11 Solving Steady-State Heat Conduction Problems by Means of Numerical Methods

This chapter is devoted to numerical methods, which are used to determine steady-state temperature fields. It contains detailed description of the following numerical methods: finite-difference method, finite-volume method (control volume), finite element method (FEM) and pseudo-transient method for solving stationary problems, based on the method of lines. Linear and non-linear problems, both simple and inverse, are solved here. Specific computational programs are developed for determining steady-state temperature fields, while Gauss elimination method, Gauss-Seidel iterative method or over-relaxation method are applied to integrate an algebraic equation system. Ordinary differential equation system in the pseudo-transient method is solved using Rung-Kutta method of 4th order. Finite element method, based on Galerkin method, is discussed in great detail, as well as the two methods for creating global equation system in FEM. Basic matrixes and vectors, which occur in FEM for one-dimensional and two-dimensional triangular and rectangular elements, are also developed. Furthermore, authors present their own solutions to FEM problems. The obtained results are compared with analytical solutions or the solutions acquired by means of finite volume method. The application of the ANSYS program is presented in Ex. 11.20, 11.21 and 11.22. Hexagonal fin efficiency is determined in Ex. 11.21, while the effect the shape of pins on the heating surface of the cast iron heating boiler has on the temperature distribution and pin-transferred heat flow is analyzed in Ex. 11.22.

Exercise 11.1 Description of the Control Volume Method

Describe how transient heat conduction problems are solved by means of the control volume method; assume that thermal conductivity can be temperature dependent. Write heat balance equation for control volume in the Cartesian and cylindrical coordinate system for two-dimensional problems.

Solution

Control volume method, also called elementary balance method or finite volume method, is a universal and effective method for solving heat conduction problems. If the thickness of an analyzed area is d and thermal properties c, ρ, λ and power densityof internal heat sources $\dot{q}_v$ are temperature dependent, then heat conduction equation can be written in the form

$$c(T)\rho(T)\frac{\partial T}{\partial t} = -\mathrm{div}\,\dot{\mathbf{q}} + \dot{q}_v . \tag{1}$$

The area is divided into control volumes, which have the following dimensions: Δx, Δy and d in the Cartesian coordinate system (Fig. 11.1) or Δr, $\Delta\varphi$ and d (Fig. 11.2) in the cylindrical coordinate system. Once (1) is integrated over the control volume, the following equation for a single cell (control volume) is obtained:

$$\int_{CV} c(T)\rho(T)\frac{\partial T}{\partial t} dV = -\int_{CV} \mathrm{div}\,\dot{\mathbf{q}}\, dV + \int_{CV} \dot{q}_v\, dV , \tag{2}$$

where *CV* stands for the *control volume.*

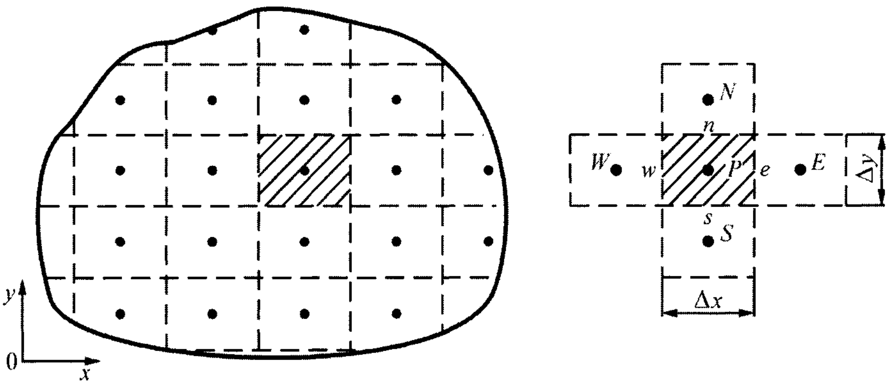


Fig. 11.1. A diagram of an area divided into finite volumes in the Cartesian coordinate system

If we apply Green-Gauss-Ostrogradski theorem to the first term on the right-hand-side of (2), the equation will assume the form

$$\int_{CV} c(T)\rho(T)\frac{\partial T}{\partial t} dV = -\int_{S} \mathbf{n}\cdot\dot{\mathbf{q}} dS + \int_{CV} \dot{q}_v\, dV , \tag{3}$$

where S is the control volume surface, while **n** a normal unit surface vector directed to the outside of the control volume. From expression

$$\mathbf{n}\cdot\dot{\mathbf{q}} = 1\cdot|\dot{\mathbf{q}}|\cos(\mathbf{n},\dot{\mathbf{q}}) = \dot{q}_n \tag{4}$$

it is evident that when the heat flows up to the control volume, the heat flux vector $\dot{\mathbf{q}}$ is directed to the inside of the control volume and the angle between vector **n** and $\dot{\mathbf{q}}_n$ is 180°. The scalar product (4) is then negative, while the surface integral in (3) is positive. If ΔV denotes the volume of a control cell, then individual terms in (3) can be approximated in the following way:

$$\int_{CV} c(T)\rho(T)\frac{\partial T}{\partial t}dV \approx \Delta V c(T_P)\rho(T_P)\frac{dT_P}{dt}, \tag{5}$$

$$-\int_S \mathbf{n}\cdot\dot{\mathbf{q}}\, dS = \sum_{i=1}^{4}\dot{Q}_i, \tag{6}$$

$$\int_{CV}\dot{q}_v dV = \Delta V\,\dot{q}_v(T_P), \tag{7}$$

where $\dot{Q}_i$ is the heat flow that flows in from the neighbouring cell. Substituting equalities (5)–(7) in (3), one obtains the following heat balance equation

$$\Delta V c(T_P)\rho(T_P)\frac{dT_P}{dt} = \sum_{i=1}^{4}\dot{Q}_i + \Delta V\dot{q}_v(T_P), \tag{8}$$

which will be written in a greater detail in the Cartesian and cylindrical coordinate system.

a) *Heat balance equation- Cartesian coordinates*

A division of an area into control volumes and a control volume are shown in Fig.11.1. The volume of a single cell is $\Delta V = (\Delta x)(\Delta y)d$. Heat flows, which inflow from nodes W, N, E and S to node P are expressed by the following formulas:

$$\dot{Q}_{W-P} = (\Delta y)d\,\dot{q}_{W-P} = (\Delta y)d\frac{\lambda(T_W)+\lambda(T_P)}{2}\cdot\frac{T_W - T_P}{\Delta x}, \tag{9}$$

$$\dot{Q}_{N-P} = (\Delta x)d\,\dot{q}_{N-P} = (\Delta x)d\frac{\lambda(T_N)+\lambda(T_P)}{2}\cdot\frac{T_N - T_P}{\Delta y}, \tag{10}$$

$$\dot{Q}_{E-P}=(\Delta y)d\,\dot{q}_{E-P}=(\Delta y)d\frac{\lambda(T_E)+\lambda(T_P)}{2}\cdot\frac{T_E-T_P}{\Delta x}\,, \tag{11}$$

$$\dot{Q}_{S-P}=(\Delta x)d\,\dot{q}_{S-P}=(\Delta x)d\frac{\lambda(T_S)+\lambda(T_P)}{2}\cdot\frac{T_S-T_P}{\Delta y}\,. \tag{12}$$

Substituting the expressions (9)–(12) in (8), one obtains

$$\begin{aligned}(\Delta x)(\Delta y)d\,c(T_P)\rho(T_P)\frac{dT_P}{dt}&=(\Delta y)d\frac{\lambda(T_W)+\lambda(T_P)}{2}\cdot\frac{T_W-T_P}{\Delta x}+\\&+(\Delta x)d\frac{\lambda(T_N)+\lambda(T_P)}{2}\cdot\frac{T_N-T_P}{\Delta y}+(\Delta y)d\frac{\lambda(T_E)+\lambda(T_P)}{2}\cdot\\&\cdot\frac{T_E-T_P}{\Delta x}+(\Delta x)d\frac{\lambda(T_S)+\lambda(T_P)}{2}\cdot\frac{T_S-T_P}{\Delta y}+(\Delta x)(\Delta y)d\cdot\dot{q}_V(T_P)\,.\end{aligned} \tag{13}$$

Assuming constant properties

$$\lambda_P=\lambda(T_P),\qquad c_P=c(T_P),\quad \dots\ ,\qquad a_P=\frac{\lambda_P}{c_P\cdot\rho_P}$$

Equation (13) is simplified to the following equation

$$\begin{aligned}\frac{dT_P}{dt}=a_P\Bigg[&\frac{\lambda_W+\lambda_P}{2\lambda_P}\cdot\frac{T_W-T_P}{(\Delta x)^2}+\frac{\lambda_N+\lambda_P}{2\lambda_P}\cdot\frac{T_N-T_P}{(\Delta y)^2}+\\&+\frac{\lambda_E+\lambda_P}{2\lambda_P}\cdot\frac{T_E-T_P}{(\Delta x)^2}+\frac{\lambda_S+\lambda_P}{2\lambda_P}\cdot\frac{T_S-T_P}{(\Delta y)^2}\Bigg]+\frac{\dot{q}_{v,P}}{c_P\cdot\rho_P}\,.\end{aligned} \tag{14}$$

When steady-state problem is analyzed, $dT_p/dt = 0$. For a uniform grid $\Delta x = \Delta y$ and for constant and temperature independent thermal properties and heat source power, (14) is simplified to a form

$$T_W+T_N+T_E+T_S-4T_P+\frac{\dot{q}_v(\Delta x)^2}{\lambda}=0\,. \tag{15}$$

b) *Heat balance equation-cylindrical coordinates*

Heat balance (8) can be transformed into a form similar to (14) after calculating of the following quantities (Fig. 11.2):

$$\Delta V=\pi\left(r_n^2-r_s^2\right)\frac{\Delta\varphi}{2\pi}d=\frac{\left(r_n^2-r_s^2\right)\Delta\varphi}{2}d\,, \tag{16}$$

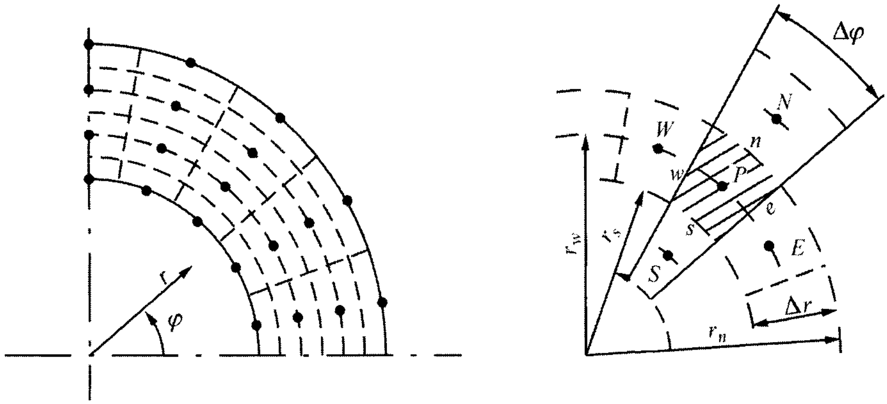


Fig. 11.2. A division of an area into finite volumes in the cylindrical coordinate system

$$\dot{Q}_{W-P} = (\Delta r) d \frac{\lambda_W + \lambda_P}{2} \cdot \frac{T_W - T_P}{r_w (\Delta \varphi)}, \tag{17}$$

$$\dot{Q}_{N-P} = r_n (\Delta \varphi) d \frac{\lambda_N + \lambda_P}{2} \cdot \frac{T_N - T_P}{\Delta r}, \tag{18}$$

$$\dot{Q}_{E-P} = (\Delta r) d \frac{\lambda_E + \lambda_P}{2} \cdot \frac{T_E - T_P}{r_w (\Delta \varphi)}, \tag{19}$$

$$\dot{Q}_{S-P} = r_s (\Delta \varphi) d \frac{\lambda_S + \lambda_P}{2} \cdot \frac{T_S - T_P}{\Delta r}, \tag{20}$$

From (8), one obtains

$$\frac{dT_P}{dt} = \frac{2a_P}{\left(r_n^2 - r_s^2\right) \cdot \Delta \varphi} \left[\frac{\lambda_W + \lambda_P}{2\lambda_P} \cdot \frac{\Delta r}{r_w (\Delta \varphi)} (T_W - T_P) + \right.$$
$$+ \frac{\lambda_N + \lambda_P}{2\lambda_P} \cdot \frac{r_n (\Delta \varphi)}{\Delta r} (T_N - T_P) + \frac{\lambda_E + \lambda_P}{2\lambda_P} \cdot \frac{\Delta r}{r_w (\Delta \varphi)} (T_E - T_P) + \tag{21}$$
$$\left. + \frac{\lambda_S + \lambda_P}{2\lambda_P} \cdot \frac{r_s \cdot \Delta \varphi}{\Delta r} (T_S - T_P) \right] + \frac{\dot{q}_{v,P}}{c_P \cdot \rho_P}.$$

In the case of steady-state problems, one should assume that $dT_P/dt = 0$. Heat balance (14) in the Cartesian coordinates or (21) in cylindrical coordinates is written for all nodes, including the nodes in the control volumes that abut to a boundary. Appropriate boundary conditions should be allowed for in the equations for boundary-adjacent control volumes. In order to determine node temperature in the cases of transient problems, one should solve the ordinary differential equation system by means of the Runge-Kutta method, for instance. In steady-state problems, one can obtain an algebraic equation system, which can be solved by direct methods, e.g. Gauss elimination method, or by iterative methods like Gauss-Seidel method.

Exercise 11.2 Determining Temperature Distribution in a Square Cross-Section of a Long Rod by Means of the Finite Volume Method

Determine temperature distribution in a square cross-section of an infinitely long rod with prescribed temperature on lateral surfaces (Fig. 11.3). In order to solve the problem, apply the control volume method, while the obtained algebraic equation system solve by means of the iterative Gauss-Seidel method. Write a computational program for the determination of temperature in nodes 1–4.

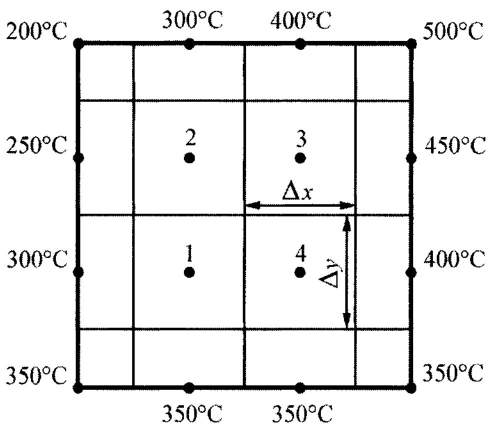


Fig. 11.3. Square cross-section of an infinitely long rod with prescribed surface temperature

Solution

Equation (15) from Ex. 11.1 will be used to solve the above stated problem; in this case, it has the form

$$T_W + T_N + T_E + T_S - 4T_P = 0 \ . \qquad (1)$$

From the equation above, we have, respectively, for nodes 1 to 4:

- node 1

$$\begin{gathered} 300 + T_2 + T_4 + 350 - 4T_1 = 0 , \\ 4T_1 - T_2 - T_4 = 650; \end{gathered} \qquad (2)$$

- node 2

$$\begin{gathered} 250 + 300 + T_3 + T_1 - 4T_2 = 0 , \\ -T_1 + 4T_2 - T_3 = 550; \end{gathered} \qquad (3)$$

- node 3

$$\begin{gathered} T_2 + 400 + 450 + T_4 - 4T_3 = 0 , \\ -T_2 + 4T_3 - T_4 = 850; \end{gathered} \qquad (4)$$

- node 4

$$\begin{gathered} T_1 + T_3 + 400 + 350 - 4T_4 = 0 , \\ -T_1 - T_3 + 4T_4 = 750. \end{gathered} \qquad (5)$$

According to the Gauss-Seidel method, (2)–(5) are transformed in a way that the temperature in the first node is determined from the first equation, in the second node from the second equation, in the third node from the third equation and in the fourth node from the fourth equation:

$$T_1 = \frac{1}{4}\left(650 + T_2 + T_4\right) , \qquad (6)$$

$$T_2 = \frac{1}{4}\left(550 + T_1 + T_3\right) , \qquad (7)$$

$$T_3 = \frac{1}{4}\left(850 + T_2 + T_4\right) , \qquad (8)$$

$$T_4 = \frac{1}{4}\left(750 + T_1 + T_3\right) . \qquad (9)$$

Next, initial approximation is assumed, e.g.

$T_1^{(0)} = 250°\text{C}$, $T_2^{(0)} = 250°\text{C}$, $T_3^{(0)} = 250°\text{C}$, $T_4^{(0)} = 250°\text{C}$

and, in turn, individual temperatures are determined from (6)–(9). Temperature determined in this way is substituted into the subsequent equation, i.e.

$$T_1^{(1)} = \frac{1}{4}\left(650 + T_2^{(0)} + T_4^{(0)}\right) = \frac{1}{4}(650 + 250 + 250) = 287.5\,°\text{C},$$

$$T_2^{(1)} = \frac{1}{4}\left(550 + T_1^{(1)} + T_3^{(0)}\right) = \frac{1}{4}(550 + 287.5 + 250) = 271.875\,°\text{C},$$

$$T_3^{(1)} = \frac{1}{4}\left(850 + T_2^{(1)} + T_4^{(0)}\right) = \frac{1}{4}(850 + 271.875 + 250) = 342.969\,°\text{C},$$

$$T_4^{(1)} = \frac{1}{4}\left(750 + T_1^{(1)} + T_3^{(1)}\right) = \frac{1}{4}(750 + 287.5 + 342.969) = 345.117\,°\text{C}.$$

The second approximation is done in a similar way

$$T_1^{(2)} = \frac{1}{4}\left(650 + T_2^{(1)} + T_4^{(1)}\right) = \frac{1}{4}(650 + 271.875 + 345.117) = 316.748\,°\text{C},$$

$$T_2^{(2)} = \frac{1}{4}\left(550 + T_1^{(2)} + T_3^{(1)}\right) = \frac{1}{4}(550 + 316.748 + 342.969) = 302.429\,°\text{C},$$

$$T_3^{(2)} = \frac{1}{4}\left(850 + T_2^{(2)} + T_4^{(1)}\right) = \frac{1}{4}(850 + 302.429 + 345.117) = 374.387\,°\text{C},$$

$$T_4^{(2)} = \frac{1}{4}\left(750 + T_1^{(2)} + T_3^{(2)}\right) = \frac{1}{4}(750 + 316.748 + 374.387) = 360.284\,°\text{C}.$$

From the third approximation, one obtains

$$T_1^{(3)} = \frac{1}{4}\left(650 + T_2^{(2)} + T_4^{(2)}\right) = \frac{1}{4}(650 + 302.429 + 360.284) = 328.178\,°\text{C},$$

$$T_2^{(3)} = \frac{1}{4}\left(550 + T_1^{(3)} + T_3^{(2)}\right) = \frac{1}{4}(550 + 328.178 + 374.387) = 313.141\,°\text{C},$$

$$T_3^{(3)} = \frac{1}{4}\left(850 + T_2^{(3)} + T_4^{(2)}\right) = \frac{1}{4}(850 + 313.141 + 360.284) = 380.856\,°\text{C},$$

$$T_4^{(3)} = \frac{1}{4}\left(750 + T_1^{(3)} + T_3^{(3)}\right) = \frac{1}{4}(750 + 328.178 + 380.856) = 364.759\,°\text{C}.$$

After fourth iteration, one has

$$T_1^{(4)} = \frac{1}{4}\left(650 + T_2^{(3)} + T_4^{(3)}\right) = \frac{1}{4}(650 + 313.141 + 364.759) = 331.975\,°\text{C},$$

$$T_2^{(4)} = \frac{1}{4}\left(550 + T_1^{(4)} + T_3^{(3)}\right) = \frac{1}{4}\left(550 + 331.975 + 380.856\right) = 315.708^\circ\text{C},$$

$$T_3^{(4)} = \frac{1}{4}\left(850 + T_2^{(4)} + T_4^{(3)}\right) = \frac{1}{4}\left(850 + 315.708 + 364.759\right) = 382.617^\circ\text{C},$$

$$T_4^{(4)} = \frac{1}{4}\left(750 + T_1^{(4)} + T_3^{(4)}\right) = \frac{1}{4}\left(750 + 331.975 + 382.617\right) = 366.148^\circ\text{C}.$$

Following that, iterative calculations are conducted in a way that satisfies the inequality below:

$$\left|T_i^{(k+1)} - T_i^{(k)}\right| < \varepsilon; \qquad i = 1, 2, 3, 4; \qquad k = 0, 1 \ldots. \tag{10}$$

For ε = 0.00001°C after k = 14 iterations, the following temperature values are obtained:

T_1 = 333.333°C, T_2 = 316.667°C,

T_3 = 383.333°C,

T_4 = 366.667°C.

Calculations were carried out by means of the FORTRAN program. In spite of the fact that a large number of iterations was done, calculation time is very short, since the formulas are very simple in form.

Program for temperature determination in nodes 1–4

```
c     Calculating two-D temperature field in a flat rod
c     by means of Gauss-Seidel method
      program seidel
      dimension t(50),tt(50)
      logical inaccurate
      open(unit=1,file='seidel.in')
      open(unit=2,file='seidel.out')
      read(1,*)n,toler,niter,t_pocz
      write(2,'(a)') "CALCULATING TWO-DIMENSIONAL TEMPERATURE
     &FIELD IN A FLAT & ROD "
      write(2,'(/a)') "DATA ENTERED"
      write(2,'(a,i10)')    "equation number  n=",n
      write(2,'(a,e10.5,a)')"calcul. toler=",toler,"[C]"
      write(2,'(a,i10)') "max. iteration number niter=",niter
      write(2,'(a,e10.5,a)')"init.temp.t_pocz=",t_pocz," [C]"
      do i=1,n
         t(i)=t_pocz
```

```
         tt(i)=t_pocz
      enddo
      i=0
      inaccurate=.true.
      do while ((i.le.niter).and.inaccurate)
         t(1)=(650.+t(2)+t(4))/4.
         t(2)=(550.+t(1)+t(3))/4.
         t(3)=(850.+t(2)+t(4))/4.
         t(4)=(750.+t(1)+t(3))/4.
         inaccurate=.false.
         do j=1,n
            if (abs(tt(j)-t(j)).gt.toler) inaccurate=.true.
         enddo
         if  (inaccurate) then
            do j=1,n
               tt(j)=t(j)
            enddo
         endif
         i=i+1
      enddo
......write(2,'(/a)')"CALCULATED TEMPERATURE"
      write(2,'(a)')"   Lp      T[C] "
      do j=1,n
         write(2,'(i5,3x,e11.6)')j,t(j)
      enddo
      write(2,'(a,i10)') "final iteration number=",i
      end  program seidel
```

```
data(seidel.in)
4 0.00001 10000000 250.
```

```
results(seidel.out)
CALCULATING TWO-DIMENSIONAL TEMPERATURE FIELD IN A FLAT ROD

DATA ENTERED
equation number n= 4
calcul. toler=.10000E-04 [C]
max. iteration number niter=  10000000
init.temp.t_pocz=.25000E+03 [C]

CALCULATED TEMPERATURE
   Lp      T[C]
    1    .333333E+03
    2    .316667E+03
    3    .383333E+03
    4    .366667E+03
final iteration number=  14
```

Exercise 11.3 A Two-Dimensional Inverse Steady-State Heat Conduction Problem

Solve an inverse heat conduction problem. Temperature is measured at a point inside a body. The unknown is the temperature of a node, which lies on the edge of that body. Consider two cases (Fig. 11.4):

a) Temperature is measured in node 1, while the unknown is the temperature in node B, which lies on the body edge.

b) Temperature is measured in node 3, while the unknown is the temperature in node B, which lies on the body edge.

As measurement values f_1 and f_3 adopt temperatures determined in the previous exercise (Ex. 11.2), for nodes 1 and 3, respectively, i.e.

$$f_1 = T_1 = 333.333°\text{C},$$

$$f_3 = T_3 = 383.333°\text{C}.$$

How the calculation results are going to change, if measurement values contain a measurement error $\Delta T = +1.0°\text{C}$ i.e.

$$f_1 = T_1 + \Delta T = 333.333 + 1.0 = 334.333°\text{C},$$

$$f_3 = T_3 + \Delta T = 383.333 + 1.0 = 384.333°\text{C}.$$

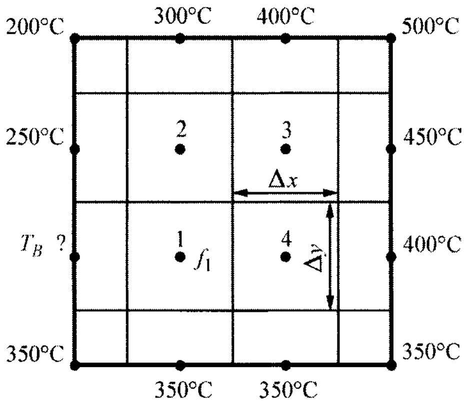


Fig. 11.4. Inverse heat conduction problem; temperature f_1 is measured in node 1, while the unknown temperature at point T_B lies on the body edge

Solution

In general, temperatures in volume nodes or finite elements are formulated by the equation system

$$\begin{aligned} a_{11}T_1 + a_{12}T_2 + a_{13}T_3 + ... + a_{1n}T_n &= b_1 \\ a_{21}T_1 + a_{22}T_2 + a_{23}T_3 + ... + a_{2n}T_n &= b_2 \\ &\cdots\cdots \\ a_{n1}T_1 + a_{n2}T_2 + a_{n3}T_3 + ... + a_{nn}T_n &= b_n \end{aligned} \tag{1}$$

Parameters that appear in the boundary conditions are expressed by the terms on the right side of the system, i.e. in vector $\mathbf{b} = (b_1, b_2, ..., b_n)^T$, while coefficients a_{ij}, $i = 1, ..., n$, $j = 1, ..., n$, i.e. the coefficient matrix $\mathbf{A}$ is known. If the equation system (1) is written in the matrix form

$$\mathbf{AT} = \mathbf{b}, \tag{2}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} T_1 \\ T_2 \\ \cdots \\ T_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}, \tag{3}$$

then the solution of the system (2) has the form

$$\mathbf{T} = \mathbf{A}^{-1}\mathbf{b}, \tag{4}$$

where $\mathbf{A}^{-1}$ is the inverse matrix to $\mathbf{A}$.

