot{Q} = 2 \cdot \pi \cdot 0.2365 \cdot 70 \cdot 0.5782 \cdot \left(400 - 20\right) = 22855\ \text{W}.$$

b) In order to compute the heat transfer coefficient for the pipeline's inner surface, one should begin by adding up all temperature decreases from the inner surface. Total temperature drop equals the sum of temperature decreases connected with, correspondingly, convectional inner surface heat exchange, pipeline heat conduction, heat conduction in an insulation and the insulation's convectional outer surface heat exchange:

$$\frac{\dot{q}_w}{k_w} = \frac{\dot{q}_w}{\alpha_w} + \frac{\dot{q}_w r_w}{\lambda_r} \ln\frac{r_z}{r_w} + \frac{\dot{q}_w r_w}{\lambda_{iz}} \ln\frac{r_{iz}}{r_z} + \frac{\dot{q}_w}{\alpha_z}\frac{r_w}{r_{iz}}.$$

Thus, the overall heat transfer coefficient related to inside tube surface is

$$k_w = \frac{1}{\frac{1}{\alpha_w} + \frac{r_w}{\lambda_r}\ln\frac{r_z}{r_w} + \frac{r_w}{\lambda_{iz}}\ln\frac{r_{iz}}{r_z} + \frac{1}{\alpha_z}\frac{r_w}{r_{iz}}}.$$

Thus

$$k_w = \frac{1}{\frac{1}{500} + \frac{0.1205}{45}\ln\frac{0.1365}{0.1205} + \frac{0.1205}{0.08}\ln\frac{0.2365}{0.1365} + \frac{1}{10}\frac{0.1205}{0.2365}} =$$

$$= 1.1348\ \mathrm{W/(m^2 \cdot K)}.$$

Heat flow transferred by the pipeline and the insulation (heat loss) is

$$\dot{Q} = A_w k_w \left(T_w - T_z\right) = 2\pi r_w L k_w \left(T_w - T_z\right).$$

Substitution of the numerical values gives

$$\dot{Q} = 2 \cdot \pi \cdot 0.1205 \cdot 70 \cdot 1.1348 \cdot \left(400 - 20\right) = 22855\ \mathrm{W}.$$

c) Heat transfer coefficient related to the tube's length can be calculated from the following equation:

$$\dot{Q} = L k_L \left(T_w - T_z\right).$$

The simple transformation gives

$$k_L = \frac{\dot{Q}}{L\left(T_w - T_z\right)} = \frac{22855}{70 \cdot \left(400 - 20\right)} = 0.8592\ \mathrm{W/(m \cdot K)}.$$

Exercise 6.5 Critical Thickness of an Insulation on an Outer Surface of a Pipe

The aim is to calculate thermal loss within the length of 1 m long copper pipe (Fig. 6.6) whose outer diameter measures $d_z = 12$ mm and wall thickness 1 mm. Water with a temperature of 90°C flows inside the pipe. Thermal conductivity of an insulating material equals $\lambda_{iz} = 0.05$ W/(m·K). Temperature of surroundings is 20°C. Heat transfer coefficient from the outer surface of the pipe, or an insulation, to surroundings is the same as above and measures $\alpha_z = 5$ W/(m^2·K). The aim is to calculate the following quantities:

a) critical thickness of the insulation,

b) heat loss in the function of insulation thickness (draw a diagram).

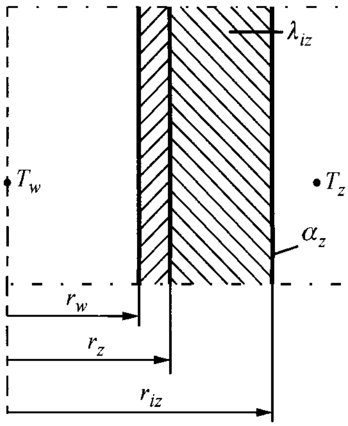

Fig. 6.6. Longitudinal cross-section of a pipeline

In both cases, inner surface thermal resistance and the copper wall resistance should be neglected.

Solution

$$\frac{\dot{q}_{iz}}{k_{iz}} = \frac{\dot{q}_{iz} r_{iz}}{\lambda_{iz}} \ln\frac{r_{iz}}{r_z} + \frac{\dot{q}_{iz}}{\alpha_z}.$$

Heat loss per unit of length:

$$\frac{\dot{Q}}{L} = 2k_{iz}\pi r_{iz}\left(T_w - T_z\right),$$

$$\frac{\dot{Q}}{L} = \frac{2\pi r_{iz}\left(T_w - T_z\right)}{\frac{r_{iz}}{\lambda_{iz}}\ln\left(\frac{r_{iz}}{r_z}\right) + \frac{1}{\alpha_z}}$$

or

$$\frac{\dot{Q}}{L} = \frac{2\pi\left(T_w - T_z\right)}{\frac{1}{\lambda_{iz}}\ln\left(\frac{r_{iz}}{r_z}\right) + \frac{1}{r_{iz}\alpha_z}}.$$

Heat loss $\dot{Q}/L$ will reach its peak, when denominator will reach a minimal value

$$M = \frac{1}{\lambda_{iz}}\ln\left(\frac{r_{iz}}{r_z}\right) + \frac{1}{r_{iz}\alpha_z}$$

From the necessary minimum condition, one obtains a critical inner insulation surface radius

$$\frac{dM}{dr_{iz}} = 0,$$

$$\frac{1}{\lambda_{iz}} \frac{r_z}{r_{iz}} \frac{1}{r_z} - \frac{1}{r_{iz}^2 \alpha_z} = 0,$$

$$r_{iz}^c = \frac{\lambda_{iz}}{\alpha_{iz}}.$$

a) critical insulation thickness:

$$g_{iz}^c = r_{iz}^c - r_z = \frac{\lambda_{iz}}{\alpha_z} - r_z,$$

$$g_{iz}^c = \frac{0.05}{5} - 0.006 = 0.01 - 0.006 = 0.004\ \text{m} = 4\ \text{mm}.$$

b) Fig. 6.7. shows the relevant graph.

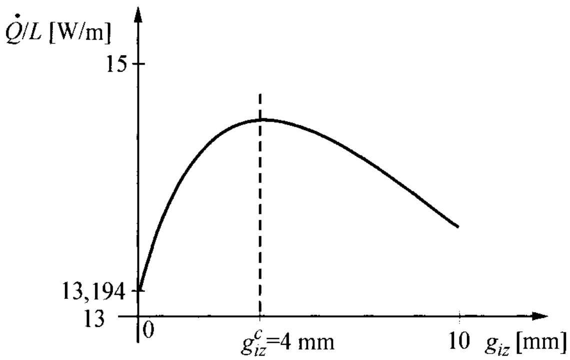

Fig. 6.7. Heat loss through the insulation-thickness function

One should emphasize here that the problem of critical insulation thickness, marked by the largest thermal loss, occurs only in pipes with very small diameters, for example, when heat transfer coefficients on an outer surfaces of an insulation are small and when thermal conductivity for insulation materials are relatively large. In other cases, thermal loss decreases when the thickness of an insulation increases.

Exercise 6.6 Radiant Tube Temperature

The aim is to calculate the temperature of a steel-made radiant tube with a thermal conductivity λ = 40 W/(m·K) and the following dimensions: d_z = = 32 mm, g = 6 mm, t = 39.6 mm (Fig. 6.8). The temperature of a medium inside the tube is T_w = 350°C. Heat transfer coefficient from an inner surface of the tube to the medium is α_w = 20000 W/(m^2·K). Thermal load of the tube (heat flux transferred by the tube at point P) is $\dot{q}$ = 350000 W/m^2. The temperature at point P should be calculated in a simplified way under the assumption that the tube is uniformly heated. Also, an accurate temperature should be calculated on the basis of the provided diagram in Fig. 6.9, with a consideration given to a heat flow from the front-part of the pipe to its unheated rear-side.

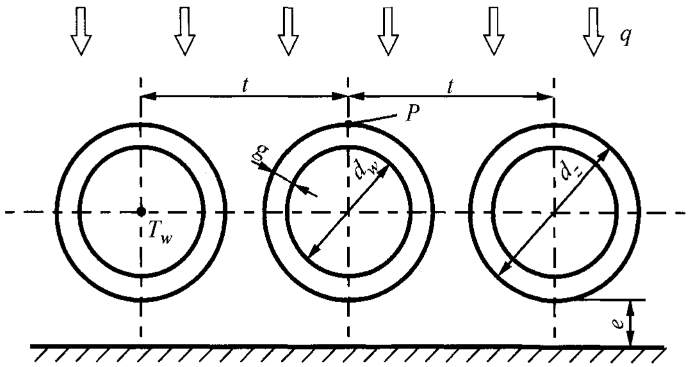

Fig. 6.8. Diagram of a smooth radiant (water-wall) tube

Solution

Tube wall temperature is described by the equation below

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0 \tag{1}$$

and by boundary conditions

$$\lambda\left.\frac{dT}{dr}\right|_{r=r_w}=\alpha\left(T|_{r=r_w}-T_w\right), \tag{2}$$

$$\lambda\left.\frac{dT}{dr}\right|_{r=r_z}=\dot{q}\,. \tag{3}$$

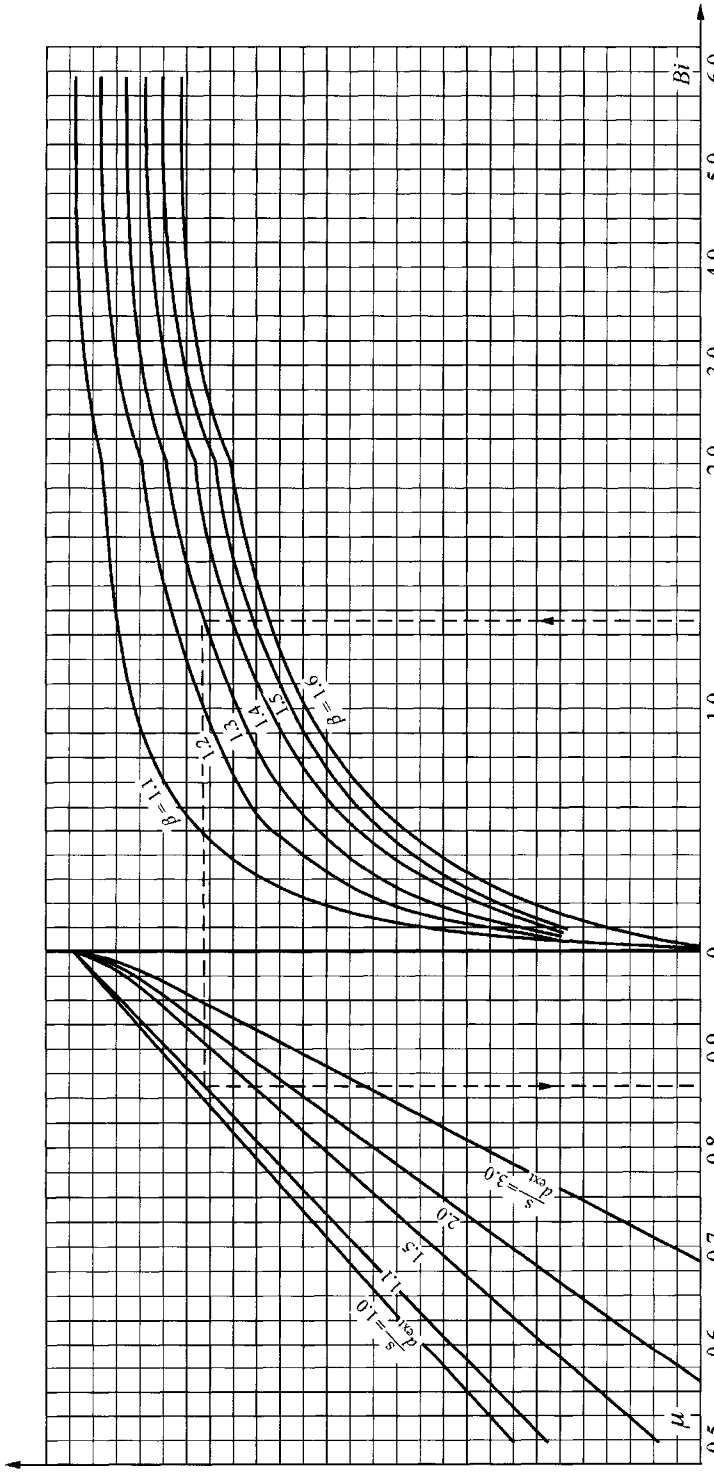

Fig. 6.9. Heat dissipation coefficient μ for radiant smooth tubes [6], where $\beta = d_z/d_w$, $s/d_{ext} = t/d_z$

The solution is obtained by integrating (1) twice in r:

$$T = C_1 \ln r + C_2 . \tag{4}$$

After substituting (4) for boundary condition (2) and (3) and determining constants, temperature distribution is formulated as

$$T = T_w + \frac{\dot{q} r_z}{\lambda}\left(\ln\frac{r}{r_w} + \frac{1}{Bi}\right), \tag{5}$$

where $Bi = \alpha\, r_w/\lambda$.

The tube temperature at point P is calculated by means of an approximate formula (5)

$$Bi = \frac{20000 \cdot 0.01}{40} = 5,$$

$$T_p = T\big|_{r=r_z} = 350\,^\circ\mathrm{C} + \frac{350000 \cdot 0.016}{40}\left(\ln\frac{0.016}{0.01} + \frac{1}{5}\right) = 443.8\,^\circ\mathrm{C}.$$

Real temperature at point P is lower, since heat flows from the tube's front-side to its unheated rear-side from the brickwork side. According to paper [6], the radiant tube's real temperature at point P can be calculated from the formula below

$$T_p' = T_w + \mu \dot{q}\frac{r_z}{r_w}\left(\frac{1}{\alpha_w} + \frac{2}{1 + \frac{r_z}{r_w}}\frac{g}{\lambda}\right),$$

where μ is a so called *heat dissipation coefficient*, which is determined from Fig. 6.9. For $Bi = 5$, $\beta = r_z/r_w = 32/20 = 1.6$ and $t/d_z = 1.2375$, one obtains $\mu = 0.89$. Thus, temperature T_p' is

$$T_p' = 350\,^\circ\mathrm{C} + 0.89 \cdot 350000\frac{0.016}{0.01}\left(\frac{1}{20000} + \frac{2}{1 + \frac{0.016}{0.01}} \cdot \frac{0.006}{40}\right) = 432\,^\circ\mathrm{C}.$$

Temperature calculated by means of the approximate formula (5) equals $T_p = 443.8$°C. It is, therefore, higher than the real temperature $T_p' = 432$°C. The difference, however $(T_p - T_p')$ is small.

Exercise 6.7 Quasi-Steady-State of Temperature Distribution and Stresses in a Pipeline Wall

The aim is to calculate the difference between inner surface temperature and average temperature across the thickness of a steel pipe wall with an outer diameter of $d_z = 324$ mm and wall thickness $g = 65$ mm, made of a ferritic steel 10CrMo910 with thermal conductivity $\lambda = 35.5$ W/(m·K) and thermal diffusivity $a = \lambda / c\rho = 7.137 \cdot 10^{-6}$ m^2/s. The outer surface of the pipe is thermally insulated. The pipe heating (the steam superheater chamber) takes place at constant temperature rate equal to $v_T = 10$ K/min. Lets assume that a quasi-stationary state forms itself in the pipe wall (Fig. 6.10) and is characterized by a stable heating rate equal to v_T. Quasi-stationary state usually occurs for $Fo = at/g^2 > 0.5$ during the heating or cooling of an element, if a temperature change rate of a medium or of an inner surface wall temperature remains constant. We will also calculate thermal stresses (axial) on an inner surface of the pipe under the assumption that the pipe ends can be easily elongated (are free). The following material constants apply for the computation: elastic modulus $E = 181600$ MPa, thermal expansion coefficient $\beta = 1.35 \cdot 10^{-5}$ 1/K, Poisson ratio $\nu = 0.301$.

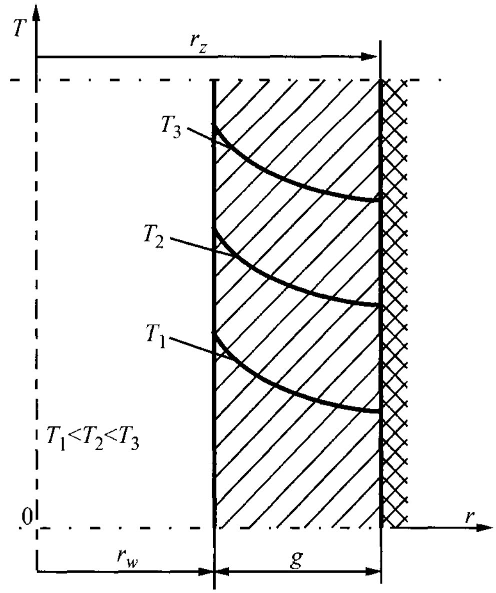

Fig. 6.10. Quasi-steady-state temperature field in the pipe wall (cylindrical chamber)

Solution

Due to a stable temperature change rate within a whole body volume equal to $\partial T/\partial t = v_T$, the heat conduction equation assumes the following form:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)=\frac{v_T}{a}. \tag{1}$$

Equation (1) will be solved using the following boundary conditions:

$$T\big|_{r=r_w}=T_w=v_T t, \tag{2}$$

$$\left.\frac{dT}{dr}\right|_{r=r_z}=0. \tag{3}$$

The solution is obtained by integrating (1) twice in r:

$$T=\frac{1}{4}\frac{v_T}{a}r^2+C_1\ln r+C_2. \tag{4}$$

Constants C_1 and C_2 are determined from the boundary conditions (2) and (3). Substitution of the C_1 and C_2 into (4) yields

$$T(r,t)=v_T t+\frac{v_T}{4a}\left(r^2-r_w^2-2r_z^2\ln\frac{r}{r_w}\right). \tag{5}$$

Average temperature $T_m(t)$ across the wall thickness is given by

$$T_m(t)=\frac{2}{r_z^2-r_w^2}\int_{r_w}^{r_z} rT\,dr, \tag{6}$$

from which, after substitution of (5) for (6) and subsequent integration, one gets

$$T_m(t)=v_T t+\frac{v_T}{a}\left[\frac{1}{8}\left(3r_z^2-r_w^2\right)-\frac{1}{2}r_z^4\frac{\ln\frac{r_z}{r_w}}{r_z^2-r_w^2}\right]. \tag{7}$$

Equation (7) can be also transformed into the following form:

$$T_m(t)=v_T t+\frac{v_T g^2}{a}\frac{\left(3u^2-1\right)\left(u^2-1\right)-4u^4\ln u}{8\left(u^2-1\right)\left(u-1\right)^2}. \tag{8}$$

The unknown temperature difference

$$\Delta T = T_m(t) - T\big|_{r=r_w} \tag{9}$$

is

$$\Delta T = \frac{v_T g^2}{a} \frac{\left(3u^2-1\right)\left(u^2-1\right)-4u^4 \ln u}{8\left(u^2-1\right)\left(u-1\right)^2},$$

where $u = r_z / r_w$.

The inner surface axial stress is given by

$$\sigma_T = \frac{E\beta}{1-\nu}\Delta T,$$

where ΔT is expressed using (9).

Temperature difference ΔT is

$$u = 324/194 = 1.6701,$$

$$\Delta T = \frac{10}{60} \frac{(0.065)^2}{7.137 \cdot 10^{-6}} \frac{\left(3 \cdot 1.67^2 - 1\right)\left(1.67^2-1\right) - 4 \cdot 1.67^4 \ln 1.67}{8\left(1.67^2-1\right)\left(1.67-1\right)^2} =$$

$$= -42.6444 \text{ K}.$$

The inner surface axial stress is

$$\sigma_T = \frac{E\beta}{1-\nu}\Delta T = \frac{181600 \cdot 1.35 \cdot 10^{-5}}{1-0.301}\left(-42.6444\right) = -149.57 \text{ MPa}.$$

Exercise 6.8 Temperature Distribution in a Flat Wall with Constant and Temperature Dependent Thermal Conductivity

The aim is to determine temperature distribution in a flat wall with thickness g heated by a heat flow with density $\dot{q}$ and cooled on the opposite side by water at temperature T_{cz} (Fig. 6.11). Heat transfer coefficient from the plate surface to water α is constant. Lets assume that the thermal conductivity of the plate material changes with temperature in a linear manner:

$$\lambda(T) = a + bT, \tag{1}$$

where a and b are constants; temperature T is expressed in °C.

We will also calculate temperature distribution with an assumption that the thermal conductivity is constant and equals

$$\lambda_m = \frac{1}{2}\left[\lambda\left(T\big|_{x=0}\right) + \lambda\left(T\big|_{x=g}\right)\right], \tag{2}$$

where $T\big|_{x=0}$ and $T\big|_{x=g}$ are front-side and rear-side surface temperatures, calculated using temperature-dependent thermal conductivity. The following values are adopted for the calculation: $g = 0.016$ m, $\dot{q} = 274800$ W/m^2, $\alpha = 2400$ W/(m^2·K), $T_{cz} = 20$°C, $a = 14.64$ W/(m·K), $b = 0.0144$ W/(m·K^2). Calculation results will be presented in a tabular and graphical form.

