5 Variable Change

The solution to many transient heat conduction problems, in particular problems with movable boundaries or heat sources, can be simplified by introducing new variables. Figure 5.1 shows an example of a heat source moving along x axis at velocity v, while Fig. 5.2 shows an ablation that occurs at velocity $v(t)$. One can transform transient heat conduction equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{a}\frac{\partial T}{\partial t} \tag{5.1}$$

in an motionless coordinate system with origin 0 under the condition that either the coordinate system, which moves along with the heat source, or a movable boundary will be introduced. Once the new coordinate ξ is introduced, expressed by equation (Fig. 5.1 and Fig. 5.2)

$$x=\int_0^t v(t)\,dt+\xi=s(t)+\xi \tag{5.2}$$

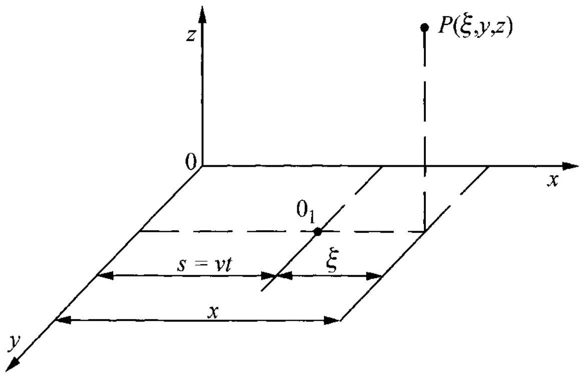

Fig. 5.1. Movable coordinate system that moves in the direction of x axis together with the heat source at velocity v

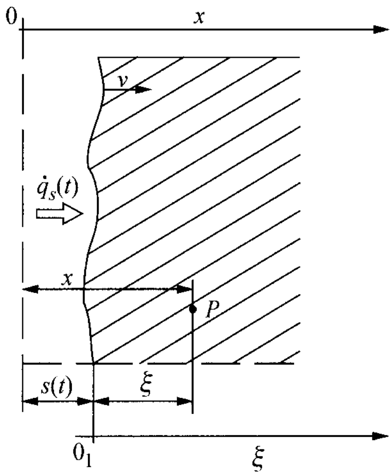

Fig. 5.2. Movable boundary (ablation) heated by a heat flow with density $\dot{q}_s(t)$

one obtains

$$\xi = x - \int_0^t v(t)\,dt = x - s(t). \tag{5.3}$$

If velocity v is independent of time, then

$$s(t) = vt, \qquad \xi = x - vt. \tag{5.4}$$

Taking into account that $T = T[\xi(t), t]$ and

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial T}{\partial \xi}, \tag{5.5}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T}{\partial \xi^2}, \tag{5.6}$$

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial T}{\partial t} = -v(t)\frac{\partial T}{\partial \xi} + \frac{\partial T}{\partial t} \tag{5.7}$$

Equation (5.1) can be written in the following form:

$$\frac{\partial T}{\partial t} - v(t)\frac{\partial T}{\partial \xi} = a\left(\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right). \tag{5.8}$$

In the new coordinate system (ξ, y, z) with the point of origin 0_1, a heat source or ablation boundary remain constant and do not change in time. Equation (5.8) has the same form as the heat conduction equation for a body that moves in the direction of axis ξ at velocity $-v(t)$.

Another example that testifies to the practicability of changing the variables is the determination of a transient temperature field for a semi-infinite body (Fig. 5.3).

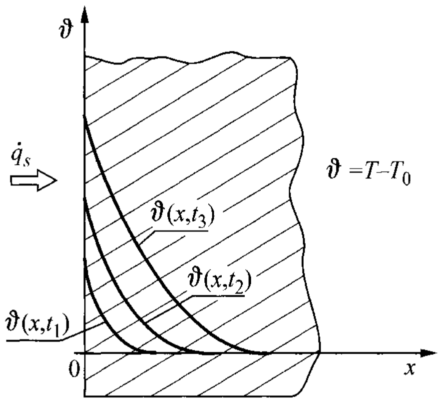

Fig. 5.3. Heating semi-infinite body with a heat flow at constant density

Temperature distribution in the semi-infinite body is formulated by the heat conduction equation

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{a}\frac{\partial T}{\partial t}, \tag{5.9}$$

using boundary conditions

$$-\lambda \left.\frac{\partial T}{\partial x}\right|_{x=0} = \dot{q}_s, \tag{5.10}$$

$$T\big|_{x\to\infty} = T_0 \tag{5.11}$$

and initial condition

$$T(x,0)=T_0 . \tag{5.12}$$

Once dimensionless variable [1] is introduced

$$\eta=\frac{x}{\sqrt{2at}} \tag{5.13}$$

and the following solution form assumed

$$T(x,t)=T_0-\frac{\dot{q}_s x}{\lambda}\frac{f(\eta)}{\eta} \tag{5.14}$$

Equation (5.9) can be reduced to a form

$$\frac{d^2 f}{d\eta^2}+\eta\frac{df}{d\eta}-f=0 . \tag{5.15}$$

Conditions (5.10) and (5.11) have the form

$$\left.\frac{df(\eta)}{d\eta}\right|_{\eta=0}=1 , \tag{5.16}$$

$$f(\eta)\big|_{\eta\to\infty}=0 . \tag{5.17}$$

Due to the fact that variable (5.13) is introduced, partial equation (5.9) was substituted by an ordinary differential equation (5.15).

Literature

1. Slattery JC (1999) Advanced Transport Phenomena. University Press, Cambridge