Once we determine the inverse matrix

$$\mathbf{C} = \mathbf{A}^{-1} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nn} \end{bmatrix} \tag{5}$$

we can determine node temperatures from (4)

$$\begin{aligned} T_1 &= c_{11}b_1 + c_{12}b_2 + c_{13}b_3 + ... + c_{1n}b_n \\ T_2 &= c_{21}b_1 + c_{22}b_2 + c_{23}b_3 + ... + c_{2n}b_n \\ &\cdots\cdots \\ T_i &= c_{i1}b_1 + c_{i2}b_2 + c_{i3}b_3 + ... + c_{i,j-1}b_{j-1} + c_{i,j}b_j + c_{i,j+1}b_{j+1} + ... + c_{in}b_n \\ &\cdots\cdots \\ T_n &= c_{n1}b_1 + c_{n2}b_2 + c_{n3}b_3 + ... + c_{nn}b_n \end{aligned} \tag{6}$$

If temperature f_i in node i is known from measurements taken, while coefficient b_j is unknown, then from the equality condition of measured temperature f_i and calculated T_i, one obtains

$$f_i = T_i, \tag{7}$$

from where, after accounting for (6), one is able to determine coefficient b_j

$$b_j = \frac{f_j - c_{i1}b_1 - c_{i2}b_2 - c_{i3}b_3 - \ldots - c_{i,j-1}b_{j-1} - c_{i,j+1}b_{j+1} - \ldots - c_{in}b_n}{c_{i,j}}. \tag{8}$$

If the measurement data contains an error, then (8) assumes the form

$$b_j = \frac{\left(f_j + \Delta T\right) - c_{i1}b_1 - c_{i2}b_2 - c_{i3}b_3 - \ldots - c_{i,j-1}b_{j-1} - c_{i,j+1}b_{j+1} - \ldots - c_{in}b_n}{c_{i,j}}.$$

Should the problem formulated in this exercise appear (Fig. 11.4), then the balance equation system has the following form

$$4T_1 - T_2 - T_4 = T_B + 350$$
$$-T_1 + 4T_2 - T_3 = 550$$
$$-T_2 + 4T_3 - T_4 = 850$$
$$-T_1 - T_3 + 4T_4 = 750$$

Hence, the coefficient matrix has the form

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 0 & -1 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ -1 & 0 & -1 & 4 \end{bmatrix}.$$

Inverse matrix, determined by means of MATINV program (see Appendix E), is

$$\mathbf{C} = \mathbf{A}^{-1} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \ldots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \ldots & c_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{n1} & c_{n2} & c_{n3} & \ldots & c_{nn} \end{bmatrix} = \begin{bmatrix} 0.292 & 0.083 & 0.042 & 0.083 \\ 0.083 & 0.292 & 0.083 & 0.042 \\ 0.042 & 0.083 & 0.292 & 0.083 \\ 0.083 & 0.042 & 0.083 & 0.292 \end{bmatrix}.$$

(7) assumes the form

$$f_1 = c_{11}\left(T_B + 350\right) + c_{12} \cdot 550 + c_{13} \cdot 850 + c_{14} \cdot 750 \qquad (i = 1)$$

Temperature T_B is

$$T_B = \frac{f_1 - 350c_{11} - 550c_{12} - 850c_{13} - 750c_{14}}{c_{11}} =$$

$$= \frac{333.333 - 350 \cdot 0.292 - 550 \cdot 0.083 - 850 \cdot 0.042 - 750 \cdot 0.083}{0.292} =$$

$$= 299.770\,^\circ\mathrm{C}$$

In an instance when data is burdened with errors, one gets

$$T'_B = \frac{(f_1 + \Delta T) - 350c_{11} - 550c_{12} - 850c_{13} - 750c_{14}}{c_{11}} =$$

$$= \frac{334.333 - 350 \cdot 0.292 - 550 \cdot 0.083 - 850 \cdot 0.042 - 750 \cdot 0.083}{0.292} =$$

$$= 303.195\,^\circ\mathrm{C}.$$

If temperature is measured in node 3, then (7) assumes the form

$$f_3 = c_{31}(T_B + 350) + c_{32} \cdot 550 + c_{33} \cdot 850 + c_{34} \cdot 750,$$

hence,

$$T_B = \frac{f_3 - 350c_{31} - 550c_{32} - 850c_{33} - 750c_{34}}{c_{31}} =$$

$$= \frac{383.333 - 350 \cdot 0.042 - 550 \cdot 0.083 - 850 \cdot 0.292 - 750 \cdot 0.083}{0.042} =$$

$$= 298.405\,^\circ\mathrm{C}.$$

For measured temperature f_3 with error ΔT, temperature T'_B is

$$T'_B = \frac{f_3 - 350c_{31} - 550c_{32} - 850c_{33} - 750c_{34}}{c_{31}} =$$

$$= \frac{384.333 - 350 \cdot 0.042 - 550 \cdot 0.083 - 850 \cdot 0.292 - 750 \cdot 0.083}{0.042} =$$

$$= 322.214\,^\circ\mathrm{C}.$$

From the analysis of the given results, one can see that in the case of temperature measured at point 1, the accuracy of the obtained results is greater for both, the accurate temperature f_1 and the error-burdened data $(f_1 + \Delta T)$. Temperature changes in T_B have a larger effect on temperature T_1. There seems to be a distinct cause and effect relationship between T_B and T_1 than there is between T_B and T_3. If temperature changes in T_B affect, to a small degree, temperature changes at the point of measurement, as it happens in the case of node 3, it is difficult to accurately determine temperature T_B on the basis of measured temperature f_3. Small temperature measurement error in f_3 triggers a very large change in temperature T_B. The explanation for this is that the inverse problem has been ill conditioned. In practice the problem can be avoided, if the sensor or temperature measuring sensors are placed within a close proximity to a surface on which the boundary condition is being determined. This is, however, not always possible. If the temperature sensor is located far from the surface, on which the boundary conditions are identified, one should expect the obtained results to be far less accurate. This is precisely what this exercise has demonstrated; adding error $\Delta T = 1°C$ to the "accurate" measurement value at point 3 causes the determined temperature $T_B' = 322.214°C$ to be significantly different from the real temperature $T_B = 300°C$.

Program inv

```
C      Inverse matrix calculation
       program inv
       dimension a(50,50),c(50,50)
       open(unit=1,file='inv.in')
       open(unit=2,file='inv.out')
       read(1,*)n
       read(1,*) ((a(i,j),j=1,n),i=1,n)
       write(2,'(a)')"INVERSE MATRIX CALCULATION"
       write(2,'(a)') "DATA ENTERED"
       write(2,'(a,i10)')"matrix A dimension  n=",n
       write(2,'(a)') "matrix A"
       write(2,'(4f8.2)') ((a(i,j),j=1,n),i=1,n)
       call matinv(a,n,c)
       write(2,'(a)') "CALCULATED MATRIX A^-1"
       write(2,'(4f9.3)') ((c(i,j),j=1,n),i=1,n)
       end program inv
```

```
data(inv.in)
4
 4. -1.  0. -1.
-1.  4. -1.  0.
 0. -1.  4. -1.
-1.  0. -1.  4.
```

```
results(inv.out)
INVERSE MATRIX CALCULATION
DATA ENTERED
matrix A dimension  n=         4
matrix A
    4.00   -1.00     .00   -1.00
   -1.00    4.00   -1.00     .00
     .00   -1.00    4.00   -1.00
   -1.00     .00   -1.00    4.00
```

```
CALCULATED MATRIX A^-1
      .292      .083      .042      .083
      .083      .292      .083      .042
      .042      .083      .292      .083
      .083      .042      .083      .292
```

Exercise 11.4 Gauss-Seidel Method and Over-Relaxation Method

Describe Gauss-Seidel method and over-relaxation method, which are frequently employed when solving a system of algebraic equations obtained from the control volume method. Write a computational program in the FORTRAN language for the calculation using the over-relaxation method. Show how the equation system obtained in Ex. 11.2 can be solved by means of this program.

Solution

In the Gauss-Seidel method, the system of algebraic equations, which are the heat balance equations for the control volume,

$$\begin{aligned} a_{11}T_1 + a_{12}T_2 + a_{13}T_3 + \ldots + a_{1n}T_n &= b_1 \\ a_{21}T_1 + a_{22}T_2 + a_{23}T_3 + \ldots + a_{2n}T_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}T_1 + a_{n2}T_2 + a_{n3}T_3 + \ldots + a_{nn}T_n &= b_n, \end{aligned} \tag{1}$$

is transformed to a form

$$\begin{aligned} T_1 &= 1/a_{11} \cdot \left(b_1 - a_{12}T_2 - a_{13}T_3 - \ldots - a_{1n}T_n\right) \\ T_2 &= 1/a_{22} \cdot \left(b_2 - a_{21}T_1 - a_{23}T_3 - \ldots - a_{2n}T_n\right) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ T_n &= 1/a_{nn} \cdot \left(b_n - a_{n1}T_1 - a_{n2}T_2 - \ldots - a_{n,n-1}T_{n-1}\right). \end{aligned} \tag{2}$$

Gauss-Seidel method is an iterative method. One begins calculations by selecting the initial approximation first: $T_1^{(0)}$, $T_2^{(0)}$, ..., $T_n^{(0)}$; more often than not it is assumed that all temperature values equal zero. Quite often, moreover, temperatures $T_i^{(0)} = b_i/a_{ii}$, $i = 1, \ldots, n$ are selected as initial values. By substituting $T_2 = T_3 = \ldots = T_n = 0$ into the first equation in the system (2), one is able to calculate the first approximation of $T_1^{(1)}$. Temperature $T_1^{(1)}$ is automatically taken into account in the second equation of the system (2).

The remaining temperature values, which are present on the right-hand-side of the second equation, are assumed to be as follow: $T_3 = T_4 = ... = T_n = 0$. This is how $T_2^{(1)}$ is calculated from the second equation. By using the same method to determine temperature in the remaining nodes, the following approximation is obtained: $T_1^{(1)}, T_2^{(1)}, ..., T_n^{(1)}$. The determination of node temperature in the iterative k-step progresses as follows:

$$T_1^{(k+1)} = \frac{1}{a_{11}} \cdot \left(b_1 - a_{12}T_2^{(k)} - a_{13}T_3^{(k)} - ... - a_{1n}T_n^{(k)}\right)$$
$$T_2^{(k+1)} = \frac{1}{a_{22}} \cdot \left(b_2 - a_{21}T_1^{(k+1)} - a_{23}T_3^{(k)} - ... - a_{2n}T_n^{(k)}\right)$$
$$\cdots\cdots\cdots$$
$$T_i^{(k+1)} = \frac{1}{a_{ii}} \cdot \left(b_i - a_{i1}T_1^{(k+1)} - a_{i2}T_2^{(k+1)} - ... - a_{i,i-1}T_{i-1}^{(k+1)} - a_{i,i+1}T_{i+1}^{(k)} - - ... - a_{in}T_n^{(k)}\right) \quad (3)$$
$$\cdots\cdots\cdots$$
$$T_n^{(k+1)} = \frac{1}{a_{nn}} \cdot \left(b_n - a_{n1}T_1^{(k+1)} - a_{n2}T_2^{(k+1)} - ... - a_{n,n-1}T_{n-1}^{(k+1)}\right).$$

This calculation method, expressed by the equations in (3), was applied in a program presented in Ex.11.2. It does not require of one to use coefficients a_{ij}, $i = 1, ..., n$, $j = 1, ..., n$ in the calculation. The calculation process is the same when coefficients a_{ij} are temperature dependent, if the thermal conductivity, for instance, is temperature dependent. A drawback to this method is the fact that one is forced to rewrite all balance equations anew, when the new problem must be analyzed.

In order to make the program more universally applicable, a formula for $T_i^{(k+1)}$ in the system (3) will be used for the calculation; it can be written in the slightly different form

$$T_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}T_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}T_j^{(k)}\right), \qquad i = 1, ..., n. \quad (4)$$

Over-relaxation method is a form of modification of the Gauss-Seidel method; it aims to accelerate the iterative process

$$T_i^{(k+1)} = T_i^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}T_j^{(k+1)} - \sum_{j=i}^{n} a_{ij}T_j^{(k)}\right), \quad (5)$$

where $1 \le \omega \le 2$ is an over-relaxation coefficient. If $\omega = 1$, then over-relaxation method is identical to Gauss-Seidel method. Both, Gauss-Seidel method and over-relaxation method are convergent when

$$\sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}| < |a_{ii}|, \qquad i = 1, 2, ..., n. \quad (6)$$

Iterative process is continued until the criterion below is satisfied

$$\left|T_i^{(k+1)} - T_i^{(k)}\right| < \varepsilon, \qquad i = 1, ..., n, \tag{7}$$

where ε is the assigned calculation tolerance or

$$\left\|\mathbf{T}^{(k+1)} - \mathbf{T}^{(k)}\right\| < \varepsilon_1, \tag{8}$$

where ε_1 is the assigned calculation tolerance, e.g. $\varepsilon_1 = 0.001$ K. Square norm is calculated from formula

$$\left\|\mathbf{T}^{(k+1)} - \mathbf{T}^{(k)}\right\| = \left[\sum_{i=1}^{n}\left(T_i^{(k+1)} - T_i^{(k)}\right)^2\right]^{\frac{1}{2}}. \tag{9}$$

Sub-program SOR for solving equation system (1) by means of the over-relaxation method is shown in Appendix F. Equation system (2)–(5) in Ex. 11.2 has the following form:

$$\begin{bmatrix} 4 & -1 & 0 & -1 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ -1 & 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 650 \\ 550 \\ 850 \\ 750 \end{Bmatrix}, \tag{10}$$

The broken brackets are column vectors.

A program for solving system (10) with the help of sub-program SOR is shown below. The value of over-relaxation coefficient is assumed to equal 1,2. Obtained results are the same as they are in Ex. 11.2.

Program content, data and the solution of equation system (10)

```
C      Solution of an equation system by means of over-
C      relaxation method and sub-program SOR
       program nadrel
       dimension a(50,51),xi(50)
       nmax=50
       mmax=nmax+1
       open(unit=1,file='nadrel.in')
       open(unit=2,file='nadrel.out')
       read(1,*)n, w, niter, toler
       m=n+1
       read(1,*)((a(i,j),j=1,m),i=1,n)
       read(1,*)(xi(i),i=1,n)
       write(2,'(a)') "SOLUTION OF EQUATION SET
     &BY OVER-RELAXATION METHOD"
       write(2,'(/a)') "DATA ENTERED"
       write(2,'(a,i10)')    "equation number  n=",n
       write(2,'(a,e10.5)')  "relaxation coefficient w=",w
```

```
      write(2,'(a,i10)')    "max. iter. number niter=",niter
      write(2,'(a,e10.5,a)') "calc.toler.toler=",toler,"[C]"
      write(2,'(a)') "matrix A"
      write(2,'(5f8.2)') ((a(i,j),j=1,m),i=1,n)
      write(2,'(a)') "initial vector XI"
      write(2,'(4f8.2)') (xi(i),i=1,n)
      call sor(a,nmax,mmax,n,xi,w,niter,toler,k)
      write(2,'(/a)')"CALCULATION RESULTS"
      write(2,'(a)')"   Lp      X "
      do j=1,n
         write(2,'(i5,3x,4e11.6)')j,xi(j)
      enddo
      write(2,'(a,i10)') "final iteration number=",k
      end  program nadrel
```

```
data(nadrel.in)
 4  1.2  30  1.0E-3
 4. -1.  0. -1.  650.
-1.  4. -1.  0.  550.
 0. -1.  4. -1.  850.
-1.  0. -1.  4.  750.
 0.  0.  0.  0.
```

```
results(nadrel.out)
SOLUTION OF EQUATION SET BY OVER-RELAXATION METHOD
DATA ENTERED
equation number   n=         4
relaxation coefficient w=.12000E+01
max. iteration number niter=        30
calc.toler.toler=.10000E-02 [C]
matrix A
    4.00   -1.00     .00   -1.00  650.00
   -1.00    4.00   -1.00     .00  550.00
     .00   -1.00    4.00   -1.00  850.00
   -1.00     .00   -1.00    4.00  750.00
initial vector XI
     .00     .00     .00     .00
CALCULATION RESULTS
   Lp      X
    1   .333333E+03
    2   .316667E+03
    3   .383333E+03
    4   .366667E+03
final iteration number=        11
```

Exercise 11.5 Determining Two-Dimensional Temperature Distribution in a Straight Fin with Uniform Thickness by Means of the Finite Volume Method

Determine temperature distribution in a fin presented in Fig. 11.5 by means of the control volume method. For the calculation adopt the values given in Ex. 7.3. Also calculate heat flow at the fin base and its efficiency.

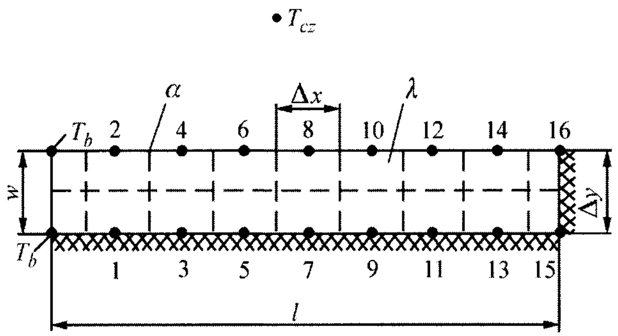


Fig. 11.5. A division of half of the fin into control volumes

Solution

Heat balance equations for control volumes have the form:

- node 1

$$\lambda \frac{\Delta y}{2} \frac{T_b - T_1}{\Delta x} + \lambda \frac{\Delta y}{2} \frac{T_3 - T_1}{\Delta x} + \lambda \Delta x \frac{T_2 - T_1}{\Delta y} = 0, \tag{1}$$

hence,

$$\left(\frac{\Delta y}{\Delta x} + \frac{\Delta x}{\Delta y} \right) T_1 - \frac{\Delta x}{\Delta y} T_2 - \frac{\Delta y}{2\Delta x} T_3 = \frac{\Delta y}{2\Delta x} T_b. \tag{2}$$

- node 2

$$\lambda \frac{\Delta y}{2} \frac{T_b - T_2}{\Delta x} + \lambda \frac{\Delta y}{2} \frac{T_4 - T_2}{\Delta x} + \lambda \Delta x \frac{T_1 - T_2}{\Delta y} + \alpha \Delta x \left(T_{cz} - T_2 \right) = 0, \tag{3}$$

from where, after transformations, one obtains

$$-\frac{\Delta x}{\Delta y} T_1 + \left(\frac{\Delta y}{\Delta x} + \frac{\Delta x}{\Delta y} + \frac{\alpha \Delta x}{\lambda} \right) T_2 - \frac{\Delta y}{2\Delta x} T_4 = \frac{\Delta y}{2\Delta x} T_b + \frac{\alpha \Delta x}{\lambda} T_{cz}. \tag{4}$$

• nodes 3, 5, 7, 9, 11, 13

Heat balance equation for i-node has the form

$$\lambda\frac{\Delta y}{2}\frac{T_{i-2}-T_i}{\Delta x}+\lambda\frac{\Delta y}{2}\frac{T_{i+2}-T_i}{\Delta x}+\lambda\Delta x\frac{T_{i+1}-T_i}{\Delta y}=0, \tag{5}$$

hence,

$$-\frac{\Delta y}{2\Delta x}T_{i-2}+\left(\frac{\Delta y}{\Delta x}+\frac{\Delta x}{\Delta y}\right)T_i-\frac{\Delta x}{\Delta y}T_{i+1}-\frac{\Delta y}{2\Delta x}T_{i+2}=0, \tag{6}$$

$i=3, 5, 7, 9, 11, 13$

• nodes 4, 6, 8, 10, 12, 14

Heat balance equation for i-node has the form

$$\lambda\frac{\Delta y}{2}\frac{T_{i-2}-T_i}{\Delta x}+\lambda\frac{\Delta y}{2}\frac{T_{i+2}-T_i}{\Delta x}+\lambda\Delta x\frac{T_{i-1}-T_i}{\Delta y}+\alpha\Delta x\left(T_{cz}-T_i\right)=0, \tag{7}$$

hence,

$$-\frac{\Delta y}{2\Delta x}T_{i-2}-\frac{\Delta x}{\Delta y}T_{i-1}+\left(\frac{\Delta y}{\Delta x}+\frac{\Delta x}{\Delta y}+\frac{\alpha\Delta x}{\lambda}\right)T_i-\frac{\Delta y}{2\Delta x}T_{i+2}=\frac{\alpha\Delta x}{\lambda}, \tag{8}$$

$i=2, 4, 6, 8, 10, 12.$

• node 15

$$\lambda\frac{\Delta y}{2}\frac{T_{13}-T_{15}}{\Delta x}+\lambda\frac{\Delta x}{2}\frac{T_{16}-T_{15}}{\Delta y}=0, \tag{9}$$

from where, after simple transformations, one obtains

$$-\frac{\Delta y}{\Delta x}T_{13}+\left(\frac{\Delta y}{\Delta x}+\frac{\Delta x}{\Delta y}\right)T_{15}-\frac{\Delta x}{\Delta y}T_{16}=0. \tag{10}$$

• node 16

$$\lambda\frac{\Delta y}{2}\frac{T_{14}-T_{16}}{\Delta x}+\lambda\frac{\Delta x}{2}\frac{T_{15}-T_{16}}{\Delta y}+\alpha\frac{\Delta x}{2}\left(T_{cz}-T_{16}\right)=0; \tag{11}$$

from where, one obtains

$$-\frac{\Delta y}{\Delta x}T_{14}-\frac{\Delta x}{\Delta y}T_{15}+\left(\frac{\Delta y}{\Delta x}+\frac{\Delta x}{\Delta y}+\frac{\alpha\Delta x}{\lambda}\right)T_{16}=\frac{\alpha\Delta x}{\lambda}T_{cz}. \tag{12}$$

After substitution of $T_b = 95°C$, $T_{cz} = 20°C$, $\Delta x = \Delta y = 0.003$ m, $\lambda = 50$ W/(m·K), $\alpha = 100$ W/(m^2·K), (2), (4), (6), (8), (10) and (12) assume the form

$$\begin{aligned}
&2T_1 - T_2 - 0.5T_3 = 47.5 && -T_1 + 2.006T_2 - 0.5T_4 = 47.62 \\
&-0.5T_{i-2} + 2T_i - T_{i+1} - 0.5T_{i+2} = 0; && i = 3, 5, 7, 9, 11, 13 \\
&-0.5T_{i-2} - T_{i-1} + 2.006T_i - 0.5T_{i+2} = 0.12; && i = 2, 4, 6, 8, 10, 12, 14 \\
&-T_{13} + 2T_{15} - T_{16} = 0 && -T_{14} - T_{15} + 2.006T_{16} = 0.12.
\end{aligned} \tag{13}$$

Gauss-Seidel method will be used to solve equation system (13). A program similar to the one in Ex. 11.2 will be used for the calculation. For that reason, one should rewrite (13), so that one could determine temperature T_i from i-equation. Equation system (13) assumes the form

$$\begin{array}{ll}
\text{Node no.} & \text{Equation} \\
1 & T_1 = 0.5\left(T_2 + 0.5T_3 + 47.5\right) \\
2 & T_2 = \dfrac{1}{2.006}\left(T_1 + 0.5T_4 + 47.62\right) \\
3 & T_3 = 0.5\left(0.5T_1 + T_4 + 0.5T_5\right) \\
4 & T_4 = \dfrac{1}{2.006}\left(0.5T_2 + T_3 + 0.5T_6 + 0.12\right) \\
5 & T_5 = 0.5\left(0.5T_3 + T_6 + 0.5T_7\right) \\
6 & T_6 = \dfrac{1}{2.006}\left(0.5T_4 + T_5 + 0.5T_8 + 0.12\right) \\
7 & T_7 = 0.5\left(0.5T_5 + T_8 + 0.5T_9\right) \\
8 & T_8 = \dfrac{1}{2.006}\left(0.5T_6 + T_7 + 0.5T_{10} + 0.12\right) \\
9 & T_9 = 0.5\left(0.5T_7 + T_{10} + 0.5T_{11}\right) \\
10 & T_{10} = \dfrac{1}{2.006}\left(0.5T_8 + T_9 + 0.5T_{12} + 0.12\right) \\
11 & T_{11} = 0.5\left(0.5T_9 + T_{12} + 0.5T_{13}\right) \\
12 & T_{12} = \dfrac{1}{2.006}\left(0.5T_{10} + T_{11} + 0.5T_{14} + 0.12\right) \\
13 & T_{13} = 0.5\left(0.5T_{11} + T_{14} + 0.5T_{15}\right) \\
14 & T_{14} = \dfrac{1}{2.006}\left(0.5T_{12} + T_{13} + 0.5T_{16} + 0.12\right) \\
15 & T_{15} = 0.5\left(T_{13} + T_{16}\right) \\
16 & T_{16} = \dfrac{1}{2.006}\left(T_{14} + T_{15} + 0.12\right).
\end{array} \tag{14}$$

The system was solved with the assumed tolerance that equals ε = 0.00001°C. As an initial solution, the following was assumed:

$$T_1^{(0)} = T_2^{(0)} = \ ... \ = T_{16}^{(0)} = 20\,^{\circ}\mathrm{C}\,. \qquad (15)$$