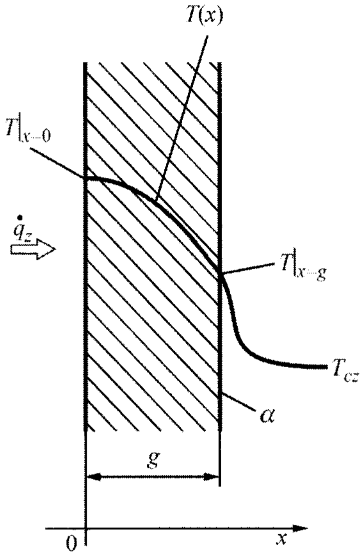

Fig. 6.11. Plate heating

Solution

First apply Fourier Law:

$$\dot{q} = -\lambda(T)\frac{dT}{dx} = -(a + bT)\frac{dT}{dx}. \tag{3}$$

Note that heat flux $\dot{q}$ is constant within the entire plate thickness, since heat flow $\dot{Q} = A\dot{q}$ is constant for steady-state heat conduction. Separation of variables in (3) gives

$$\dot{q}dx = -(a + bT)dT. \tag{4}$$

By integrating (4), one obtains a quadratic equation with respect to T

$$\dot{q}x = -aT - \frac{1}{2}bT^2 + C, \tag{5}$$

whose solution is the function

$$T(x) = -\frac{a}{b} + \sqrt{\left(\frac{a}{b}\right)^2 - \frac{2(\dot{q}x - C)}{b}}. \tag{6}$$

Constant C is determined from condition

$$T\big|_{x=g} = \frac{\dot{q}}{\alpha} + T_{cz}. \tag{7}$$

After substituting (7) for (5), one obtains

$$C = \dot{q}g + aT\big|_{x=g} + \frac{1}{2}b\left(T\big|_{x=g}\right)^2,$$

where $T\big|_{x=g}$ is expressed by (7).

In order to determine temperature distribution, constant C is calculated first:

$$T\big|_{x=g} = \frac{274800}{2400} + 20 = 129.5°\text{C},$$

$$C = 274800 \cdot 0.016 + 14.65 \cdot 129.5 + \frac{1}{2}0.0144 \cdot 129.5^2 = 6414.7208 \text{ W/m}.$$

Temperature distribution is expressed by the following function:

$$T(x) = -\frac{14.65}{0.0144} + \sqrt{\left(\frac{14.65}{0.0144}\right)^2 - \frac{2(274800x - 6414.7208)}{0.0144}}.$$

Table 6.1 and Fig. 6.12 shows the determined $T(x)$ distribution. Mean thermal conductivity λ_m determined from (2) is:

$$T\big|_{x=0} = 370.4274°\text{C}, \quad T\big|_{x=g} = 129.5°\text{C},$$

$$\lambda\left(T\big|_{x=0}\right) = \lambda\left(370.4274°\text{C}\right) = 19.9842 \text{ W/(m} \cdot \text{K)},$$

$$\lambda\left(T\big|_{x=g}\right) = \lambda\left(129.5°\text{C}\right) = 16.5148 \text{ W/(m} \cdot \text{K)},$$

$$\lambda_m = 0.5 \cdot (19.9842 + 16.5148) = 18.2495 \text{ W/(m} \cdot \text{K)}.$$

Table 6.1. Temperature distribution $T(x)$ across the plate thickness

No.	x [m]	Non-linear Problem T [°C]	Linear Problem T [°C]
1	0.000	370.4274	370.4271
2	0.001	356.6077	355.3692
3	0.002	342.6476	340.3112
4	0.003	328.5427	325.2533
5	0.004	314.2884	310.1953
6	0.005	299.8799	295.1374
7	0.006	285.3119	280.0794
8	0.007	270.5793	265.0215
9	0.008	255.6761	246.9635
10	0.009	240.5964	234.9056
11	0.010	225.3337	219.8476
12	0.011	209.8812	204.7897
13	0.012	194.2317	189.7317
14	0.013	178.3773	174.6738
15	0.014	162.3099	159.6158
16	0.015	146.0206	144.5579
17	0.016	129.5000	129.5000

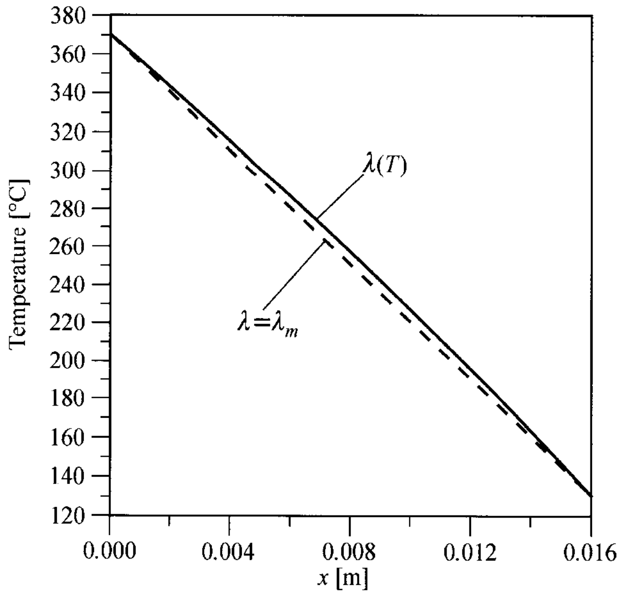

Fig. 6.12. Temperature distribution across a wall thickness for constant and temperature dependent thermal conductivity

In the case of a linear problem, when the thermal conductivity is constant and equals λ_m, temperature distribution across the plate thickness is formulated as

$$T(x)=\frac{\dot{q}(g-x)}{\lambda_m}+\frac{\dot{q}}{\alpha}+T_{cz}.$$

After substitution of the numerical values, function $T(x)$ has the form

$$T(x)=-15057.9468\cdot x+370.4271°\mathrm{C}.$$

Temperature distribution of $T(x)$ for the linear problem is presented in Table 6.1 and Fig. 6.12. It is clear from the analysis of results presented there that the variable conductivity causes discernible temperature differences with regard to the linear problem.

Exercise 6.9 Determining Heat Flux on the Basis of Measured Temperature at Two Points Using a Flat and Cylindrical Sensor

Flat or cylindrical conductometric sensors are used to measure heat flux $\dot{q}$. The sensors operate by measuring temperature at two selected points: P_1 i P_2 (see Fig. 6.13). The known values are the thermal conductivity λ (T), coordinates of measured temperature points x_1 and x_2 (Fig. 6.13a) or (Fig. 6.13b) and measured temperatures T_1 and T_2. The aim is to determine unknown heat fluxes, assuming that the thermal conductivity of the sensor's material is a function of temperature and formulated as

$$\lambda(T)=a+bT+cT^2+dT^3, \qquad (1)$$

where a, b, c and d are known coefficients.
The following values are assumed for the computation:
$g = 0.016$ m; $x_1 = 0.002$ m; $x_2 = 0.012$ m;
$a = 14.99$ W/(m·K);
$b = 1.35\cdot10^{-2}$ W/(m·K^2);
$c = -4.51\cdot10^{-6}$ W/(m·K^3);
$d = 3.59\cdot10^{-9}$ W/(m·K^4);
$T_1 = 330$°C; $T_2 = 180$°C.
In the case of a cylindrical wall:
$r_2 = 0.020$ m; $r_1 = 0.030$ m and $r_z = 0.032$ m.
Remaining values are the same as the values for a flat wall.

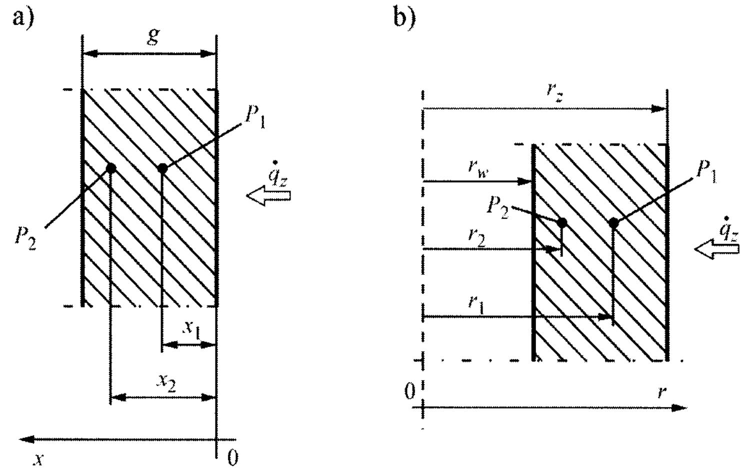

Fig. 6.13. Conductometric sensors: (**a**) flat sensor, (**b**) cylindrical sensor

Solution

a) *Flat sensor* (Fig. 6.13a)
Once the Fourier Law variables are separated

$$\dot{q} = -\lambda(T)\frac{dT}{dx}, \tag{2}$$

and subsequently the integration is carried out from x_1 to x_2, one obtains

$$\int_{x_1}^{x_2} \dot{q}dx = -\int_{T_1}^{T_2} \lambda(T)dT. \tag{3}$$

Introducing the mean thermal conductivity

$$\lambda_m = \frac{1}{T_1 - T_2}\int_{T_2}^{T_1} \lambda(T)dT, \tag{4}$$

and transforming (3), one obtains an expression which can be used to calculate heat flux $\dot{q}$

$$\dot{q} = \lambda_m \frac{T_1 - T_2}{x_2 - x_1}. \tag{5}$$

After substitution of (1) into (4) and integration, one obtains

$$\lambda_m = \frac{1}{T_1 - T_2}\left[a(T_1 - T_2) + \frac{1}{2}b\left(T_1^2 - T_2^2\right) + \frac{1}{3}c\left(T_1^3 - T_2^3\right) + \frac{1}{4}d\left(T_1^4 - T_2^4\right)\right]. \tag{6}$$

Mean thermal conductivity λ_m is

$$\lambda_m = \frac{1}{330-180}\left[14.99\cdot(330-180)+\frac{1}{2}\cdot 1.35\cdot 10^{-2}\cdot\left(330^2-180^2\right)+\right.$$

$$\left.+\frac{1}{3}\cdot\left(-4.51\cdot 10^{-6}\right)\cdot\left(330^3-180^3\right)+\frac{1}{4}\cdot 3.59\cdot 10^{-9}\cdot\left(330^4-180^4\right)\right]= \quad (7)$$

$$=14.99+3.4425-0.301719+0.1184796=18.25\ \mathrm{W/(m\cdot K)}.$$

Thus, heat flux $\dot{q}$ is

$$\dot{q}=18.25\frac{330-180}{0.012-0.002}=273750\ \mathrm{W/m^2}. \quad (8)$$

b) Cylindrical sensor (Fig. 6.13b)

Heat flow $\dot{Q}$ is given by

$$\dot{Q}=\lambda(T)\frac{dT}{dr}A(r), \qquad \dot{Q}=2\pi r L\lambda(T)\frac{dT}{dr}, \quad (9)$$

where L is the length of the sensor. After the separation of variables

$$\frac{\dot{Q}}{2\pi L}\frac{dr}{r}=\lambda(T)dT \quad (10)$$

and integration (10), one obtains

$$\frac{\dot{Q}}{2\pi L}\int_{r_2}^{r_1}\frac{dr}{r}=\int_{T_2}^{T_1}\lambda(T)dT, \quad (11)$$

$$\frac{\dot{Q}}{2\pi L}\ln\frac{r_1}{r_2}=\lambda_m\left(T_1-T_2\right), \quad (12)$$

where λ_m is expressed by (4). Since on the outer surface, heat flux $\dot{q}_z$ is given by the expression

$$\dot{q}_z=\frac{\dot{Q}}{2\pi r_z L},$$

one obtains from (12):

$$\dot{q}_z=\frac{\lambda_m\left(T_1-T_2\right)}{r_z\ln(r_1/r_2)},$$

$$\dot{q}_z=\frac{18.25\cdot(330-180)}{0.032\cdot\ln(0.030/0.020)}=210984.55\ \mathrm{W/m^2}.$$

Exercise 6.10 Determining Heat Flux By Means of Gardon Sensor with a Temperature Dependent Thermal Conductivity

A thin-walled measuring device called the Gardon sensor was applied to measure heat flux in a furnace chamber of a boiler; the sensor is a cylindrical plate insulated on the back surface Fig. 6.14). Since the plate is constantly utilized to measure heat flux, it is cooled on the edges by water (Fig. 6.15).

Assume that the circular measuring plate is made of austenitic steel (18% Cr, 8% Ni) with thermal conductivity dependent on temperature

$$\lambda(T) = 15.1 + 0.0136 \cdot T\,, \tag{1}$$

Where λ is expressed in W/(m·K) and temperature T in °C. The thickness of the measuring plate is 1.8 mm. Coordinates of the installation points of thermoelements are $r_1 = 0$ mm and $r_2 = 10$ mm. Measured temperatures at points r_1 and r_2 are, respectively $T(r_1) = T_1 = 420$°C and $T(r_2) = T_2 = 250$°C. The aim is to derive a formula for calculating temperature distribution in Gardon sensor and heat flux $\dot{q}$ on the basis of measured temperatures T_1 and T_2.

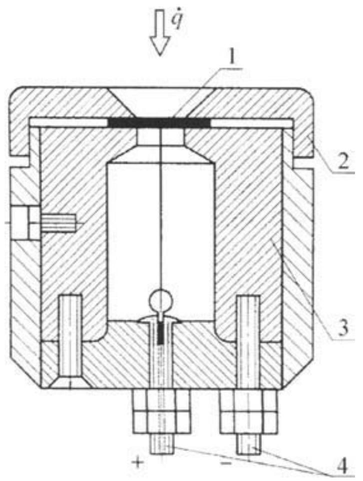

Fig. 6.14. The longitudinal section of Gardon's measuring device: 1 – constantan foil, 2 – protective shield, 3 – copper block, 4 – copper ends

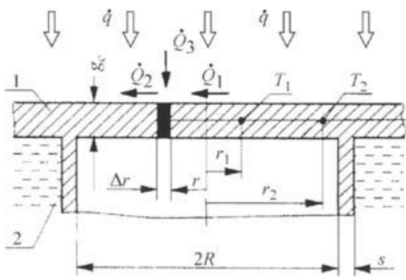

Fig. 6.15. The operation principle of Gardon's measuring device: 1 – measuring plate, 2 – water coolant, $\dot{q}$ – heat flux, T_1 and T_2 – measured temperatures

Solution

In order to derive a differential equation, which describes heat conduction in a sensor of Gardon's measuring device, the energy balance equation will be written for an elementary volume $dV = 2\ rg_c\ r$ (Fig. 6.15)

$$c(T)\rho(T)2\pi r g_c \Delta r \frac{\partial T}{\partial t} = \dot{Q}_1 - \dot{Q}_2 + \dot{Q}_3, \tag{2}$$

where

$$\dot{Q}_1 = -2\pi r g_c \left[\lambda(T) \frac{\partial T}{\partial r} \right] \Bigg|_r, \quad \dot{Q}_2 = -2\pi (r + \Delta r) g_c \left[\lambda(T) \frac{\partial T}{\partial r} \right] \Bigg|_{r+\Delta r}, \tag{3}$$

$$\dot{Q}_3 = -2\pi r \Delta r \dot{q}.$$

By substituting (3) into (2), one obtains the following for $\Delta r \to 0$

$$c(T)\rho(T)\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[\lambda(T) r \frac{\partial T}{\partial r} \right] + \frac{\dot{q}}{g_c}. \tag{4}$$

In the case of steady-state problems $\partial T/\partial t = 0$, and temperature distribution is only a function of a single variable r; therefore,

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\lambda(T) r \frac{\partial T}{\partial r} \right] = -\frac{\dot{q}}{g_c}. \tag{5}$$

Boundary conditions have the following form:

$$\left.\frac{dT}{dr}\right|_{r=0} = 0, \qquad T\big|_{r=r_1} = T_1, \qquad T\big|_{r=r_2} = T_2. \tag{6}$$

In order to linearize problem (5)–(6), Kirchhoff's transformation will be used

$$U = \int_0^T \lambda(T)\, dT. \tag{7}$$

Since

$$\frac{dU}{dr} = \frac{dU}{dT}\frac{dT}{dr} = \lambda(T)\frac{dT}{dr}, \tag{8}$$

Equation (5) and boundary conditions (6) become linear

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dU}{dr} \right) = -\frac{\dot{q}}{g_c}, \tag{9}$$

$$\left.\frac{dU}{dr}\right|_{r=0} = 0, \qquad U\big|_{r=r_1} = U_1, \qquad U\big|_{r=r_2} = U_2, \tag{10}$$

where

$$U_1 = \int_0^{T_1} \lambda(T)\,dT, \qquad U_2 = \int_0^{T_2} \lambda(T)\,dT. \tag{11}$$

From (9) with boundary conditions (10), one obtains

$$U = -\frac{1}{4}\frac{\dot{q}\left(r^2 - r_1^2\right)}{g_c} + U_1. \tag{12}$$

From the third boundary condition (10) it follows that

$$U_2 = -\frac{1}{4}\frac{\dot{q}\left(r_2^2 - r_1^2\right)}{g_c} + U_1. \tag{13}$$

After simple transformations of (13) we obtain a formula for heat flux $\dot{q}$

$$\dot{q} = \frac{4\left(U_1 - U_2\right)g_c}{r_2^2 - r_1^2}. \tag{14}$$

Since

$$U_1 - U_2 = \int_0^{T_1} \lambda(T)\,dT - \int_0^{T_2} \lambda(T)\,dT = \int_{T_2}^{T_1} \lambda(T)\,dT = \lambda_m\left(T_1 - T_2\right), \tag{15}$$

where

$$\lambda_m = \frac{\int_{T_2}^{T_1} \lambda(T)\,dT}{T_1 - T_2} \tag{16}$$

Equation (14) can be written in the following form:

$$\dot{q} = \frac{4\lambda_m\left(T_1 - T_2\right)g_c}{r_2^2 - r_1^2}. \tag{17}$$

If $\lambda(T) = a + bT$, then the average thermal conductivity given by (16) is

$$\lambda_m = \frac{\int_{T_2}^{T_1} (a + bT)\,dT}{T_1 - T_2} = \frac{a\left(T_1 - T_2\right) + \frac{1}{2}b\left(T_1^2 - T_2^2\right)}{T_1 - T_2} = \tag{18}$$

$$= a + \frac{1}{2}b\left(T_1 + T_2\right) = \lambda\left(T_m\right),$$

where $T_m = \frac{1}{2}\left(T_1 + T_2\right)$.

On the basis of the derived formulae, one can determine the heat flux value. Average temperature T_m measures

$$T_m = \frac{1}{2}(T_1 + T_2) = \frac{1}{2}(420 + 250) = 335\ ^\circ\text{C}.$$

After calculating the average thermal conductivity

$$\lambda_m = \lambda(T_m) = a + bT_m = 15{,}1 + 0.0136 \cdot 335 = 19.656\ \text{W/(m·K)},$$

one can calculate heat flux from (17):

$$\dot{q} = \frac{4\lambda_m (T_1 - T_2) g_c}{r_2^2 - r_1^2} = \frac{4 \cdot 19.656 \cdot (420 - 250) \cdot 0.0018}{0.010^2 - 0^2} =$$

$$= 240589.44\ \text{W/m}^2.$$

Exercise 6.11 One-Dimensional Steady-State Plate Temperature Distribution Produced by Uniformly Distributed Volumetric Heat Sources

A round plate with a diameter $d_z = 200$ mm and thickness $g = 20$ mm is electrically heated. The aim is to calculate the upper and lower surface temperature of the plate under the assumption that heat is uniformly generated within the entire plate volume. The bottom and side parts of the plate surface are thermally insulated.

Heat transfer coefficient on the plate surface is $\alpha = 300$ W/(m^2·K). Surrounding air temperature is $T_p = 20$°C. The plate's thermal conductivity is $\lambda = 30$ W/(m·K). The second aim is to calculate temperature distribution within the entire plate thickness under the assumption that the plate's thermal power is $\dot{Q} = 6$ kW.

Solution

The problem under consideration is shown in Fig. 6.16.

Since the power of internal heat sources remains constant

$$\dot{q}_v = \frac{\dot{Q}}{V} = \frac{4\dot{Q}}{\pi d_z^2 g}, \tag{1}$$

temperature field is determined using the heat conduction equation

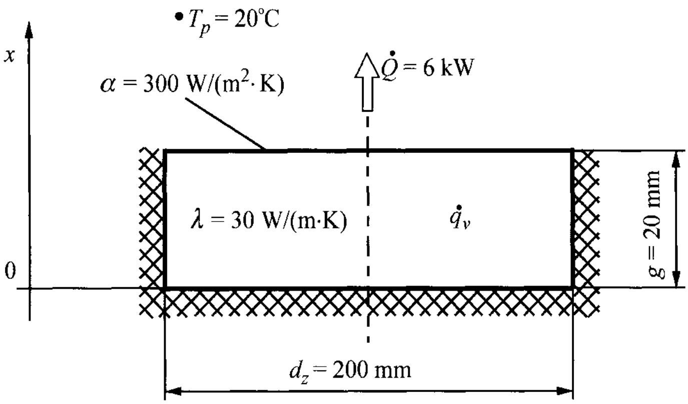

Fig. 6.16. Plate heating

$$\frac{d^2T}{dx^2} = -\frac{\dot{q}_v}{\lambda} \tag{2}$$

and boundary conditions

$$\left.\frac{dT}{dx}\right|_{x=0} = 0, \tag{3}$$

$$-\lambda \left.\frac{dT}{dx}\right|_{x=g} = \alpha\left(T|_{x=g} - T_p\right). \tag{4}$$

The solution is obtained by integrating (2) twice in x

$$T = -\frac{1}{2}\frac{\dot{q}_v}{\lambda}x^2 + C_1 x + C_2. \tag{5}$$

Once constants C_1 and C_2 are determined from boundary conditions (3) and (4) and substituted into (5), one obtains

$$T(x) = \frac{\dot{q}_v g^2}{2\lambda}\left[1-\left(\frac{x}{g}\right)^2\right] + \frac{\dot{q}_v g}{\alpha} + T_p. \tag{6}$$

Insulated surface temperature ($x = 0$) is

$$T(0) = \frac{\dot{q}_v g^2}{2\lambda} + \frac{\dot{q}_v g}{\alpha} + T_p. \tag{7}$$

The upper part plate surface temperature ($x = g$) is

$$T(g) = \frac{\dot{q}_v g}{\alpha} + T_p. \tag{8}$$

Substitution of the numerical values into (1), (7) and (8) yields

$$\dot{q}_v = \frac{4 \cdot 6000}{\pi \cdot 0.2^2 \cdot 0{,}02} = 9.5493 \cdot 10^6 \,\text{W/m}^3 .$$

Temperature of the lower part of the plate surface is

$$T(0) = \frac{9.5493 \cdot 10^6 \cdot (0.02)^2}{2 \cdot 30} + \frac{9.5493 \cdot 10^6 \cdot 0.02}{300} + 20 = 720.28\,^\circ\text{C} .$$

Temperature of the upper part is slightly lower:

$$T(0.02) = \frac{9.5493 \cdot 10^6 \cdot 0.02}{300} + 20 = 656.62^\circ\text{C} .$$

Exercise 6.12 One-Dimensional Steady-State Pipe Temperature Distribution Produced by Uniformly Distributed Volumetric Heat Sources

Electric current flows at an intensity of 300 A through a pipe made of an alloy steel pipe (Fig. 6.17) with an inner diameter $d_w = 7.2$ mm and outer diameter $d_z = 8$ mm. Thermal conductivity of this steel is $\lambda = 18.4$ W/(m·K), while its specific resistance $\rho = 0.85$ $(\Omega \cdot \text{mm}^2)$/m. The aim is to calculate temperature distribution within the entire wall thickness under the assumption that the outer surface of the pipe is thermally insulated and that total heat generated by the pipe flows inside it. The inner surface temperature of the pipe is $T_w = 300$°C. The second aim is to calculate the heat transfer coefficient on the inner surface of the pipe, if the temperature of a medium is $T_{cz} = 20$°C.