Computational Program Content

```
C     Calculating two-dimensional fin temp. field (Fig.11.5)
C     by means of control volume method equation system
C     solved by Gauss-Seidel method
      program seidel2
      dimension t(50),tt(50)
      logical inaccurate
      open(unit=1,file='seidel2.in')
      open(unit=2,file='seidel2.out')
      read(1,*)n,toler,niter,t_pocz
      write(2,'(a)')
     &"CALCULATING TWO-DIMENSIONAL FIN TEMPERATURE FIELD "
      write(2,'(/a)') "DATA ENTERED"
      write(2,'(a,i10)')    "equation number  n=",n
      write(2,'(a,e10.5,a)') "cal.toler toler=",toler," [C]"
      write(2,'(a,i10)') "max. iteration number niter=",niter
      write(2,'(a,e10.5,a)')"init temp. t_pocz=",t_pocz,"[C]"
      do i=1,n
         t(i)=t_pocz
         tt(i)=t_pocz
      enddo

      i=0
      inaccurate=.true.
      do while ((i.le.niter).and.inaccurate)
      t(1)=(t(2)+0.5*t(3)+47.5)*0.5
      t(2)=(t(1)+0.5*t(4)+47.62)/2.006
      t(3)=(0.5*t(1)+t(4)+0.5*t(5))*0.5
      t(4)=(0.5*t(2)+t(3)+0.5*t(6)+0.12)/2.006
      t(5)=(0.5*t(3)+t(6)+0.5*t(7))*0.5
      t(6)=(0.5*t(4)+t(5)+0.5*t(8)+0.12)/2.006
      t(7)=(0.5*t(5)+t(8)+0.5*t(9))*0.5
      t(8)=(0.5*t(6)+t(7)+0.5*t(10)+0.12)/2.006
      t(9)=(0.5*t(7)+t(10)+0.5*t(11))*0.5
      t(10)=(0.5*t(8)+t(9)+0.5*t(12)+0.12)/2.006
      t(11)=(0.5*t(9)+t(12)+0.5*t(13))*0.5
      t(12)=(0.5*t(10)+t(11)+0.5*t(14)+0.12)/2.006
      t(13)=(0.5*t(11)+t(14)+0.5*t(15))*0.5
      t(14)=(0.5*t(12)+t(13)+0.5*t(16)+0.12)/2.006
      t(15)=(t(13)+t(16))*0.5
      t(16)=(t(14)+t(15)+0.12)/2.006
      inaccurate=.false.
```

```
do j=1,n
    if (abs(tt(j)-t(j)).gt.toler) inaccurate=.true.
enddo
  if (inaccurate) then
    do j=1,n
      tt(j)=t(j)
    enddo
  endif
 i=i+1
enddo
write(2,'(/a)')"CALCULATED TEMPERATURE"
write(2,'(a)')"   Lp      T[C] "
do j=1,n
 write(2,'(i5,3x,e11.6)')j,t(j)
enddo
write(2,'(a,i10)') "final iteration number=",i
end  program seidel2
```

```
data(seidel.in)
16 0.00001 100000 20.
```

```
results(seidel.out)
DATA ENTERED
equation number  n=        16
cal.toler toler =.10000E-04 [C]
max. iteration number niter=    100000
init temp. t_pocz=.20000E+02 [C]
CALCULATED TEMPERATURE
Lp      T[C]
 1   .921152E+02
 2   .919377E+02
 3   .895855E+02
 4   .893836E+02
 5   .874594E+02
 6   .872585E+02
 7   .857353E+02
 8   .855386E+02
 9   .844046E+02
10   .842117E+02
11   .834597E+02
12   .832696E+02
13   .828950E+02
14   .827066E+02
15   .827072E+02
16   .825193E+02
final iteration number=       536
```

Table 11.1. Temperatures in control volume nodes shown in Fig. 11.5

Node no.	Temperature		Node no.	Temperature	
	Control Volume Method	Analytical Method		Control Volume Method	Analytical Method
1	92.11	92.09	9	84.40	84.37
2	91.94	91.88	10	84.21	84.18
3	89.58	89.55	11	83.46	83.42
4	89.38	89.34	12	83.27	83.23
5	87.46	87.42	13	82.89	82.86
6	87.26	87.22	14	82.71	82.67
7	85.73	85.70	15	82.71	82.67
8	85.54	85.50	16	82.52	82.48

Calculation results are presented in Table 11.1.

One can see from the table above that temperatures calculated by means of control volume method are almost the same as the values determined by means of the analytical formula ((2), Ex.7.3). On the basis of temperature distribution, one can calculate fin-base heat flow, which equals the fin-to-surroundings transferred heat flow. If a fin perpendicular to the diagram surface (fin length) measures 1 m in length, then the heat flow at the fin-base is expressed by formula

$$\dot{Q}_b = 2\left(\frac{w}{2}\cdot 1\cdot\lambda\frac{T_b - T_2}{\Delta x} + \frac{w}{2}\cdot 1\cdot\lambda\frac{T_b - T_1}{\Delta x}\right) =$$

$$= \frac{2w\lambda}{2\Delta x}\left(2T_b - T_1 - T_2\right) = 2\lambda\left(T_b - \frac{T_1 + T_2}{2}\right),$$

hence,

$$\dot{Q}_b = 2\cdot 50\left(95 - \frac{92.11 + 91.94}{2}\right) = 297.5\ \mathrm{W}.$$

Multiplier 2 was placed in front of the square brackets, since the heat flow given off by the fin is twice as large; only half of the fin was taken into consideration in Fig.11.5. In order to calculate efficiency, one needs to know what the value of heat flow $\dot{Q}_{\max}$ is, given off by an isothermal fin with temperature T_b within its entire volume

$$\dot{Q}_{\max} = 2\cdot l\cdot 1\cdot\alpha\left(T_b - T_{cz}\right) = 2\cdot 0.024\cdot 1\cdot 100\cdot(95 - 20) = 360\ \mathrm{W}.$$

Therefore, fin efficiency determined by means of the control volume method is

$$\eta = \frac{\dot{Q}}{\dot{Q}_{\max}} = \frac{297.5}{360} = 0.826\,.$$

Fin efficiency η_e calculated by means of the analytical method is formulated as

$$\eta_e = \frac{\dot{Q}_e}{\dot{Q}_{\max}} = \frac{2w\overline{\dot{q}_x}(0)}{\dot{Q}_{\max}} = \frac{2w\overline{\dot{q}_x}(0)}{360}\,.$$

Hence,

$$\eta_e = \frac{2\cdot 0.003\cdot 53253}{360} = 0.887\,.$$

Relative error from the efficiency calculation above is

$$\Delta\eta = \frac{\eta-\eta_e}{\eta_e}\cdot 100\% = \frac{0.826-0.887}{0.887}\cdot 100\% = -6.8\%.$$

In spite of the coarse control volume grid, a good agreement was established between temperature distribution (Table 11.1) and two-dimensional analytical solution. Calculation of heat flux at the fin-base, as indicated by the calculated efficiency value, is less accurate.

In order to improve accuracy, fin efficiency will be calculated in a different way. Fin-dissipated heat flow can also be calculated by determining heat flow received by the lateral surfaces of the fin first:

$$\dot{Q} = 2\alpha\Delta x\left[\frac{1}{2}\left(T_b - T_{cz}\right) + \left(T_2 - T_{cz}\right) + \left(T_4 - T_{cz}\right) + \left(T_6 - T_{cz}\right) + \left(T_8 - T_{cz}\right) + \right.$$

$$\left. + \left(T_{10} - T_{cz}\right) + \left(T_{12} - T_{cz}\right) + \left(T_{14} - T_{cz}\right) + \frac{1}{2}\left(T_{16} - T_{cz}\right)\right] =$$

$$= 2\alpha\Delta x\left[\frac{1}{2}T_b + T_2 + T_4 + T_6 + T_8 + T_{10} + T_{12} + T_{14} + \frac{1}{2}T_{16} - 8T_{cz}\right] =$$

$$= 2\cdot 100\cdot 0.003\left(\frac{95}{2} + 91.94 + 89.38 + 87.26 + 85.54 + 84.21 + 83.27 + \right.$$

$$\left. + 82.71 + \frac{82.52}{2} - 8\cdot 20\right) = 319.842\ \mathrm{W/m}.$$

Fin efficiency, then, determined by means of the control volume method is

$$\eta = \frac{\dot{Q}}{\dot{Q}_{\max}} = \frac{319.842}{360} = 0.888\,.$$

Relative error is at

$$\Delta\eta = \frac{\eta - \eta_e}{\eta_e} \cdot 100\% = \frac{0.888 - 0.887}{0.887} \cdot 100\% = 0.112\% .$$

Exercise 11.6 Determining Two-Dimensional Temperature Distribution in a Square Cross-Section of a Chimney

Determine temperature distribution in a chimney cross-section presented in Fig. 11.6. External dimensions of the chimney are $2b \times 2b$. Internal canal has a square cross-section and the length of its side is $2a$. For the calculation assume that $b = 0.375$ m and $a = 0.125$ m. Thermal conductivity of the chimney's material is $\lambda = 1.25$ W/(m·K). Heat transfer coefficient from emissions to inner surface is $\alpha_w = 60$ W/(m^2·K), while from outer surface to surroundings is $\alpha_z = 20$ W/(m^2·K). Emissions temperature measures $T_w = 250$°C, while air temperature of surroundings $T_z = 10$°C. Solve the problem using control volume method.

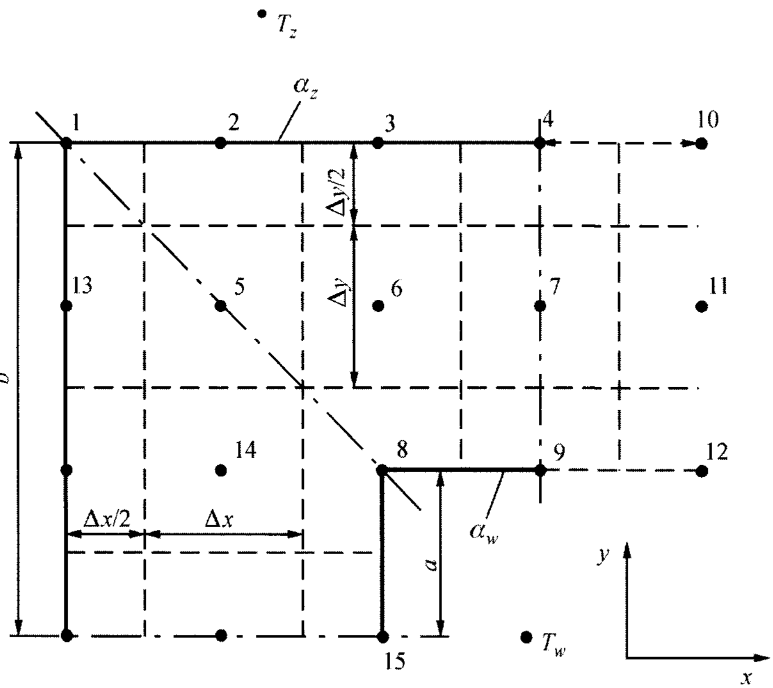


Fig. 11.6. A division of ¼ of a chimney into control volumes

Solution

Due to the symmetry of temperature field, only 1/8 of the chimney cross-section will be analyzed below. Temperature will be determined for nodes from node 1 to 9 (Fig. 11.6). Control volumes are squares with a side that measures $\Delta x = \Delta y = (b - a)/2 = (0.375 - 0.125)/2 = 0.125$ m.

Heat balance equations for control volumes have the following form:

- Node 1

$$\alpha_z\left(\frac{\Delta x}{2}+\frac{\Delta y}{2}\right)\left(T_z - T_1\right) + \lambda\frac{\Delta x}{2}\frac{T_{13}-T_1}{\Delta y} + \lambda\frac{\Delta y}{2}\frac{T_2 - T_1}{\Delta x} = 0, \tag{1}$$

from where, one obtains (when $\Delta y = \Delta x$)

$$T_1 = \frac{1}{1+\Delta Bi_z}\left[\frac{T_2 + T_{13}}{2} + \left(\Delta Bi_z\right)T_z\right]. \tag{2}$$

When $T_{13} = T_2$, (2) assumes the form

$$T_1 = \frac{1}{1+\Delta Bi_z}\left[T_2 + \left(\Delta Bi_z\right)T_z\right], \tag{3}$$

where $\Delta Bi_z = \alpha(\Delta x)/\lambda$.

- Node 2

$$\alpha_z\Delta x\left(T_z - T_2\right) + \lambda\frac{\Delta y}{2}\frac{T_1 - T_2}{\Delta x} + \lambda\frac{\Delta y}{2}\frac{T_3 - T_2}{\Delta x} + \lambda\Delta x\frac{T_5 - T_2}{\Delta y} = 0, \tag{4}$$

hence,

$$T_2 = \frac{1}{2+\Delta Bi_z}\left[T_5 + \frac{T_1 + T_3}{2} + \left(\Delta Bi_z\right)T_z\right]. \tag{5}$$

- Node 3

$$\alpha_z\Delta x\left(T_z - T_3\right) + \lambda\frac{\Delta y}{2}\frac{T_2 - T_3}{\Delta x} + \lambda\frac{\Delta y}{2}\frac{T_4 - T_3}{\Delta x} + \lambda\Delta x\frac{T_6 - T_3}{\Delta y} = 0, \tag{6}$$

where from, after transformations, one obtains

$$T_3 = \frac{1}{2+\Delta Bi_z}\left[T_6 + \frac{T_2 + T_4}{2} + \left(\Delta Bi_z\right)T_z\right]. \tag{7}$$

- Node 4

$$\alpha_z\Delta x\left(T_z - T_4\right) + \lambda\frac{\Delta y}{2}\frac{T_3 - T_4}{\Delta x} + \lambda\frac{\Delta y}{2}\frac{T_{10} - T_4}{\Delta x} + \lambda\Delta x\frac{T_7 - T_4}{\Delta y} = 0, \tag{8}$$

hence, when $T_{10} = T_3$, one obtains

$$T_4 = \frac{1}{2 + \Delta Bi_z}\left[T_7 + T_3 + \left(\Delta Bi_z\right)T_z\right]. \tag{9}$$

• Node 5

$$\lambda \Delta x \frac{T_2 - T_5}{\Delta y} + \lambda \Delta x \frac{T_{14} - T_5}{\Delta y} + \lambda \Delta y \frac{T_{13} - T_5}{\Delta x} + \lambda \Delta y \frac{T_6 - T_5}{\Delta x} = 0, \tag{10}$$

hence, when $T_{13} = T_2$ and $T_{14} = T_6$, one gets

$$T_5 = \frac{1}{2}\left(T_2 + T_6\right). \tag{11}$$

• Node 6

$$\lambda \Delta y \frac{T_5 - T_6}{\Delta x} + \lambda \Delta y \frac{T_7 - T_6}{\Delta x} + \lambda \Delta x \frac{T_3 - T_6}{\Delta y} + \lambda \Delta x \frac{T_8 - T_6}{\Delta y} = 0, \tag{12}$$

hence, one obtains

$$T_6 = \frac{1}{4}\left(T_3 + T_5 + T_7 + T_8\right). \tag{13}$$

• Node 7

$$\lambda \Delta y \frac{T_6 - T_7}{\Delta x} + \lambda \Delta y \frac{T_{11} - T_7}{\Delta x} + \lambda \Delta x \frac{T_4 - T_7}{\Delta y} + \lambda \Delta x \frac{T_9 - T_7}{\Delta y} = 0, \tag{14}$$

hence, when $T_{11} = T_6$, one obtains

$$T_7 = \frac{1}{4}\left(T_4 + 2T_6 + T_9\right). \tag{15}$$

• Node 8

$$\alpha_w \left(\frac{\Delta x}{2} + \frac{\Delta y}{2}\right)\left(T_w - T_8\right) + \lambda \Delta x \frac{T_6 - T_8}{\Delta y} + \lambda \Delta y \frac{T_{14} - T_8}{\Delta x} + \lambda \frac{\Delta y}{2} \frac{T_9 - T_8}{\Delta x} + \\ + \lambda \frac{\Delta x}{2} \frac{T_{15} - T_8}{\Delta y} = 0, \tag{16}$$

hence, after transformations and when $T_{15} = T_6$ and $T_{14} = T_9$, one has

$$T_8 = \frac{1}{3 + \Delta Bi_z}\left[2T_6 + T_9 + \left(\Delta Bi_w\right)T_w\right], \tag{17}$$

where $\Delta Bi_z = \alpha(\Delta x)/\lambda$.

• Node 9

$$\alpha_w \Delta x\left(T_w - T_9\right) + \lambda \frac{\Delta y}{2}\frac{T_8 - T_9}{\Delta x} + \lambda \frac{\Delta y}{2}\frac{T_{12} - T_9}{\Delta x} + \lambda \Delta x \frac{T_7 - T_9}{\Delta y} = 0,$$

hence, after transformations and when $T_{12} = T_8$, one has

$$T_9 = \frac{1}{2 + \Delta Bi_z}\left[T_7 + T_8 + \left(\Delta Bi_w\right)T_w\right].$$

One should emphasize that due to temperature field symmetry, planes 1-5-8 and 4-7-9 are thermally insulated (are adiabatic). There is no need to take additional nodes located outside area 1-4-9-8-1 into consideration, if one takes into consideration that symmetry planes are thermally insulated. The heat balance for node 8, with the plane 1-5-8 thermally insulated, has the form

$$\alpha_w \frac{\Delta x}{2}\left(T_w - T_8\right) + \lambda \Delta x \frac{T_6 - T_8}{\Delta y} + \lambda \frac{\Delta y}{2}\frac{T_9 - T_8}{\Delta x} = 0,$$

hence, after transformations, (17) is obtained. Equations for nodes located in symmetry planes 1, 5, 4, 7 and 9 can be derived in a similar way.

After substitution, one has

$$\Delta Bi_z = \frac{\alpha_z \Delta x}{\lambda} = \frac{20 \cdot 0.125}{1.25} = 2.0,$$

$$\Delta Bi_w = \frac{\alpha_w \Delta x}{\lambda} = \frac{60 \cdot 0.125}{1.25} = 6.0.$$

Heat balance equations for nodes 1 to 9 have the form

$$T_1 = \frac{1}{3}\left(T_2 + 20\right),\ T_2 = \frac{1}{4}\left(T_5 + \frac{T_1 + T_3}{2} + 20\right),\ T_3 = \frac{1}{4}\left(T_6 + \frac{T_2 + T_4}{2} + 20\right),$$

$$T_4 = \frac{1}{4}\left(T_7 + T_3 + 20\right), \quad T_5 = \frac{1}{2}\left(T_2 + T_6\right), \quad T_6 = \frac{1}{4}\left(T_3 + T_5 + T_7 + T_8\right),$$

$$T_7 = \frac{1}{4}\left(T_4 + 2T_6 + T_9\right),\ T_8 = \frac{1}{9}\left(2T_6 + T_9 + 1500\right),\ T_9 = \frac{1}{9}\left(T_7 + T_8 + 1500\right).$$

The equation system above will be solved by Gauss-Seidel method, when $\varepsilon = 0.00001\,°\mathrm{C}$. The printout of the program in FORTRAN language is presented below.

Computational program in FORTRAN language used for determining temperature distribution in a chimney cross-section

```
C     Calculating two-dim. temperature field in a chimney
C     cross-section (Fig. 11.6) by means of control volume
C     method, equation system solved by Gauss-Seidel method
      program seidel3
      dimension t(50),tt(50)
      logical inaccurate
      open(unit=1,file='seidel3.in')
      open(unit=2,file='seidel3.out')
      read(1,*)n,toler,niter,t_pocz
      write(2,'(a)') "CALCULATING TWO-DIMENSIONAL TEMPERATURE
     &FIELD IN CHIMNEY CROSS-SECTION"
      write(2,'(/a)') "DATA ENTERED"
      write(2,'(a,i10)')    "equation number  n=",n
      write(2,'(a,e10.5,a)') "calculation tolerance toler=",
     &toler, " [C]"
      write(2,'(a,i10)')    "max.iteration number niter=",
     &niter
      write(2,'(a,e10.5,a)')"initial temp. t_pocz=",t_pocz,
     &"[C]"
      do i=1,n
         t(i)=t_pocz
         tt(i)=t_pocz
      enddo
      i=0
      inaccurate=.true.
      do while ((i.le.niter).and.inaccurate)
         t(1)=(t(2)+20.)/3.
         t(2)=(t(5)+0.5*t(1)+0.5*t(3)+20.)/4.
         t(3)=(t(6)+0.5*t(2)+0.5*t(4)+20.)/4.
         t(4)=(t(7)+t(3)+20.)/4.
         t(5)=(t(2)+t(6))/2.
         t(6)=(t(3)+t(5)+t(7)+t(8))/4.
         t(7)=(t(4)+2.*t(6)+t(9))/4.
         t(8)=(2.*t(6)+t(9)+1500.)/9.
         t(9)=(t(7)+t(8)+1500.)/8.
         inaccurate=.false.
         do j=1,n
            if (abs(tt(j)-t(j)).gt.toler) inaccurate=.true.
         enddo
            if  (inaccurate) then
           do j=1,n
              tt(j)=t(j)
           enddo
         endif
         i=i+1
      enddo
```

```
write(2,'(/a)')"CALCULATED TEMPERATURE"
      write(2,'(a)')"   Lp     T[C] "
      do j=1,n
         write(2,'(i5,3x,e11.6)')j,t(j)
      enddo
write(2,'(a,i10)') "final iteration number=",i
      end  program seidel3
```

```
data (seidel3.in)
9 0.00001 100000 10.
```

```
results(seidel3.out)
CALCULATING TWO-DIMENSIONAL TEMPERATURE FIELD IN CHIMNEY
CROSS-SECTION

DATA ENTERED
equation number  n=9
calculation tolerance  toler=.10000E-04 [C]
max. iteration number niter=    100000
initial temp.    t_pocz=.10000E+02 [C]

CALCULATED TEMPERATURE
   Lp      T[C]
    1    .169665E+02
    2    .308994E+02
    3    .437345E+02
    4    .477868E+02
    5    .732471E+02
    6    .115595E+03
    7    .127413E+03
    8    .217985E+03
    9    .230675E+03
final iteration number= 26
```

The following initial values were assumed: $T_1^{(0)} = T_2^{(0)} = \ldots = T_9^{(0)} =$ 10°C. After $n = 26$ iterations, the following temperature values were obtained:

$$T_1 = 16.96\,^\circ\text{C}, \qquad T_2 = 30.90\,^\circ\text{C}, \qquad T_3 = 43.73\,^\circ\text{C},$$

$$T_4 = 47.78\,^\circ\text{C}, \qquad T_5 = 73.25\,^\circ\text{C}, \qquad T_6 = 115.59\,^\circ\text{C},$$

$$T_7 = 127.41\,^\circ\text{C}, \qquad T_8 = 217.98\,^\circ\text{C}, \qquad T_9 = 230.67\,^\circ\text{C}.$$

The accuracy of this solution can be evaluated if one calculates the heat flow, which is dissipated through the outer and inner chimney surface on the length of 1 m. Outer surface heat flow is

$$\dot{Q}_z = 8\alpha_z\left[\frac{\Delta x}{2}(T_1 - T_z) + \Delta x(T_2 - T_z) + \Delta x(T_3 - T_z) + \frac{\Delta x}{2}(T_4 - T_z)\right] =$$
$$= 8 \cdot 20 \cdot 0.125\left(\frac{1}{2}16.96 + 30.90 + 43.73 + \frac{1}{2}47.78 - 3 \cdot 10\right) = 1540 \text{ W}.$$

Outer surface heat flow can be calculated from the formula below

$$\dot{Q}_w = 8\left[\alpha_w \frac{\Delta x}{2}(T_w - T_8) + \alpha_w \frac{\Delta x}{2}(T_w - T_9)\right],$$

where from, after transformations, one has

$$\dot{Q}_w = 8\alpha_w(\Delta x)\left(T_w - \frac{T_8 + T_9}{2}\right) = 8 \cdot 60 \cdot 0.125\left(250 - \frac{217.98 + 230.67}{2}\right) =$$
$$= 1540.5 \text{ W}.$$

Heat flows $\dot{Q}_z$ and $\dot{Q}_w$ should be equal, since the heat conduction is steady-state. Relative difference $\Delta\dot{Q}$ is at

$$\Delta\dot{Q} = \frac{\dot{Q}_w - \dot{Q}_z}{\dot{Q}_w} \cdot 100\% = \frac{1540.5 - 1540}{1540.5} \cdot 100\% = 0.03\%.$$

The difference between $\dot{Q}_z$ *i* $\dot{Q}_w$ is attributed to a rather small number of control volumes.

Exercise 11.7 Pseudo-Transient Determination of Steady-State Temperature Distribution in a Square Cross-Section of a Chimney; Heat Transfer by Convection and Radiation on an Outer Surface of a Chimney

Determine steady-state temperature distribution in a cross-section of a chimney presented in Fig. 11.6; allow for both, heat transfer by convection and heat transfer by radiation. Assume that the equivalent emissivity of the chimney's interior is $\varepsilon_w = 0.9$, while the outer surface $\varepsilon_z = 0.8$. Other values remain the same as they are in Ex. 11.6. Use control volume method to determine temperature distribution.