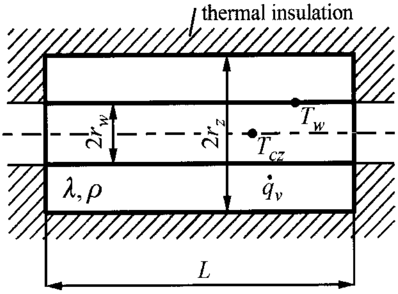

Fig. 6.17. Electrically heated steel pipe

Solution

Firstly, we will determine temperature distribution within the pipe wall under the assumption that heat is uniformly generated within the entire body volume. The heat conduction equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)=-\frac{\dot{q}_v}{\lambda} \tag{1}$$

will be solved using the following boundary conditions:

$$T\big|_{r=r_w}=T_w, \tag{2}$$

$$\left.\frac{dT}{dr}\right|_{r=r_z}=0. \tag{3}$$

The solution is obtained by integrating (1) twice in r

$$\frac{dT}{dr}=-\frac{\dot{q}_v r}{2\lambda}+\frac{C_1}{r} \tag{4}$$

and

$$T(r)=-\frac{\dot{q}_v r^2}{4\lambda}+C_1\ln r+C_2. \tag{5}$$

By substituting (5) into (2) and (4) into (3), one gets two algebraic equations

$$-\frac{\dot{q}_v r_w^2}{4\lambda}+C_1\ln r_w+C_2=T_w, \tag{6}$$

$$-\frac{\dot{q}_v r_z}{2\lambda}+\frac{C_1}{r_z}=0, \tag{7}$$

whose solution are constants C_1 and C_2:

$$C_1=\frac{\dot{q}_v r_z^2}{2\lambda}, \tag{8}$$

$$C_2=\frac{\dot{q}_v}{2\lambda}\left(\frac{r_w^2}{2}-r_z^2\ln r_w\right)+T_w. \tag{9}$$

By substituting constants C_1 and C_2 into (5), one obtains the following, after simple transformations:

$$T(r) = T_w + \frac{\dot{q}_v}{2\lambda}\left(r_z^2 \ln\frac{r}{r_w} + \frac{r_w^2 - r^2}{2} \right). \tag{10}$$

Outer surface temperature is obtained from (10) by substituting $r = r_z$

$$T(r_z) = T_w + \frac{\dot{q}_v}{2\lambda}\left(r_z^2 \ln\frac{r_z}{r_w} + \frac{r_w^2 - r_z^2}{2} \right). \tag{11}$$

The power of internal heat sources with respect to a unit of volume is

$$\dot{q}_v = \frac{\dot{Q}}{V} = \frac{I^2 R}{\pi\left(r_z^2 - r_w^2\right)L}. \tag{12}$$

Since electrical resistance R is formulated as

$$R = \frac{\rho L}{\pi\left(r_z^2 - r_w^2\right)}, \tag{13}$$

therefore, from (12) one obtains

$$\dot{q}_v = \frac{I^2 \rho}{\pi^2\left(r_z^2 - r_w^2\right)^2}. \tag{14}$$

Heat transfer coefficient on the pipe's inner surface will be calculated using the Newton's Law of Cooling:

$$\alpha = \frac{\lambda \left.\dfrac{dT}{dr}\right|_{r=r_w}}{\left(T_w - T_{cz}\right)}, \tag{15}$$

where the heat flux on the inner surface is given by

$$\dot{q}_w = \lambda \left.\frac{dT}{dr}\right|_{r=r_w} = \lambda\left(-\frac{\dot{q}_v r_w}{2\lambda} + \frac{\dot{q}_v r_z^2}{2\lambda r_w} \right) = \frac{\dot{q}_v r_w}{2}\left[\left(\frac{r_z}{r_w}\right)^2 - 1\right]. \tag{16}$$

Heat flux on the pipe's inner surface can also be calculated from energy balance equation. The entire heat flow produced inside the pipe penetrates the inner surface; hence, we get

$$\dot{q}_v \pi\left(r_z^2 - r_w^2\right)L = \dot{q}_w 2\pi r_w L. \tag{17}$$

From (17), one obtains

$$\dot{q}_w = \frac{\dot{q}_v\left(r_z^2 - r_w^2\right)}{2r_w} = \frac{\dot{q}_v r_w}{2}\left[\left(\frac{r_z}{r_w}\right)^2 - 1\right]. \tag{18}$$

Therefore, the heat transfer coefficient on the pipe's inner surface is expressed as

$$\alpha = \frac{\dot{q}_v r_w}{2}\left[\left(\frac{r_z}{r_w}\right)^2 - 1\right]\frac{1}{T_w - T_{cz}}. \tag{19}$$

By substituting the numerical values into (12), (11) and (19), one obtains

$$\dot{q}_v = \frac{300^2 \cdot 8.5 \cdot 10^{-7}}{\pi^2\left(0.004^2 - 0.0036^2\right)^2} = 8.38715 \cdot 10^8 \text{ W/m}^3,$$

$$T(r_z) = 300 + \frac{8.38715 \cdot 10^8}{2 \cdot 18.4}\left(0.004^2 \ln\frac{0.004}{0.0036} + \frac{0.0036^2 - 0.004^2}{2}\right) =$$

$$= 303.77805\,^\circ\text{C},$$

$$\alpha = \frac{8.38715 \cdot 10^8 \cdot 0.0036}{2}\left[\left(\frac{0.004}{0.0036}\right)^2 - 1\right]\frac{1}{300 - 20} = 1264.73 \text{ W/(m}^2\text{K)}.$$

Exercise 6.13 Inverse Steady-State Heat Conduction Problem in a Pipe

The aim here is to solve the problem formulated in Ex. 6.12. In contrast to Ex. 6.12, both conditions, i.e. heat flux and temperature $T_z = 303.77805$°C, are set on an outer surface. It is thus an inverse steady-state heat conduction problem characterized by the fact that temperature distribution across the wall thickness can be determined on the basis of known temperature values and heat flux at a single body point.

Solution

In a given case, heat conduction equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{\dot{q}_v}{\lambda} \tag{1}$$

will be solved when both conditions are assigned on the outer surface

$$T\big|_{r=r_z} = T_z\,, \tag{2}$$

$$\left.\frac{dT}{dr}\right|_{r=r_z} = 0\,. \tag{3}$$

Integrating (1) twice, yields

$$\frac{dT}{dr} = -\frac{\dot{q}_v r}{2\lambda} + \frac{C_1}{r} \tag{4}$$

$$T(r) = -\frac{\dot{q}_v r^2}{4\lambda} + C_1 \ln r + C_2\,. \tag{5}$$

By substituting (4) into (3) and (5) into (2), one gets

$$C_1 = \frac{\dot{q}_v r_z^2}{2\lambda}\,, \tag{6}$$

$$C_2 = T_z + \frac{\dot{q}_v r_z^2}{4\lambda} - \frac{\dot{q}_v r_z^2}{2\lambda}\ln r_z\,. \tag{7}$$

Substituting (6) and (7) into (5), gives the temperature distribution $T(r)$

$$T(r) = T_z + \frac{\dot{q}_v}{2\lambda}\left(\frac{r_z^2 - r^2}{2} - r_z^2 \ln\frac{r_z}{r}\right). \tag{8}$$

Inner surface temperature is

$$T_w = T(r_w) = T_z + \frac{\dot{q}_v}{2\lambda}\left(\frac{r_z^2 - r_w^2}{2} - r_z^2 \ln\frac{r_z}{r_w}\right). \tag{9}$$

Temperature drop across within the wall thickness, determined from (9) is given by

$$T_z - T_w = \frac{\dot{q}_v}{2\lambda}\left(r_z^2 \ln\frac{r_z}{r_w} + \frac{r_w^2 - r_z^2}{2}\right). \tag{10}$$

The same result is obtained from (11) in Ex. 6.12.

Inner surface temperature determined from (9) is

$$T_w = 303.77805 + \frac{8.38715\cdot 10^8}{2\cdot 18.4}\left(-\,0.004^2 \ln\frac{0.004}{0.0036} + \right.$$

$$\left. + \frac{0.004^2 - 0.0036^2}{2}\right) = 300\,^{\circ}\mathrm{C}.$$

The remaining results are identical to the results from Ex. 6.12.

Exercise 6.14 General Equation of Heat Conduction in Fins

The aim is to derive a differential equation to describe heat transfer in fins with arbitrary shapes (Fig. 6.18) under the assumption that temperature across the fin thickness is constant. In other words, one should disregard temperature drop across the fin thickness and derive a formula for fin efficiency.

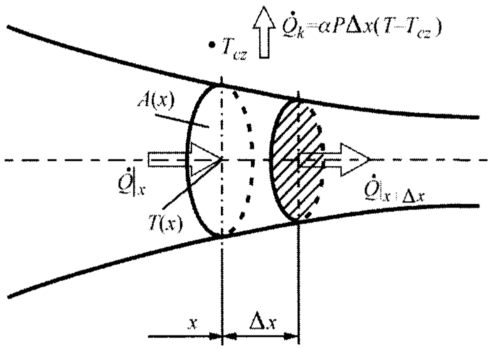

Fig. 6.18. Heat flow through fins with arbitrary shapes

Solution

By assuming that fin temperature remains constant within the fin's cross-section and changes only in the direction of x axis, the heat balance for control volume $A(x)\Delta x$ has the form

$$\dot{Q}\Big|_x = \dot{Q}\Big|_{x+\Delta x} + \dot{Q}_k . \tag{1}$$

Heat flows $\dot{Q}$ are expressed by the following formulas:

$$\dot{Q}\Big|_x = -\lambda A \frac{dT}{dx}\Big|_x , \quad \dot{Q}\Big|_{x+\Delta x} = -\lambda A \frac{dT}{dx}\Big|_{x+\Delta x} , \quad \dot{Q}_k = \alpha P \Delta x \left(T - T_{cz}\right). \tag{2}$$

By substituting (2) for (1), one obtains

$$\frac{\lambda A \dfrac{dT}{dx}\Big|_{x+\Delta x} - \lambda A \dfrac{dT}{dx}\Big|_x}{\Delta x} - \alpha P \left(T - T_{cz}\right) = 0 , \tag{3}$$

where $A(x)$ is a cross-section area of a fin perpendicular to the direction of heat flow through a fin, P is the fin circumference at a point with x coordinate. If $\Delta x \to 0$, then (3) assumes the form:

$$\frac{d}{dx}\left(\lambda A\frac{dT}{dx}\right)-\alpha P\left(T-T_{cz}\right)=0\,. \tag{4}$$

For a constant thermal conductivity λ and constant cross-section A, (4) can be written in the form

$$\frac{d^2T}{dx^2}-m^2\left(T-T_{cz}\right)=0\,, \tag{5}$$

where

$$m^2=\frac{\alpha P}{\lambda A}\,. \tag{6}$$

Two boundary conditions are necessary in order to determine temperature distribution in a fin of height L. The first condition is assigned at point $x = 0$ in the base of the fin, the second at the end of the fin at point $x = L$. In practical computations, it is usually assumed that fin base temperature T_b is constant and equal to a temperature of a surface on which the fin is mounted; i.e. it is assumed that fins do not disturb temperature distribution in a construction element to which they are attached. The fin tip is usually regarded as being thermally insulated, since the surface area of the tip is considerably smaller than the area of fin's side surfaces; therefore, one can neglect the heat flow transmitted by the tip. Assuming that heat exchange takes place on the tip of the fin, the boundary conditions have the form

$$T\big|_{x=0}=T_b\,,$$

$$-\lambda\frac{dT}{dx}\bigg|_{x=L}=\alpha_w\left(T\big|_{x=L}-T_{cz}\right),$$

where α_w is the heat transfer coefficient from the tip to surroundings, while temperature T_{cz} is the temperature of a medium that surrounds the fin. It is usually assumed that $\alpha_w = \alpha$ or $\alpha_w = 0$, when a fin tip is thermally insulated.

Fin efficiency is a ratio of a heat flow $\dot{Q}$, transferred by an actual fin, to a maximal heat flow $\dot{Q}_{\max}$, which the fin could transfer. Maximal heat flow $\dot{Q}_{\max}$ occurs when temperature of the fin is uniform within its entire volume and is equal to the base temperature T_b. Heat flow $\dot{Q}$ can be calculated as a flow that is conducted through the base of the fin or as a dissipated flow by lateral surfaces of the fin and the tip:

$$\dot{Q} = -\left(\lambda A \frac{dT}{dx} \right)\bigg|_{x=0} = \int_0^L \alpha \left(T - T_{cz} \right) P \, dx ,$$

$$\dot{Q}_{\max} = \int_0^L \alpha \left(T_b - T_{cz} \right) P \, dx .$$

Fin efficiency is formulated as

$$\eta = \frac{\dot{Q}}{\dot{Q}_{\max}} .$$

Assuming that α, P and T_{cz} are independent of their position, formulas for efficiency in fins with standard shapes are not very complicated.

Exercise 6.15 Temperature Distribution and Efficiency of a Straight Fin with Constant Thickness

The aim is to determine temperature distribution and efficiency of a straight fin with constant thickness under the assumption that fin tip is thermally insulated. Next, to consider heat exchange through the fin tip, the fin height will be increased by half of the fin's thickness. Then, the fin temperature, heat flow dissipated by the fin and fin efficiency should also be calculated. The following values are assumed for the calculation: fin material – copper with the thermal conductivity λ = 390 W/(m·K), fin thickness t = 0.5 mm, height L = 7.5 cm, width w = 0.7 m, fin base temperature T_b = 80°C, air temperature T_{cz} = 20°C, heat transfer coefficient α = 10 W/(m^2·K).

Solution

Differential (5) from Ex. 6.14, which describes thermal exchange in a fin, has the following form:

$$\frac{d^2T}{dx^2} - m^2 \left(T - T_{cz} \right) = 0 . \tag{1}$$

If fin base temperature is T_b, while fin tip is thermally insulated, the boundary conditions have the form

$$T\big|_{x=0} = T_b , \tag{2}$$

$$\left.\frac{dT}{dx}\right|_{x=L} = 0. \tag{3}$$

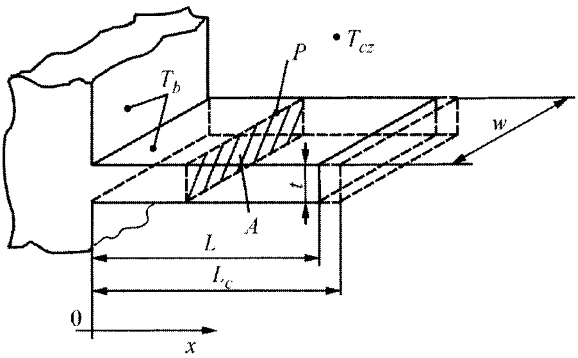

Fig. 6.19. Straight fin of constant cross-section

In the case of a straight fin, shown in Fig. 6.19, circumference P, on which thermal exchange takes place, is $P = 2(w + t)$, while the cross-section area with regard to the direction of thermal conduction is $A = wt$. Fin parameter m^2 ((6), Ex. 6.14) has, in the given case, the form

$$m^2 = \frac{2\alpha(w+t)}{\lambda wt} \approx \frac{2\alpha}{\lambda t}, \tag{4}$$

since usually $t << w$. Once the new variable is introduced

$$\theta = T - T_{cz} \tag{5}$$

Equation (1) and boundary conditions (2), (3) can be rewritten in the following way

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0, \tag{6}$$

$$\left.\theta\right|_{x=0} = \theta_b, \qquad \theta_b = T_b - T_{cz}, \tag{7}$$

$$\left.\frac{d\theta}{dx}\right|_{x=L} = 0. \tag{8}$$

Solution of the homogenous (6) has the following form:

$$\theta = C_1 e^{mx} + C_2 e^{-mx}. \tag{9}$$

From boundary conditions (7)and (8), two algebraic equations are obtained

$$C_1 + C_2 = \theta_b, \tag{10}$$

$$mC_1 e^{mL} - mC_2 e^{-mL} = 0. \tag{11}$$

Once constants C_1 and C_2 are determined from (10) and (11) and substituted into (9), one obtains the following after transformations:

$$\frac{\theta(x)}{\theta_b} = \frac{T(x) - T_{cz}}{T_b - T_{cz}} = \frac{\cosh m(L-x)}{\cosh mL}. \tag{12}$$

Flow $\dot{Q}$ and $\dot{Q}_{\max}$ will be determined in order to define fin efficiency:

$$\begin{aligned}\dot{Q} &= -\lambda A \frac{dT}{dx}\bigg|_{x=0} = -\left[\lambda wt\left(T_b - T_{cz}\right)\frac{-m\sinh m(L-x)}{\cosh mL}\right]\bigg|_{x=0} = \\ &= \lambda wt\sqrt{\frac{2\alpha}{\lambda t}}\left(T_b - T_{cz}\right)\operatorname{tgh} mL,\end{aligned} \tag{13}$$

$$\dot{Q}_{\max} = \alpha 2wL\left(T_b - T_{cz}\right). \tag{14}$$

Efficiency of a straight fin with constant cross-section is

$$\eta = \frac{\dot{Q}}{\dot{Q}_{\max}} = \frac{\lambda wt\sqrt{\frac{2\alpha}{\lambda t}}\left(T_b - T_{cz}\right)\operatorname{tgh} mL}{\alpha 2wL\left(T_b - T_{cz}\right)} = \frac{\operatorname{tgh} mL}{mL}, \tag{15}$$

where m is formulated in (4).

Hyperbolic trigonometric functions are expressed using the following formulas:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}, \qquad \operatorname{tgh} x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

In (12) and (15), fin tip heat exchange is not taken into consideration. It can be considered only when the fin height is increased by $t/2$ (Fig. 6.19). Once the fin height substitute is introduced, L_c

$$L_c = L + \frac{t}{2}$$

temperature distribution and fin efficiency are expressed as

$$\frac{\theta(x)}{\theta_b} = \frac{T(x) - T_{cz}}{T_b - T_{cz}} = \frac{\cosh m(L_c - x)}{\cosh mL_c},$$

$$\eta = \frac{\text{tgh}\, mL_c}{mL_c},$$

where $m = \sqrt{2\alpha / \lambda t}$.

After substitution of the numerical values, one obtains the following results:

$$m = \sqrt{\frac{2 \cdot 10}{390 \cdot 0.0005}} = 10.1274 \text{ 1/m},$$

$$L_c = L + \frac{t}{2} = 0.075 + \frac{1}{2} \cdot 0.0005 = 0.07525 \text{ m},$$

- fin tip temperature

$$T(L_c) = (T_b - T_{cz}) \frac{1}{\cosh mL_c} + T_{cz} = (80 - 20) \frac{1}{\cosh(10.1274 \cdot 0.07525)} +$$
$$+ 20 = 65.97\,^\circ\text{C}.$$

- fin efficiency

$$\eta = \frac{\text{tgh}\, mL_c}{mL_c} = \frac{\text{tgh}(10.1274 \cdot 0.07525)}{10.1274 \cdot 0.07525} = 0.8428.$$

- heat flow to-surrounding air

$$\dot{Q} = \eta \dot{Q}_{\max} = \eta \alpha 2wL(T_b - T_{cz}) = 0.8428 \cdot 10 \cdot 2 \cdot 0.7 \cdot 0.07525 \cdot (80 - 60) =$$
$$= 53.275 \text{ W}$$

Exercise 6.16 Temperature Measurement Error Caused by Thermal Conduction Through Steel Casing that Contains a Thermoelement as a Measuring Device

The aim is to calculate temperature measurement error of combustion gases by means of a thermoelement installed inside a steel casing (Fig. 6.20). The following data are used for the calculation: d_z = 12 mm, g_o = 3 mm, L = 120 mm, λ_o = 50 W/(m·K), T_p = 20°C, T_{sp} = 210°C. Heat transfer coefficient from combustion gases to casing is α_o = 45 W/(m^2·K). Heat transfer coefficient on an outer and inner side of the combustion channel is α_p = 15 W/(m^2·K) and α_{sc} = 30 W/(m^2·K), respectively. Lets assume that the base temperature of a steel casing T_b is equal to an average temperature T_{sc} of a wall, which the thermometer casing is welded onto (Fig. 6.20).