Solution

The presence of radiation renders this problem to be non-linear. If the problem is solved as a steady-state problem, one obtains a non-linear algebraic equation system for node temperature, which can be solved by Newton-Raphson method or by other iteration methods. The problem in question can be also solved as a transient problem, since there are a number of well developed methods, which can be used for solving non-linear ordinary differential equation systems, for e.g., Rung- Kutta method. Temperature distribution is determined after a sufficiently long duration; that is, the unknown steady-state temperature distribution is determined. Heat balance equations for individual control volumes (Fig. 11.6) have the form (when $\Delta x = \Delta y$):

- Node 1

$$\frac{1}{2}(\Delta x)\frac{1}{2}(\Delta x)c\rho\frac{dT_1}{dt} = \alpha_z\cdot(\Delta x)(T_z - T_1) + \varepsilon_z\sigma\cdot(\Delta x)(T_z^4 - T_1^4) + \\ +\lambda\frac{\Delta x}{2}\frac{T_{13} - T_1}{\Delta x} + \lambda\frac{\Delta x}{2}\frac{T_2 - T_1}{\Delta x}, \tag{1}$$

hence, one obtains

$$\frac{dT_1}{dt} = \frac{4a}{(\Delta x)^2}\left[(\Delta Bi_z)T_z - \frac{\varepsilon_z\sigma\cdot(\Delta x)}{\lambda}(T_z^4 - T_1^4) - (1 + \Delta Bi_z)T_1 + \right. \\ \left. + \frac{T_2}{2} + \frac{T_{13}}{2}\right], \tag{2}$$

following that, when $T_{13} = T_2$

$$\frac{dT_1}{dt} = \frac{4a}{(\Delta x)^2}\left[(\Delta Bi_z)T_z - \frac{\varepsilon_z\sigma\cdot(\Delta x)}{\lambda}(T_z^4 - T_1^4) - (1 + \Delta Bi_z)T_1 + T_2\right], \tag{3}$$

where temperatures are expressed in Kelvin, while $\sigma = 5.67\cdot 10^{-8}$ W/(m$^2\cdot$K^4) is the Stefan-Boltzmann constant.

- Node 2

$$\frac{1}{2}(\Delta x)(\Delta x)c\rho\frac{dT_2}{dt} = \alpha_z\cdot(\Delta x)(T_z - T_2) + \varepsilon_z\sigma\cdot(\Delta x)(T_z^4 - T_2^4) + \\ +\lambda\frac{\Delta x}{2}\frac{T_1 - T_2}{\Delta x} + \lambda\frac{\Delta x}{2}\frac{T_3 - T_2}{\Delta x} + \lambda(\Delta x)\frac{T_5 - T_2}{\Delta x}, \tag{4}$$

which results in

$$\frac{dT_2}{dt} = \frac{2a}{(\Delta x)^2}\left[(\Delta Bi_z)T_z + \frac{\varepsilon_z\sigma\cdot(\Delta x)}{\lambda}(T_z^4 - T_2^4) - (2 + \Delta Bi_z)T_2 + \right. \\ \left. + \frac{T_1}{2} + \frac{T_3}{2} + T_5\right]. \tag{5}$$

• Node 3

$$\frac{1}{2}(\Delta x)(\Delta x)c\rho\frac{dT_3}{dt}=\alpha_z\cdot(\Delta x)(T_z-T_3)+\varepsilon_z\sigma\cdot(\Delta x)(T_z^4-T_3^4)+ \\ +\lambda\frac{\Delta x}{2}\frac{T_2-T_3}{\Delta x}+\lambda\frac{\Delta x}{2}\frac{T_4-T_3}{\Delta x}+\lambda\cdot(\Delta x)\frac{T_6-T_3}{\Delta x}, \tag{6}$$

$$\frac{dT_3}{dt}=\frac{2a}{(\Delta x)^2}\left[(\Delta Bi_z)T_z+\frac{\varepsilon_z\sigma\cdot(\Delta x)}{\lambda}(T_z^4-T_3^4)-(2+\Delta Bi_z)T_3+ \\ +\frac{T_2}{2}+\frac{T_4}{2}+T_6\right]. \tag{7}$$

• Node 4

$$(\Delta x)\frac{1}{2}(\Delta x)c\rho\frac{dT_4}{dt}=\alpha_z(\Delta x)(T_z-T_4)+\varepsilon_z\sigma\cdot(\Delta x)(T_z^4-T_4^4)+ \\ +\lambda\frac{\Delta x}{2}\frac{T_{10}-T_4}{\Delta x}+\lambda\frac{\Delta x}{2}\frac{T_3-T_4}{\Delta x}+\lambda\cdot(\Delta x)\frac{T_7-T_4}{\Delta x}, \tag{8}$$

hence, when $T_{10}=T_3$,

$$\frac{dT_4}{dt}=\frac{2a}{(\Delta x)^2}\left[(\Delta Bi_z)T_z+\frac{\varepsilon_z\sigma\cdot(\Delta x)}{\lambda}(T_z^4-T_4^4)-(2+\Delta Bi_z)T_4+T_3+T_7\right]. \tag{9}$$

• Node 5

$$(\Delta x)(\Delta x)c\rho\frac{dT_5}{dt}=\lambda\cdot(\Delta x)\frac{T_2-T_5}{\Delta x}+\lambda\cdot(\Delta x)\frac{T_6-T_5}{\Delta x}+ \\ +\lambda\cdot(\Delta x)\frac{T_{13}-T_5}{\Delta x}+\lambda\cdot(\Delta x)\frac{T_{14}-T_5}{\Delta x}, \tag{10}$$

where from, after transformations and when $T_{13}=T_2$ and $T_{14}=T_6$, one has

$$\frac{dT_5}{dt}=\frac{2a}{(\Delta x)^2}(T_2+T_6-2T_5), \tag{11}$$

• Node 6

$$(\Delta x)^2 c\rho\frac{dT_6}{dt}=\lambda\cdot(\Delta x)\frac{T_3-T_6}{\Delta x}+\lambda\cdot(\Delta x)\frac{T_5-T_6}{\Delta x}+ \\ +\lambda\cdot(\Delta x)\frac{T_7-T_6}{\Delta x}+\lambda\cdot(\Delta x)\frac{T_8-T_6}{\Delta x}, \tag{12}$$

where from, after transformations, one obtains

$$\frac{dT_6}{dt} = \frac{2a}{(\Delta x)^2}\left(T_3 + T_5 + T_7 + T_8 - 4T_6\right). \tag{13}$$

- Node 7

$$(\Delta x)^2 c\rho \frac{dT_7}{dt} = \lambda \cdot (\Delta x)\frac{T_4 - T_7}{\Delta x} + \lambda \cdot (\Delta x)\frac{T_6 - T_7}{\Delta x} + \lambda \cdot (\Delta x)\frac{T_9 - T_7}{\Delta x} + \\ + \lambda \cdot (\Delta x)\frac{T_{11} - T_7}{\Delta x}, \tag{14}$$

hence, when $T_{11} = T_6$, one has

$$\frac{dT_7}{dt} = \frac{a}{(\Delta x)^2}\left(T_4 + 2T_6 + T_9 - 4T_7\right), \tag{15}$$

- Node 8

$$\frac{3}{4}(\Delta x)^2 c\rho \frac{dT_8}{dt} = \varepsilon_w \sigma \cdot (\Delta x)\left(T_w^4 - T_8^4\right) + \alpha_w \cdot (\Delta x)\left(T_w - T_8\right) + \\ + \lambda \cdot (\Delta x)\frac{T_6 - T_8}{\Delta x} + \lambda \frac{\Delta x}{2}\frac{T_9 - T_8}{\Delta x} + \lambda \frac{\Delta x}{2}\frac{T_{18} - T_8}{\Delta x} + \lambda \cdot (\Delta x)\frac{T_{14} - T_8}{\Delta x}, \tag{16}$$

hence, after transformations and when $T_{14} = T_6$ and $T_{15} = T_9$, one has

$$\frac{dT_8}{dt} = \frac{4}{3}\frac{a}{(\Delta x)^2}\left[\left(\Delta Bi_w\right)T_w + 2T_6 + T_9 - \left(3 + \Delta Bi_w\right)T_8 \right. \\ \left. + \frac{\varepsilon_w \sigma \cdot (\Delta x)}{\lambda}\left(T_w^4 - T_8^4\right)\right]. \tag{17}$$

- Node 9

$$\frac{1}{2}(\Delta x)^2 c\rho \frac{dT_9}{dt} = \varepsilon_w \sigma \cdot (\Delta x)\left(T_w^4 - T_9^4\right) + \alpha_w \cdot (\Delta x)\left(T_w - T_9\right) + \\ + \lambda \cdot (\Delta x)\frac{T_7 - T_9}{\Delta x} + \lambda \frac{\Delta x}{2}\frac{T_8 - T_9}{\Delta x} + \lambda \frac{\Delta x}{2}\frac{T_{12} - T_9}{\Delta x}, \tag{18}$$

hence, when $T_{12} = T_8$, one gets

$$\frac{dT_9}{dt} = \frac{2a}{(\Delta x)^2}\left[\left(\Delta Bi_w\right)T_w + \frac{\varepsilon_w \sigma \cdot (\Delta x)}{\lambda}\left(T_w^4 - T_9^4\right) + T_7 + T_8 - \left(2 + \Delta Bi_w\right)T_9\right] \tag{19}$$

After assuming for $\Delta x = \Delta y = (b - a)/2 = (0.375 - 0.125)/2 = 0.125$ m, $\lambda = 1.25$ W/(m·K), $\alpha_w = 60$ W/(m^2·K), $\alpha_z = 20$ W/(m^2·K), $T_w = 250 + 273.15 = 523.15$ K, $T_z = 10 + 273.15 = 283.15$ K , one can calculate ΔBi_w and ΔBi_z:

$$\Delta Bi_w = \frac{\alpha_w \cdot (\Delta x)}{\lambda} = \frac{60 \cdot 0.125}{1.25} = 6.0,$$

$$\Delta Bi_z = \frac{\alpha_z \cdot (\Delta x)}{\lambda} = \frac{20 \cdot 0.125}{1.25} = 2.0$$

and

$$\frac{\varepsilon_w \sigma \cdot (\Delta x)}{\lambda} = \frac{0.9 \cdot 5.67 \cdot 10^{-8} \cdot 0.125}{1.25} = 5.103 \cdot 10^{-9} \ 1/\mathrm{K}^3,$$

$$\frac{\varepsilon_z \sigma \cdot (\Delta x)}{\lambda} = \frac{0.8 \cdot 5.67 \cdot 10^{-8} \cdot 0.125}{1.25} = 4.536 \cdot 10^{-9} \ 1/\mathrm{K}^3.$$

In order to calculate temperature distribution, one needs to know what the value of heat diffusivity $a = \lambda/c\rho$ is. To quickly reach a steady-state, a should have a large value, e.g. $a = 1.5625 \cdot 10^{-5}$ m/s^2; then $a/(\Delta x)^2 = 1.5625 \cdot 10^{-5}/(0.125)^2 = 0.001$ 1/s . If we were to assume that $a = 5.2 \cdot 10^{-7}$ m^2/s, we would significantly lengthen the whole calculation, since the transient state (chimney heating) would last longer. At an initial moment when $t = 0$ s, chimney temperature is uniform and is of 10°C, thus $T_1(0) = T_2(0) = \ldots = T_9(0) = 283.15$ K. Once all the data is taken into consideration, (3), (5), (7), (9), (11), (13), (15), (17) and (19) assume, respectively, the following forms:

$$\frac{dT_1}{dt} = 0.004\left[566.3 + 4.536 \cdot 10^{-9}\left(283.15^4 - T_1^4\right) - 3T_1 + T_2\right], \tag{20}$$

$$\frac{dT_2}{dt} = 0.002\left[566.3 + 4.536 \cdot 10^{-9}\left(283.15^4 - T_2^4\right) - 4T_2 + \frac{T_1}{2} + \frac{T_3}{2} + T_5\right], \tag{21}$$

$$\frac{dT_3}{dt} = 0.002\left[566.3 + 4.536 \cdot 10^{-9}\left(283.15^4 - T_3^4\right) - 4T_3 + \frac{T_2}{2} + \frac{T_4}{2} + T_6\right], \tag{22}$$

$$\frac{dT_4}{dt} = 0.002\left[566.3 + 4.536 \cdot 10^{-9}\left(283.15^4 - T_4^4\right) - 4T_4 + T_7 + T_3\right], \tag{23}$$

$$\frac{dT_5}{dt} = 0.002\left(T_2 - 2T_5 + T_6\right), \tag{24}$$

$$\frac{dT_6}{dt} = 0.001\left(T_3 + T_5 + T_7 + T_8 - 4T_6\right), \tag{25}$$

$$\frac{dT_7}{dt} = 0.001\left(T_4 + 2T_6 + T_9 - 4T_7\right), \tag{26}$$

$$\frac{dT_8}{dt} = \frac{0.004}{3}\left[3138.9 + 5.103\cdot 10^{-9}\left(523.15^4 - T_8^4\right) + 2T_6 + T_9 - 9T_8\right], \tag{27}$$

$$\frac{dT_9}{dt} = 0.002\left[3138.9 + 5.103\cdot 10^{-9}\left(523.15^4 - T_9^4\right) + T_7 + T_8 - 8T_9\right]. \tag{28}$$

Initial conditions have the form

$$T_1(0) = T_2(0) = \ \ldots\ = T_9(0) = 283.15\ \text{K} \tag{29}$$

Problem (20)–(29) will be solved by the Rung-Kutta method. A subprogram for the integration of the equation system by means of the Rung-Kutta method is presented in Appendix G. If we assume that temperature is already in a steady-state after t = 9120 s, then the following temperature values will be obtained:

$T_1 = 288.36\ \text{K} = 15.21^\circ\text{C}$, $T_2 = 300.99\ \text{K} = 27.84^\circ\text{C}$,
$T_3 = 312.55\ \text{K} = 39.40^\circ\text{C}$, $T_4 = 315.99\ K = 42.84^\circ\text{C}$,
$T_5 = 345.27\ \text{K} = 72.12^\circ\text{C}$, $T_6 = 389.56\ \text{K} = 116.41^\circ\text{C}$,
$T_7 = 401.19\ \text{K} = 128.04^\circ\text{C}$, $T_8 = 499.22\ \text{K} = 226.07^\circ\text{C}$,
$T_9 = 509.66\ \text{K} = 236.51^\circ\text{C}$.

The integration step in Rung-Kutta method was assumed to equal Δt = 60 s. Heat flow transferred by an inner surface of the chimney within the length of 1 m is at

$$\dot{Q}_w = 4\Big[\alpha_w \cdot (\Delta x)(T_w - T_8) + \varepsilon_w \sigma \cdot (\Delta x)\left(T_w^4 - T_8^4\right) + \alpha_w \cdot (\Delta x)(T_w - T_9) +$$
$$+\varepsilon_w \sigma \cdot (\Delta x)\left(T_w^4 - T_9^4\right)\Big] = 4\big[60 \cdot 0.125(523.15 - 499.22) + 0.9 \cdot 5.67 \times$$
$$\times 10^{-8} \cdot 0.125 \cdot \left(523.15^4 - 499.22^4\right) + 60 \cdot 0.125(523.15 - 509.66) +$$
$$+0.9 \cdot 5.67 \cdot 10^{-8} \cdot 0.125 \cdot \left(523.15^4 - 509.66^4\right)\big] = 1638.65 \text{ W}.$$

Heat flow given off by an outer surface is

$$\dot{Q}_z = 8\Big[\alpha_z \frac{\Delta x}{2}(T_1 - T_z) + \frac{1}{2}\varepsilon_z \sigma \cdot (\Delta x)\left(T_1^4 - T_z^4\right) + \alpha_z \cdot (\Delta x)(T_2 - T_z) +$$
$$+\varepsilon_z \sigma \cdot (\Delta x)\left(T_2^4 - T_z^4\right) + \alpha_z \cdot (\Delta x)(T_3 - T_z) + \varepsilon_z \sigma \cdot (\Delta x)\left(T_3^4 - T_z^4\right) +$$
$$+\alpha_z \frac{\Delta x}{2}(T_4 - T_z) + \frac{1}{2}\varepsilon_z \sigma \cdot (\Delta x)\left(T_4^4 - T_z^4\right)\Big] = 8\Big[\alpha_z \cdot (\Delta x) \cdot \Big[\frac{T_1}{2} + T_2 +$$
$$+T_3 + \frac{T_4}{2} - 3T_z\Big] + \varepsilon_z \sigma \cdot (\Delta x)\left(\frac{T_1^4}{2} + T_2^4 + T_3^4 + \frac{T_2^4}{2} - 3T_z^4\right)\Big],$$

hence, after substitution of the numerical values, one has

$$\dot{Q}_z = 8\Big[20 \cdot 0.125 \cdot \left(\frac{288.36}{2} + 300.99 + 312.55 + \frac{315.99}{2} - 3 \cdot 283.15\right) +$$
$$+0.8 \cdot 5.67 \cdot 10^{-8} \cdot 0.125\left(\frac{288.36^4}{2} + 300.99^4 + 312.55^4 +\right.$$
$$\left.\frac{300.99^4}{2} - 3 \cdot 283.15^4\right)\Big] = 1598.71 \text{ W}$$

Heat flows $\dot{Q}_w$ and $\dot{Q}_z$ should be equal. Relative difference

$$\Delta\dot{Q} = \frac{\dot{Q}_w - \dot{Q}_z}{\dot{Q}_w} \cdot 100\% = \frac{1638.65 - 1598.71}{1638.65} \cdot 100\% = 2.44\%$$

results from the small number of control volumes. By increasing the number of control volumes, one can improve the accuracy of the obtained results and at the same time decrease the difference between $\dot{Q}_w$ and $\dot{Q}_z$. If heat exchange by radiation is neglected in (20)–(28), i.e. when $\varepsilon_w = \varepsilon_z = 0$, then the following node temperature values are obtained (after $t = 9780$ s):

$T_1 = 290.12\text{ K} = 16.97°\text{C},$ $T_2 = 304.05\text{ K} = 30.90°\text{C},$
$T_3 = 316.88\text{ K} = 43.73°\text{C},$ $T_4 = 320.94\text{ K} = 47.79°\text{C},$
$T_5 = 346.40\text{ K} = 73.25°\text{C},$ $T_6 = 388.74\text{ K} = 115.59°\text{C},$
$T_7 = 400.56\text{ K} = 127.41°\text{C},$ $T_8 = 491.13\text{ K} = 217.98°\text{C},$
$T_9 = 503.82\text{ K} = 230.67°\text{C},$

They are almost identical to temperatures calculated in Ex.11.6. Due to the fact that steady-state was treated as a particular case of transient state, when $t \to \infty$, temperature distribution was calculated in a relatively simple way, as it was not necessary to apply iteration and to select approximate, initial temperature values at the beginning of the iteration process.

Computational program in the FORTRAN language used for determining temperature distribution in a cross-section of a chimney

```
c     Calculating two-dim. temperature field in a chimney
c     cross-section (Fig. 11.6) by means of control volume
c     method, equation system solved by Runge- Kutta method
      program rkutta
      integer co_ile_druk
      dimension y(6000),f(6000)
      open(unit=1,file='rkutta.in')
      open(unit=2,file='rkutta.out')
      read(1,*) t,dt,m,n_row,n_time
      read(1,*) t_init
      read(1,*) co_ile_druk
      write(2,'(a)')"CALCULATING 2-D TEMP FIELD IN A CHIMNEY"
      write(2,'(/a)') "DATA ENTERED"
      write(2,'(a,e10.5,a)')  "initial time=",t," [s]"
      write(2,'(a,e10.5,a)')  "time step=",dt," [s]"
      write(2,'(a,i10)')    "parameter m=",m
      write(2,'(a,i10)')    "equation number  n_row=",n_row
      write(2,'(a,i10)')"time step number  n_time=",n_time
      write(2,'(a,e10.5,a)')"initial temp. t_init=",t_init,
     &" [C]"
      write(2,'(a,i10)')  "printing frequency=", co_ile_druk
      z_w=5.103E-9
      z_z=4.536E-9
      write(2,'(/a)')"CALCULATED TEMPERATURE [K]"
      write(2,'(a,a)')"t[s]   T(1)    T(2)    T(3)    T(4)",
     &"    T(5)    T(6)    T(7)    T(8)    T(9)"
      numerator=0
      kolejny=1
      to i=1,n_row
      y(i)= t_init
      enddo
```

```
      write(2,'(f9.0,9f8.3)') t,((y(i)),i=1,n_row)

    8 if( (kolejny-n_time).le.0.0 ) then
               kolejny= kolejny+1
    6    licznik = licznik+1
               call runge (n_row,y,f,t,dt,m,k)
               goto (10,20),k
   10 f(1)=0.004*(566.3+z_z*(283.15**4-y(1)**4)-3.*y(1)+y(2))
      f(2)=0.002*(566.3+z_z*(283.15**4-y(2)**4)-4.*y(2)+
     &0.5*y(1)+0.5*y(3)+y(5))
      f(3)=0.002*(566.3+z_z*(283.15**4-y(3)**4)-4.*y(3)+
     &0.5*y(2)+0.5*y(4)+y(6))
      f(4)=0.002*(566.3+z_z*(283.15**4-y(4)**4)-
     &4.*y(4)+y(7)+y(3))
      f(5)=0.002*(y(2)-2.*y(5)+y(6))
      f(6)=0.001*(y(3)+y(5)+y(7)+y(8)-4.*y(6))
      f(7)=0.001*(y(4)+2.*y(6)+y(9)-4.*y(7))
      f(8)=0.004*(3138.9+z_w*(523.15**4-y(8)**4)+2.*y(6)+
     &y(9)-9.*y(8))/3.
      f(9)=0.002*(3138.9+z_w*(523.15**4-y(9)**4)+y(7)+
     &y(8)-8.*y(9))
         goto 6
   20    continue
      if((float(licznik / co_ile_druk)* co_ile_druk)
     & .ne. licznik) goto 6
         write(2,'(f9.0,9f8.3)') t,((y(i)),i=1,n_row)
              goto 8
       endif
      stop
      end
```

```
data(rkutta.in)
0.0 60. 0 9 200
283.15
1
```

```
results(rkutta.out - set part)
CALCULATING 2-D TEMP FIELD IN A CHIMNEY

DATA ENTERED
initial time=.00000E+00 [s]
time step=.60000E+02 [s]
parameter m=         0
equation number  n_row=         9
time step number  n_time=       200
initial temp.    t_init=.28315E+03 [C]
```

```
printing frequency=        1

CALCULATED TEMPERATURE [K]
t[s]   T(1)    T(2)    T(3)    T(4)    T(5)    T(6)    T(7)
T(8)    T(9)
0. 283.150 283.150 283.150 283.150 283.150 283.150 283.150
283.150 283.150
60. 283.150 283.158 283.268 283.306 283.277 286.477 287.810
387.704 418.660
120. 283.155 283.217 283.892 284.131 283.975 293.481 296.517
439.175 469.283
180. 283.178 283.394 285.038 285.567 285.357 301.487 305.863
463.512 488.055
240. 283.231 283.722 286.546 287.393 287.323 309.332 314.766
475.076 495.437
     .................
9000. 288.361 300.990 312.555 315.994 345.275 389.560 401.193
499.218 509.656
9060. 288.361 300.990 312.555 315.994 345.275 389.560 401.193
499.218 509.656
9120. 288.361 300.990 312.555 315.995 345.275 389.560 401.193
499.218 509.656
9180. 288.361 300.990 312.555 315.995 345.275 389.560 401.193
499.218 509.656
9240. 288.361 300.990 312.555 315.995 345.275 389.560 401.193
499.218 509.656
     .................
11820. 288.361 300.990 312.555 315.995 345.275 389.560
401.193 499.218 509.656
11880. 288.361 300.990 312.555 315.995 345.275 389.560
401.193 499.218 509.656
11940. 288.361 300.990 312.555 315.995 345.275 389.560
401.193 499.218 509.656
12000. 288.361 300.990 312.555 315.995 345.275 389.560
401.193 499.218 509.656
```

Exercise 11.8 Finite Element Method

Describe the procedure for calculating temperature fields by means of the finite element method (FEM). List main advantages and disadvantages of FEM.

Historical Development of FEM

The precursor of the FEM method was a mathematician by the name of Courant, who in 1943 employed the segmental approximation by polynomial method in combination with the variational method in order to solve the torsion problem [3]. The method was developed and its present name, namely the *finite element method* appeared in 1950s [2, 7]. Traditional analytical approximation methods, such as variational methods or Galerkin methods [5] have many limitations, which arise from the approximation of solution within the entire analyzed area by means of a single function. It is almost impossible to apply analytical approximation method in an instance when the shape of an analyzed region is complex and its boundary conditions change in a time and position. Similar limitations, with respect to shape, characterize the classical finite difference method, in which partial derivatives in differential equations are approximated by means of differentce quotients. Finite difference method allows one to analyze different boundary conditions; the shape of a body, however, should be isometric, e.g. a rectangle, prism, cylinder, sphere, or a flat, cylindrical or spherical wall. The universal applicability of the finite difference method, such as the *control volume method*, also known as *finite volume method*, allows to find a solution in construction elements or in the complex shape regions [1]. Capabilities of this method are very similar to the capabilities of FEM. A region, whose temperature distribution we are trying to establish, can be divided into control volumes (cells) of arbitrary shapes; due to this reason, one can analyze curvilinear boundaries or other complex-shape boundaries.

The application of finite element method in heat transfer and fluid mechanics also has certain limitations. At the boundary of a given element, heat or mass that flows from one element can differ from the heat that flows towards adjacent element, in spite of the fact that the same section of the boundary is in both instances analyzed. This is due to the fact that a discontinuity of heat flux occurs in FEM at the boundary of adjacent elements. To eliminate this problem, a so called *finite element balance method* was developed. Another difference between FEM and the control (finite) volume method is the approximation method for a temperature derivative after time in transient problems. In FEM, thermal capacity of an element is distributed among the element nodes at appropriate weights; in finite volume method, however, thermal capacity of a cell (element) is concentrated in a single node that lies inside the cell. From the comparison of calculation results obtained by means of FEM and finite volume method, it is evident that the concentration of thermal capacity in a single point not only does not tamper with calculation accuracy but increases it. In inverse problems, concentration of thermal capacity in a single point, which lies inside the control volume, improves the solution stability.

Finally, one can conclude that although finite difference method and FEM were treated initially as two separate methods, the discrepancy is almost invisible between the finite volume method, which derives from finite difference method, and the FEM balance method, with a concentrated thermal capacity in finite elements.

Also the functions, which interpolate temperature distribution (or other unknown quantities) inside the finite volume or a finite element can be employed in both methods.

Solution

FEM consists of the following calculation steps:

1. Division of an area into finite elements (a grid generation), Fig. 11.7.
2. Mathematical formulation of Galerkin or variational method (Ritz method) for the analyzed boundary or initial-boundary problem within the area of a single element.

3. Selection of functions, which interpolate temperature distribution inside the element (shape function).
4. Determination of an algebraic equation system for a steady-state problem or of an ordinary differential equation system for transient problems in a single element by means of Galerkin or variational method, formulated in step 2. The equation number equals the node number in a given element, since node temperatures are the unknown quantities in the element.
5. Summing up of equation systems for individual elements, with an aim to create a single universal node-temperature equation system for the whole analyzed region.
6. Allowing for the parameters present in the boundary conditions of the global equation system.
7. Solving the algebraic equation system in the case of a steady-state problem or the ordinary differential equation system in the case of a transient problem.
8. Calculation of heat flux, heat flow and other secondary quantities and graphical representation of the calculation results (*post-processing*).

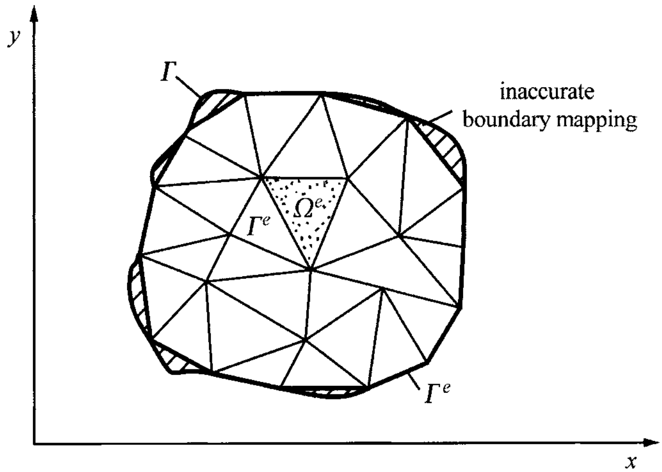


Fig. 11.7. Division of an area into finite elements

The procedure outlined above is typical of large commercial programs, designed to provide solutions to problems from different disciplines. In terms of individual solutions to specific problems and a development of ones own computational program, the procedure steps can have a different sequence; for instance, boundary conditions can already be accounted for when creating algebraic equations for a given element, i.e. at point 4. Also the global equation system can be created in a different way by summing up, for instance, algebraic equations (or ordinary differential equations) for node i, for example, obtained for elements with the same shape coefficient N_i around the node in Galerkin method.

Assuming that few elements share a common node i (Fig. 11.8), the shape coefficient N_i assumed in Galerkin method occurs only in elements with common node i. In other nodes, the value of such coefficient equals zero. This is the reason why we can add up the equations obtained for all elements with the same shape coefficient N_i around node i when creating a global equation system.

If the same is done with respect to all other nodes, a global algebraic equation system or ordinary differential equation system is obtained for node temperatures that can be solved by means of different methods.

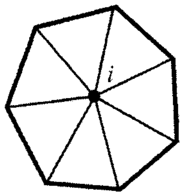


Fig. 11.8. Elements taken into consideration when determining an algebraic equation for steady-state problems or an ordinary differential equation for transient problems in a single node

The second method for creating a global equation system is the same as in the finite volume method, in which heat balance is written for node i.

Finite element method has the following advantages:

- It is suited for problem analysis in complex shape bodies.
- Boundary conditions can be non-linear and time and location-dependent .
- There are number of very good software programs that enable one to quickly solve numerous problems, including problems related to deformation mechanics, heat transfer and fluid mechanics.
- One can solve non-linear problems, when thermo-physical properties of a material are temperature-dependent, and problems in heterogeneous bodies with location-dependent properties in, for instance, composite or anisotropic materials.
- A division of an area into finite elements is automatically carried out (using commonly available software programs), which makes it easier to evaluate accuracy of obtained results by increasing density of an element grid.
- Calculation results are obtained in a graphical and numerical form, which make it easier for a user to quickly analyze obtained results (majority of software programs can be installed in personal computers). The software costs are gradually decreasing.

Finite element method is not, however, free of drawbacks; its main drawbacks are as follow:

- The initial installation costs of the FEM software program are very high.
- Source programs are usually not included in the software set; thus, there is no possibility for program modification or improvement.
- If the problem under analysis is part of a larger problem, it is difficult then to combine one software set with one's own programs or with other sets, especially if the problem is to be solved in an on-line mode.
- Particular attention should be paid to the accuracy of results obtained by means of FEM. The apparent ease, which the results are obtained with in a graphical form is rather deceptive. Even when boundary conditions are prescribed incorrect, e.g. when the end conditions for a construction element are incorrectly set during the determination of thermal stresses, the obtained results seem to be correct at a first glance.

Exercise 11.9 Linear Functions That Interpolate Temperature Distribution (Shape Functions) Inside Triangular and Rectangular Elements

Describe the simplest forms of temperature-distribution-interpolating functions inside triangular and rectangular elements (*shape functions*).

Solution

First, we will discuss triangular elements (Fig. 11.9).

Temperature distribution in a triangular element will be approximated by a linear function

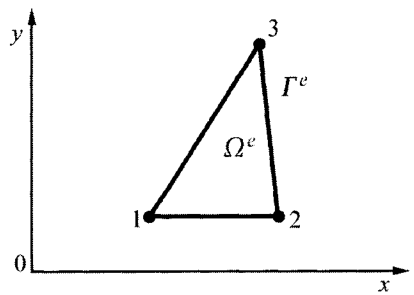


Fig. 11.9. Triangular finite element

$$T^e(x,y) = \alpha_1^e + \alpha_2^e x + \alpha_3^e y. \tag{1}$$

Constants α_1^e, α_2^e, α_3^e will be determined from conditions

$$T^e(x_1,y_1) = T_1^e, \qquad T^e(x_2,y_2) = T_2^e, \qquad T^e(x_3,y_3) = T_3^e. \tag{2}$$

By substituting (1) into (2), one obtains the following equation system

$$\begin{aligned} \alpha_1^e + \alpha_2^e x_1 + \alpha_3^e y_1 &= T_1^e \\ \alpha_1^e + \alpha_2^e x_2 + \alpha_3^e y_2 &= T_2^e \\ \alpha_1^e + \alpha_2^e x_3 + \alpha_3^e y_3 &= T_3^e, \end{aligned} \tag{3}$$

where from, one has

$$\begin{aligned} \alpha_1^e &= \frac{1}{2A^e}\left[(x_2y_3 - x_3y_2)T_1^e + (x_3y_1 - x_1y_3)T_2^e + (x_1y_2 - x_2y_1)T_3^e\right], \\ \alpha_2^e &= \frac{1}{2A^e}\left[(y_2 - y_3)T_1^e + (y_3 - y_1)T_2^e + (y_1 - y_2)T_3^e\right], \\ \alpha_3^e &= \frac{1}{2A^e}\left[(x_3 - x_2)T_1^e + (x_1 - x_3)T_2^e + (x_2 - x_1)T_3^e\right], \end{aligned} \tag{4}$$

where

$$2A^e = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \quad (A^e\text{–a surface area of triangle 1-2-3 from Fig. 11.9}). \tag{5}$$

After substituting (4) into (1) and ordering, one has

$$T^e(x,y) = N_1^e \cdot T_1^e + N_2^e \cdot T_2^e + N_3^e \cdot T_3^e, \tag{6}$$

where N_1^e, N_2^e N_3^e are, so called, *shape functions*, formulated as

$$\begin{aligned} N_1^e &= \frac{1}{2A^e}\left(a_1^e + b_1^e x + c_1^e y\right), \\ N_2^e &= \frac{1}{2A^e}\left(a_2^e + b_2^e x + c_2^e y\right), \\ N_3^e &= \frac{1}{2A^e}\left(a_3^e + b_3^e x + c_3^e y\right), \end{aligned} \tag{7}$$

where

$$\begin{aligned} a_1^e &= x_2y_3 - x_3y_2, \quad b_1^e = y_2 - y_3, \quad c_1^e = x_3 - x_2, \\ a_2^e &= x_3y_1 - x_1y_3, \quad b_2^e = y_3 - y_1, \quad c_2^e = x_1 - x_3, \\ a_3^e &= x_1y_2 - x_2y_1, \quad b_3^e = y_1 - y_2, \quad c_3^e = x_2 - x_1. \end{aligned} \tag{8}$$

On the basis of temperature distribution inside the element, one can determine heat flux vector

$$\dot{\mathbf{q}} = -\left(\lambda_x \frac{\partial T}{\partial x}\mathbf{i} + \lambda_y \frac{\partial T}{\partial y}\mathbf{j}\right). \tag{9}$$

Derivative $\partial T/\partial x$ is formulated as

$$\frac{\partial T}{\partial x} = \frac{\partial N_1^e}{\partial x}T_1^e + \frac{\partial N_2^e}{\partial x}T_2^e + \frac{\partial N_3^e}{\partial x}T_3^e, \tag{10}$$

hence, after substituting into (7), one gets

$$\begin{aligned}\frac{\partial T}{\partial x} &= \frac{1}{2A^e}\left(b_1^e \cdot T_1^e + b_2^e \cdot T_2^e + b_3^e \cdot T_3^e\right) = \frac{1}{2A^e}\Big[(y_2 - y_3)T_1^e + \\ &+ (y_3 - y_1)T_2^e + (y_1 - y_2)T_3^e\Big].\end{aligned} \tag{11}$$

Derivative $\partial T/\partial y$ can be calculated in a similar way

$$\frac{\partial T}{\partial y} = \frac{1}{2A^e}\left[(x_3 - x_2)T_1^e + (x_1 - x_3)T_2^e + (x_2 - x_1)T_3^e\right]. \tag{12}$$

It is evident, thus, that components of the heat flux vector

$$\dot{q}_x = -\lambda_x \frac{\partial T}{\partial x} \quad \text{and} \quad \dot{q}_y = -\lambda_y \frac{\partial T}{\partial y} \tag{13}$$

are constant and position-independent inside the element. It seems, therefore, that heat flux equality does not occur on the element boundary, since heat flux is constant yet different in every element. The condition of temperature continuity, however, is preserved. It is easy to demonstrate that the calculated temperature transient is the same for two adjacent elements with common side, regardless of the element in which the transient is calculated. The lack of continuity on the element boundary impairs the accuracy of the solution obtained by means of FEM. In order to calculate heat flux at a specific point inside the analyzed region or to determine heat flow using a boundary segment with an assigned temperature, one should employ a denser element mesh so that a satisfactory result in terms of accuracy could be obtained.

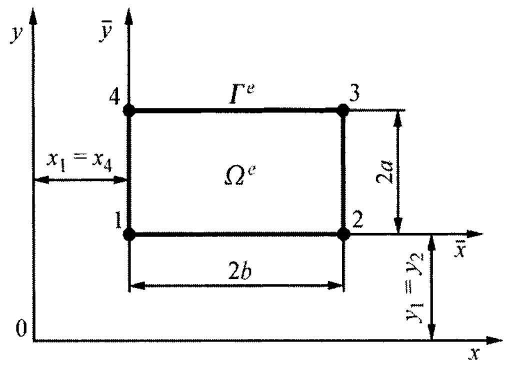


Fig. 11.10. Linear rectangular finite element

A linear tetragonal element, presented in Fig. 11.10, will be discussed below.

Temperature distribution inside the element will be approximated by function

$$T^e = \beta_1^e + \beta_2^e \cdot \bar{x} + \beta_3^e \cdot \bar{y} + \beta_4^e \cdot \bar{x} \cdot \bar{y} . \tag{14}$$

Constants $\beta_1^e, \dots, \beta_4^e$ will be determined from conditions

$$\begin{aligned} T^e(0,0) &= T_1^e, \qquad T^e(2b,0) = T_2^e, \\ T^e(2b,2a) &= T_3^e, \qquad T^e(0,2a) = T_4^e, \end{aligned} \tag{15}$$

from which, once (14) is substituted, the following equation system is obtained:

$$\begin{aligned} &\beta_1^e = T_1^e, \qquad \beta_1^e + 2b\beta_2^e = T_2^e, \\ &\beta_1^e + 2b\beta_2^e + 2a\beta_3^e + 4ab\beta_4^e = T_3^e, \qquad \beta_1^e + 2a\beta_3^e = T_4^e . \end{aligned} \tag{16}$$

Once the above equation system is solved, obtains

$$\begin{aligned} &\beta_1^e = T_1^e, \qquad \beta_2^e = \frac{1}{2b}\left(T_2^e - T_1^e\right), \\ &\beta_3^e = \frac{1}{2a}\left(T_4^e - T_1^e\right), \qquad \beta_4^e = \frac{1}{4ab}\left(T_1^e - T_2^e + T_3^e - T_4^e\right). \end{aligned} \tag{17}$$

After substituting (17) into (14) and after transformations, one obtains

$$T^e = N_1^e \cdot T_1^e + N_2^e \cdot T_2^e + N_3^e \cdot T_3^e + N_4^e \cdot T_4^e , \tag{18}$$

where shape functions are formulated as follow:

$$N_1^e = \left(1 - \frac{\overline{x}}{2b}\right)\left(1 - \frac{\overline{y}}{2a}\right), \qquad N_2^e = \frac{\overline{x}}{2b}\left(1 - \frac{\overline{y}}{2a}\right),$$
$$N_3^e = \frac{\overline{x} \cdot \overline{y}}{4ab}, \qquad N_4^e = \frac{\overline{y}}{2a}\left(1 - \frac{\overline{x}}{2b}\right). \tag{19}$$

Shape functions N_i^e have the following properties:

- $N_i^e\left(x_j, y_j\right) = \delta_{i,j}$ ($i, j = 1, 2, 3$) for a triangular element (20)

and

- $N_i^e\left(\overline{x}_j, \overline{y}_j\right) = \delta_{i,j}$ ($i, j = 1, 2, 3, 4$) for a tetragonal element (21)

where $\delta_{i,j}$ is the Kronecker delta, which satisfies

$$\delta_{i,j} = \begin{cases} 1, & \text{for} \quad i = j \\ 0, & \text{for} \quad i \neq j. \end{cases} \tag{22}$$

From properties (20) and (21), it follows that

$$\sum_{i=1}^{n} N_i^e = 1, \tag{23}$$

where n stands for the number of nodes in an element.

Therefore, in i-node the shape function $N_i^e = 1$, in other nodes, however, it equals zero. Aside from linear functions discussed above, one can apply other interpolation functions, for instance, the square functions.

Exercise 11.10 Description of FEM Based on Galerkin Method

Derive basic equations in FEM for a single element using Galerkin method. Assume that two-dimensional temperature field is source-based, while three different boundary conditions of 1st, 2nd and 3rd order are assigned on the body's edge. Allow for the fact that the medium is anisotropic, i.e. $\lambda_x \neq \lambda_y$.

Solution

We need to find a solution for the heat conduction equation

$$\frac{\partial}{\partial x}\left(\lambda_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y \frac{\partial T}{\partial y}\right) + \dot{q}_v = 0 \tag{1}$$

when boundary conditions are (Fig. 11.11):

$$T\big|_{\Gamma_T} = T_b, \tag{2}$$

$$\left(\lambda_x \frac{\partial T}{\partial x} n_x + \lambda_y \frac{\partial T}{\partial y} n_y\right)\Bigg|_{\Gamma_q} = \dot{q}_B \tag{3}$$

and

$$\left(\lambda_x \frac{\partial T}{\partial x} n_x + \lambda_y \frac{\partial T}{\partial y} n_y\right)\Bigg|_{\Gamma_\alpha} = \alpha\left(T_{cz} - T\big|_{\Gamma_\alpha}\right). \tag{4}$$

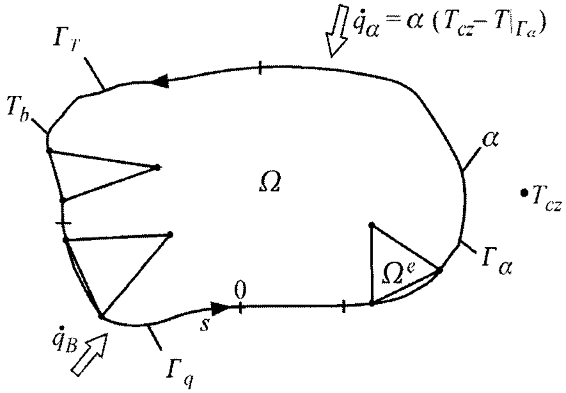


Fig. 11.11. A diagram with different boundary conditions

In (1)–(4), the following symbols are used:

λ_x – thermal conductivity of a material in *x-axis* direction
λ_y – thermal conductivity of a material in *y-axis* direction
T_b – temperature set on the body boundary Γ_T,
$\dot{q}_B$ – heat flux on the body boundary Γ_q,
α – heat transfer coefficient on the body boundary Γ_α,
T_{cz} – temperature of a medium.

In order to make these calculations more widely applicable, three different types of boundary conditions are assumed:

- 1st order kind condition, section Γ_T,
- 2nd order kind condition, section Γ_q,
- 3rd order kind condition, section Γ_α.

It is also assumed that heat flux $\dot{q}$ in (3) is positive, i.e. the body is being heated. Normal to boundary **n** is a unit vector directed to the outside of the region, while its components are equal to directional cosinuses

$$n_x = \cos\varphi, \qquad n_y = \cos\left(\frac{\pi}{2} - \varphi\right) = \sin\varphi, \tag{5}$$

where φ is the slope angle of normal to a horizontal plane.

Boundary conditions (3) and (4) can be written in a slightly different form, if conductivity matrix is entered into the equation

$$[\lambda] = \begin{bmatrix} \lambda_x & 0 \\ 0 & \lambda_y \end{bmatrix} \tag{6}$$

and column vector of temperature gradient

$$\{g\} = \begin{Bmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \end{Bmatrix}. \tag{7}$$

Heat flux components $\dot{q}_x$ and $\dot{q}_y$ can be written then in the form

$$\begin{Bmatrix} \dot{q}_x \\ \dot{q}_y \end{Bmatrix} = -[\lambda]\{g\}. \tag{8}$$

If we take into account, moreover, that

$$\dot{\mathbf{q}} = -\lambda_x \frac{\partial T}{\partial x}\mathbf{i} - \lambda_y \frac{\partial T}{\partial y}\mathbf{j} = \dot{q}_x\mathbf{i} + \dot{q}_y\mathbf{j}, \qquad \mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j} \tag{9}$$

and

$$-\left(-\lambda_x \frac{\partial T}{\partial x} n_x - \lambda_y \frac{\partial T}{\partial y} n_y\right) = -\,\dot{\mathbf{q}} \cdot \mathbf{n} = \dot{q}_n, \tag{10}$$

then boundary conditions (3) and (4) can be correspondingly written in the form

$$\dot{q}_n\big|_{\Gamma_q^e} = \dot{q}_B, \tag{11}$$

$$\dot{q}_n\big|_{\Gamma_\alpha^e} = \alpha\left(T_{cz} - T\big|_{\Gamma_\alpha^e}\right), \tag{12}$$

where $\dot{q}_n$ is the component value of the normal heat flux. One also assumes that the body thickness in the direction perpendicular to the plane of the diagram is of 1m.

Boundary problem (1)–(4) was formulated for the whole region Ω. In FEM, Galerkin method is first formulated for a single element Ω^e. It is assumed that three types of boundary conditions are assigned, as they are for the whole region, on the boundary of a single element. One needs to apply such a formulation to elements adjacent to body boundary (Fig. 11.11). It is not necessary, however, to consider boundary conditions for elements, which lie inside the body. Temperature distribution inside the element Ω^e is approximated by function

$$T^e(x,y)=\sum_{j=1}^{n}T_j^e\cdot N_j^e(x,y)=\left[N^e\right]\left\{T^e\right\}, \tag{13}$$

where n is the number of nodes in the element, T_j^e – temperature in j-node and $N_j^e(x, y)$ the shape function (interpolation function).

Galerkin method will be used to determine an approximate temperature T_j^e in nodes, j = 1, ..., n.

$$\int_{\Omega^e}\left[\frac{\partial}{\partial x}\left(\lambda_x\frac{\partial T^e}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda_y\frac{\partial T^e}{\partial y}\right)+\dot{q}_v\right]N_i^e(x,y)\,dxdy=0. \tag{14}$$

Green theorem will be applied in order to transform (14):

$$\int_{\Omega^e}\left(\frac{\partial G}{\partial x}-\frac{\partial F}{\partial y}\right)dxdy=\oint_{\Gamma^e}\left(Fdx+Gdx\right). \tag{15}$$

Integration on the boundary Γ^e is anti-clockwise.

If one assumes that

$$F=-\lambda_y\frac{\partial T^e}{\partial y}N_i^e \quad\text{and}\quad G=\lambda_x\frac{\partial T^e}{\partial x}N_i^e, \tag{16}$$

then on the basis of (15), one obtains

$$\begin{aligned}&\int_{\Omega^e}\left[\frac{\partial}{\partial x}\left(\lambda_x\frac{\partial T^e}{\partial x}N_i^e\right)+\frac{\partial}{\partial y}\left(\lambda_y\frac{\partial T^e}{\partial y}N_i^e\right)\right]dxdy=\\&=\int_{\Gamma^e}N_i^e\left(-\lambda_y\frac{\partial T^e}{\partial y}dx+\lambda_x\frac{\partial T^e}{\partial x}dy\right).\end{aligned} \tag{17}$$

Once the left-hand-side of (17) is transformed, the equation can be written in the form

$$\int_{\Omega^e}\left[\frac{\partial}{\partial x}\left(\lambda_x\frac{\partial T^e}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda_y\frac{\partial T^e}{\partial y}\right)\right]N_i^e dxdy=-\int_{\Omega^e}\left(\lambda_x\frac{\partial T^e}{\partial x}\frac{\partial N_i^e}{\partial x}+\right.$$
$$\left.+\lambda_y\frac{\partial T^e}{\partial y}\frac{\partial N_i^e}{\partial y}\right)dxdy+\int_{\Gamma^e}N_i^e\left(-\lambda_y\frac{\partial T^e}{\partial y}dx+\lambda_x\frac{\partial T^e}{\partial x}dy\right). \tag{18}$$

By substituting (18) into (14), one has

$$\int_{\Omega^e}\left(\lambda_x\frac{\partial T^e}{\partial x}\frac{\partial N_i^e}{\partial x}+\lambda_y\frac{\partial T^e}{\partial y}\frac{\partial N_i^e}{\partial y}\right)dxdy=\int_{\Omega^e}N_i^e\dot{q}_v dxdy+$$
$$+\int_{\Gamma^e}N_i^e\left(-\lambda_y\frac{\partial T^e}{\partial y}dx+\lambda_x\frac{\partial T^e}{\partial x}dy\right). \tag{19}$$

Because of (Fig. 11.12)

$$-dx=ds\cdot\sin\varphi=n_y ds, \tag{20}$$

$$dy=ds\cdot\cos\varphi=n_x ds \tag{21}$$

and on the basis of (10), the expression in the brackets in the curvilinear integral in (19) can be transformed in the following way:

$$-\lambda_y\frac{\partial T^e}{\partial y}dx+\lambda_x\frac{\partial T^e}{\partial x}dy=\lambda_y\frac{\partial T}{\partial y}n_y ds+\lambda_x\frac{\partial T}{\partial x}n_x ds=-\dot{\mathbf{q}}\cdot\mathbf{n}=\dot{q}_n ds. \tag{22}$$

Hence, from the above and the boundary conditions (11) and (12) in (19), one gets

$$\int_{\Omega^e}\left(\lambda_x\frac{\partial T^e}{\partial x}\frac{\partial N_i^e}{\partial x}+\lambda_y\frac{\partial T^e}{\partial y}\frac{\partial N_i^e}{\partial y}\right)dxdy=\int_{\Omega^e}N_i^e\dot{q}_v dxdy+\int_{\Gamma_q^e}N_i^e\dot{q}_B ds+$$
$$+\int_{\Gamma_\alpha^e}N_i^e\alpha\left(T_{cz}-T^e\right)ds. \tag{23}$$

After substituting (13) into (23), one gets

$$\int_{\Omega^e}\left(\lambda_x\frac{\partial N_i^e}{\partial x}\sum_{j=1}^{n}T_j^e\frac{\partial N_j^e}{\partial x}+\lambda_y\frac{\partial N_i^e}{\partial y}\sum_{j=1}^{n}T_j^e\frac{\partial N_j^e}{\partial y}\right)dxdy=\int_{\Omega^e}N_i^e\dot{q}_v dxdy+$$
$$+\int_{\Gamma_q^e}N_i^e\dot{q}_B ds-\int_{\Gamma_\alpha^e}N_i^e\alpha\sum_{j=1}^{n}T_j^e N_j^e ds+\int_{\Gamma_\alpha^e}N_i^e\alpha T_{cz}ds. \tag{24}$$

Equation (24) for i-node can be written in the form

$$\sum_{j=1}^{n}\left(K_{c,ij}^{e}+K_{\alpha,ij}^{e}\right)\cdot T_{j}^{e}=f_{Q,i}^{e}+f_{q,i}^{e}+f_{\alpha,i}^{e}, \tag{25}$$

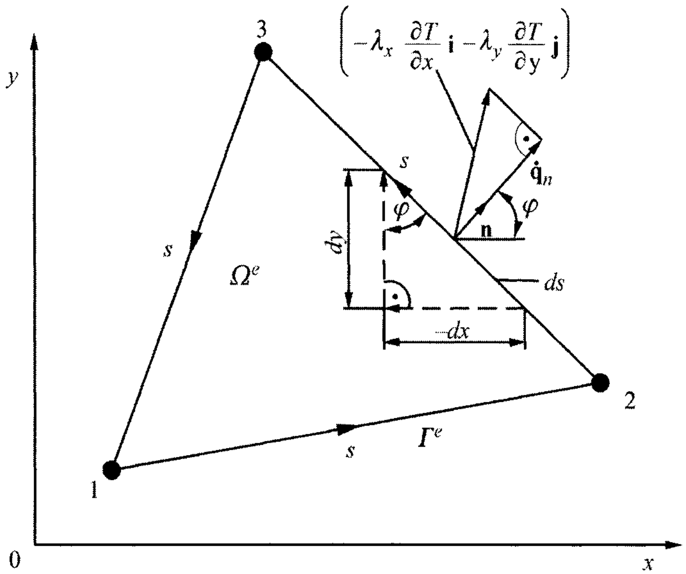


Fig. 11.12. A diagram with a calculation of a curvilinear integral on the element's perimeter (in an anti-clockwise direction)

where

$$K_{c,ij}^{e}=\int_{\Omega^{e}}\left(\lambda_{x}\frac{\partial N_{i}^{e}}{\partial x}\frac{\partial N_{j}^{e}}{\partial x}+\lambda_{y}\frac{\partial N_{i}^{e}}{\partial y}\frac{\partial N_{j}^{e}}{\partial y}\right)dxdy, \tag{26}$$

$$K_{\alpha,ij}^{e}=\int_{\Gamma_{\alpha}^{e}}\alpha N_{i}^{e}N_{j}^{e}ds, \tag{27}$$

$$f_{Q,i}^{e}=\int_{\Omega^{e}}\dot{q}_{v}N_{i}^{e}dxdy, \tag{28}$$

$$f_{q,i}^{e}=\int_{\Gamma_{q}^{e}}\dot{q}_{B}N_{i}^{e}ds \tag{29}$$

and

$$f_{\alpha,i}^{e} = \int_{\Gamma_{\alpha}^{e}} \alpha T_{cz} N_{i}^{e} ds . \tag{30}$$

If in (14), and by that in (25) N_j^e is assumed to be the shape function for the consecutive nodes of a finite element, then one obtains for a given element a system of n equations, which can be written in the form

$$\left(\left[K_{c}^{e}\right]+\left[K_{\alpha}^{e}\right]\right)\left\{T^{e}\right\}=\left\{f_{Q}^{e}\right\}+\left\{f_{q}^{e}\right\}+\left\{f_{\alpha}^{e}\right\}, \tag{31}$$

where matrix coefficient and elements of column vectors are expressed by (26)–(30), while vector $\{T^e\}$ has the form $\{T^e\} = [\ T_1^e, \ldots , T_n^e]$, where n is the number of nodes in an element. In contrast to other exercises where bold type designates matrixes and column vectors, the traditional notation used in FEM is preserved in (31). [ ] stands for a matrix or row vector, while { } a column vector. Matrix $[K_c^e]$ is called *stiffnes* or *conductivity matrix*. Matrix $[K_c^e]$ is symmetrical, since $K_{c,ij}^{e} = K_{c,ji}^{e}$. Equation (31) forms the basis of FEM for (1).