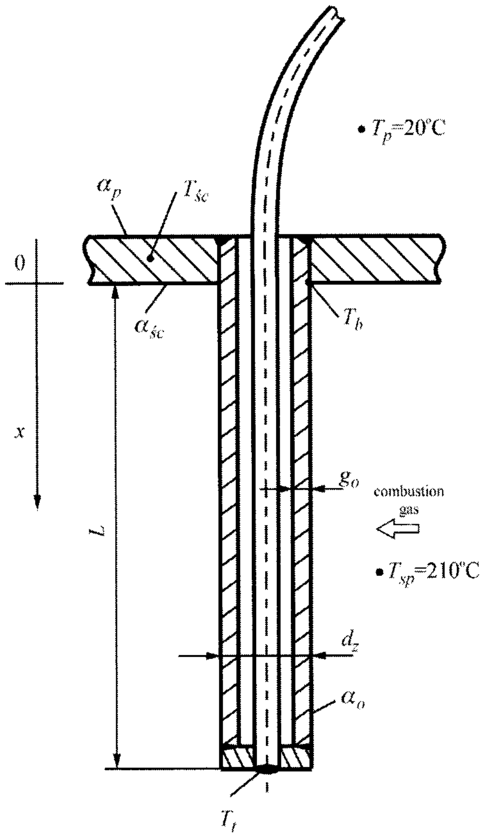

Fig. 6.20. Thermoelement installation

Solution

First wall temperature T_{sc} will be calculated. From the heat flux equality condition on an inner and outer surface of the combustion channel, one obtains

$$\dot{q}_{sc} = \alpha_{sc}\left(T_{sp} - T_{sc}\right) = \alpha_p\left(T_{sc} - T_p\right), \tag{1}$$

hence,

$$T_{sc} = \frac{\alpha_{sc}T_{sp} + \alpha_p T_p}{\alpha_{sc} + \alpha_p} = \frac{30 \cdot 210 + 15 \cdot 20}{30 + 15} = 146.7\,°\text{C}. \tag{2}$$

Thermoelement-indicated temperature T_t can be calculated in the same way as the fin tip temperature (insulated on the tip). Fin temperature distribution is expressed by the function ((12), Ex. 6.15):

$$\frac{\theta}{\theta_b} = \frac{T - T_{sp}}{T_b - T_{sp}} = \frac{\cosh m(L - x)}{\cosh mL}, \tag{3}$$

where

$T_b = T_{sc}$ – wall temperature, which the casing is welded onto,

T_{sp} – temperature of combustion gases,

$T = T(x)$ – temperature of a casing within distance x from the channel wall,

m – fin parameter, defined as follows

$$m = \sqrt{\alpha_o P / \lambda_o A},$$

where α_o – heat transfer coefficient on an outer surface of the casing, $P = \pi d_z$ – outer surface of the casing,

$$A = \frac{\pi}{4}\left(d_z^2 - d_w^2\right) = \pi d_{sr} g_o, \qquad d_{sr} = \frac{d_w + d_z}{2} = d_z - g.$$

Parameter m:

$$m = \sqrt{\frac{\alpha_o P}{\lambda_o A}} = \sqrt{\frac{\alpha_o \pi d_z}{\lambda_o \pi d_{sr} g_o}} = \sqrt{\frac{\alpha_o d_z}{\lambda_o d_{sr} g_o}}.$$

After substitution, one obtains

$$m = \sqrt{\frac{45 \cdot 0.012}{50 \cdot (0.012 - 0.006) \cdot 0.003}} = 20 \text{ 1/m.}$$

Tip temperature of the casing T_t, indicated by the thermoelement, is determined from (3) for $x = L$

$$\frac{T_t - T}{T_b - T_{sp}} = \frac{1}{\cosh mL}, \qquad T_t = T + \frac{T_b - T_{sp}}{\cosh mL}.$$

Since

$$mL = 20 \cdot 0.12 = 2.4,$$

then

$$T_t = 210 + \frac{146.7 - 210}{\cosh(2.4)} = 210 - \frac{63.3}{5.5569} = 198.61\,^\circ\text{C}.$$

Relative temperature measurement error for combustion gases is

$$\varepsilon = \frac{T_{sp} - T_t}{T_{sp}} \cdot 100\,\% = \frac{210 - 198.61}{210} 5.42\%.$$

Exercise 6.17 Temperature Distribution and Efficiency of a Circular Fin of Constant Thickness

The aim is to derive an equation for a circular fin of constant thickness from a general equation of heat transfer in fins (Ex. 6.14) and to determine formulas for temperature distribution in fins, for a fin-transferred heat flow and for fin efficiency. Following that calculate fin tip temperature, fin efficiency and dissipate heat flow using the following data: fin-base temperature $T_b = 90°C$, temperature of surroundings $T_{cz} = 20°C$, $r_1 = 12.5$ mm, $r_2 = 28.5$ mm, $t = 0.4$ mm, material of a fin – aluminium with thermal conductivity $\lambda = 205$ W/(m·K), heat transfer coefficient on the fin surface α = 70 W/(m^2·K). Take into account heat exchange on a fin tip by increasing fin height L to $L_c = L + t/2$.

Solution

In the case of a circular fin shown in Fig. 6.21, surface area of a fin cross-section is $A = 2\pi rt$, while circumference, on which thermal exchange occurs is $P = 4\pi r$. Parameter m is defined as

$$m = \sqrt{\frac{\alpha P}{\lambda A}} = \sqrt{\frac{\alpha 4\pi r}{\lambda 2\pi rt}} = \sqrt{\frac{2\alpha}{\lambda t}} . \tag{1}$$

Differential equation (4) in Ex. 6.14 assumes the following form:

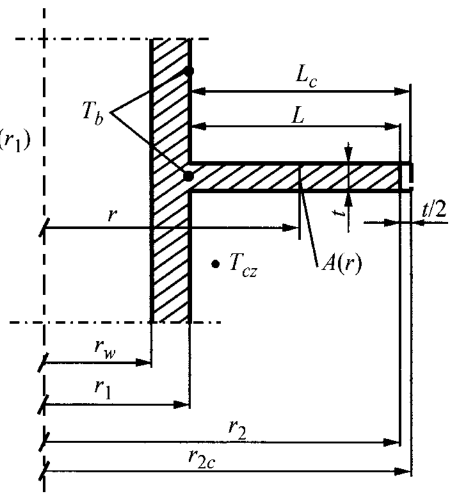

Fig. 6.21. Circular fin with constant thickness

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)-\frac{2\alpha}{\lambda t}\left(T-T_{cz}\right)=0\,. \tag{2}$$

Once excess temperature $\theta = T - T_{cz}$ and parameter m given by (1) are introduced, (2) can be written in a form

$$\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}-m^2\theta=0\,. \tag{3}$$

It is a modified Bessel equation, for which general solution has the form

$$\theta(r)=C_1 I_0(mr)+C_2 K_0(mr)\,. \tag{4}$$

Constants C_1 and C_2 will be determined from boundary conditions

$$\theta\Big|_{r=r_1}=\theta_b,\ \theta_b=T_b-T_{cz}, \tag{5}$$

$$\left.\frac{d\theta}{dr}\right|_{r=r_2}=0\,. \tag{6}$$

Once constants are determined and substituted into (4), one obtains a formula for temperature distribution $\theta(r)$ in a fin:

$$\frac{\theta}{\theta_b}=\frac{T(r)-T_{cz}}{T_b-T_{cz}}=\frac{K_0(mr)I_1(mr_2)+I_0(mr)K_1(mr_2)}{I_0(mr_1)K_1(mr_2)+K_0(mr_1)I_1(mr_2)}\,. \tag{7}$$

Heat flow $\dot{Q}$ dissipated by the fin

$$\dot{Q}=-\lambda A_b\left.\frac{dT}{dr}\right|_{r=r_1}=2\pi\lambda r_1 t\theta_b m\frac{K_1(mr_1)I_1(mr_2)-I_1(mr_1)K_1(mr_2)}{K_0(mr_1)I_1(mr_2)+I_0(mr_1)K_1(mr_2)}\,. \tag{8}$$

Since maximal flow $\dot{Q}_{\max}$ is

$$\dot{Q}_{\max}=\alpha A_z\left(T_b-T_{cz}\right)=\alpha 2\pi\left(r_2^2-r_1^2\right)\theta_b\,, \tag{9}$$

fin efficiency, then, can be determined as:

$$\eta=\frac{\dot{Q}}{\dot{Q}_{\max}}=\frac{2r_1}{m\left(r_2^2-r_1^2\right)}\frac{K_1(mr_1)I_1(mr_2)-I_1(mr_1)K_1(mr_2)}{K_0(mr_1)I_1(mr_2)+I_0(mr_1)K_1(mr_2)}\,. \tag{10}$$

Fin tip heat exchange can be taken into account by substituting radius r_2 in (7)–(10) for a slightly larger radius $r_{2c} = r_2 + t/2$. As in the case of a straight fin, fin length is larger: $L_c = L + t/2$.

After substitution of the numerical values, one obtains

$$mr_1 = 41.32 \cdot 0.0125 = 0.5165,$$

$$r_{2c} = r_2 + \frac{t}{2} = 0.0285 + 0.0002 = 0.0287 \text{ m},$$

$$mr_{2c} = 1.1859$$

The values of the Bessel functions are [5]

$$I_0(mr_{2c}) = I_0(1.1859) = 1.3837, \qquad K_1(mr_{2c}) = K_1(1.1859) = 0.4443,$$

$$K_0(mr_{2c}) = K_0(1.1859) = 0.3247,$$

$$I_1(mr_{2c}) = I_1(1.1859) = 0.7035, \qquad I_0(mr_1) = I_0(0.5165) = 1.0678,$$

$$K_0(mr_1) = K_0(0.5165) = 0.8977,$$

one obtains

$$\frac{T(r_{2c}) - 20}{90 - 20} = \frac{0.3247 \cdot 0.7035 + 1.3837 \cdot 0.4443}{1.0678 \cdot 0.4443 + 0.8977 \cdot 0.7035} = 0.7624,$$

$$T(r_{2c}) = 73.38^\circ\text{C}.$$

In order to determine fin efficiency, two additional Bessel functions are needed:

$$K_1(mr_1) = K_1(0.5165) = 1.5887,$$

$$I_1(mr_1) = I_1(0.5165) = 0.2670.$$

Heat flow dissipated by the fin is

$$\eta = \frac{2 \cdot 0.0125}{41.32\left(0.0287^2 - 0.0125^2\right)} \frac{1.5887 \cdot 0.7035 - 0.2670 \cdot 0.4443}{0.8977 \cdot 0.7035 + 1.0678 \cdot 0.4443} = 0.8188 .$$

Fin-diffused heat flow:

$$\dot{Q} = \eta \dot{Q}_{max} = \eta \alpha 2\pi \left(r_{2c}^2 - r_1^2\right)\left(T_b - T_{cz}\right) =$$

$$= 0.8188 \cdot 70 \cdot 2 \cdot \pi \left(0.0287^2 - 0.0125^2\right)(90 - 20) = 16.82 \text{ W}.$$

Bessel function values, present in formulas for temperature distribution and circular fin efficiency, can be read from table [5] or calculated by means of library procedures [4, 7].

Exercise 6.18 Approximated Calculation of a Circular Fin Efficiency

Calculate circular fin efficiency from Ex. 6.17 using the following approximation formulas:

a) according to Schmidt [11, 12]

$$\eta_S = \frac{\operatorname{tgh} mL_c\varphi}{mL_c\varphi}, \tag{1}$$

where

$$\varphi = 1 + 0.35\ln\left(1 + \frac{L_c}{r_1}\right), \qquad L_c = r_{2c} - r_1, \qquad m = \sqrt{\frac{2\alpha}{\lambda t}}. \tag{2}$$

If $\eta > 0.5$, then efficiency η calculated using (1) does not differ more than ±1% from the real value.

b) according to Brandt [2]

$$\eta_B = \frac{2r_1}{2r_1 + L_c}\frac{\operatorname{tgh} mL_c}{mL_c}\left[1 + \frac{\operatorname{tgh} mL_c}{2mr_1} - C\frac{(\operatorname{tgh} mL_c)^p}{(mr_1)^n}\right], \tag{3}$$

where

$$C = 0.071882, \qquad p = 3.7482, \qquad n = 1.4810. \tag{4}$$

Maximal error from efficiency determination by means of (3) is smaller than 0.6% from an error made when determining efficiency by means of (1).

c) formula according to [3]

$$\eta_H = \frac{1}{1 + \frac{1}{3}(mL_c)^2\sqrt{r_{2c}/r_1}}. \tag{5}$$

Equation (5) gives good results, when $\eta > 0.75$. For the calculation, use the values from Ex. 6.17.

Solution

a) according to Schmidt

$$m = 41.32\ 1/\mathrm{m}, \qquad r_1 = 0.0125\ \mathrm{m}, \qquad r_{2c} = 0.0287\ \mathrm{m},$$

$$L_c = r_{2c} - r_1 = 0.0162\ \mathrm{m}, \quad \varphi = 1.2909,$$

$$\eta_S = \frac{\text{tgh}(41.32 \cdot 0.0162 \cdot 1.2909)}{41.32 \cdot 0.0162 \cdot 1.2909} = 0.8082\,.$$

Since a real value of the efficiency is η = 0.8188, relative error, then, comes to

$$\varepsilon_S = \frac{(\eta_S - \eta)}{\eta} 100\% = \frac{0.8082 - 0.8188}{0.8188} \cdot 100\% = -1.295\%\,.$$

b) according to Brandt [2]

$$\eta_B = \frac{2 \cdot 0.0125}{2 \cdot 0.0125 + 0.0162} \cdot \frac{0.584575}{41.32 \cdot 0.0162} \times$$

$$\times \left(1 + \frac{0.584575}{2 \cdot 41.32 \cdot 0.0125} - 0.071882 \cdot \frac{(0.584575)^{3.7482}}{(41.32 \cdot 0.0125)^{1.4810}} \right) = 0.8162.$$

Relative error is:

$$\varepsilon_B = \frac{\eta_B - \eta}{\eta} \cdot 100\% = \frac{0.8162 - 0.8188}{0.8188} \cdot 100\% = -0.312\%\,.$$

c) formula according to [3]

$$\eta_H = \frac{1}{1 + \frac{1}{3}(41.32 \cdot 0.0162)^2 \sqrt{\frac{0.0287}{0.0125}}} = 0.8155\,.$$

Relative error is:

$$\varepsilon_H = \frac{\eta_H - \eta}{\eta} \cdot 100\% = \frac{0.8155 - 0.8188}{0.8188} \cdot 100\% = -0.409\%\,.$$

From the comparison of the results presented above, one can see that the least accurate result is given by Schmidt formula. Brandt formula allows for the most accurate calculation of circular fin efficiency; however, the amount of work required to obtain the results is not much smaller than in the case of the analytical formula. Equation (5) is both simple, yet accurate.

Exercise 6.19 Calculating Efficiency of Square and Hexagonal Fins

Calculate fin efficiency (equivalent fins) in a fin-plate exchanger made of pipes with an outer diameter of d_z = 10 mm and wall thickness g_r = 1 mm.

Distance between pipes in a hexagonal system is constant and is equal to $2s$ (Fig. 6.22b). Perpendicular and longitudinal pitch 2s, with pipes arranged in rows, is assumed constant (Fig. 6.22a). Calculations are to be made for a

a) in-line
b) hexagonal

pipe configuration for two fin-plates of different thickness: t = 0.33 mm and t = 0.13 mm, assuming that $2s$ = 25 mm. Fin plates are made of aluminium alloy with thermal conductivity of λ = 165 W/(m·K). Heat transfer coefficient is α = 50 W/(m²·K).

Solution

Pipe configuration is shown in Fig. 6.22. Fin efficiency will be calculated using (5) from Ex. 6.18.

$$\eta = \frac{1}{1+\frac{1}{3}(mL)^2\sqrt{\frac{r_2^*}{r_1}}},$$

where $L = r_2^* - r_1$, $r_1 = d_z/2$, r_2^* – equivalent radius of a circular fin, calculated from a condition of equality of a conventional and circular fin surface area, as shown in Fig. 6.22. Fin parameter is calculated using formula

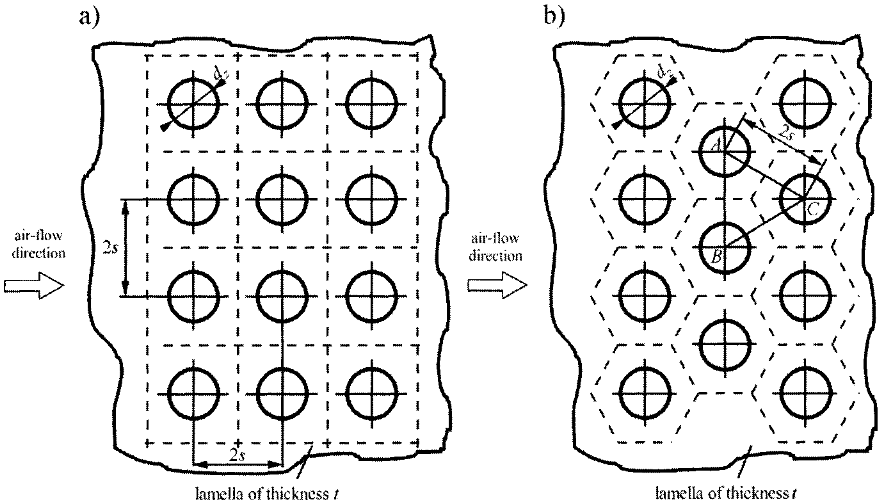

Fig. 6.22. Pipe lay-out in a fin-plate exchanger: (**a**) in-line pipe configuration, (**b**) hexagonal (staggered) pipe configuration

$$m = \sqrt{\frac{2\alpha}{\lambda t}}\,.$$

a) *In-line pipe configuration* is shown in Fig. 6.22a

Equivalent radius r_2^* is calculated from a condition of equality of circular and square fin surface area, whose side is $2s$

$$2s \cdot 2s - \pi r_1^2 = \pi \left(r_2^*\right)^2 - \pi r_1^2,$$

$$r_2^* = \frac{2s}{\sqrt{\pi}}.$$

After substitution, one obtains

$$r_2^* = \frac{2s}{\sqrt{\pi}} = \frac{0.025}{\sqrt{\pi}} = 0.0141 \text{ m}\,.$$

Parameter m:

when $t = 0.33$ mm, $$m = \sqrt{\frac{2 \cdot 50}{165 \cdot 0.00033}} = 42.85 \text{ 1/m}\,,$$

when $t = 0.13$ mm, $$m = \sqrt{\frac{2 \cdot 50}{165 \cdot 0.00013}} = 68.28 \text{ 1/m}\,.$$

Since,

$$L = r_2^* - r_1 = 0.0141 - 0.005 = 0.0091 \text{ m}\,,$$

fin efficiency is

when $t = 0{,}33$ mm, $$\eta = \frac{1}{1 + \frac{1}{3}(42.85 \cdot 0.0091)^2 \sqrt{\frac{0.0141}{0.005}}} = 0.9216,$$

when $t = 0.13$ mm, $$\eta = \frac{1}{1 + \frac{1}{3}(68.28 \cdot 0.0091)^2 \sqrt{\frac{0.0141}{0.005}}} = 0.8223\,.$$

b) *Hexagonal pipe configuration* is shown in Fig. 6.22b

Area of triangle ABC :

$$S_{ABC} = \frac{1}{2} 2s \sqrt{(2s)^2 - s^2} = \sqrt{3} s^2\,.$$

Fin surface area: $S_{\dot z} = 2S_{ABC}$ and

$$2S_{ABC} - \pi r_1^2 = \pi\left(r_2^*\right)^2 - \pi r_1^2, \qquad 2\sqrt{3}s^2 = \pi\left(r_2^*\right)^2;$$

One obtains then

$$r_2^* = \sqrt{\frac{2\sqrt{3}}{\pi}s} = \left(\frac{2\sqrt{3}}{\pi}\right)^{1/2} s.$$

Equivalent radius:

$$r_2^* = \left(\frac{2\sqrt{3}}{\pi}\right)^{1/2} \cdot s = \left(\frac{2\sqrt{3}}{\pi}\right)^{1/2} \cdot 0.0125 = 0.0131\,\text{m}.$$

Since,

$$L = r_2^* - r_1 = 0.0131 - 0.005 = 0.0081\,\text{m},$$

one obtains the following efficiency values:

when, $t = 0.33\,\text{mm}$, $$\eta = \frac{1}{1 + \frac{1}{3}\left(42.85 \cdot 0.0081\right)^2 \sqrt{\frac{0.0131}{0.005}}} = 0.9390,$$

when, $t = 0.13\,\text{mm}$, $$\eta = \frac{1}{1 + \frac{1}{3}\left(68.28 \cdot 0.0081\right)^2 \sqrt{\frac{0.0131}{0.005}}} = 0.8583.$$

From the comparison of results given, it is clear that hexagonal configuration ensures greater fin efficiency; therefore, the flow of transferred heat is larger than it is in the case of a in-line pipe configuration.

Exercise 6.20 Calculating Efficiency of Hexagonal Fins by Means of an Equivalent Circular Fin Method and Sector Method

The aim is to calculate efficiency of an equivalent fin in a fin-plate exchanger with a staggered fin configuration (Fig. 6.23). Pipes with an outer diameter of d_z = 7.59 mm arranged together with the following pitches: P_t = 21 mm and P_l = 12.7 mm (Fig. 6.23). The thickness of aluminium alloy based fin plates is t = 0.115 mm. Thermal conductivity of the fin

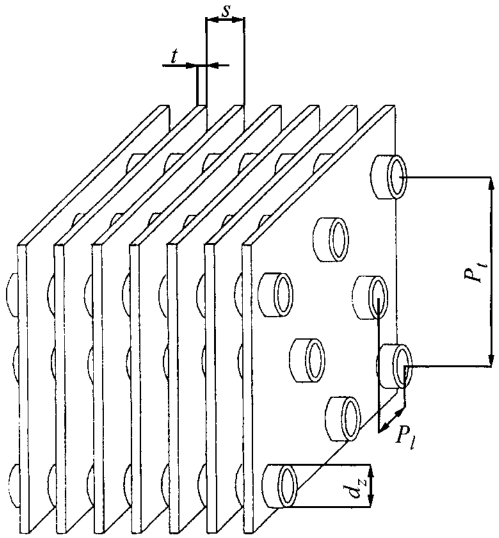

Fig. 6.23. Fin-plate heat exchanger with a staggered pipe configuration

material is $\lambda = 165$ W/(m·K). Heat transfer coefficient on a fin-plate surface is equal to α = 40 (W/m^2·K). Calculate efficiency of fins with an equivalent outer radius by means of a simplified Brandt formula and sector method.

Solution

a) Method I

First, we will calculate an equivalent outer radius of a circular fin, whose surface area equals a surface area of a equivalent fin shown in Fig. 6.24. The equivalent fin surface area is

$$A_{\dot{z}} = \left[\left(2A_{OAB} + 4A_{OBC} \right) - \pi r_1^2 \right] \cdot 2 ,$$

where A_{OBC} is an area of triangle OBC.

The above formula took into account the fact that thermal exchange occurs on both sides of fin plates. Surface area of a circular fin A_o with an equivalent outer r_2^* and inner radius substitute and r_1, respectively, is

$$A_{\dot{z}o} = 2\pi \left[\left(r_2^* \right)^2 - r_1^2 \right].$$

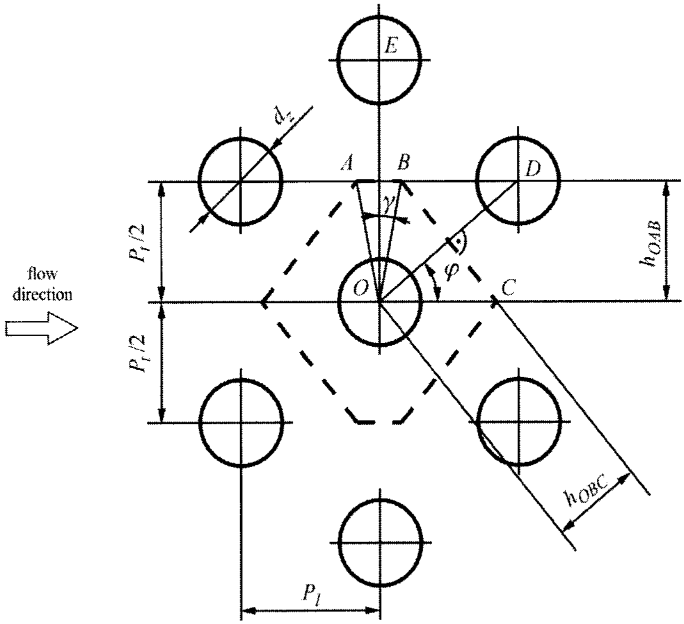

Fig. 6.24. Calculation of an equivalent fin surface area

From the equality

$$A_{\dot{z}o} = A_{\dot{z}}$$

One obtains,

$$2\left[\left(2A_{OAB} + 4A_{OBC}\right) - \pi r_1^2\right] = 2\pi\left[\left(r_2^*\right)^2 - r_1^2\right],$$

hence,

$$r_2^* = \sqrt{\frac{2A_{OAB} + 4A_{OBC}}{\pi}}.$$

Area of a triangle *OAB* is formulated as (Fig. 6.24)

$$A_{OAB} = \frac{1}{2}|AB| \cdot h_{OAB},$$

where

$$|AB| = 2\frac{P_t}{2}\mathrm{tg}\gamma = P_t\mathrm{tg}\gamma, \qquad h_{OAB} = \frac{P_t}{2}, \qquad \gamma = \frac{\pi}{2} - 2\varphi,$$

$$\sin\varphi = \frac{P_t}{2|OD|}, \qquad \varphi = \arcsin\frac{P_t}{2|OD|}, \qquad |OD| = \sqrt{\left(\frac{P_t}{2}\right)^2 + P_l^2}.$$

Area of a triangle *OBC* is formulated as

$$A_{OBC} = \frac{1}{2}|BC|h_{OBC} = \frac{1}{2}|BC|\frac{1}{2}|OD| = \frac{1}{4}|BC||OD|,$$

$$|BC| = 2\frac{|OD|}{2}\mathrm{tg}\varphi = |OD|\mathrm{tg}\varphi.$$

According to a simplified Brandt formula, fin efficiency is expressed using (3) from Ex. 6.18

$$\eta = \frac{2r_1}{2r_1 + L_c}\frac{\mathrm{tgh}mL_c}{mL_c}\left[1 + \frac{\mathrm{tgh}mL_c}{2mr_1} - C\frac{(\mathrm{tgh}mL_c)^p}{(mr_1)^n}\right],$$

where

$$C = 0.071882, \quad p = 3.7482, \quad n = 1.481,$$

$$L = r_2^* - r_1, \qquad m = \sqrt{\frac{2\alpha}{\lambda t}}.$$

After substitution of the numerical values, one obtains

$$|OD| = \sqrt{\left(\frac{P_t}{2}\right)^2 + P_l^2} = \sqrt{\left(\frac{0.021}{2}\right)^2 + 0.0127^2} = 0.01648 \text{ m},$$

$$\sin\varphi = \frac{P_t}{2|OD|} = \frac{0.021}{2\cdot 0.01648} = 0.6372,$$

$$\varphi = \arcsin\frac{P_t}{2|OD|} = 0.6909 \text{ rad},$$

$$\gamma = \frac{\pi}{2} - 2\varphi = \frac{\pi}{2} - 2\cdot 0.6909 = 0.1890 \text{ rad},$$

$$|AB| = P_t\mathrm{tg}\gamma = 0.021\cdot 0.19128 = 0.004017 \text{ m},$$

$$h_{OAB} = \frac{P_t}{2} = \frac{0.021}{2} = 0.0105 \text{ m},$$

$$A_{OAB} = \frac{1}{2}|AB|h_{OAB} = \frac{1}{2}\cdot 0.004017\cdot 0.0105 = 2.1089\cdot 10^{-5} \text{ m}^2,$$

$$|BC| = |OD|\cdot\mathrm{tg}\varphi = 0.01648\cdot\mathrm{tg}0.6909 = 0.01363,$$

$$A_{OBC} = \frac{1}{4}|BC||OD| = \frac{1}{4}\cdot 0.01363\cdot 0.01648 = 5.6156\cdot 10^{-5} \text{ m}^2,$$

$$r_2^* = \sqrt{\frac{2A_{OAB} + 4A_{OBC}}{\pi}} = \sqrt{\frac{2 \cdot 2.1089 \cdot 10^{-5} + 4 \cdot 5.6156 \cdot 10^{-5}}{\pi}} =$$

$$= 9.2155 \cdot 10^{-3}\,\text{m}, \quad m = \sqrt{\frac{2\alpha}{\lambda t}} = \sqrt{\frac{2 \cdot 40}{165 \cdot 0.000115}} = 64.93\ 1/\text{m},$$

$$L = r_2^* - r_1 = 9.2155 \cdot 10^{-3} - 3.795 \cdot 10^{-3} = 5.4205 \cdot 10^{-5}\,\text{m},$$

$$mL = 0.35195,$$

$$\eta_e = \frac{2 \cdot 3.795 \cdot 10^{-3}}{2 \cdot 3.795 \cdot 10^{-3} + 5.4205 \cdot 10^{-3}} \frac{\text{tgh}\ 0.35195}{0.35195} \times$$

$$\times \left[1 + \frac{\text{tgh}\ 0.35195}{2 \cdot 64.93 \cdot 3.795 \cdot 10^{-3}} - 0.071882 \frac{\text{tgh}\ 0.35195^{3.7482}}{\left(64.93 \cdot 3.795 \cdot 10^{-3}\right)^{1.481}} \right] =$$

$$= 0.5834 \cdot 0.96066 \cdot \left(1 + 0.68606 - 9.8256 \cdot 10^{-3}\right) = 0.9394.$$

b) *Sector method*

In keeping with the sector method, a triangular fin *OAB* will be substituted by an equivalent circular fin sector (with an equivalent surface area).

$$A_{OAB} - \frac{2\gamma}{2\pi} r_1^2 = \frac{2\gamma}{2\pi} \pi r_{2,1}^2 - \frac{2\gamma}{2\pi} \pi r_1^2,$$

hence,

$$r_{2,1} = \sqrt{\frac{A_{OAB}}{\gamma}} = \sqrt{\frac{2.1089 \cdot 10^{-5}}{0.189}} = 0.010563\ \text{m}.$$

Circular fin efficiency with an outer radius $r_{2,1}$ and inner radius r_1 amounts to

$$L_1 = r_{2,1} - r_1 = 0.010563 - 3.795 \cdot 10^{-3} = 6.768 \cdot 10^{-3}\ \text{m},$$

$$mL_1 = 0.43945,$$

$$\eta_1 = \frac{2 \cdot 3.795 \cdot 10^{-3}}{2 \cdot 3.795 \cdot 10^{-3} + 6.768 \cdot 10^{-3}} \frac{\text{tgh}0.43945}{0.43945} \times$$

$$\times \left[1 + \frac{\text{tgh}0.43945}{2 \cdot 64.93 \cdot 3.795 \cdot 10^{-3}} - 0.071882 \frac{\text{tgh}0.43945^{3.7482}}{\left(64.93 \cdot 3.795 \cdot 10^{-3}\right)^{1.481}} \right] =$$

$$= 0.5834 \cdot 0.94024 \cdot \left(1 + 0.83842 - 0.020836\right) = 0.9034.$$

Radius $r_{2,2}$ of a circular fin sector with a surface area identical to a surface area of a triangular fin *OBC* is calculated in a similar way.

$$L_2 = r_{2,2} - r_1 = 5.2205 \cdot 10^{-3} \text{ m},$$

$$mL_2 = 64.93 \cdot 5.2205 \cdot 10^{-3} = 0.33897,$$

$$\eta_2 = \frac{2 \cdot 3.795 \cdot 10^{-3}}{2 \cdot 3.795 \cdot 10^{-3} + 5.2205 \cdot 10^{-3}} \frac{\text{tgh}0.33897}{0.33897} \times$$

$$\times \left[1 + \frac{\text{tgh}0.33897}{2 \cdot 64.93 \cdot 3.795 \cdot 10^{-3}} - 0.071882 \frac{(\text{tgh}0.33897)^{3.7482}}{(64.93 \cdot 3.795 \cdot 10^{-3})^{1.481}} \right] =$$

$$= 0.59248 \cdot 0.96338 \cdot (1 + 0.66263 - 8.6258 \cdot 10^{-3}) = 0.9441.$$

Fin efficiency calculated by means of a sector method comes to

$$\eta_s = \frac{2\left(A_{OAB} - \frac{2\gamma}{2\pi}\pi r_1^2\right)\eta_1 + 4\left(A_{OBC} - \frac{2\varphi}{2\pi}\pi r_1^2\right)\eta_2}{2\left(A_{OAB} - \frac{2\gamma}{2\pi}\pi r_1^2\right) + 4\left(A_{OBC} - \frac{2\varphi}{2\pi}\pi r_1^2\right)} =$$

$$= \frac{2\left(A_{OAB} - \gamma r_1^2\right)\eta_1 + 4\left(A_{OBC} - \varphi r_1^2\right)\eta_2}{2\left(A_{OAB} - \gamma r_1^2\right) + 4\left(A_{OBC} - \varphi r_1^2\right)}.$$

After substitution of the numerical values, one obtains

$$\eta_s = \frac{2(2.1089 \cdot 10^{-5} - 0.189 \cdot 0.003795^2)0.9034 + 4(5.6156 \cdot 10^{-5} - 0.6909}{2(2.1089 \cdot 10^{-5} - 0.189 \cdot 0.003795^2) + 4(5.6156 \cdot 10^{-5} - 0.6909} \times$$

$$\times \frac{\cdot 0.003795^2)0.9441}{\cdot 0.003795^2)} = \frac{2 \cdot 1.8367 \cdot 10^{-5} \cdot 0.9034 + 4 \cdot 4.62056 \cdot 10^{-5} \cdot 0.9441}{3.6734 \cdot 10^{-5} + 1.84823 \cdot 10^{-4}} =$$

$$= 0.9373.$$

From the comparison of results, with $\eta_e = 0.9394$ and $\eta_s = 0.9373$, it is clear that method I, in which the whole fin is substituted by an equivalent circular fin, and sector method generate almost identical results. However, method I requires less computational work.

Exercise 6.21 Calculating Rectangular Fin Efficiency

The aim is to calculate efficiency of an equivalent fin in a fin-plate exchanger with a serial pipe configuration (Fig. 6.25). Pipes with outer diameters d_z = 10 mm are arranged with the following pitches: P_t = 25 mm and P_l = 22 mm. The thickness of fin-plates from aluminium alloy with thermal conductivity λ = 165 W/(m·K) is t = 0.13 mm. Heat transfer coefficient on a fin-plate surface equals α = 35 (W/m²·K). Calculate efficiency of an equivalent circular fin with radius r_2^* by means of a simplified (5) from Ex. 6.18.

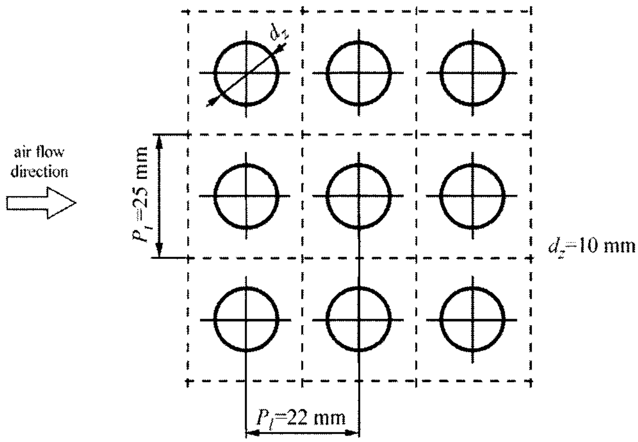

Fig. 6.25. Serial pipe configuration in a fin-plate exchanger

Solution

Equivalent radius r_2^* of a circular fin will be calculated from the condition of equality of a circular and rectangular fin surface area (Fig. 6.25):

$$2\left[\pi\left(r_2^*\right)^2 - \pi r_1^2\right] = 2\left(P_t P_l - \pi r_1^2\right).$$

From the above formula, one obtains

$$r_2^* = \sqrt{P_t P_l / \pi}\ .$$

The efficiency of a circular fin is calculated from (5), Ex. 6.18

$$\eta = \frac{1}{1 + \frac{1}{3}(mL)^2 \sqrt{r_2^* / r_1}},$$

where

$$m = \sqrt{2\alpha / \lambda t}, \qquad L = r_2^* - r_1, \qquad r_1 = d_z / 2.$$

After substitution, one obtains

$$r_2^* = \sqrt{\frac{(0.025 \cdot 0.022)}{\pi}} = 0.01323 \text{ m}, \qquad r_1 = 0.5 d_1 = 0.005 \text{ m},$$

$$L = r_2^* - r_1 = 0.01323 - 0.005 = 0.008231 \text{ m},$$

$$m = \sqrt{\frac{2 \cdot 35}{165 \cdot 0.00013}} = 57.13 \text{ 1/m}, \ mL = 57.13 \cdot 0.008231 = 0.47023,$$

$$\eta = \frac{1}{1 + \frac{1}{3} \cdot 0.47023^2 \sqrt{\frac{0.01323}{0.005}}} = 0.8929.$$

Exercise 6.22 Heat Transfer Coefficient in Exchangers with Extended Surfaces

Derive a formula for an overall heat transfer coefficient k for thermal exchangers with extended surfaces (Fig. 6.26). Compare coefficients to an inner surface k_{rw} and outer surface k_{rz} of a smooth pipe and to an entire outer surface k_c (fins + surfaces between fins). Take into account the presence of scale on an inner fin-free surface, assuming that coefficient α_g from the finned side is determined in a way that it already accounts for the resistance from external scale.

Solution

In order to derive a formula for a heat transfer coefficient, an equivalent heat transfer coefficient α_{zr} will be introduced with regard to smooth surface area A_g, on which fins are mounted. It takes into consideration both heat transfer by fins and heat transfer by inter-fin spaces. It is assumed, at the same time, that fins do not disturb a one-dimensional temperature field in a wall of an element to which they are attached. If T_w stands for a wall surface temperature from a liquid's side, while T_z is an outer surface temperature from the gas side (Fig. 6.26), then heat flow transferred by an outer surface

$$\dot{Q}_s = \alpha_{zr} A_g \left(T_z - T_g\right) = \alpha_g A_{m\dot{z}} \left(T_z - T_g\right) + \alpha_g A_{\dot{z}} \left(T_z - T_g\right) \eta, \qquad (1)$$

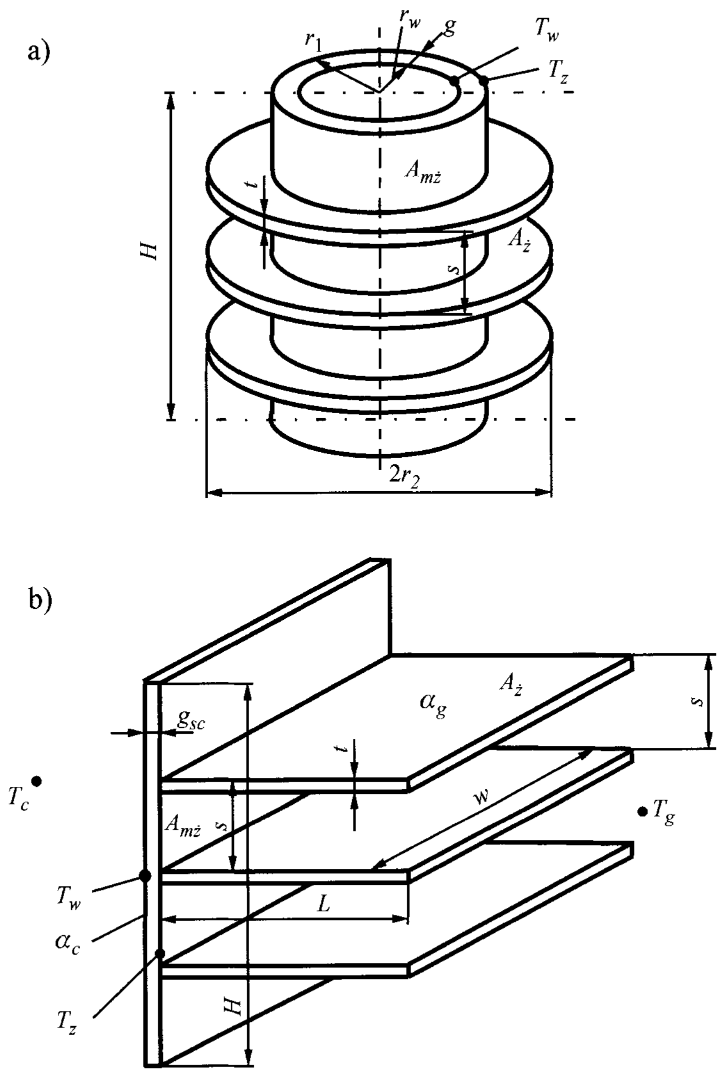

Fig. 6.26. Extended heat transfer surfaces: (**a**) finned pipe, (**b**) flat plate with fins

where $\dot{Q}_s$ stands for a heat flow that flows from a liquid of temperature T_c to a gas of temperature T_g that corresponds to a single pitch s, while $T_c > T_s$. Surface area A_g stands for a surface area of a smooth pipe or fin-free plate that corresponds to a single pitch s:

$$A_g = A_{rz} = 2\pi r_1 s \quad \text{for a smooth pipe,}$$

$$A_g = A = w \cdot s \quad \text{for a smooth plate.}$$

Surface area between fins, which corresponds to single pitch s, can be calculated in a similar way:

$$A_{m\dot{z}} = 2\pi r_1 (s - t) \quad \text{for a finned pipe,}$$

$$A_{m\dot{z}} = (s - t) w \quad \text{for a finned plate.}$$

The surface of a single fin, which corresponds to a single pitch is given by

$$A_{\dot{z}} = 2\pi \left(r_{2c}^2 - r_1^2 \right), \quad \text{where } r_{2c} = r_2 + \frac{t}{2}, \text{ for a finned pipe,}$$

$$A_{\dot{z}} = 2wL_c, \quad \text{where } L_c = L + \frac{t}{2}, \text{ for a finned plate.}$$

From (1), one obtains a formula for an equivalent heat transfer coefficient

$$\alpha_{zr} = \alpha_g \left(\frac{A_{m\dot{z}}}{A_g} + \frac{A_{\dot{z}}}{A_g} \eta \right). \tag{2}$$