Equation system (31) is frequently written in a slightly different form. Once (13) is substituted into (7), temperature gradient vector can be written in the form

$$\{g\}=\begin{bmatrix} \dfrac{\partial N_1^e}{\partial x} & \dfrac{\partial N_2^e}{\partial x} & \cdots & \dfrac{\partial N_n^e}{\partial x} \\ \dfrac{\partial N_1^e}{\partial y} & \dfrac{\partial N_2^e}{\partial y} & \cdots & \dfrac{\partial N_n^e}{\partial y} \end{bmatrix}\left\{T^e\right\} . \tag{32}$$

If we denote by [B] the matrix in the square bracket:

$$[B]=\begin{bmatrix} \dfrac{\partial N_1^e}{\partial x} & \dfrac{\partial N_2^e}{\partial x} & \cdots & \dfrac{\partial N_n^e}{\partial x} \\ \dfrac{\partial N_1^e}{\partial y} & \dfrac{\partial N_2^e}{\partial y} & \cdots & \dfrac{\partial N_n^e}{\partial y} \end{bmatrix}, \tag{33}$$

Then $\{g\}$ can be expressed in a shortened form

$$\{g\}=\left[B^e\right]\left\{T^e\right\}. \tag{34}$$

Conductivity matrix $[K_c^e]$ can be written then in the form

$$\left[K_c^e\right]= \int_{\Omega^e}\left[B^e\right]^T[\lambda]\left[B^e\right]dxdy . \tag{35}$$

The remaining matrixes and column vectors present in (31) can be expressed in the following way:

$$\left[K_\alpha^e\right] = \int_{\Gamma_\alpha^e} \alpha\left[N^e\right]^T\left[N^e\right]ds,$$
$$\left\{f_Q^e\right\} = \int_{\Omega^e} \dot{q}_v\left[N^e\right]^T dxdy, \tag{36}$$

$$\left\{f_q^e\right\} = \int_{\Gamma_q^e} \dot{q}_B\left[N^e\right]^T d\Gamma$$
$$\left\{f_\alpha^e\right\} = \int_{\Gamma_\alpha^e} \alpha T_{cz}\left[N^e\right]^T ds, \tag{37}$$

where $\left[N^e\right] = \left[N_1^e, N_2^e, ..., N_n^e\right]$

$\left[N^e\right]^T = \begin{Bmatrix} N_1^e \\ N_2^e \\ \vdots \\ N_n^e \end{Bmatrix}$, while n is the number of nodes in an element.

Exercise 11.11 Determining Conductivity Matrix for a Rectangular and Triangular Element

Determine conductivity matrix $[K_c^e]$ for a rectangular and triangular element.

Solution

Equation (26) from Ex. 11.10 and formulas for shape functions shown in Ex. 11.9. will be used to calculate the elements of a conductivity matrix.

a) *Conductivity matrix $[K_c^e]$ for a finite rectangular element*

Matrix elements $[K_c^e]$ are expressed by (26) in Ex. 11.10

$$K_{c,ij}^e = \int_{\Omega^e}\left(\lambda_x \frac{\partial N_i^e}{\partial x}\frac{\partial N_j^e}{\partial x} + \lambda_y \frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial y}\right)dxdy. \tag{1}$$

Only two matrix elements will be calculated from (19) in Ex. 11.9 for the shape functions $K^e_{c,11}$ and $K^e_{c,12}$. Let x, y be local coordinates.

After determining derivatives

$$\frac{\partial N_1^e}{\partial x} = -\frac{1}{2b}\left(1-\frac{y}{2a}\right) \text{ and } \frac{\partial N_1^e}{\partial y} = -\frac{1}{2a}\left(1-\frac{x}{2b}\right) \tag{2}$$

and

$$\frac{\partial N_2^e}{\partial x} = \frac{1}{2b}\left(1-\frac{y}{2a}\right) \text{ and } \frac{\partial N_2^e}{\partial y} = -\frac{x}{4ab}\ , \tag{3}$$

element $K^e_{c,11}$ will be calculated first

$$K^e_{c,11} = \int_{\Omega^e}\left(\lambda_x \frac{\partial N_1^e}{\partial x}\frac{\partial N_1^e}{\partial x} + \lambda_y \frac{\partial N_1^e}{\partial y}\frac{\partial N_1^e}{\partial y}\right)dxdy =$$

$$= \lambda_x \int_0^{2a}\left[\int_0^{2b}\frac{1}{4b^2}\left(1-\frac{y}{2a}\right)^2 dx\right]dy + \lambda_y \int_0^{2a}\left[\int_0^{2b}\frac{1}{4a^2}\left(1-\frac{x}{2b}\right)^2 dx\right]dy = \tag{4}$$

$$= \frac{\lambda_x}{3}\frac{a}{b} + \frac{\lambda_y}{3}\frac{b}{a}\ .$$

Element $K^e_{c,12}$ is calculated in a similar way:

$$K^e_{c,12} = \int_{\Omega^e}\left(\lambda_x \frac{\partial N_1^e}{\partial x}\frac{\partial N_2^e}{\partial x} + \lambda_y \frac{\partial N_1^e}{\partial y}\frac{\partial N_2^e}{\partial y}\right)dxdy =$$

$$= -\lambda_x \int_0^{2a}\left[\int_0^{2b}\frac{1}{4b^2}\left(1-\frac{y}{2a}\right)^2 dx\right]dy + \lambda_y \int_0^{2a}\left[\int_0^{2b}\frac{1}{8a^2 b}\left(x-\frac{x^2}{2b}\right)dx\right]dy = \tag{5}$$

$$= -\frac{\lambda_x}{3}\frac{a}{b} + \frac{\lambda_y}{6}\frac{b}{a}\ .$$

Also the remaining elements of the conductivity matrix $[K_c^e]$ can be determined in a similar way, namely

$$\left[K_c^e\right]=\frac{\lambda_x}{6}\frac{a}{b}\begin{bmatrix}2 & -2 & -1 & 1\\ -2 & 2 & 1 & -1\\ -1 & 1 & 2 & -2\\ 1 & -1 & -2 & 2\end{bmatrix}+\frac{\lambda_y}{6}\frac{b}{a}\begin{bmatrix}2 & 1 & -1 & -2\\ 1 & 2 & -2 & -1\\ -1 & -2 & 2 & 1\\ -2 & -1 & 1 & 2\end{bmatrix}. \tag{6}$$

b) *Conductivity matrix $[K_c^e]$ for a finite triangular element*

Matrix $[B]$ for a triangular element is formulated as

$$\left[B^e\right]=\begin{bmatrix}\frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial x} & \frac{\partial N_3^e}{\partial x}\\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_3^e}{\partial y}\end{bmatrix}=\frac{1}{2A^e}\begin{bmatrix}b_1^e & b_2^e & b_3^e\\ c_1^e & c_2^e & c_3^e\end{bmatrix}, \tag{7}$$

where A^e is the surface area of a triangle, while coefficients b_i^e, c_i^e, $i = 1, 2, 3$, are expressed by formulas in (8), Ex. 11.9. Since the coefficients in matrix $[B^e]$ are constants and λ_x and λ_y are material constants independent of position and temperature inside the element, the conductivity matrix can be easily determined, since

$$\left[K_c^e\right]=\int_{\Omega^e}\left[B^e\right]^T\left[\lambda^e\right]\left[B^e\right]dxdy=\left[B^e\right]^T\left[\lambda^e\right]\left[B^e\right]\int_{\Omega^e}dxdy$$

or

$$\left[K_c^e\right]=\left[B^e\right]^T\left[\lambda^e\right]\left[B^e\right]A^e. \tag{8}$$

Once (7) is substituted into (8) and the appropriate operations carried out, one obtains

$$\left[K_c^e\right]=\frac{\lambda_x^e}{4A^e}\begin{bmatrix}\left(b_1^e\right)^2 & b_1^e b_2^e & b_1^e b_3^e\\ b_1^e b_2^e & \left(b_2^e\right)^2 & b_2^e b_3^e\\ b_1^e b_3^e & b_2^e b_3^e & \left(b_3^e\right)^2\end{bmatrix}+\frac{\lambda_x^e}{4A^e}\begin{bmatrix}\left(c_1^e\right)^2 & c_1^e c_2^e & c_1^e c_3^e\\ c_1^e c_2^e & \left(c_2^e\right)^2 & c_2^e c_3^e\\ c_1^e c_3^e & c_2^e c_3^e & \left(c_3^e\right)^2\end{bmatrix}. \tag{9}$$

It can easily verify that the same results are obtained when calculating matrix coefficient with (1). When a body is isotropic, i.e. $\lambda_x^e = \lambda_x^e = \lambda^e$, the conductivity matrix $[K_c^e]$ for a triangle expressed by (9) can be written in a simpler form, by introducing the notation shown in Fig. 11.13.

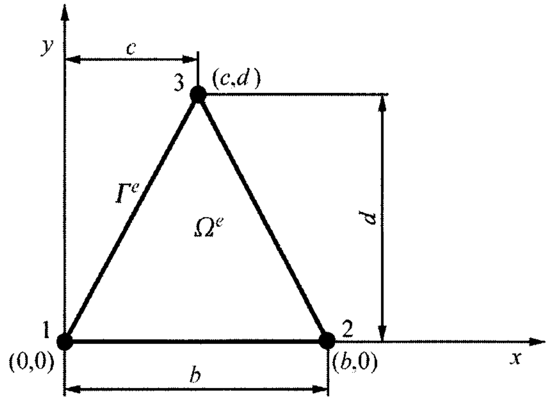


Fig. 11.13. Triangular finite element

On the basis of formulas (8) from Ex. 11.9 and the notations from Fig. 11.13, (9) for $\lambda_x^e = \lambda_x^e = \lambda^e$ can be written in the form

$$\left[K_c^e\right] = \frac{\lambda^e}{4A^e}\begin{bmatrix} d^2+(c-b)^2 & -d^2-c(c-b) & b(c-b) \\ -d^2-c(c-b) & d^2+c^2 & -cb \\ b(c-b) & -cb & b^2 \end{bmatrix}. \quad (10)$$

In a case when a triangle is rectangular in shape (Fig. 11.14), conductivity matrix assumes the form

$$\left[K_c^e\right] = \frac{\lambda^e}{2}\begin{bmatrix} \frac{d}{b} & -\frac{d}{b} & 0 \\ -\frac{d}{b} & \frac{d}{b}+\frac{b}{d} & -\frac{b}{d} \\ 0 & -\frac{b}{d} & \frac{b}{d} \end{bmatrix}. \quad (11)$$

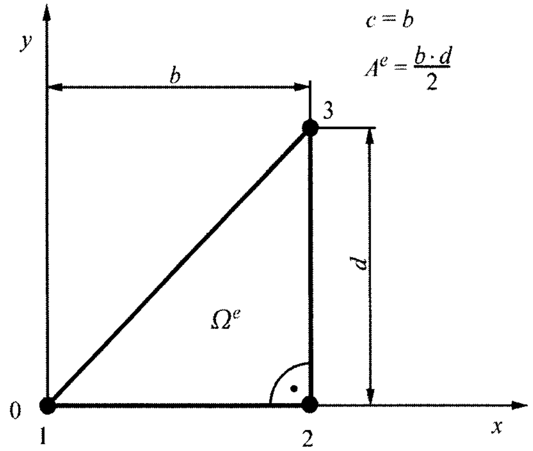


Fig. 11.14. Finite element in a rectangular triangle form

Exercise 11.12 Determining Matrix $[K_\alpha^e]$ in Terms of Convective Boundary Conditions for a Rectangular and Triangular Element

Determine matrix $[K_\alpha^e]$ for a rectangular and triangular element.

Solution

Matrix $[K_\alpha^e]$ present in (25) [Ex. 11.10], whose coefficients are expressed by (27) [Ex. 11.10], arises from 3rd order boundary conditions assigned on the boundary of an element. Matrix $[K_\alpha^e]$ can be also determined by means of (36) from Ex. 11.10. Coefficients $K_{\alpha,ij}^e$ will be calculated by means of (27) from Ex. 11.10

$$K_{\alpha,ij}^e = \int_{\Gamma_\alpha^e} \alpha N_i^e N_j^e ds . \tag{1}$$

The determination of integrals in FEM is discussed, among others, in articles [4, 6].

a) *Rectangular finite element*

If convective heat transfer is assigned on all sides of an element with a heat transfer coefficient α, then matrix $[K_\alpha^e]$ is formulated as

$$\left[K_\alpha^e\right]=\int\limits_{\Gamma_\alpha^e}\alpha\begin{bmatrix}\left(N_1^e\right)^2 & N_1^eN_2^e & N_1^eN_3^e & N_1^eN_4^e\\ N_1^eN_2^e & \left(N_2^e\right)^2 & N_2^eN_3^e & N_2^eN_4^e\\ N_1^eN_3^e & N_2^eN_3^e & \left(N_3^e\right)^2 & N_3^eN_4^e\\ N_1^eN_4^e & N_2^eN_4^e & N_3^eN_4^e & \left(N_4^e\right)^2\end{bmatrix}ds\,. \tag{2}$$

In practice, convective heat transfer is usually set on one or two element sides, which constitute a fragment of a body boundary. If convective heat transfer takes place on the side 1-2 of a rectangular element (Fig. 11.15), then in (2) one should assume that $N_3 = N_4 = 0$. Equation (2) assumes the form

$$\left[K_\alpha^e\right]=\int\limits_0^{2b}\alpha\begin{bmatrix}\left(N_1^e\right)^2 & N_1^eN_2^e & 0 & 0\\ N_1^eN_2^e & \left(N_2^e\right)^2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}ds\,. \tag{3}$$

Since $ds = dx$ and on the basis of (19) from Ex. 11.9, individual integrals in (3) are

$$\int\limits_0^{2b}\left(N_1^e\right)^2dx=\frac{2b}{3}=\frac{L_{12}}{3}\,, \tag{4}$$

where L_{12} is the length of the side 1-2 of the element in question.

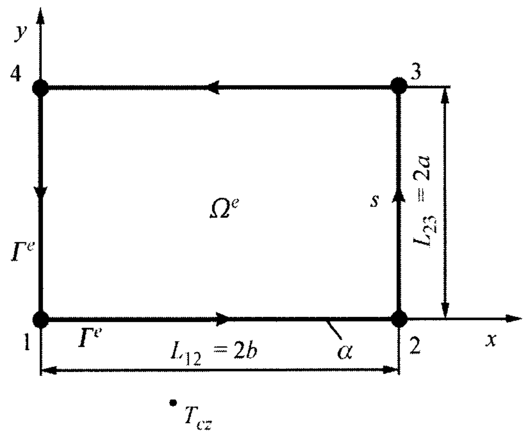


Fig. 11.15. Convective heat transfer is prescribed on the boundary 1-2 of a rectangular element

Furthermore, once the following is determined

$$\int_0^{2b} N_1^e N_2^e dx = \frac{2b}{6} = \frac{L_{12}}{6} \tag{5}$$

and

$$\int_0^{2b} \left(N_2^e\right)^2 dx = \frac{2b}{3} = \frac{L_{12}}{3} \tag{6}$$

matrix (3) assumes the form

$$\left[K_\alpha^e\right] = \frac{\alpha L_{12}}{6} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{7}$$

Similar results are obtained for the remaining sides of the element

$$\left[K_\alpha^e\right] = \frac{\alpha L_{23}}{6} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{8}$$

$$\left[K_\alpha^e\right] = \frac{\alpha L_{34}}{6} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \tag{9}$$

$$\left[K_\alpha^e\right] = \frac{\alpha L_{41}}{6} \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}, \tag{10}$$

where L_{23}, L_{34}, L_{41} are the lengths of the sides on which the convective heat transfer takes place.

b) *Triangular finite element*

For a triangular element, matrix $[K_\alpha^e]$ has the form

$$\left[K_\alpha^e\right] = \int_{\Gamma_\alpha^e} \alpha \begin{bmatrix} \left(N_1^e\right)^2 & N_1^e N_2^e & N_1^e N_3^e \\ N_2^e N_1^e & \left(N_2^e\right)^2 & N_2^e N_3^e \\ N_3^e N_1^e & N_3^e N_2^e & \left(N_3^e\right)^2 \end{bmatrix} ds. \tag{11}$$

The above matrix refers to a case when convective heat transfer takes place on all three sides of a triangular element. When heat transfer occurs only on the side 1-2, one assumes in (11) that $N_3^e = 0$, while after integration, one has

$$\left[K_\alpha^e\right] = \frac{\alpha L_{12}}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{12}$$

Formulas for the remaining sides are obtained in a similar way

$$\left[K_\alpha^e\right] = \frac{\alpha L_{23}}{6} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \tag{13}$$

$$\left[K_\alpha^e\right] = \frac{\alpha L_{31}}{6} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \tag{14}$$

where L_{12}, L_{23}, L_{31} are the respective side lengths of the triangular element.

When calculating curvilinear integrals, present in (11), for a triangular element, needed in order to determine (12)–(14), (1) was used:

$$\int_0^L N_i^m(s) \cdot N_j^n(s)\, ds = L \frac{m!n!}{(m+n+1)!}. \tag{15}$$

It is easy to calculate the integrals in (11) by means of (15); e.g. to calculate integral

$$\int_0^{L_{12}} \left(N_i^e\right)^2 ds,$$

in (15), one assumes that $m = 2$, $n = 0$, hence

$$\int_0^{L_{12}} \left(N_i^e\right)^2 ds = L_{12} \frac{2!0!}{(2+0+1)!} = L_{12} \frac{1 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = \frac{L_{12}}{3}.$$

Exercise 11.13 Determining Vector $\{f_Q^e\}$ with Respect to Volumetric and Point Heat Sources in a Rectangular and Triangular Element

Determine vector $\{f_Q^e\}$ for a rectangular and triangular element, when unit heat source power is constant within the area of the element and constant for a point heat source.

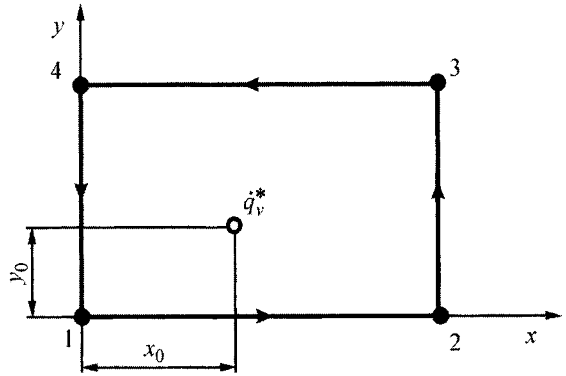


Fig. 11.16. Point heat source inside a rectangular element

Solution

Vector components $\{f_Q^e\}$ will be calculated according to (28) from Ex. 11.10

$$f_{Q,i}^e = \int_{\Omega^e} \dot{q}_v N_i^e dxdy . \tag{1}$$

a) *Rectangular element*

If power density of a heat source is constant, it is easy to calculate $\left\{f_{Q,i}^e\right\}$

$$f_{Q,i}^e = \int_0^{2a}\left(\int_0^{2b} \dot{q}_v N_i^e dx\right)dy ; \tag{2}$$

hence, after substituting into (19) from Ex. 11.9, one obtains

$$\{f_Q^e\} = \frac{\dot{q}_v A_e}{4}\begin{Bmatrix}1\\1\\1\\1\end{Bmatrix} = \dot{q}_v ab\begin{Bmatrix}1\\1\\1\\1\end{Bmatrix}, \tag{3}$$

where A_e is the surface area of an element, equal to $4ab$.

It follows from (3) that 1/4 of total body heat flow is allotted to every node in a tetragonal element.

In the case of the point heat source (Fig. 11.16), (1) assumes the form

$$\dot{q}_v = \dot{q}_v^* \delta(x - x_0)\delta(y - y_0), \tag{4}$$

where $\dot{q}_v^*$ [W/m] is the heat flow emitted at point (x_0, y_0), with respect to a unit of length as the heat source is infinitely long in the direction perpendicular to the diagram plane. Function δ is a Dirac delta, which approaches infinity at point (x_0, y_0); at the remaining points, however, it equals zero.

By substituting (4) into (1), one has

$$\{f_Q^e\} = \dot{q}_v^* \begin{Bmatrix} N_1^e(x_0, y_0) \\ N_2^e(x_0, y_0) \\ N_3^e(x_0, y_0) \\ N_4^e(x_0, y_0) \end{Bmatrix}. \tag{5}$$

b) *Triangular element*

If density $\dot{q}_v$ is constant, then from (1), one obtains

$$f_{Q,i}^e = \dot{q}_v \int_{\Omega^e} N_i^e dx dy. \tag{6}$$

In order to calculate the integral on the surface of a triangular element, a formula from reference [4] will be used here:

$$\int_{\Omega^e} (N_1^e)^l (N_2^e)^m (N_3^e)^n dA = \frac{l!m!n!}{(l+m+n+2)!} 2A_e. \tag{7}$$

Since in the given case $m = 0$, $n = 0$, $l = 1$, then from (6), one has

$$f_{Q,i}^e = \frac{\dot{q}_v A_e}{3}. \tag{8}$$

Therefore, vector $\{f_Q^e\}$ has the form

$$\{f_Q^e\} = \frac{\dot{q}_v A_e}{3}\begin{Bmatrix}1\\1\\1\end{Bmatrix}. \tag{9}$$

It follows from (9) that 1/3 of the total heat flow in an element is allotted to every node in that element. In the case of point heat source, vector $\{f_Q^e\}$ has the form

$$\{f_Q^e\} = \dot{q}_v^* \begin{Bmatrix}N_1^e(x_0,y_0)\\N_2^e(x_0,y_0)\\N_3^e(x_0,y_0)\end{Bmatrix}. \tag{10}$$

Equations (5) and (10) refer to a case when the point heat source is located inside an element.

When a heat source is located in a node common to several elements (Fig. 11.17), then source power $\dot{q}_v^*$ per unit of length can be divided among individual elements proportionally to angle φ at the tip of a given element. For a triangular element, vector $\{f_Q^e\}$ has the form

$$\{f_Q^e\} = \frac{\varphi\,\dot{q}_v^*}{2\pi}\begin{Bmatrix}0\\1\\0\end{Bmatrix}, \tag{11}$$

where angle φ is expressed in radians.

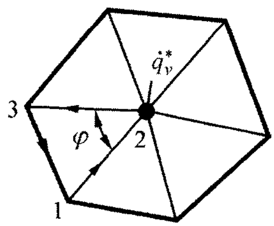


Fig. 11.17. Point heat source in a node common to several elements

In practice, the location of a heat source is of no great significance, since in the global equation system, with the heat balance equations for individual nodes, total power $\dot{q}_v^*$ is present in this equation of a node, which has a point heat source inside.