If an equivalent heat transfer coefficient is known, then the calculation of an overall heat transfer coefficient k_{rz}, with regard to a plate surface or an outer surface of a smooth pipe, is conducted in an identical way as the calculation of a smooth pipe or plate with a heat transfer coefficient equal to α_{zr}. Heat flow $\dot{Q}_s$, which corresponds to a single pitch, can be calculated for a pipe (Fig. 6.27a) using the following formula:

$$\dot{Q}_s = k_{rz} A_g \left(T_c - T_g \right) = \dot{q}_1 A_g . \tag{3}$$

a) b)

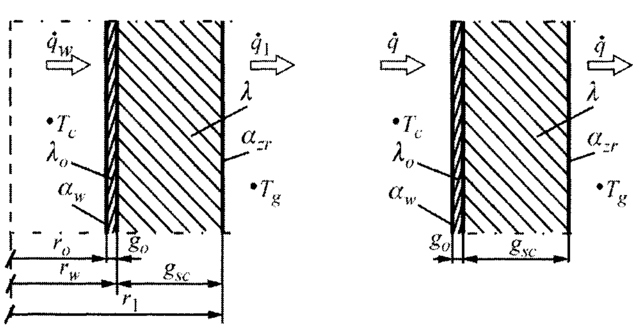

Fig. 6.27. Auxiliary diagram for determining overall heat transfer coefficient k for finned surfaces with a scale layer of thickness g_o: (**a**) pipe, (**b**) plate

Overall heat transfer coefficient k_{rz}, which pertains to a surface area of a smooth pipe with an outer radius of r_1 is calculated in the same way as the coefficient for a smooth pipe, although here one allows for the fact that four temperature decreases contribute to a temperature drop of $T_c - T_g$; these are:

• Temperature decrease between a temperature of a liquid T_c and an inner scale surface temperature T_o formulated as

$$T_c - T_o\big|_{r=r_o} = \frac{\dot{q}_o}{\alpha_c}. \tag{4}$$

Since the following equality occurs in a pipe

$$\dot{Q} = \dot{q}_o 2\pi r_o H = \dot{q}_w 2\pi r_w H = \dot{q}_1 2\pi r_1 H, \tag{5}$$

then

$$\dot{q}_o r_o = \dot{q}_w r_w = \dot{q}_1 r_1, \tag{6}$$

hence $\dot{q}_o$ can be calculated as shown below:

$$\dot{q}_o = \frac{\dot{q}_1 r_1}{r_o} \approx \frac{\dot{q}_1 r_1}{r_w}. \tag{7}$$

Equation (4) can be written then in the following way:

$$T_c - T\big|_{r=r_o} = \frac{\dot{q}_1 r_1}{r_w \alpha_c}. \tag{8}$$

• Temperature decrease inside a scale layer is formulated as

$$T\big|_{r=r_o} - T\big|_{r=r_w} \approx \frac{\dot{q}_o g_o}{\lambda_o} \approx \frac{\dot{q}_1 r_1}{r_w} \frac{g_o}{\lambda_o}. \tag{9}$$

Hence, after introducing thermal contact resistance

$$R_o = \frac{g_o}{\lambda_o} \left[(\mathrm{m}^2 \cdot \mathrm{K})/\mathrm{W}\right] \tag{10}$$

one obtains

$$T\big|_{r=r_o} - T\big|_{r=r_w} = \frac{\dot{q}_1 r_1}{r_w} R_o. \tag{11}$$

• Temperature decrease across a thickness of a pipe wall is

$$T\big|_{r=r_w} - T\big|_{r=r_1} = \frac{\dot{q}_1 r_1}{\lambda} \ln \frac{r_1}{r_w}. \tag{12}$$

• Temperature decrease between an outer surface of a pipe and gas temperature T_g is

$$T\big|_{r=r_1} - T_g = \frac{\dot{q}_1}{\alpha_{zr}}. \tag{13}$$

After summation of four temperature decreases formulated by (8)–(13), one obtains temperature difference $(T_c - T_g)$

$$T_c - T_g = \frac{\dot{q}_1 r_1}{r_w \alpha_c} + \frac{\dot{q}_1 r_1}{r_w} R_o + \frac{\dot{q}_1 r_1}{\lambda} \ln\frac{r_1}{r_w} + \frac{\dot{q}_1}{\alpha_{zr}}. \tag{14}$$

From the definition of an overall heat transfer coefficient k_{rz}, the following results from (3):

$$T_c - T_g = \frac{\dot{q}_1}{k_{rz}}. \tag{15}$$

From the dependence (14) and (15), one obtains

$$\frac{\dot{q}_1}{k_{rz}} = \frac{\dot{q}_1 r_1}{r_w \alpha_c} + \frac{\dot{q}_1 r_1}{r_w} R_o + \frac{\dot{q}_1 r_1}{\lambda} \ln\frac{r_1}{r_w} + \frac{\dot{q}_1}{\alpha_{zr}}, \tag{16}$$

hence,

$$\frac{1}{k_{rz}} = \frac{r_1}{r_w \alpha_c} + \frac{r_1}{r_w} R_o + \frac{r_1}{\lambda} \ln\frac{r_1}{r_w} + \frac{1}{\alpha_{zr}}. \tag{17}$$

One can determine overall heat transfer coefficient k_w with regard to inner surface A_w and coefficient k_c with regard to total surface $A_c = A + A_{mc}$ from the following equality:

$$\dot{Q}_s = k_{zr} A_g \left(T_c - T_g\right) = k_w A_w \left(T_c - T_g\right) = k_c A_c \left(T_c - T_g\right),$$

hence,

$$\frac{1}{k_w} = \frac{1}{k_{rz}} \frac{A_w}{A_g}, \qquad \frac{1}{k_c} = \frac{1}{k_{rz}} \frac{A_c}{A_g},$$

where: $A_w = 2\pi r_w s$, $\quad A_g = 2\pi r_1 s$, $\quad A_c = A + A_m$.

In the case of a finned flat wall (Fig. 6.27b), overall heat transfer coefficient k_g is determined, with regard to a smooth surface, from a formula

$$\frac{\dot{q}}{k_g} = \frac{\dot{q}}{\alpha_w} + \frac{\dot{q} g_o}{\lambda_o} + \frac{\dot{q} g_{sc}}{\lambda} + \frac{\dot{q}}{\alpha_{zr}},$$

from which the following equality results

$$\frac{1}{k_g}=\frac{1}{\alpha_w}+R_o+\frac{g_{sc}}{\lambda}+\frac{1}{\alpha_{zr}},$$

where $R_o = g_o/\lambda_o$ [(m^2·K)/W] is a thermal contact resistance of a scale layer.

The unknown overall heat transfer coefficient with regard to total surface $A_c = A + A_m$, can be determined from the condition

$$\dot{Q}_s = k_g A_g \left(T_c - T_g\right) = k_c A_c \left(T_c - T_g\right),$$

which results in

$$\frac{1}{k_c}=\frac{1}{k_g}\frac{A_c}{A_g}.$$

Exercise 6.23 Calculating Overall Heat Transfer Coefficient in a Fin Plate Exchanger

Calculate overall heat transfer coefficient in a fin-plate exchanger analyzed in Ex. 6.20. The thickness of an aluminium made pipe wall with thermal conductivity λ_r = 165 W/(m·K) measures g_r = 1 mm. Heat transfer coefficient on an inner surface of the pipe is α_c = 5000 W/(m^2·K). The pitch of fin spacing is s = 1.6 mm. Other values remain the same as they were in Ex. 6.20. For the calculation, assume fin efficiency of $\eta = \eta_e = 0.9394$, calculated by means of method I (Ex. 6.20). How many times does an overall heat transfer coefficient increase in comparison to a smooth pipe?

Solution

First, surface areas will be calculated:

- fin surface area (Fig. 6.24)

$$A_{\dot{z}} = \left(2A_{OAB} + 4A_{OBC}\right) = 2\left(2\cdot 2.1089\cdot 10^{-5} + 4\cdot 5.6156\cdot 10^{-5}\right) = \\ = 5.33604\cdot 10^{-4}\ \text{m}^2. \qquad (1)$$

- surface area between fins

$$A_{m\dot{z}} = \pi d_z \left(s - t\right) = \pi\cdot 7.59\cdot 10^{-3}\left(1.6 - 0.115\right)\cdot 10^{-3} = 3.54094\cdot 10^{-5}\ \text{m}^2.$$

- smooth surface area

$$A_g = \pi d_z s = \pi\cdot 7.59\cdot 10^{-3}\cdot 1.6\cdot 10^{-3} = 3.81515\cdot 10^{-5}\text{m}^2.$$

- surface area of a fin and a pipe between fins

$$A_c = A_{\dot{z}} + A_{m\dot{z}} = 5.33604\cdot 10^{-4} + 3.54094\cdot 10^{-5} = 5.690134\cdot 10^{-4}\ \text{m}^2.$$

An equivalent heat transfer coefficient is

$$\alpha_{zr} = \alpha_g \left(\frac{A_{m\dot{z}}}{A_g} + \frac{A_{\dot{z}}}{A_g}\eta \right) = 40 \left(\frac{3.54094 \cdot 10^{-5}}{3.81515 \cdot 10^{-5}} + \frac{5.33604 \cdot 10^{-4}}{3.81515 \cdot 10^{-5}} \cdot 0.9394 \right) = \quad (2)$$

$$= 596.58 \text{ W}/\left(\text{m}^2 \cdot \text{K}\right)$$

Overall heat transfer coefficient (k_{rz}), with regard to an outer surface of a smooth pipe, is calculated in the following way:

$$\frac{1}{k_{rz}} = \frac{r_1}{r_w \alpha_c} + \frac{r_1}{\lambda_r} \ln \frac{r_1}{r_w} + \frac{1}{\alpha_{zr}} = \frac{7.59 \cdot 10^{-3}}{2\left(\left(7.59 \cdot 10^{-3}/2\right) - 0.001\right)} \frac{1}{5000} +$$

$$+ \frac{7.59 \cdot 10^{-3}}{2 \cdot 165} \times \ln \frac{7.59 \cdot 10^{-3}}{2\left(\frac{7.59 \cdot 10^{-3}}{2} - 0.001 \right)} + \frac{1}{596.58} = 2.71556 \cdot 10^{-4} +$$

$$+7.0346 \cdot 10^{-6} + 1.676221 \cdot 10^{-3} = 1.9548 \cdot 10^{-3} \text{ (m}^2 \cdot \text{K)/W},$$

$$k_{rz} = 511.56 \text{ W/(m}^2 \cdot \text{K)}.$$

Overall heat transfer coefficient (k) for smooth fin-free pipes with respect to an outer surface of a pipe is:

$$\frac{1}{k} = \frac{r_1}{r_w \alpha_c} + \frac{r_1}{\lambda_r} \ln \frac{r_1}{r_w} + \frac{1}{\alpha} = 2.71556 \cdot 10^{-4} + 7.0346 \cdot 10^{-6} + 0.025 =$$

$$= 0.0252786 \text{ (m}^2 \cdot \text{K)/W},$$

$$k = 39.559 \text{ W/(m}^2 \cdot \text{K)}.$$

Due to the application of flat fins (fin plates), heat transfer coefficient increased n times:

$$n = \frac{k_{rz}}{k} = \frac{511.56}{39.559} = 12.93.$$

Exercise 6.24 Overall Heat Transfer Coefficient for a Longitudinally Finned Pipe with a Scale Layer on an Inner Surface

A *dowtherm* was applied in a steam boiler. It flows inside a ring-shaped gap between two pipes. In order to increase heat transfer coefficient, the center pipe is longitudinally finned on its outer surface (Fig. 6.28). A water-vapour mixture flows inside the pipe. The following data is used for the calculation:

- outer surface pipe diameter $d_l = 48.3$ mm (Fig. 6.29),
- inner surface pipe diameter $d_w = 40.94$ mm,
- fin height $L = 12.7$ mm ($d_2 = d_1 + 2L = 73.7$ mm),
- fin thickness $t = 0.889$ mm,
- fin number on the pipe's perimeter $N = 24$,
- thermal conductivity of the fins and pipe material: $\lambda = 55$ W/(m·K) (carbon steel),

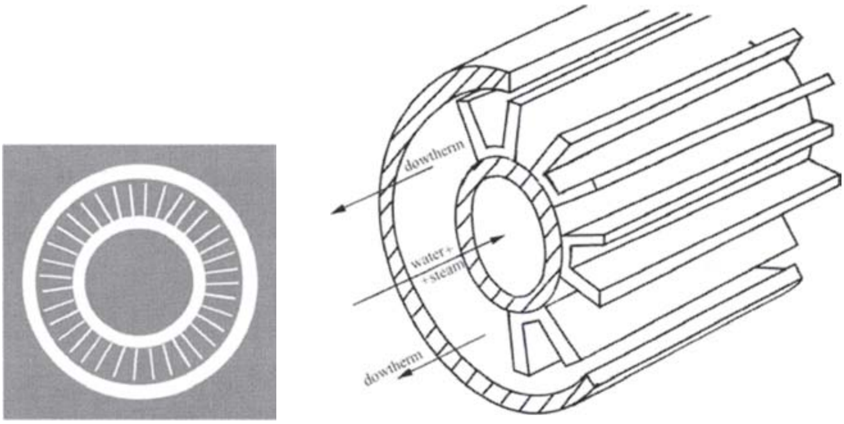

Fig. 6.28. A diagram of a heat exchanger of pipe-in-pipe type with an internal pipe finned on an outer surface

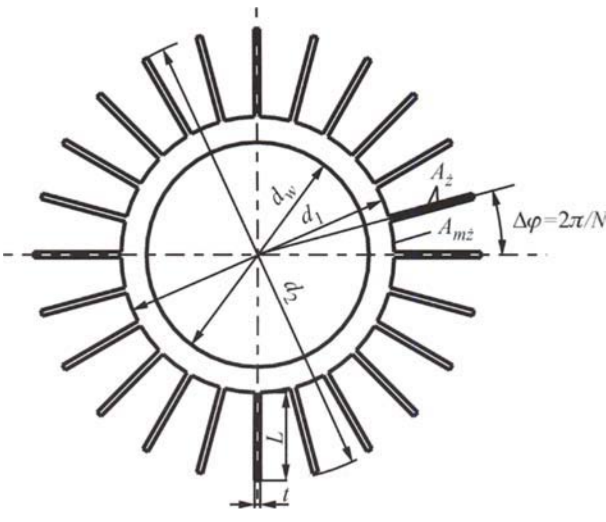

Fig. 6.29. A diagram of a longitudinally finned pipe

- heat transfer coefficient from the dowtherm's side: α_d = 1500 W/(m²·K),
- heat transfer coefficient on an inner surface of the pipe: α_w = 10000 W/(m²·K).

How much will a heat transfer coefficient decrease when a layer of a boiler scale, g_o = 0.1 mm thick with a thermal conductivity of λ_o = 1 W/(m·K) accumulates on an inner surface of the pipe?

Solution

$$L_c = L + \frac{t}{2} = 12.7\cdot 10^{-3} + \frac{1}{2}\cdot 0.889\cdot 10^{-3} = 13.1445\cdot 10^{-3}\,\text{m}.$$

Fin parameter comes to

$$m = \sqrt{\frac{2\alpha_d}{\lambda t}} = \sqrt{\frac{2\cdot 1500}{55\cdot 0.889\cdot 10^{-3}}} = 247.7 \ \ 1/\text{m},$$

$$mL_c = 247.7\cdot 13.1445\cdot 10^{-3} = 3.256.$$

Fin efficiency is formulated as

$$\eta = \frac{\text{tgh}(mL_c)}{mL_c} = \frac{\text{tgh}(3.256)}{3.256} = 0.3062.$$

In order to calculate an equivalent heat transfer coefficient α_{zr}, surface area A, A_m and A_g, which correspond to 1 m of pipe and one pitch $\Delta\varphi$, will be calculated first (Fig. 6.29).

$$A_{ż} = 2\cdot L_c\cdot 1 = 2\cdot 13.1445\cdot 10^{-3}\cdot 1 = 2.6289\cdot 10^{-2}\,\text{m}^2,$$

$$A_g = \Delta\varphi\cdot\frac{d_1}{2}\cdot 1 = \frac{2\pi}{24}\cdot\frac{48.3\cdot 10^{-3}}{2} = 6.3225\cdot 10^{-3}\,\text{m}^2,$$

$$A_{mż} = A_g - t\cdot 1 = 6.3225\cdot 10^{-3} - 0.889\cdot 10^{-3} = 5.4335\cdot 10^{-3}\,\text{m}^2.$$

An equivalent heat transfer coefficient α_{zr} calculated with respect to an outer surface of a smooth pipe is

$$\alpha_{zr} = \alpha_d\left(\frac{A_{mż}}{A_g} + \frac{A_{ż}}{A_g}\eta\right) = 1500\cdot\left(\frac{5.4335\cdot 10^{-3}}{6.3225\cdot 10^{-3}} + \frac{2.6289\cdot 10^{-2}}{6.3225\cdot 10^{-3}}\cdot 0.3062\right) =$$

$$= 3198.85 \ \ \text{W/(m}^2\cdot\text{K)}.$$

Overall heat transfer coefficient k_{rz} calculated with respect to an outer surface of a smooth pipe is formulated as

$$\frac{1}{k_{rz}} = \frac{r_1}{r_w}\frac{1}{\alpha_w} + \frac{r_1}{r_w}R_o + \frac{r_1}{\lambda}\ln\frac{r_1}{r_w} + \frac{1}{\alpha_{zr}}.$$

First coefficient k_{rz} is calculated, for the situation when an inner pipe surface is scale free. Since $R_o = 0$ and

$$r_1 = \frac{d_1}{2} = \frac{48.3\cdot 10^{-3}}{2} = 24.15\cdot 10^{-3}\,\text{m},\quad r_w = \frac{40.94\cdot 10^{-3}}{2} = 20.47\cdot 10^{-3}\,\text{m},$$

then,

$$\frac{1}{k_{rz}} = \frac{24.15\cdot 10^{-3}}{20.47\cdot 10^{-3}}\frac{1}{10000} + \frac{24.15\cdot 10^{-3}}{55}\ln\frac{24.15\cdot 10^{-3}}{20.47\cdot 10^{-3}} + \frac{1}{3198.85} =$$

$$= 1.17978\cdot 10^{-4} + 7.2592\cdot 10^{-5} + 3.1264\cdot 10^{-4} = 5.0321\cdot 10^{-4}\ (\text{m}^2\cdot\text{K})/\text{W}.$$

hence,

$$k_{rz} = 1987.2\ \text{W}/(\text{m}^2\cdot\text{K}).$$

It can be seen that here overall heat transfer coefficient is only slightly bigger than a dowtherm heat transfer coefficient, equal to α_d = 1500 $\text{W}/(\text{m}^2\cdot\text{K})$. This is caused by a large heat transfer coefficient α_w, for which fin efficiency is rather small. In order to consider the effect of scale layer on a calculated coefficient k_{rz} , it is sufficient to add calculated value $1/k_{rz}$ to a scale resistance equal to

$$\frac{r_1}{r_w}R_o = \frac{r_1}{r_w}\cdot\frac{g_o}{\lambda_o} = \frac{24.15\cdot 10^{-3}}{20.47\cdot 10^{-3}}\cdot\frac{0.0001}{1} = 1.1789\cdot 10^{-4}\,(\text{m}^2\cdot\text{K})/\text{W}.$$

Overall heat transfer coefficient k'_{rz} for a pipe with a scale layer on an inner surface is

$$\frac{1}{k'_{rz}} = 5.0321\cdot 10^{-4} + 1.1798\cdot 10^{-4} = 6.2119\cdot 10^{-4}\ (\text{m}^2\cdot\text{K})/\text{W},$$

therefore,

$$k'_{rz} = 1609.82\,\text{W}/(\text{m}^2\cdot\text{K}).$$

Due to a scale accumulation on an inner surface, overall heat transfer coefficient became smaller by

$$\Delta k_{rz} = k_{rz} - k'_{rz} = 1987.23 - 1609.82 = 377.4\,\text{W}/(\text{m}^2\cdot\text{K}).$$

Exercise 6.25 Overall Heat Transfer Coefficient for a Longitudinally Finned Pipe

Calculate overall heat transfer coefficient, with respect to a smooth outer surface of a pipe, for finned pipes with an outer diameter d_1 = 16 mm and thickness g_r = 2 mm, made of aluminium with thermal conductivity λ = 205 W/(m·K). The outer diameter of a fin,with thickness t = 0.1 is d_2 = 35 mm. Pitch of the fin spacing is s = 1.1 mm. A hot water of temperature t_w = 80°C flows inside the pipe at a speed of w_1 = 0.5 m/s; it warms the air, which circulates the finned pipes crosswise. Heat transfer coefficient from the air side is equal to α = 60 W/(m^2·K). How is the overall heat transfer coefficient going to change when the velocity of flowing water increases to w_2 = 2 m/s? Calculate heat transfer coefficient from water to pipe using the following Stender and Merkel formulas

$$\alpha_w = 2040\left(1+0.015t_w\right)w^{0.87}d_w^{-0.13}, \tag{1}$$

in which the units are as follow :

α_w [W/(m^2·K)], t_w [°C], w [m/s], d_w [m].