Exercise 11.14 Determining Vectors $\{f_q^e\}$ and $\{f_\alpha^e\}$ with Respect to Boundary Conditions of 2nd and 3rd Kind on the Boundary of a Rectangular or Triangular Element

Determine vectors $\{f_q^e\}$ and $\{f_\alpha^e\}$ for a finite rectangular and triangular element.

Solution

Elements of column vectors $\{f_q^e\}$ and $\{f_\alpha^e\}$ are determined from (29) and (30), Ex. 11.10

$$f_{q,i}^e = \int_{\Gamma_q^e} \dot{q}_B N_i^e ds \,, \tag{1}$$

$$f_{\alpha,i}^e = \int_{\Gamma_\alpha^e} \alpha T_{cz} N_i^e ds \,, \tag{2}$$

therefore, from almost identical integrals. If we assume that $\dot{q}_B = \alpha T_{cz}$ in the first integral, then we obtain (2). This is why only vector $\{f_q^e\}$ will be determined below.

a) *Finite rectangular element*

If heat flux is given on the boundary 1–2 of a finite element (Fig. 11.18) with thickness 1, then vector $\{f_q^e\}$ is formulated as

$$\left\{f_q^e\right\} = \int_{\Gamma_q^e} \dot{q}_B \left[N^e\right]^T ds = \int_0^{2b} \dot{q}_B \begin{Bmatrix} N_1^e \\ N_2^e \\ N_3^e \\ N_4^e \end{Bmatrix} dx \,. \tag{3}$$

Since also $N_3^e = N_4^e = 0$ on the side 1-2, (3) assumes the form

$$\left\{f_q^e\right\} = \dot{q}_B \int_0^{2b} \begin{Bmatrix} N_1^e(x,0) \\ N_2^e(x,0) \\ 0 \\ 0 \end{Bmatrix} dx \,, \tag{4}$$

where shape functions N_1^e and N_2^e are expressed by (19), in Ex. 11.9.

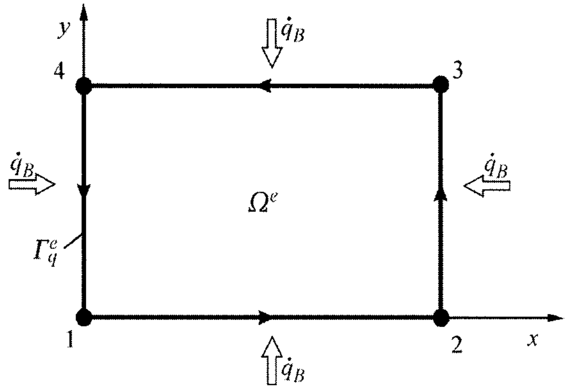


Fig. 11.18. Rectangular element heated by heat flux $\dot{q}_B$

Once the integrals are calculated

$$\int_0^{2b} N_1^e(x,0)dx = \int_0^{2b}\left(1-\frac{x}{2b}\right)dx = x-\frac{x^2}{4b}\Bigg|_0^{2b} = b = \frac{L_{12}}{2}, \tag{5}$$

$$\int_0^{2b} N_2^e(x,0)dx = \int_0^{2b}\frac{x}{2b}dx = \frac{x^2}{4b}\Bigg|_0^{2b} = b = \frac{L_{12}}{2}, \tag{6}$$

vector $\{f_q^e\}$ can be written in the form

$$\{f_q^e\} = \frac{\dot{q}_B L_{12}}{2}\begin{Bmatrix}1\\1\\0\\0\end{Bmatrix}. \tag{7}$$

It is evident from the analysis of (7) that the term $\dot{q}_B L_{12}/2$ for node no. 1 will appear on the right-hand-side of the (25), Ex. 11.10, as it will for node no. 2. This means that half of the heat, which flows through the lateral surface of an element with length L_{12} and thickness 1, flows to node no. 1. The second half flows to node no. 2. Vector $\{f_q^e\}$ can be calculated in a similar way when the heat inflows into the element through surfaces 2-3, 3-4 and 4-1; one then obtains, respectively

$$\{f_q^e\} = \frac{\dot{q}_B L_{23}}{2}\begin{Bmatrix}0\\1\\1\\0\end{Bmatrix}, \tag{8}$$

$$\{f_q^e\} = \frac{\dot{q}_B L_{34}}{2}\begin{Bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{Bmatrix}, \tag{9}$$

$$\{f_q^e\} = \frac{\dot{q}_B L_{41}}{2}\begin{Bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{Bmatrix}. \tag{10}$$

If the heat flow at density $\dot{q}_B$ inflows through all lateral surfaces of an element (Fig. 11.18), then vectors (7)–(10) should be added; hence

$$\{f_q^e\} = \frac{\dot{q}_B}{2}\begin{Bmatrix} L_{12} + L_{41} \\ L_{12} + L_{23} \\ L_{23} + L_{34} \\ L_{34} + L_{41} \end{Bmatrix}. \tag{11}$$

From the analysis of (11), it is evident that the term $\dot{q}_B\,(L_{12}+ L_{41})/2$ for node no. 1 will appear on the right-hand-side of an algebraic equation, term $\dot{q}_B\,(L_{12}+ L_{32})/2$ for node no. 2 on the right-hand-side of the equation, and so on. The above is, therefore, the same procedure for calculating heat balance in nodes as the one, which is used in the control volume method.

b) *Finite triangular element*

To calculate curvilinear integral (1), we will use formula

$$\int_0^L N_i^m(s) N_j^n(s)\,ds = L\frac{m!n!}{(m+n+1)!}. \tag{12}$$

Vector

$$\{f_q^e\} = \int_{\Gamma_q^e} \dot{q}_B \left[N^e\right]^T ds = \dot{q}_B \int_{\Gamma_q^e} \begin{Bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{Bmatrix} ds, \tag{13}$$

with a heat flux set on the surface 1-2 is calculated under the assumption that $N_3^e = 0$ and $m = 1$, $n = 0$ in (12). Once the integrals are calculated

$$\int_0^{L_{12}} N_1^e ds = L_{12}\frac{1!0!}{(1+0+1)!} = \frac{L_{12}}{2}, \tag{14}$$

$$\int_0^{L_{12}} N_2^e ds = \frac{L_{12}}{2}, \tag{15}$$

vector $\{f_q^e\}$ assumes the form

$$\{f_q^e\} = \frac{\dot{q}_B L_{12}}{2}\begin{Bmatrix}1\\1\\0\end{Bmatrix}. \tag{16}$$

If a heat flow with density $\dot{q}_B$ is assigned on the surface 2-3 or 3-1, then the corresponding vectors have the form

$$\{f_q^e\} = \frac{\dot{q}_B L_{23}}{2}\begin{Bmatrix}0\\1\\1\end{Bmatrix}, \tag{17}$$

$$\{f_q^e\} = \frac{\dot{q}_B L_{31}}{2}\begin{Bmatrix}1\\0\\1\end{Bmatrix}. \tag{18}$$

When heating a triangular element on all sides, an appropriate vector is obtained as a result of adding vectors (16)–(18)

$$\{f_q^e\} = \frac{\dot{q}_B}{2}\begin{Bmatrix}L_{12}+L_{31}\\L_{12}+L_{23}\\L_{23}+L_{31}\end{Bmatrix}. \tag{19}$$

As in the case of a rectangular element, a heat flow, which inflows through half of the surfaces that pass through a given node, occurs on the right-hand-side of an algebraic equation when the equation is being formulated for a given node.

Exercise 11.15 Methods for Building Global Equation System in FEM

Describe the way global equation system is created using the finite element method by summing up

a) equation systems obtained for individual finite elements [Method I],
b) algebraic equations obtained for different elements that share, nevertheless, a common node (as an analogy to finite volume method) [Method II].

Are the temperature continuity conditions and heat flux conditions satisfied on the boundary between elements?

Solution

a) In order to create a global equation system, conductivity matrix $[K_c^e]$ and the matrix that comes from the assigned 3rd kind boundary conditions $[K_\alpha^e]$, derived for individual elements, must be summated, i.e.

$$[K] = \sum_{e=1}^{N} \left(\left[K_c^e \right] + \left[K_\alpha^e \right] \right) \tag{1}$$

That includes the summation of vectors $\{f_Q^e\}$, $\{f_q^e\}$, $\{f_\alpha^e\}$, present on the right-hand-side of the (31) in Ex. 11.10

$$\{f_Q\} = \sum_{e=1}^{N} \{f_Q^e\} , \tag{2}$$

$$\{f_q\} = \sum_{e=1}^{N} \{f_q^e\} \tag{3}$$

and

$$\{f_\alpha\} = \sum_{e=1}^{N} \{f_\alpha^e\} , \tag{4}$$

where N is the finite elements integral number, which the entire analyzed region was divided to. The global equation system for temperature in element nodes has the form

$$[K]\{T\} = \{f_Q\} + \{f_q\} + \{f_\alpha\} , \tag{5}$$

where $\{T\}$ is the column vector of size N, which comprises of unknown temperatures in element nodes. Next, one has to account for parameters present in the boundary conditions in the above created global equation system (5).

The method for creating matrix $[K]$, which is sparse, should be discussed in greater detail, since only some of the coefficients present in it are other than zero. It is assumed that the flat region is divided into triangular elements (Fig. 11.19).

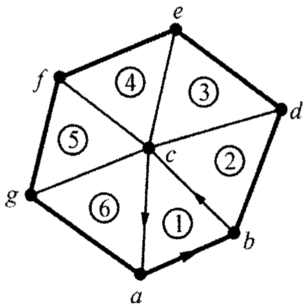


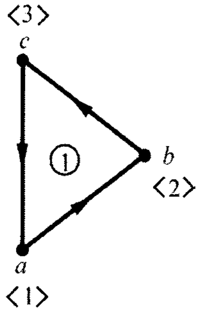
Fig. 11.19. A division of a flat region into finite triangular elements; element numbers and global node numbers are marked: ①–⑥ – finite element numbers, a–g – global node numbers

If element ① lies inside the analyzed region, thereby $[K_\alpha^{\ 1}]$ can be disregarded, then matrix $[K^1] = [K_c^{\ 1}]$ for the first element (Fig. 11.20) can be written in the following way:

$$\begin{matrix} & a & b & c \\ \langle 1 \rangle & \multirow{3}{*}{} \end{matrix}$$

$$\begin{array}{c} \\ \langle 1 \rangle \\ \langle 2 \rangle \\ \langle 3 \rangle \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 \\ K_{21}^1 & K_{22}^1 & K_{23}^1 \\ K_{31}^1 & K_{32}^1 & K_{33}^1 \end{bmatrix} \\ \begin{array}{ccc} \langle 1 \rangle & \langle 2 \rangle & \langle 3 \rangle \end{array} \end{array} . \tag{6}$$

If global node numbers of a triangular element are marked as a, b, and c, while local node numbers as ⟨1⟩, ⟨2⟩, ⟨3⟩, then one can see that coefficient K^1_{aa} corresponds to coefficient K^1_{11} in matrix $[K^1]$ (6), coefficient K^1_{bc} corresponds to coefficient K^1_{23}, etc. When creating a matrix of coefficients $[K]$ according to (1), one should be guided by global indexes, i.e. one should add coefficient K^e_{aa} that occurs in the matrix of element ⓔ to coefficient K^1_{aa} that occurs in the conductivity matrix of element ①. Coefficients that have the same global indexes in conductivity matrixes of other elements are added together. Such common coefficients appear in conductivity matrixes of elements, which share a common node, e.g. in the case of elements in Fig. 11.19, the node common to six elements is node c.



Numeration of nodes in element no. ①	
Global node number	Local node number
a	⟨1⟩
b	⟨2⟩
c	⟨3⟩

Fig. 11.20. A diagram of global (*a*, *b* and *c*) and local (⟨1⟩, ⟨2⟩, ⟨3⟩) node numeration in a triangular element

A global equation system for node temperatures can be created in another way, which resembles the way heat balance equations are developed using the control volume method. One can also create control volume in FEM around node *c* (Fig. 11.19), common to surrounding elements, by linking centers of gravity of triangular elements with the midpoints of triangle sides (Fig . 11.21). The equation number equal to the number of nodes in an element is assigned to every element. There are three equations in the case of a triangular element. When creating a global equation for node *c* (Fig. 11.19), only those equations are considered in which the shape function was selected as a weight function in the Galerkin method at point *c*. For element ① when local nodes are positioned as shown in Fig. 11.20, the third equation is the equation in question; it results from the application of Galerkin method when weight coefficient equals $N_3^1(x, y)$.

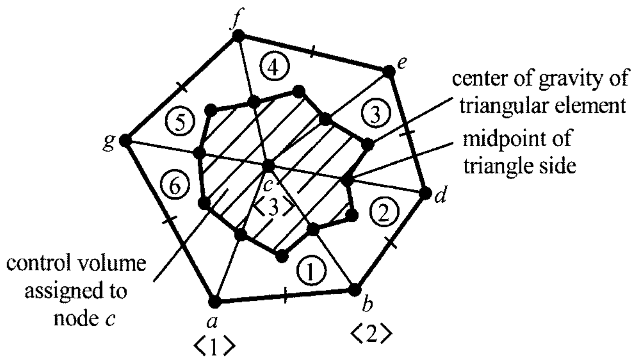


Fig. 11.21. Control volume in FEM with a region divided into triangular elements; linear functions interpolate temperature distribution inside the element

If similar local node numeration is assumed for the remaining elements shown in Fig. 11.19, then we must account for the third equation in every equation system for a given element, since in every element local node ⟨3⟩

corresponds to node *c*. In Galerkin method, function $N_3^e(x,y)$ plays a role of a weight function for element ⓔ.

b) The second method for creating global equation system, based on the formulation of an appropriate equation for a given node, indicates that there is a close relationship between FEM and the control (finite) volume method.

The following conditions should be retained when aggregating (summating) elements (Fig. 11.22):

- continuity of temperature field, including boundaries between elements;
- continuity of heat flow, also on the boundary between elements.

The first condition is satisfied in FEM; the second condition, however, is not completely satisfied. On the boundary between elements, the following temperature continuity takes place

$$T_3^1 = T_3^2 = T_4$$

and

$$T_2^1 = T_1^2 = T_2 \,. \tag{7}$$

The above indexes (7) are the element numbers. Temperature equality on the boundary between elements follows from the equation of temperature in nodes and linear character of temperature distribution inside the elements.

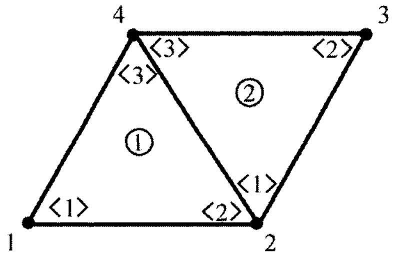


Fig. 11.22. Global and local numeration of nodes in finite elements

In agreement with (29), Ex. 11.10, the equality of integrals takes place on the boundary between elements:

$$\int_{L_{23}^1} \dot{q}_n^1 N_2^1 ds = -\int_{L_{13}^2} \dot{q}_n^2 N_1^2 ds \,, \tag{8}$$

$$\int_{L_{23}^1} \dot{q}_n^1 N_3^1 ds = -\int_{L_{13}^2} \dot{q}_n^2 N_3^2 ds \,. \tag{9}$$

Therefore, only in the case of very small elements, when the side common to both elements is very short in length, the heat flux equality is ensured on the boundary where two elements meet. Furthermore, heat flux inside an element is constant when temperature distribution is interpolated inside the element by linear functions. Therefore, heat flux step-change occurs at the point of contact of two elements. At such point, there is also no continuity among the first derivatives of function $T^e(x, y)$. This lack of continuity at the point of contact between elements negatively affects the accuracy of solution. In order to determine heat flux at a given point in an analyzed region or to calculate heat flow transmitted by a body boundary, the region should be divided into very small elements, so that the accuracy of the determined heat flux values is satisfactory.

Exercise 11.16 Determining Temperature Distribution in a Square Cross-Section of an Infinitely Long Rod by Means of FEM, in which the Global Equation System is Constructed using Method I (from Ex. 11.15)

Determine temperature distribution in a square cross-section of an infinitely long rod (Fig. 11.23). Upper and lower surfaces are thermally insulated. Left vertical surface is heated by a heat flow whose density is $\dot{q}_B = 200\ 000$ W/m^2, while the surface on the opposite side is cooled by water at temperature $T_{cz} = 20$°C with a heat transfer coefficient equal to $\alpha = 1000$ W/(m^2·K). Thermal conductivity of the rod's material $\lambda_x = \lambda_y = 50$ W/(m·K). The length of the side within the square cross-section of the rod is $a = 2$ cm.

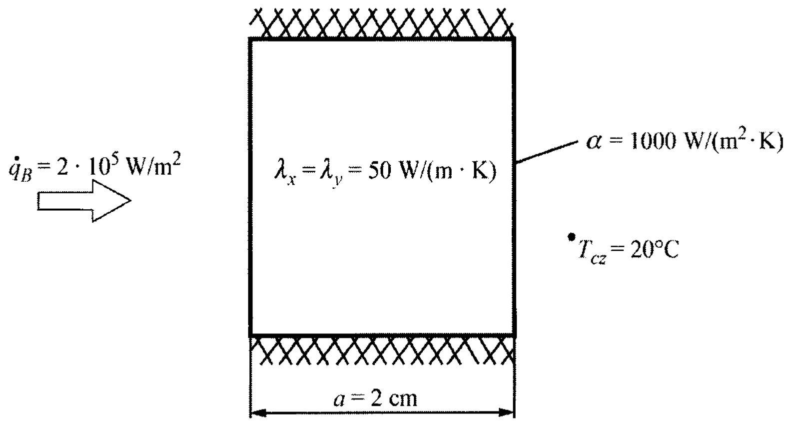


Fig. 11.23. Cross-section of an infinitely long rod

Solution

Temperature field will be treated as two-dimensional. Cross-section of the rod will be divided into four elements. Local and global node numeration is given in Fig. 11.24. and Table 11.2.

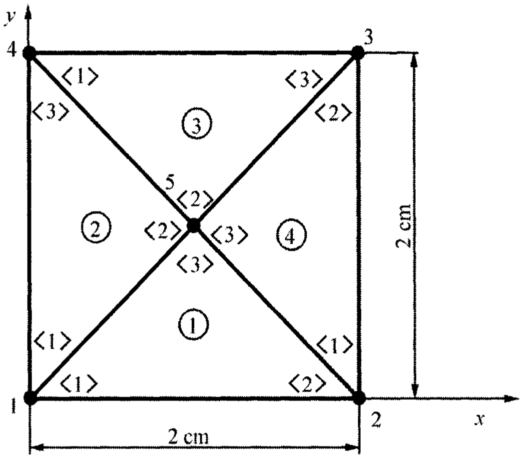


Fig. 11.24. A division of a rod's cross-section into four elements with a marked global and local node numeration; local numeration is given in brackets ⟨ ⟩

Table 11.2. Local and global node numeration

Element No.	Local Node Number	Global Node Number
	⟨1⟩	1
①	⟨2⟩	2
	⟨3⟩	5
	⟨1⟩	1
②	⟨2⟩	5
	⟨3⟩	4
	⟨1⟩	4
③	⟨2⟩	5
	⟨3⟩	3
	⟨1⟩	2
④	⟨2⟩	3
	⟨3⟩	5

First we will determine the quantities in individual elements of the conductivity matrix $[K_c^e]$ and in the element of matrix element $[K_\alpha^4]$, since a convection heat transfer is assigned on the side 2-3 (⟨1⟩-⟨2⟩). Conductivity matrix for a triangular element is formulated by (9) in Ex. 11.11, which in the given case for $\lambda_x = \lambda_y = \lambda$ has the form

$$\left[K_c^e\right] = \frac{\lambda}{4A^e}\begin{bmatrix} \left(b_1^e\right)^2 + \left(c_1^e\right)^2 & b_1^e b_2^e + c_1^e c_2^e & b_1^e b_3^e + c_1^e c_3^e \\ b_1^e b_2^e + c_1^e c_2^e & \left(b_2^e\right)^2 + \left(c_2^e\right)^2 & b_2^e b_3^e + c_2^e c_3^e \\ b_1^e b_3^e + c_1^e c_3^e & b_2^e b_3^e + c_2^e c_3^e & \left(b_3^e\right)^2 + \left(c_3^e\right)^2 \end{bmatrix}, \tag{1}$$

where matrix coefficients are formulated in (8), Ex. 11.9.

Matrix $[K_\alpha^4]$ for element ④ is formulated in (12), Ex. 11.12

$$\left[K_\alpha^4\right] = \frac{\alpha L_{12}}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2}$$

where $L_{12} = 0$ is the length of the side in the square cross-section. From matrices $[K_c^e]$ and $[K_\alpha^4]$, one obtains:

- Element ①

$$x_1^1 = 0\,\text{m}, \qquad y_1^1 = 0\,\text{m},$$
$$x_2^1 = 2\cdot 10^{-2}\,\text{m}, \qquad y_2^1 = 0\,\text{m},$$
$$x_3^1 = 1\cdot 10^{-2}\,\text{m}, \qquad y_3^1 = 1\cdot 10^{-2}\,\text{m},$$
$$b_1^1 = 0 - 1\cdot 10^{-2} = -1\cdot 10^{-2}\,\text{m}, \qquad b_2^1 = 1\cdot 10^{-2} - 0 = 1\cdot 10^{-2}\,\text{m},$$
$$b_3^1 = 0 - 0 = 0\,\text{m},$$
$$c_1^1 = 1\cdot 10^{-2} - 2\cdot 10^{-2} = -1\cdot 10^{-2}\,\text{m}, \qquad c_2^1 = 0 - 1\cdot 10^{-2} = -1\cdot 10^{-2}\,\text{m},$$
$$c_3^1 = 2\cdot 10^{-2} - 0 = 2\cdot 10^{-2}\,\text{m},$$
$$A^1 = \frac{1}{2}\cdot 0.02\cdot 0.01 = 1\cdot 10^{-4}\,\text{m}^2.$$

Conductivity matrix $[K_c^1]$ calculated with (1) is as follows:

$$\begin{matrix} & 1 & 2 & 5 & \\ \left[K_c^1\right] = 50 & \left[\begin{matrix} 0.5 \\ 0 \\ -0.5 \end{matrix}\right. & \begin{matrix} 0 \\ 0.5 \\ -0.5 \end{matrix} & \left.\begin{matrix} -0.5 \\ -0.5 \\ 1.0 \end{matrix}\right] & \begin{matrix} 1 \\ 2, \\ 5 \end{matrix} \end{matrix} \tag{3}$$

The numbers above and next to the matrix are the global node numbers.

• Element ②

$x_1^2 = 0 \text{ m}, \quad y_1^2 = 0 \text{ m}, \quad x_2^2 = 0.01 \text{ m}, \quad y_2^2 = 0.01 \text{ m},$

$x_3^2 = 0.0 \text{ m}, \quad y_3^2 = 0.02 \text{ m},$

$b_1^2 = 0.01 - 0.02 = -1 \cdot 10^{-2} \text{ m}, \quad b_2^2 = 0.02 - 0 = 2 \cdot 10^{-2} \text{ m},$

$b_3^2 = 0 - 0.01 = -1 \cdot 10^{-2} \text{ m},$

$c_1^2 = 0 - 0.01 = -1 \cdot 10^{-2} \text{ m}, \quad c_2^2 = 0 - 0 = 0 \text{ m},$

$c_3^2 = 0.01 - 0 = 1 \cdot 10^{-2} \text{ m},$

$A^2 = \frac{1}{2} \cdot 0.02 \cdot 0.01 = 1 \cdot 10^{-4} \text{ m}^2.$

Conductivity matrix is

$$\begin{matrix} & \begin{matrix} 1 & \quad 5 & \quad 4 \end{matrix} & \\ \left[K_c^2\right] = 50 & \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1.0 & -0.5 \\ 0.0 & -0.5 & 0.5 \end{bmatrix} & \begin{matrix} 1 \\ 5. \\ 4 \end{matrix} \end{matrix} \tag{4}$$

• Element ③

$x_1^3 = 0 \text{ m}, \quad y_1^3 = 2 \cdot 10^{-2} \text{ m},$

$x_2^3 = 1 \cdot 10^{-2} \text{ m}, \quad y_2^3 = 1 \cdot 10^{-2} \text{ m},$

$x_3^3 = 2 \cdot 10^{-2} \text{ m}, \quad y_3^3 = 2 \cdot 10^{-2} \text{ m},$

$b_1^3 = 1 \cdot 10^{-2} - 2 \cdot 10^{-2} = -1 \cdot 10^{-2} \text{ m}, \quad b_2^3 = 2 \cdot 10^{-2} - 2 \cdot 10^{-2} = 0 \text{ m},$

$b_3^3 = 2 \cdot 10^{-2} - 1 \cdot 10^{-2} = 1 \cdot 10^{-2} \text{ m},$

$c_1^3 = 2 \cdot 10^{-2} - 1 \cdot 10^{-2} = 1 \cdot 10^{-2} \text{ m}, \quad c_2^3 = 0 - 2 \cdot 10^{-2} = -2 \cdot 10^{-2} \text{ m},$

$c_3^3 = 1 \cdot 10^{-2} - 0 = 1 \cdot 10^{-2} \text{ m}.$

Matrix $[K_c^3]$ is as follows:

$$\begin{matrix} & \begin{matrix} 4 & \quad 5 & \quad 3 \end{matrix} & \\ \left[K_c^3\right] = 50 & \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1.0 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix} & \begin{matrix} 4 \\ 5. \\ 3 \end{matrix} \end{matrix} \tag{5}$$

• Element ④

$x_1^4 = 2\cdot10^{-2}$ m, $y_1^4 = 0$ m, $x_2^4 = 2\cdot10^{-2}$ m, $y_2^4 = 2\cdot10^{-2}$ m,

$x_3^4 = 1\cdot10^{-2}$ m, $y_3^4 = 1\cdot10^{-2}$ m,

$b_1^4 = 2\cdot10^{-2} - 1\cdot10^{-2} = 1\cdot10^{-2}$ m, $b_2^4 = 1\cdot10^{-2} - 0 = 1\cdot10^{-2}$ m,

$b_3^4 = 0 - 2\cdot10^{-2} = -2\cdot10^{-2}$ m,

$c_1^4 = 1\cdot10^{-2} - 2\cdot10^{-2} = -1\cdot10^{-2}$ m, $c_2^4 = 2\cdot10^{-2} - 1\cdot10^{-2} = 1\cdot10^{-2}$ m,

$c_3^4 = 2\cdot10^{-2} - 2\cdot10^{-2} = 0$ m.