Solution

A diagram of the finned pipe is presented in Fig. 6.30. First, we will calculate fin efficiency from formula

$$\eta = \frac{1}{1+\frac{1}{3}\left(mL_c\right)^2\sqrt{\frac{r_{2c}}{r_1}}}, \tag{2}$$

where,

$$r_1 = \frac{d_1}{2} = \frac{16\cdot10^{-3}}{2} = 8\cdot10^{-3}\ \text{m},$$

$$r_2 = \frac{d_2}{2} = \frac{35\cdot10^{-3}}{2} = 17.5\cdot10^{-3}\ \text{m},$$

$$r_{2c} = r_2 + \frac{t}{2} = 17.5\cdot10^{-3} + 0.05\cdot10^{-3} = 17.55\cdot10^{-3}\ \text{m},$$

$$L_c = L + \frac{t}{2} = r_2 - r_1 + \frac{t}{2} = 17.5\cdot10^{-3} - 8\cdot10^{-3} + \frac{0.1\cdot10^{-3}}{2} = 9.55\cdot10^{-3}\text{m}.$$

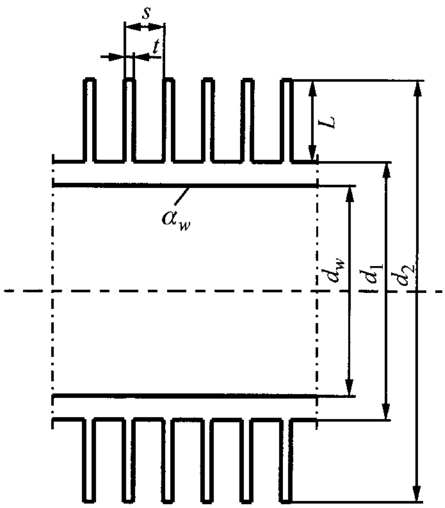

Fig. 6.30. A diagram of a finned pipe with circular fins of constant thickness

Fin parameter m is

$$m = \sqrt{\frac{2\alpha}{\lambda t}} = \sqrt{\frac{2 \cdot 60}{205 \cdot 0.0001}} = 76.51 \quad 1/\text{m},$$

$$mL_c = 76.51 \cdot 9.55 \cdot 10^{-3} = 0.7307.$$

Thus:

$$\eta = \frac{1}{1 + \frac{1}{3}(0.7307)^2 \sqrt{\frac{17.55 \cdot 10^{-3}}{8 \cdot 10^{-3}}}} = 0.7914\,.$$

Next, we will calculate fin surface area ($A_{\dot{z}}$) and smooth pipe surface area (A_g) on an outer surface and inner surface area (A_w) for a single pitch s:

$$A_{\dot{z}} = 2\pi\left(r_2^2 - r_1^2\right) = 2\pi\left[\left(17.5 \cdot 10^{-3}\right)^2 - \left(8 \cdot 10^{-3}\right)^2\right] = 1.5221 \cdot 10^{-3} \text{ m}^2,$$

$$A_g = 2\pi r_1 s = 2\pi \cdot 8 \cdot 10^{-3} \cdot 1.1 \cdot 10^{-3} = 5.5292 \cdot 10^{-5} \text{ m}^2,$$

$$A_w = 2\pi r_w s = 2\pi\left(r_1 - g_r\right)s = 2\pi\left(8 - 2\right) \cdot 10^{-3} \cdot 1.1 \cdot 10^{-3} = 4.1469 \cdot 10^{-5} \text{ m}^2.$$

Surface area between fins per single pitch is

$$A_{m\dot{z}} = A_g - \pi d_1 t = 2\pi r_1\left(s - t\right) = 2\pi \cdot 8 \cdot 10^{-3}\left(1.1 - 0.1\right)10^{-3} =$$
$$= 5.0265 \cdot 10^{-5}\text{m}^2$$

Equivalent heat transfer coefficient α_{zr}, calculated with respect to a smooth outer surface of the pipe is

$$\alpha_{zr} = \alpha\left(\frac{A_{m\dot{z}}}{A_g} + \frac{A_{\dot{z}}}{A_g}\eta\right) = 60\cdot\left(\frac{5.0265\cdot 10^{-5}}{5.5292\cdot 10^{-5}} + \frac{1.5221\cdot 10^{-3}}{5.5292\cdot 10^{-5}}\cdot 0.7914\right) =$$

$$= 1362 \ \text{W/(m}^2\cdot\text{K)}.$$

Overall heat transfer coefficient from water to inner surface of the pipe is

$$\alpha_w = 2040(1+0.015\cdot 80)0.5^{0.87}\cdot(0.012)^{-0.13} = 4363.8 \ \text{W/(m}^2\cdot\text{K)}.$$

Heat transfer coefficient with respect to smooth outer surface of the pipe is calculated using (17), from Ex. 6.22:

$$\frac{1}{k_{rz}} = \frac{r_1}{r_w\alpha_w} + \frac{r_1}{\lambda}\ln\frac{r_1}{r_w} + \frac{1}{\alpha_{zr}} = \frac{8\cdot 10^{-3}}{(8-2)\cdot 10^{-3}\cdot 4363.8} +$$

$$+\frac{8\cdot 10^{-3}}{205}\ln\frac{8\cdot 10^{-3}}{(8-2)\cdot 10^{-3}} + \frac{1}{1362} = 3.055441\cdot 10^{-4} +$$

$$+1.12266\cdot 10^{-5} + 7.34198\cdot 10^{-4} = 1.05097\cdot 10^{-3} \ \text{(m}^2\cdot\text{K)/W},$$

hence,

$$k_{rz} = 951.5 \ \text{W/(m}^2\cdot\text{K)}.$$

Next, we will calculate overall heat transfer coefficient k'_{rz} when the velocity of a flowing water equals $w_2 = 2$ m/s. Heat transfer coefficient α'_w is larger:

$$\alpha'_w = 2040(1+0.015\cdot 80)\cdot 2^{0.87}\cdot(0.012)^{-0.13} = 14576.6 \ \text{W/(m}^2\cdot\text{K)}.$$

Overall heat transfer coefficient is calculated in the following way:

$$\frac{1}{k'_{rz}} = \frac{8\cdot 10^{-3}}{(8-2)\cdot 10^{-3}\cdot 14576.6} + 1.12266\cdot 10^{-5} + 7.34198\cdot 10^{-4} =$$

$$= 8.36895\cdot 10^{-4} \ \text{W/(m}^2\cdot\text{K)},$$

$$k'_{rz} = 1194.9 \ \text{W/(m}^2\cdot\text{K)}.$$

Due to the increased water velocity, from $w_1 = 0.5$ m/s to $w_2 = 2$ m/s, the overall heat transfer coefficient has increased by the following percentage value

$$W = \frac{k'_{rz} - k_{rz}}{k_{rz}}\cdot 100\% = \frac{1194.9 - 951.5}{951.5}\cdot 100\% = 25.6\%.$$

Exercise 6.26 Determining One-Dimensional Temperature Distribution in a Flat Wall by Means of Finite Volume Method

Determine temperature distribution and heat flux on a slab surface, in which temperature field is described using the following differential equation and boundary conditions:

$$\frac{d^2T}{dx^2} = -\frac{\dot{q}_v}{\lambda}, \qquad 0 \le x \le L, \tag{1}$$

$$\left.\frac{dT}{dx}\right|_{x=0} = 0, \tag{2}$$

$$-\lambda \left.\frac{dT}{dx}\right|_{x=L} = \alpha\left[T|_{x=L} - T_{cz}\right]. \tag{3}$$

Determine temperature distribution by means of finite volume method for the following data: λ = 15 W/(m·K), α = 300 W/(m^2·K), L = 0.06 m, $\dot{q}_v$ = 1·10^6 W/m^3, T_{cz} = 20°C. Compare determined temperature distribution with the accurate analytical solution (Ex. 6.11)

$$T(x) = T_{cz} + \frac{\dot{q}_v L}{\alpha} + \frac{\dot{q}_v L^2}{2\lambda}\left[1 - \left(\frac{x}{L}\right)^2\right], \tag{4}$$

when the wall thickness is divided into five finite volumes (Fig. 6.31).

Solution

Heat balance equation for the first finite volume has the form

$$\frac{\lambda}{\Delta x}\left(T_2 - T_1\right) + \dot{q}_v \frac{\Delta x}{2} = 0, \tag{5}$$

from which one obtains

$$2T_2 - 2T_1 = -\frac{\dot{q}_v (\Delta x)^2}{\lambda}, \tag{6}$$

where $\Delta x = L/4$.

Heat balance equation for finite volumes with numbers i = 2, 3, 4 has the form

$$\frac{\lambda}{\Delta x}\left(T_{i-1}-T_i\right)+\frac{\lambda}{\Delta x}\left(T_{i+1}-T_i\right)+\dot{q}_v\Delta x=0, \tag{7}$$

hence,

$$T_{i-1}-2T_i+T_{i+1}=-\frac{\dot{q}_v\left(\Delta x\right)^2}{\lambda}, \qquad i=2,\,3,\,4. \tag{8}$$

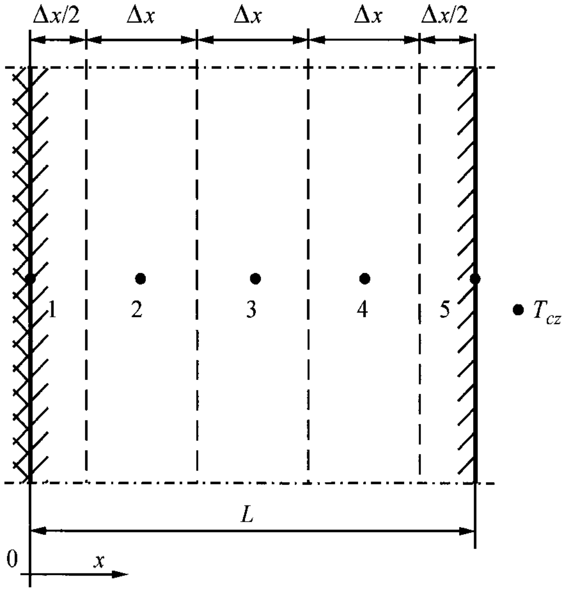

Fig. 6.31. Wall division into control volumes

Heat balance equation for control volume no. 5 has the form

$$\frac{\lambda}{\Delta x}\left(T_4-T_5\right)+\alpha\left(T_{cz}-T_5\right)+\dot{q}_v\frac{\Delta x}{2}=0, \tag{9}$$

hence, one obtains

$$2T_4-\left(2+\frac{2\alpha\Delta x}{\lambda}\right)T_5=-\frac{\dot{q}_v\left(\Delta x\right)^2}{\lambda}-\frac{2\alpha\Delta x}{\lambda}T_{cz}. \tag{10}$$

By substituting the numerical values in (6), (8) and (10), one obtains the following system:

$$\begin{aligned} -2T_1 + 2T_2 &= -15 \\ T_1 - 2T_2 + T_3 &= -15 \\ T_2 - 2T_3 + T_4 &= -15 \\ T_3 - 2T_4 + T_5 &= -15 \\ 2T_4 - 2.6T_5 &= -27. \end{aligned} \tag{11}$$

This system can also be written in the matrix form

$$\mathbf{AT} = \mathbf{b}, \tag{12}$$

where

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 2 & -2.6 \end{bmatrix}, \quad \mathbf{T} = \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} -15 \\ -15 \\ -15 \\ -15 \\ -27 \end{Bmatrix}.$$

Equation system (12) has been solved using Gauss elimination method. Solution can be written in the form

$$\mathbf{T} = \mathbf{A}^{-1}\mathbf{b},$$

where

$$\mathbf{T} = \begin{Bmatrix} 340.0 \\ 332.5 \\ 310.0 \\ 272.5 \\ 220.0 \end{Bmatrix}, \quad \mathbf{A}^{-1} = \begin{bmatrix} -3.67 & -6.33 & -5.33 & -4.33 & -1.67 \\ -3.17 & -6.33 & -5.33 & -4.33 & -1.67 \\ -2.67 & -5.33 & -5.33 & -4.33 & -1.67 \\ -2.17 & -4.33 & -4.33 & -4.33 & -1.67 \\ -1.67 & -3.33 & -3.33 & -3.33 & -1.67 \end{bmatrix}.$$

It is important to know the inverse matrix $\mathbf{A}^{-1}$, since it allows to define the relationship between temperature in a given node and the heat source unit power $\dot{q}_v$ and the parameters α and T_{cz}, which prevail in the boundary condition (3). In order to compare obtained temperature values with analytical solution, temperature values in nodes were also calculated using (4) (Table 6.2).

Heat flux on a slab surface is

$$\dot{q}\big|_{x=L} = \alpha\left(T\big|_{x=L} - T_{cz}\right).$$

Table 6.2. Comparison of temperature values calculated numerically nodes with exact values

Node no.	Coordinate x [m]	Finite volume method T [°C]	Exact solution T [°C]
1	0	340.0	340.0
2	0.015	332.5	332.5
3	0.030	310.0	310.0
4	0.045	272.5	272.5
5	0.060	220.0	220.0

Accurate value:

$$\dot{q}\Big|_{x=L} = 300(220-20) = 60000 \text{ W/m}^2 .$$

Approximated value $\dot{q}_p(L)$ calculated from formula

$$\dot{q}_p\Big|_{x=L} = \alpha\left(T_5 - T_{cz}\right)$$

is

$$\dot{q}_p\Big|_{x=L} = 300(220-20) = 60000 \text{ W/m}^2 .$$

Approximated value $\dot{q}_p\Big|_{x=L}$ can also be determined by means of Fourier Law, while first derivative $dT/dx\big|_{x=L}$ will be calculated by means of a backward finite difference differential quotient with an accuracy of 2nd order

$$\dot{q}_p\Big|_{x=L} = -\lambda \frac{dT}{dx}\bigg|_{x=L} \approx -\lambda\left(\frac{3T_N - 4T_{N-1} + T_{N-2}}{2\Delta x}\right)\bigg|_{N=5} .$$

After substituting node temperature values calculated by means of the finite volume method, one obtains

$$\dot{q}_p\Big|_{x=L} = -15\frac{3\cdot 220 - 4\cdot 272.5 + 310}{2\cdot 0.015} = 60000 \text{ W/m}^2 .$$

From the analysis of the obtained results, it is clear that the accuracy of the finite volume method is very good.

Program for calculating temperature distribution in slabs

```
 program mat
dimension a(50,50),b(50),c(50,50),t(50)
open(unit=1,file='mat.in')
```

```
      open(unit=2,file='mat.out')
      read(1,*)n
       read(1,*) ((a(i,j),j=1,n),i=1,n), (b(i),i=1,n)
      write(2,*)'A'
      write(2,66) ((a(i,j),j=1,n),i=1,n)
      write(2,*)'B'
      write(2,66) (b(i),i=1,n)
      call matinv(a,n,c)
      write(2,*)'A^-1'
      write(2,66) ((c(i,j),j=1,n),i=1,n)
      do 20 i=1,n
      sum=0.0
      do 10 j=1,n
   10 sum=sum+c(i,j)*b(j)
   20 t(i)=sum
      write(2,*)''
      write(2,50) (i,t(i),i=1,n)
      stop     7
   50 format(5('T(',i2,')=',f8.2,2x))
   66 format(5f8.2)
      end
```

```
data(mat.in)
5
-2. 2. 0. 0. 0.
1. -2. 1. 0. 0.
0. 1. -2. 1. 0.
0. 0. 1. -2. 1.
0. 0. 0. 2. -2.6
-15. -15. -15. -15. -27.
results(mat.out)
A
   -2.00    2.00     .00     .00     .00
    1.00   -2.00    1.00     .00     .00
     .00    1.00   -2.00    1.00     .00
     .00     .00    1.00   -2.00    1.00
     .00     .00     .00    2.00   -2.60
B
  -15.00  -15.00  -15.00  -15.00  -27.00
A^-1
   -3.67   -6.33   -5.33   -4.33   -1.67
   -3.17   -6.33   -5.33   -4.33   -1.67
   -2.67   -5.33   -5.33   -4.33   -1.67
   -2.17   -4.33   -4.33   -4.33   -1.67
   -1.67   -3.33   -3.33   -3.33   -1.67

T( 1)=  340.00  T( 2)=  332.50  T( 3)=  310.00
T( 4)=  272.50  T( 5)=  220.00
```

Exercise 6.27 Determining One-Dimensional Temperature Distribution in a Cylindrical Wall By Means of Finite Volume Method

Solve the problem formulated in Ex. 6.12 using the finite volume method. Calculate tube wall temperature in five uniformly spaced points (Fig. 6.32). Compare the obtained numerical solution with the accurate analytical solution.

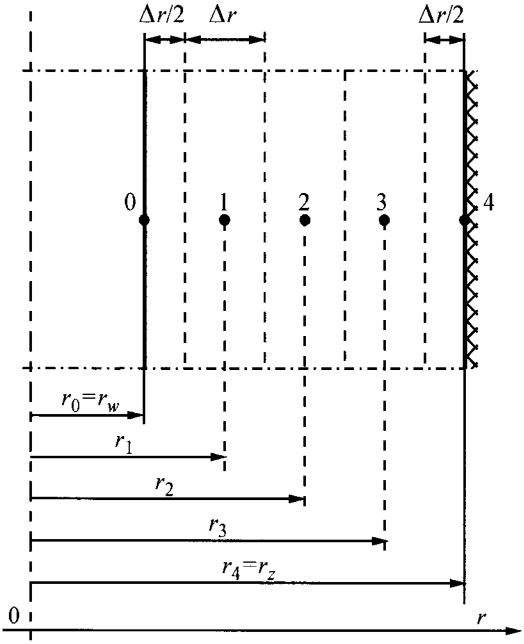

Fig. 6.32. Division of a cylindrical wall into finite volumes

Solution

Heat balance equation for finite volumes, adjacent to the inner surface, is not required, since the inner surface temperature is known: $T_0 = T_w$. Heat balance equations for nodes i = 1, 2, 3 can be written in the condensed form

$$2\pi\left(r_i - \frac{\Delta r}{2}\right)\lambda\frac{T_{i-1}-T_i}{\Delta r} + 2\pi\left(r_i + \frac{\Delta r}{2}\right)\lambda\frac{T_{i+1}-T_i}{\Delta r} + 2\pi r_i \Delta r \dot{q}_v = 0, \quad (1)$$

$i = 1, 2, 3,$

where $\Delta r = (r_z - r_w)/4$.

On the basis of (1), one obtains the following equations for nodes numbered 1, 2 and 3:

$$\frac{r_1 - \frac{\Delta r}{2}}{\Delta r}\left(T_0 - T_1\right) + \frac{r_1 + \frac{\Delta r}{2}}{\Delta r}\left(T_2 - T_1\right) + \frac{\dot{q}_v r_1 \Delta r}{\lambda} = 0\,, \tag{2}$$

$$\frac{r_2 - \frac{\Delta r}{2}}{\Delta r}\left(T_1 - T_2\right) + \frac{r_2 + \frac{\Delta r}{2}}{\Delta r}\left(T_3 - T_2\right) + \frac{\dot{q}_v r_2 \Delta r}{\lambda} = 0\,, \tag{3}$$

$$\frac{r_3 - \frac{\Delta r}{2}}{\Delta r}\left(T_2 - T_3\right) + \frac{r_3 + \frac{\Delta r}{2}}{\Delta r}\left(T_4 - T_3\right) + \frac{\dot{q}_v r_3 \Delta r}{\lambda} = 0\,. \tag{4}$$

An equation for node 4 has the form

$$2\pi\left(r_4 - \frac{\Delta r}{2}\right)\lambda\frac{T_3 - T_4}{\Delta r} + \pi\left[r_4^2 - \left(r_4 - \frac{\Delta r}{2}\right)^2\right]\dot{q}_v = 0\,. \tag{5}$$

After simple transformations (2)–(5) can be written in the following form:

$$-2T_1 + \frac{r_1 + \frac{\Delta r}{2}}{r_1}T_2 = -\frac{r_1 - \frac{\Delta r}{2}}{r_1}T_0 - \frac{\dot{q}_v\left(\Delta r\right)^2}{\lambda}\,, \tag{6}$$

$$\frac{r_2 - \frac{\Delta r}{2}}{r_2}T_1 - 2T_2 + \frac{r_2 + \frac{\Delta r}{2}}{r_2}T_3 = -\frac{\dot{q}_v\left(\Delta r\right)^2}{\lambda}\,, \tag{7}$$

$$\frac{r_3 - \frac{\Delta r}{2}}{r_3}T_2 - 2T_3 + \frac{r_3 + \frac{\Delta r}{2}}{r_3}T_4 = -\frac{\dot{q}_v\left(\Delta r\right)^2}{\lambda}\,, \tag{8}$$

$$2T_3 - 2T_4 = -\frac{r_4 - \frac{\Delta r}{4}}{r_4 - \frac{\Delta r}{2}}\frac{\dot{q}_v\left(\Delta r\right)^2}{\lambda}\,. \tag{9}$$

After substitution of the numerical values

$$\Delta r = \frac{r_z - r_w}{4} = \frac{(8-7.2)\cdot 10^{-3}}{2\cdot 4} = 1\cdot 10^{-4}\ \text{m},$$

$$r_1 = r_w + \Delta r = 3.6\cdot 10^{-3} + 1\cdot 10^{-4} = 3.7\cdot 10^{-3}\ \text{m},$$

$$r_2 = r_w + 2(\Delta r) = 3.6\cdot 10^{-3} + 2\cdot 1\cdot 10^{-4} = 3.8\cdot 10^{-3}\ \text{m},$$

$$r_3 = r_w + 3(\Delta r) = 3.6\cdot 10^{-3} + 3\cdot 1\cdot 10^{-4} = 3.9\cdot 10^{-3}\ \text{m}, \tag{10}$$

$$\dot{q}_v = 8.38715\cdot 10^{8}\ \text{W/m}^3,$$

$$\lambda = 18.4\ \text{W/(m}\cdot\text{K)},$$

$$T_0 = T_w = 300\,^\circ\text{C},$$

the equation system (7)–(10) assumes the form

$$\begin{aligned}
&-2T_1 + 1.0135135T_2 = -296.4017693,\\
&0.986842T_1 - 2T_2 + 1.01315789T_3 = -\ 0.4558234,\\
&0.9871795T_2 - 2T_3 + 1.0128205T_4 = -\ 0.4558234,\\
&2T_3 - 2T_4 = -0.458708.
\end{aligned} \tag{11}$$

Equation system (11) can be written in the matrix form

$$\mathbf{AT} = \mathbf{b}, \tag{12}$$

where

$$\mathbf{A} = \begin{bmatrix} -2 & 1.0135135 & 0 & 0 \\ 0.986842 & -2 & 1.01315789 & 0 \\ 0 & 0.9871795 & -2 & 1.0128205 \\ 0 & 0 & 2 & -2 \end{bmatrix},$$

$$\mathbf{T} = \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix},\quad \mathbf{b} = \begin{Bmatrix} -296.4017693 \\ -0.4558234 \\ -0.4558234 \\ -0.458708 \end{Bmatrix}. \tag{13}$$

The solution of the equation system (12), obtained by means of the Gauss elimination method, has the form

$$\mathbf{T} = \mathbf{A}^{-1}\mathbf{b}, \tag{14}$$

where

$$\mathbf{T}=\begin{Bmatrix}301.6718\\302.8493\\303.5464\\303.7757\end{Bmatrix},\ \mathbf{A}^{-1}=\begin{bmatrix}-1.0137 & -1.0411 & -1.0685 & -0.5411\\-1.0137 & -2.0544 & -2.1085 & -1.0678\\-1.0137 & -2.0544 & -3.1215 & -1.5807\\-1.0137 & -2.0544 & -3.1215 & -2.0807\end{bmatrix}. \quad (15)$$

The comparison of the obtained numerical solution with analytical solution from Ex. 6.12

$$T(r)=T_0+\frac{\dot{q}_v}{2\lambda}\left(r_z^2\ln\frac{r}{r_w}+\frac{r_w^2-r^2}{2}\right) \quad (16)$$

is presented in Table 6.3. Heat flux on the inner surface of the pipe will be determined from formula

$$\dot{q}_w=\lambda\frac{dT}{dr}\bigg|_{r=r_w}\approx\lambda\frac{-3T_0+4T_1-T_2}{2\Delta r}. \quad (17)$$

Table 6.3. The comparison of numerically and analytically calculated node temperature values

Node no.	Coordinate x [m]	Finite Volume Method T [°C]	Exact Method T [°C]
1	0.0036	300	300.0
2	0.0037	301.6718	301.6725
3	0.0038	302.8493	302.8506
4	0.0039	303.5464	303.5482
5	0.0040	303.7757	303.7781

Heat transfer coefficient on an inner surface of the pipe will be determined from formula

$$\alpha=\frac{\dot{q}_w}{T|_{r=r_w}-T_{cz}}=\frac{\dot{q}_w}{T_0-T_{cz}},$$

where $\dot{q}_w$ is expressed using (17).

After substitution, one obtains

$$\dot{q}_w=18.4\frac{-3\cdot300+4\cdot301.6718-302.8493}{2\cdot1\cdot10^{-4}}=353086.8\ \text{W/m}^2$$

and

$$\alpha = \frac{353086.8}{300-20} = 1261.024 \text{ W/(m}^2\text{·K)}.$$

It is clear that the obtained results are almost identical to the results obtained by means of the analytical formulas from Ex. 6.12.

Exercise 6.28 Inverse Steady-State Heat Conduction Problem for a Pipe Solved by Space-Marching Method

Electrical current with an intensity of 300 A flows through a stainless steel pipe with an inner diameter d_w = 7.2 mm and outer diameter d_z = 8 mm. Thermal conductivity of this steel is λ = 18.4 W/(m·K), while its specific resistance ρ = 0.85 $(\Omega \cdot \text{mm}^2)$/m. Assuming that the outer surface of the pipe is thermally insulated and the temperature of this surface is known and equals T_z = 303.77805°C, calculate temperature distribution across the wall thickness. Assume that total heat rate generated inside the wall flows to the interior of the pipe. Calculate heat transfer coefficient on the inner surface of the pipe, if the temperature of a medium measures T_{cz} = 20°C. Perform calculations using finite volume method. Analytical solution of the problem formulated above is presented in Ex. 6.13.

Solution

A division of the wall into finite volumes is shown in Fig. 6.32. An equation system for temperature in finite volume nodes has the same form as (6)–(9) in Ex. 6.27, therefore

$$\begin{aligned}
&\frac{r_1 - \Delta r/2}{r_1} T_0 - 2T_1 + \frac{r_1 + \Delta r/2}{r_1} T_2 = -\frac{\dot{q}_v (\Delta r)^2}{\lambda},\\
&\frac{r_2 - \Delta r/2}{r_2} T_1 - 2T_2 + \frac{r_2 + \Delta r/2}{r_2} T_3 = -\frac{\dot{q}_v (\Delta r)^2}{\lambda},\\
&\frac{r_3 - \Delta r/2}{r_3} T_2 - 2T_3 + \frac{r_3 + \Delta r/2}{r_3} T_4 = -\frac{\dot{q}_v (\Delta r)^2}{\lambda},\\
&2T_3 - 2T_4 = -\frac{r_4 - \Delta r/4}{r_4 - \Delta r/2} \frac{\dot{q}_v (\Delta r)^2}{\lambda}.
\end{aligned} \tag{1}$$

After substitution of the numerical values, equation system (1) assumes the form

$$\begin{aligned}
&0.986486T_0 - 2T_1 + 1.0135135T_2 = -\ 0.4558234 \\
&0.98684T_1 - 2T_2 + 1.01315789T_3 = -\ 0.4558234 \\
&0.9871795T_2 - 2T_3 + 1.0128205T_4 = -\ 0.4558234 \\
&2T_3 - 2T_4 = -\ 0.458708\,.
\end{aligned} \tag{2}$$

This is different from what we obtained in Ex. 6.27, in which the system in question is solved using the Gauss elimination method, since in this case we can easily determine the solution of system (1) by starting calculations from the last equation. If we take into account that

$$T_4 = T_z = 303.77805\,^\circ\mathrm{C} \tag{3}$$

from the fourth equation of system (2), we get

$$T_3 = T_4 - \frac{0.458708}{2} = 303.77805 - 0.229354\,^\circ\mathrm{C} = 303.5487\,^\circ\mathrm{C}\,. \tag{4}$$

From the third equation of system (2), we get

$$\begin{aligned}
T_2 &= \frac{1}{0.9871795}\left(2T_3 - 1.0128205T_4 - 0.4558234\right) = \\
&= \frac{1}{0.9871795}\left(2\cdot 303.5487 - 1.0128205\cdot 303.77805 - 0.4558234\right) = \\
&= 302.8516\,^\circ\mathrm{C}.
\end{aligned}$$

From the second equation we have temperature at node 1:

$$\begin{aligned}
T_1 &= \frac{1}{0.98684}\left(2T_2 - 1.01315789T_3 - 0.4558234\right) = \\
&= \frac{1}{0.98684}\left(2\cdot 302.8516 - 1.01315789\cdot 303.5487 - 0.4558234\right) = \\
&= 301.6746^\circ\mathrm{C}.
\end{aligned}$$

Inner surface temperature T_0 is determined from the first equation

$$\begin{aligned}
T_0 &= \frac{1}{0.986486}\left(2T_1 - 1.0135135T_2 - 0.4558234\right) = \\
&= \frac{1}{0.986486}\left(2\cdot 301.6746 - 1.0135135\cdot 302.8516 - 0.4558234\right) = \\
&= 300.00344^\circ\mathrm{C}.
\end{aligned}$$

Inner surface heat flux calculated from the approximate formula

$$\dot{q}_w = \lambda \left.\frac{dT}{dr}\right|_{r=r_w} \approx \lambda \frac{-3T_0 + 4T_1 - T_2}{2\Delta r} \tag{5}$$

is

$$\dot{q}_w = 18.4 \frac{-3 \cdot 300.00344 + 4 \cdot 301.6746 - 302.8516}{2 \cdot 1 \cdot 10^{-4}} = 352956 \text{ W/m}^2 .$$

Inner surface heat transfer coefficient is

$$\alpha = \frac{\dot{q}_w}{T_0 - T_{cz}} = \frac{352956}{300.00344 - 20} = 1260.5 \text{ W/(m}^2 \cdot \text{K)} .$$

Inner surface heat flux can be also calculated from an energy balance for node 0

$$\dot{q}_w \cdot 2\pi r_w = \lambda 2\pi \left(r_w + \frac{\Delta r}{2} \right) \frac{T_1 - T_0}{\Delta r} , \tag{6}$$

hence,

$$\dot{q}_w = \lambda \left(1 + \frac{\Delta r}{2r_w} \right) \frac{T_1 - T_0}{\Delta r} = 18.4 \left(1 + \frac{1 \cdot 10^{-4}}{2 \cdot 3.6 \cdot 10^{-3}} \right) \times \\ \times \left(\frac{301.6746 - 300.00344}{1 \cdot 10^{-4}} \right) = 311764.2 \text{ W/m}^2. \tag{7}$$

Heat transfer coefficient is:

$$\alpha = \frac{\dot{q}_w}{T_0 - T_{cz}} = \frac{311764.2}{300.00344 - 20} = 1113.4 \text{ W/(m}^2 \cdot \text{K)}. \tag{8}$$

It is clear, therefore, that the second method for calculating heat transfer coefficient is less accurate, since value of α calculated in Ex. 6.12 by using the exact method is α = 1264.73 W/(m^2·K). This is due to the accurate calculation of heat flux on the inner surface by means of difference quotients. The accuracy of (5) is of the second order, while of (7) – of the first order.

A problem, in which both assumed conditions are set on a single boundary, is an inverse heat conduction problem. A method applied in this exercise is called the space-marching method. According to this method, temperatures are determined by space marching from the location, with temperature and heat flux known, to a surface with a boundary condition unknown. One can also see that calculated node temperatures differ slightly from the "exact" values (calculated using analytical formulas) given in Table 6.3. This is mainly due to the rounding calculation errors made in this exercise. In inverse problems, rounding errors or temperature measurement errors, especially in transient problems, significantly influence the accuracy of the obtained results.

Exercise 6.29 Temperature Distribution and Efficiency of a Circular Fin with Temperature-Dependent Thermal Conductivity

Determine efficiency of a circular fin with constant thickness t by assuming that thermal conductivity of the fin material λ is a linear temperature function of temperature:

$$\lambda(T) = \lambda(T_{cz})(1+\varepsilon\theta),$$

where $\varepsilon = \left[\lambda(T_1) - \lambda(T_{cz})\right]/\lambda(T_{cz})$, $\theta = (T - T_{cz})/(T_1 - T_{cz})$.

Symbols T_1 and T_{cz} stand for fin base temperature and medium's temperature, respectively. Draw graphs show the fin efficiency as the function of the parameter M:

$$k = r_2/r_1, \qquad M = L^{3/2}\sqrt{\alpha/\left[\lambda(T_{cz})Lt\right]},$$

where r_1 and r_2 are the base and fin tip radius, $L = r_2 - r_1$ fin height, α – heat transfer coefficient on the fin surface. Show results in a tabular form for k = 2 as a function of parameter M.

Solution

Equations and boundary conditions, which describe fin temperature field, have the following form (see reference [12] and equation (4.8)–(4.12) in Part 1 of this book):

$$\frac{dQ}{d\rho} = -2\pi N^2 \rho\theta, \tag{1}$$

$$\frac{d\theta}{d\rho} = -\frac{Q}{2\pi(1+\varepsilon\theta)\rho}, \tag{2}$$

$$\theta\big|_{\rho=1} = 1, \tag{3}$$

$$Q\big|_{\rho=k} = 0, \tag{4}$$

$$Q = -2\pi(1+\varepsilon\theta)\rho\frac{d\theta}{d\rho}. \tag{5}$$

Symbol $\rho = r/r_1$ is a dimensionless radius. The following relation occurs between parameters N and M:

$$N^2 = \frac{2}{(k-1)^2}M^2. \tag{6}$$

Fin efficiency η, defined as a ratio of real-fin-dissipated heat flow to isothermal-fin-dissipated heat flow, is formulated as

$$\eta_{\dot{z}} = \frac{-\left[2\pi tr\lambda(T_{cz})(1+\varepsilon\theta)\frac{dT}{dr}\right]\Big|_{r=r_1}}{\int_{r_1}^{r_2} 4\pi\alpha(T_1 - T_\infty)rdr}. \tag{7}$$

Once (5) is substituted in (7) and subsequently transformed, one obtains

$$\eta_{\dot{z}} = \frac{Q\big|_{\rho=1}}{\pi N^2(k^2-1)} = \frac{(k-1)Q\big|_{\rho=1}}{2\pi(k+1)M^2}. \tag{8}$$

By solving two-point boundary value problem (1)–(4), one is able to determine $Q|_{\rho=1}$. Two-point boundary value problem will be solved iteratively using secant method, also called *Newton-Raphson method* [1, 8]. Boundary problem (1)–(4) will be substituted by the initial problem under the assumption that value Q is given at the base of the fin

$$Q\big|_{\rho=1} = \beta. \tag{9}$$

If we assume a certain numerical value β, we will be able to solve the initial problem, formulated by (1) and (2) and by initial conditions (3) and (9), at a given iterative step. *Runge-Kutta Method of 4th order* was applied to solve the initial problem. Value β must be chosen in such way that condition (4) is satisfied. Variable $Q|_{\rho=k}$ is, therefore, a function of parameter β

$$Q\big|_{\rho=k} = \bar{Q}(\beta). \tag{10}$$

One should find such value of parameter β^* for which $\bar{Q}(\beta^*) = 0$. Therefore, the solution of two-point boundary problems (1)–(4) is reduced to the determination of the root of the following algebraic equation:

$$\bar{Q}(\beta) = 0 \,. \tag{11}$$

Such equation will be solved by means of the secant method, according to which (Fig. 6.33)

$$\frac{\beta_n - \beta_{n-1}}{\bar{Q}(\beta_n) - \bar{Q}(\beta_{n-1})} = \frac{\beta_{n+1} - \beta_n}{\bar{Q}(\beta_{n+1}) - \bar{Q}(\beta_n)} \,. \tag{12}$$

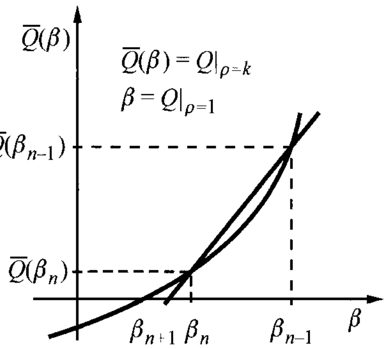

Fig. 6.33. Determining the root of a non-linear algebraic equation $\bar{Q}(\beta)$ by means of the secant method

Next, by taking into account the following condition in (12)

$$\bar{Q}(\beta_{n+1}) = 0 \tag{13}$$

one obtains,

$$\frac{\beta_n - \beta_{n-1}}{\bar{Q}(\beta_n) - \bar{Q}(\beta_{n-1})} = \frac{\beta_{n+1} - \beta_n}{0 - \bar{Q}(\beta_n)} \,, \tag{14}$$

from which it follows that

$$\beta_{n+1} = \beta_n - \frac{(\beta_n - \beta_{n-1})\bar{Q}(\beta_n)}{\bar{Q}(\beta_n) - \bar{Q}(\beta_{n-1})} \,, \qquad n = 0, 1, \ldots \tag{15}$$

At the beginning of the calculation, two values $\beta_0 > 0$ and $\beta_1 > 0$ are assumed, and following that roots β_2, β_3, ... are calculated using (15). Iterative process is continued until the following condition is met

$$|\beta_{n+1} - \beta_n| < e,$$

where e is the assigned tolerance of the calculation and equals $e = 0.001$. The results of fin efficiency calculation are presented in Fig. 6.34–6.37 for different values of $k = r_2/r_1$, ε and M.

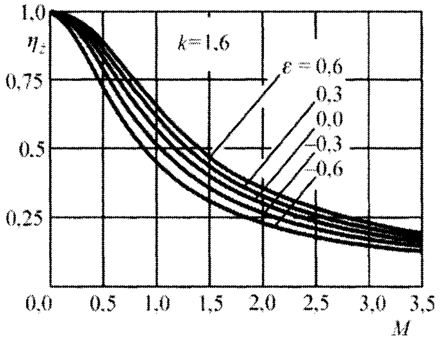

Fig. 6.34. Efficiency of a fin with constant thickness and variable thermal conductivity for $k = 1.6$

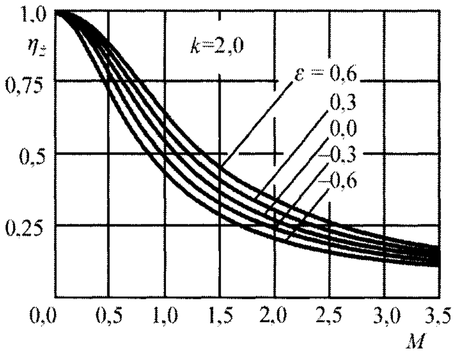

Fig. 6.35. Efficiency of a circular fin with constant thickness and variable thermal conductivity for $k = 2.0$

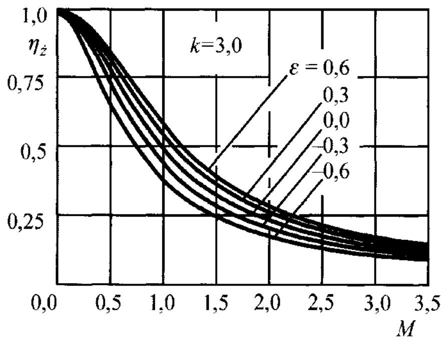

Fig. 6.36. Efficiency of a circular fin of constant thickness and variable thermal conductivity for $k = 3.0$

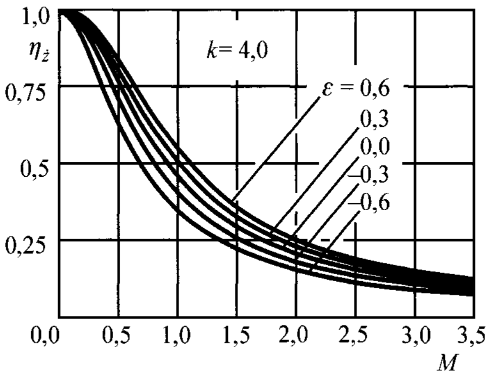

Fig. 6.37. Efficiency of a circular fin of constant thickness and variable thermal conductivity for $k = 4.0$

From the comparisons presented in Table 6.4, it is evident that the given method for calculating fin efficiency is highly accurate. In paper [12], the method described above was also used to determine circular fin efficiency with position-dependent heat transfer coefficient α.

Table 6.4. Efficiency $\eta_{\dot{z}}$ of a circular fin with constant thickness for $k = 2$; value $\eta_{\dot{z}}$ for $\varepsilon = 0$ (constant thermal conductivity) calculated by means of the analytical formula ((10), Ex. 6.17) is given in brackets

ε	Parameter M											
	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2.0	2.25	2.5	2.75
–0.6	1.0	0.8831	0.6915	0.5351	0.4217	0.3410	0.2829	0.2401	0.2078	0.1827	0.1628	0.1468
–0.3	1.0	0.9241	0.7648	0.6095	0.4868	0.3959	0.3292	0.2795	0.2419	0.2126	0.1894	0.1707
0.0	1.0	0.9445	0.8133	0.6674	0.5418	0.4440	0.3704	0.3149	0.2725	0.2395	0.2133	0.1921
	(1.0)	(.9445)	(0.8133)	(0.6674)	(0.5418)	(0.4440)	(0.3704)	(0.3149)	(0.2725)	(0.2395)	(.2133)	(0.1921)
0.3	1.0	0.9565	0.8465	0.7128	0.5885	0.4867	0.4077	0.3473	0.3007	0.2643	0.2353	0.2119
0.6	1.0	0.9642	0.8702	0.7486	0.6282	0.5247	0.4419	0.3772	0.3269	0.2874	0.2558	0.2303

Literature

1. Björck A, Dahlquist G (1987) Numerical methods. PWN, Warsaw
2. Brandt F (1985) Wärmeübertragung in Dampferzeugern und Wärmeaustauschern, FDBR Fachverband Dampfkessel, Behälter- und Rohrleitungsbau E.V. Vulkan Verlag, Essen
3. Gnielinski V, Žukauskas A, Skrinska A (1992) Banks of plain and finned tubes. Chapter 2.5.3.Hewitt G. F.: Handbook of Heat Exchanger Design. Begell House, New York: 2.5.3–1 – 2.5.3–16
4. IMSL Math/Library (1994) Special Functions, Visual Numerics. Houston, Texas, USA.

5. Janke E, Emde F, Lösch F (1960) Tafeln höherer Funktionen. B. G. Teubner, Stuttgart
6. Lokshin VA, Peterson F, Schwarz AL (1988) Standard Methods of Hydraulic Design for Power Boilers. Hemisphere-Springer, Berlin-Washington
7. MathCAD 7 Professional (1997). MathSoft Inc, Cambridge Massachusetts, USA
8. Press WH, Teukolsky SA, Vetterling WT, Flannery BP (1996) Numerical Recipies in Fortran 77. Sec. Ed. Cambridge University Press
9. Recknagel H, Sprenger E, Hönmann W, Schramek ER (1994) Handbook, Heating and air conditioning, including refrigeration engineering and hot water supply. EWFE
10. Schmidt ThE (1950) Die Wärmeleistung von berippten Oberflächen. Abh. Deutsch. Kältetechn. Verein. Nr. 4, C.F. Müller, Karlsruhe
11. Schmidt ThE (1949) Heat transfer calculations for extended surfaces. Refig. Eng., Apr.: 351–357
12. Taler J, Przybyliński P (1982) Heat transfer by circular fins of variable conduction and non-uniform heat transfer coefficient. Chemical and Process Engineering 3, No 3–4: 659–676
13. Weber RL (1994) Principles of Enhanced Heat Transfer. Wiley & Sons, New York