Matrix $[K_c^4]$ is as follows:

$$\begin{matrix} & 2 & 3 & 5 & \\ \left[K_c^4\right] = 50 & \left[\begin{matrix} 0.5 \\ 0 \\ -0.5 \end{matrix}\right. & \begin{matrix} 0 \\ 0.5 \\ -0.5 \end{matrix} & \left.\begin{matrix} -0.5 \\ -0.5 \\ 1.0 \end{matrix}\right] & \begin{matrix} 2 \\ 3. \\ 5 \end{matrix} \end{matrix} \tag{6}$$

Matrix $[K_\alpha^4]$, which results from the heat transfer on the side 2-3, has the form

$$\begin{matrix} & 2 & 3 & & \\ \left[K_\alpha^4\right] = 3.33(3) & \left[\begin{matrix} 2 \\ 1 \\ 0 \end{matrix}\right. & \begin{matrix} 1 \\ 2 \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] & \begin{matrix} 2 \\ 3, \\ \end{matrix} \end{matrix} \tag{7}$$

or

$$\begin{matrix} & 2 & 3 & & \\ \left[K_\alpha^4\right] = & \left[\begin{matrix} 6.667 \\ 3.333 \\ 0 \end{matrix}\right. & \begin{matrix} 3.333 \\ 6.667 \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] & \begin{matrix} 2 \\ 3. \\ \end{matrix} \end{matrix} \tag{8}$$

The sum of matrix $[K_c^4] + [K_\alpha^4]$ is

$$\begin{matrix} & 2 & 3 & 5 & \\ \left[K_c^4\right] + \left[K_\alpha^4\right] = & \left[\begin{matrix} 31.667 \\ 3.333 \\ -25.0 \end{matrix}\right. & \begin{matrix} 3.333 \\ 31.667 \\ -25.0 \end{matrix} & \left.\begin{matrix} -25.0 \\ -25.0 \\ 50 \end{matrix}\right] & \begin{matrix} 2 \\ 3. \\ 5 \end{matrix} \end{matrix} \tag{9}$$

Coefficient matrix in the global equation system can be obtained by adding coefficients in matrices (3)–(6) and (9):

$$K_{11} = K_{11}^1 + K_{11}^2 = 25 + 25 = 50 \text{ W/(m} \cdot \text{K)}, \quad K_{12} = 0, \quad K_{13} = 0, \quad K_{14} = 0,$$
$$K_{15} = K_{15}^1 + K_{15}^2 = -25 + (-25) = -50 \text{ W/(m} \cdot \text{K)},$$
$$K_{21} = 0, K_{22} = K_{22}^1 + \left(K_{22}^4 + K_{\alpha,22}^4\right) = 25 + 31.6667 = 56.6667 \text{ W/(m} \cdot \text{K)},$$
$$K_{23} = \left(\left[K_{23}^4\right] + \left[K_{\alpha,23}^4\right]\right) = 3.333 \text{ W/(m} \cdot \text{K)}, \quad K_{24} = 0,$$
$$K_{25} = K_{25}^1 + K_{25}^4 = -25 + (-25) = -50 \text{ W/(m} \cdot \text{K)},$$
$$K_{31} = 0, \ K_{32} = \left(K_{32}^4 + K_{\alpha,32}^4\right) = 3.333 \text{ W/(m} \cdot \text{K)},$$
$$K_{33} = K_{33}^3 + \left(K_{33}^4 + K_{\alpha,33}^4\right) = 25 + 31.6667 = 56.6667 \text{ W/(m} \cdot \text{K)},$$
$$K_{34} = K_{34}^3 = 0 \text{ W/(m} \cdot \text{K)}, K_{35} = K_{35}^3 + K_{35}^4 = -25 + (-25) = -50 \text{ W/(m} \cdot \text{K)},$$
$$K_{41} = K_{41}^2 = 0, K_{42} = 0, K_{43} = 0, K_{44} = K_{44}^2 + K_{44}^3 = 25 + 25 = 50 \text{ W/(m} \cdot \text{K)},$$
$$K_{45} = K_{45}^2 + K_{45}^3 = -25 + (-25) = -50 \text{ W/(m} \cdot \text{K)},$$
$$K_{51} = K_{51}^1 + K_{51}^2 = -25 + (-25) = -50 \text{ W/(m} \cdot \text{K)},$$
$$K_{52} = K_{52}^1 + K_{52}^4 = -25 + (-25) = -50 \text{ W/(m} \cdot \text{K)},$$
$$K_{53} = K_{53}^3 + K_{53}^4 = -25 + (-25) = -50 \text{ W/(m} \cdot \text{K)},$$
$$K_{54} = K_{54}^2 + K_{54}^3 = -25 + (-25) = -50 \text{ W/(m} \cdot \text{K)},$$
$$K_{55} = K_{55}^1 + K_{55}^2 + K_{55}^3 + K_{55}^4 = 50 + 50 + 50 + 50 = 200 \text{ W/(m} \cdot \text{K)}.$$

If all coefficients are known, one can write then the global conductivity matrix

$$[K] = \begin{bmatrix} 50 & 0 & 0 & 0 & -50 \\ 0 & 56.667 & 3.333 & 0 & -50 \\ 0 & 3.333 & 56.667 & 0 & -50 \\ 0 & 0 & 0 & 50 & -50 \\ -50 & -50 & -50 & -50 & 200 \end{bmatrix} \text{ W/(m} \cdot \text{K)}. \tag{10}$$

Following that, vector $\{f_q^2\}$ is defined according to (17), Ex. 11.14

$$\left\{f_q^2\right\} = \frac{\dot{q}_B L_{31}}{2} \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_5 \\ f_4 \end{Bmatrix}, \tag{11}$$

$$\left\{f_q^2\right\} = \begin{Bmatrix} f_1 \\ f_5 \\ f_4 \end{Bmatrix} = \begin{Bmatrix} 2000 \\ 0 \\ 2000 \end{Bmatrix}. \tag{12}$$

Vector $\{f_\alpha^4\}$ will be calculated according to (16), Ex. 11.14

$$\left\{f_\alpha^4\right\} = \frac{\alpha T_{cz}}{2} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} f_2 \\ f_3 \\ f_5 \end{Bmatrix}, \tag{13}$$

where from, after substitution, one obtains

$$\left\{f_\alpha^4\right\} = \begin{Bmatrix} f_2 \\ f_3 \\ f_5 \end{Bmatrix} = \begin{Bmatrix} 10000 \\ 10000 \\ 0 \end{Bmatrix} \text{ W/m}^2. \tag{14}$$

The right-hand-side vector has the form

$$\{f\} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{Bmatrix} = \begin{Bmatrix} 2000 \\ 10000 \\ 10000 \\ 2000 \\ 0 \end{Bmatrix} \text{ W/m}^2. \tag{15}$$

The equation system (31) from Ex. 11.10, from which node temperatures will be determined, assumes in this case the following form

$$\begin{bmatrix} 50 & 0 & 0 & 0 & -50 \\ 0 & 56.667 & 3.333 & 0 & -50 \\ 0 & 3.333 & 56.667 & & -50 \\ 0 & 0 & 0 & 50 & -50 \\ -50 & -50 & -50 & -50 & 200 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 2000 \\ 10000 \\ 10000 \\ 2000 \\ 0 \end{Bmatrix}. \tag{16}$$

Since the equation system (16) is solved by means of the Gauss elimination method and the program shown in Ex. 6.26, the following is obtained: T_1 = 280°C, T_2 = 200°C, T_3 = 200°C, T_4 = 280°C, T_5 = 240°C. Due to thermal insulation of lateral surfaces 1-2 and 3-4, temperature field is one-dimensional in the cross-section of the rod.

Temperature can be calculated from analytical formulas

$$T_1 = T_4 = \frac{\dot{q}_B a}{\lambda} + \frac{\dot{q}_B}{\alpha} = \frac{200000 \cdot 0.02}{50} + \frac{200000}{1000} = 280°\text{C},$$

$$T_5 = \frac{\dot{q}_B a}{2\lambda} + \frac{\dot{q}_B}{\alpha} = \frac{200000 \cdot 0.02}{2 \cdot 50} + \frac{200000}{1000} = 240°\text{C},$$

$$T_2 = T_3 = \frac{\dot{q}_B}{\alpha} = \frac{200000}{1000} = 200°\text{C}.$$

It is clear, therefore, that the results obtained by means of FEM and the analytical formulas are identical to each other.

Exercise 11.17 Determining Temperature Distribution in an Infinitely Long Rod with Square Cross-Section by Means of FEM, in which the Global Equation System is Constructed using Method II (from Ex. 11.15)

Solve the problem formulated in Ex. 11.16 by means of FEM; namely, the equation (of heat balance) for individual nodes. Use Method II discussed in Ex. 11.15.

Solution

• Node 1

Elements ① and ② have node 1 in common (Fig. 11.24). Equation system for element ① has the form

$$\left[K^1\right]\left\{T^1\right\} = \left\{f^1\right\}, \tag{1}$$

where $[K^1]$ is formulated in (3), Ex. 11.16. Because $\{f^1\} = [0, 0, 0]^T$, equation system (1) has the form

$$\begin{bmatrix} 25 & 0 & -25 \\ 0 & 25 & -25 \\ -25 & -25 & 50 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{2}$$

Node 1 in the global numeration is also ⟨1⟩ in the local numeration of element ①; therefore, only the first equation is taken into consideration in (2)

$$25T_1 - 25T_5 = 0. \tag{3}$$

The algebraic equation system for element ② has the form

$$\left[K^2\right]\left\{T^2\right\}=\left\{f^2\right\},$$

where $[K^2]$ is formulated in (4) while vector $\{f^2\}$ in (12), Ex. 11.16. The equation system for element ② assumes the form then

$$\begin{bmatrix} 25 & 25 & 0 \\ -25 & 50 & -25 \\ 0 & -25 & 25 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_5 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 2000 \\ 0 \\ 2000 \end{Bmatrix}. \tag{4}$$

Node 1 in the global numeration is node ⟨1⟩ in the local numeration in element ②; therefore, only the first equation in (4) is taken into consideration

$$25T_1 - 25T_5 = 2000. \tag{5}$$

Once (3) is added to (5), an algebraic equation for node 1 is obtained in global numeration

$$50T_1 - 50T_5 = 2000. \tag{6}$$

- Node 2

Node 2 is shared by element ① and ④. Node 2 in the global numeration is also node ⟨2⟩ in the local numeration. Therefore, in the equation system (2), second equation is taken into consideration

$$25T_2 - 25T_5 = 0. \tag{7}$$

Equation system for element ④ has the form

$$\left[K^4\right]\left\{T^4\right\}=\left\{f^4\right\}, \tag{8}$$

where $[K^4]$ is formulated in (6), Ex. 11.16, while $\{f^4\}$ in (14), Ex. 11.16. Thus, the equation system (8) has the form

$$\begin{bmatrix} 31{,}667 & 3{,}333 & -25 \\ 3{,}333 & 31{,}667 & -25 \\ -25 & -25 & 50 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 10000 \\ 10000 \\ 0 \end{Bmatrix}. \tag{9}$$

Node 2 in the global numeration is node ⟨1⟩ in the local numeration in element ④. In the equation system (9) only the first equation is taken into consideration

$$31.667T_2 + 3.333T_3 - 25T_5 = 10000. \tag{10}$$

Once (7) and (10) are added together, the equation for node 2 is obtained (in global numeration)

$$56.667T_2 + 3.333T_3 - 50T_5 = 10000. \tag{11}$$

• Node 3

Node 3 has a local number ⟨2⟩ in element ④. The second equation in the system (9) has the form

$$3.333T_2 + 31.667T_3 - 25T_5 = 10000. \tag{12}$$

An equation system for element ③, to which node 3 belongs, has the form

$$\left[K^3\right]\left\{T^3\right\} = \left\{f^3\right\}, \tag{13}$$

where $[K^3]$ is expressed in (4), Ex. 11.16, while $\{f^3\} = [0,0,0]^T$. The equation system (13) assumes the form

$$\begin{bmatrix} 25 & -25 & 0 \\ -25 & 50 & -25 \\ 0 & -25 & 25 \end{bmatrix} \begin{Bmatrix} T_4 \\ T_5 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{14}$$

Only the third equation from above (14) is allowed for, since node 3 has the local number ⟨3⟩

$$25T_3 - 25T_5 = 0. \tag{15}$$

Once corresponding sides of (12) and (15) are added, a global equation for node 3 is obtained

$$3.333T_2 + 56.667T_3 - 50T_5 = 10000. \tag{16}$$

• Node 4

Node 4 is shared by element ② and ③. In the equation system for element ②, the third equation is taken into consideration, since the analyzed node has a local number ⟨3⟩ ②. From (4), one obtains

$$25T_4 - 25T_5 = 2000. \tag{17}$$

In the equation system for element ③, the first equation is taken into consideration, since node 4 has the local number equal to ⟨1⟩ in element ③. From (14) one has

$$25T_4 - 25T_5 = 0. \tag{18}$$

Once corresponding sides of (17) and (18) are added, one obtains a global equation for node 4

$$50T_4 - 50T_5 = 2000. \tag{19}$$

• Node 5

This is a node common to all elements. In the equation system (2) for element ①, the third equation is accounted for, since the analyzed node has a local number ⟨3⟩ in element ①. From (2), one gets

$$-25T_1 - 25T_2 + 50T_5 = 0. \tag{20}$$

In (4) for element ②, the second equation is considered

$$-25T_1 - 25T_4 + 50T_5 = 0, \tag{21}$$

since node 5 has the local number ⟨2⟩ in this element. In the equation system for element ③, the second equation is accounted for

$$-25T_3 - 25T_4 + 50T_5 = 0, \tag{22}$$

since node 5 has the local number ⟨2⟩ in this element. In the equation system (9) for element ④, the third equation is accounted for

$$-25T_2 - 25T_3 + 50T_5 = 0, \tag{23}$$

since node 5 has the local number ⟨3⟩ in element ④. Once corresponding sides of (20), (21), (22) and (23) are added, one has

$$-50T_1 - 50T_2 - 50T_3 - 50T_4 + 200T_5 = 0. \tag{24}$$

Equations (6), (11), (16), (19) and (24) form a global equation system.

$$\begin{aligned}
&50T_1 - 50T_5 = 2000\\
&56.6667T_2 + 3.333T_3 - 50T_5 = 10000\\
&3.333T_2 + 56.667T_3 - 50T_5 = 10000\\
&50T_4 - 50T_5 = 2000\\
&-50T_1 - 50T_2 - 50T_3 - 50T_4 + 200T_5 = 0.
\end{aligned} \tag{25}$$

Equation system (25) and system (16) from Ex. 11.16 are the same. It is clear, therefore, that regardless of how the global equation system is created, it always remains the same.

Exercise 11.18 Determining Temperature Distribution by Means of FEM in an Infinitely Long Rod with Square Cross-Section, in which Volumetric Heat Sources Operate

Determine steady-state temperature distribution in a square region whose side is $2a$ = 2 cm in length. Assume that the thermal conductivity of a medium is λ = 42 W/(m·K). Heat source power per unit of volume is $\dot{q}_v = 1 \cdot 10^7$ W/m^3. Boundary conditions are illustrated in Fig.11.25. Assume the following values for the calculation: $\dot{q}_B$ = 200000 W/m^2, α = 60 W/(m^2·K), T_{cz} = 20°C, T_s = 100°C.

Solution

Boundary conditions can be written in the following way:

$$-\lambda \frac{\partial T}{\partial x}\bigg|_{x=0} = \dot{q}_B, \tag{1}$$

$$T\big|_{x=2a} = T_s, \tag{2}$$

$$-\lambda \frac{\partial T}{\partial y}\bigg|_{y=0} = \alpha\left(T_{cz} - T\big|_{y=0}\right), \tag{3}$$

$$-\lambda \frac{\partial T}{\partial y}\bigg|_{y=2a} = 0. \tag{4}$$

Temperature distribution will be determined by means of FEM with a division depicted in Fig.11.25. Local and global node numeration is presented in Table 11.3.

Conductivity matrix (rigidity) in the case of the square element and constant thermal conductivity, when $\lambda_x = \lambda_y = \lambda$, has the following form ((6), Ex. 11.11).

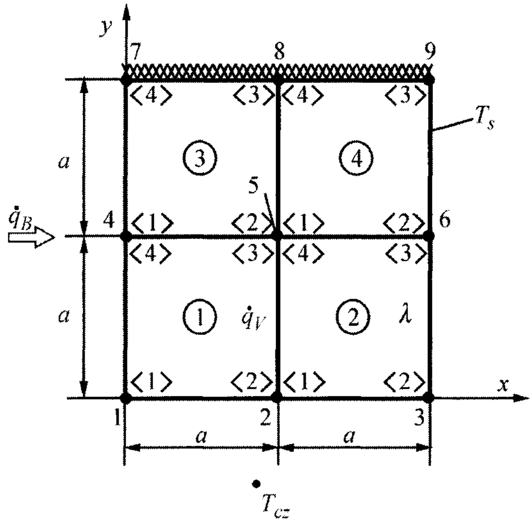


Fig. 11.25. Diagram of the analyzed region, which illustrates boundary conditions and division of an area into finite elements

Table 11.3. Local and global node numeration

Element No.	Local Node No.	Global Node No.
①	⟨1⟩	1
	⟨2⟩	2
	⟨3⟩	5
	⟨4⟩	4
②	⟨1⟩	2
	⟨2⟩	3
	⟨3⟩	6
	⟨4⟩	5
③	⟨1⟩	4
	⟨2⟩	5
	⟨3⟩	8
	⟨4⟩	7
④	⟨1⟩	5
	⟨2⟩	6
	⟨3⟩	9
	⟨4⟩	8

$$\left[K_c^e\right]=\frac{\lambda}{6}\begin{bmatrix}4 & -1 & -2 & -1\\ -1 & 4 & -1 & -2\\ -2 & -1 & 4 & -1\\ -1 & -2 & -1 & 4\end{bmatrix}. \tag{5}$$

Next, matrixes of conductivity (stiffness) $[K_c^e]$ will be written for individual elements. Global node numeration will be used.

- Element ①

$$\left[K_c^1\right]=7\begin{array}{c}\begin{matrix}1 & 2 & 5 & 4\end{matrix}\\ \begin{bmatrix}4 & -1 & -2 & -1\\ -1 & 4 & -1 & -2\\ -2 & -1 & 4 & -1\\ -1 & -2 & -1 & 4\end{bmatrix}\end{array}\begin{matrix}\\1\\2\\5\\4\end{matrix}, \tag{6}$$

- Element ②

$$\left[K_c^2\right]=7\begin{array}{c}\begin{matrix}2 & 3 & 6 & 5\end{matrix}\\ \begin{bmatrix}4 & -1 & -2 & -1\\ -1 & 4 & -1 & -2\\ -2 & -1 & 4 & -1\\ -1 & -2 & -1 & 4\end{bmatrix}\end{array}\begin{matrix}\\2\\3\\6\\5\end{matrix}, \tag{7}$$

- Element ③

$$\left[K_c^3\right]=7\begin{array}{c}\begin{matrix}4 & 5 & 8 & 7\end{matrix}\\ \begin{bmatrix}4 & -1 & -2 & -1\\ -1 & 4 & -1 & -2\\ -2 & -1 & 4 & -1\\ -1 & -2 & -1 & 4\end{bmatrix}\end{array}\begin{matrix}\\4\\5\\8\\7\end{matrix}, \tag{8}$$

- Element ④

$$\left[K_c^4\right]=7\begin{array}{c}\begin{matrix}5 & 6 & 9 & 8\end{matrix}\\ \begin{bmatrix}4 & -1 & -2 & -1\\ -1 & 4 & -1 & -2\\ -2 & -1 & 4 & -1\\ -1 & -2 & -1 & 4\end{bmatrix}\end{array}\begin{matrix}\\5\\6\\9\\8\end{matrix}, \tag{9}$$

Global conductivity matrix (stiffness) $[K_c]$ results from the summation of matrixes for individual elements. By doing so, one should also pay attention to coefficients with the same global indexes, as they should be summated as well. Coefficients of the global conductivity matrix (index c was not included in the designations of coefficients $K_{c,ij}$ in order to shorten the notation) are as follow:

$$K_{11}=K_{11}^1=28,\quad K_{12}=K_{12}^1=-7,\quad K_{14}=K_{14}^1=-7,\quad K_{15}=K_{15}^1=-14,$$

$$K_{21}=K_{21}^1=-7,\quad K_{22}=K_{22}^1+K_{22}^2=7(4+4)=56,\quad K_{23}=K_{23}^2=-7,$$

$$K_{24}=K_{24}^1=-14,\quad K_{25}=K_{25}^1+K_{25}^2=7(-1+(-1))=-14,$$

$$K_{26}=K_{26}^2=-14,\quad K_{32}=K_{32}^2=-7,\quad K_{33}=K_{33}^2=28,\quad K_{35}=K_{35}^2=-14,$$

$$K_{36}=K_{36}^2=-7,\quad K_{41}=K_{41}^1=-7,\quad K_{42}=K_{42}^1=-14,$$

$$K_{44}=K_{44}^1+K_{44}^3=56,\quad K_{45}=K_{45}^1+K_{45}^3=-7+(-7)=-14,$$

$$K_{47}=K_{47}^3=-7,\quad K_{48}=K_{48}^3=-14,$$

$$K_{48}=K_{48}^3=-14,\quad K_{51}=K_{51}^1=-14,\quad K_{52}=K_{52}^1+K_{52}^2=-7+(-7)=-14,$$

$$K_{53}=K_{53}^2=-14,\quad K_{54}=K_{54}^1+K_{54}^3=-7+(-7)=-14,$$

$$K_{55}=K_{55}^1+K_{55}^2+K_{55}^3+K_{55}^4=28+28+28+28=112,$$

$$K_{56}=K_{56}^2+K_{56}^4=-7+(-7)=-14,\quad K_{57}=K_{57}^3=-14,$$

$$K_{58}=K_{58}^3+K_{58}^4=-7+(-7)=-14,\quad K_{59}=K_{59}^4=-14,\quad K_{62}=K_{62}^2=-14,$$

$$K_{63}=K_{63}^2=-7,\quad K_{65}=K_{65}^2+K_{65}^4=-7+(-7)=-14,$$

$$K_{66}=K_{66}^2+K_{66}^4=28+28=56,$$

$$K_{68}=K_{68}^4=-14,\quad K_{69}=K_{69}^4=-7,\quad K_{74}=K_{74}^3=-7,\quad K_{75}=K_{75}^3=-14,$$

$$K_{77}=K_{77}^3=28,$$

$$K_{78}=K_{78}^3=-7,\quad K_{84}=K_{84}^3=-14,\quad K_{85}=K_{85}^3+K_{85}^4=-7+(-7)=-14,$$

$$K_{86}=K_{86}^4=-14,$$

$$K_{87}=K_{87}^3=-7,\quad K_{88}=K_{88}^3+K_{88}^4=28+28=56,\quad K_{89}=K_{89}^4=-7,$$

$$K_{95}=K_{95}^4=-14,\quad K_{96}=K_{96}^4=-7,\quad K_{98}=K_{98}^4=-7,\quad K_{99}=K_{99}^4=28.$$

The remaining coefficients equal zero. Global conductivity matrix $[K_c]$ has the form

$$
[K_c]=\begin{bmatrix}
28 & -7 & 0 & -7 & -14 & 0 & 0 & 0 & 0\\
-7 & 56 & -7 & -14 & -14 & -14 & 0 & 0 & 0\\
0 & -7 & 28 & 0 & -14 & -7 & 0 & 0 & 0\\
-7 & -14 & 0 & 56 & -14 & 0 & -7 & -14 & 0\\
-14 & -14 & -14 & -14 & 112 & -14 & -14 & -14 & -14\\
0 & -14 & -7 & 0 & -14 & 56 & 0 & -14 & -7\\
0 & 0 & 0 & -7 & -14 & 0 & 28 & -7 & 0\\
0 & 0 & 0 & -14 & -14 & -14 & -7 & 56 & -7\\
0 & 0 & 0 & 0 & -14 & -7 & 0 & -7 & 28
\end{bmatrix}. \quad (10)
$$

Next, matrix $[K_\alpha]$ is determined

$$
[K_\alpha]=\left[K_\alpha^1\right]+\left[K_\alpha^2\right], \quad (11)
$$

since heat transfer takes place on the lateral surface 1-2 of element ① and on 2-3 of element ②. Matrix $[K_\alpha^1]$ will be calculated using (7), from Ex. 11.12

$$
\left[K_\alpha^1\right]=\frac{\alpha L_{12}}{6}\begin{array}{c}\begin{matrix}1 & 2 & 5 & 4\end{matrix}\\ \begin{bmatrix}2 & 1 & 0 & 0\\ 1 & 2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\end{array}\begin{matrix}1\\2\\5\\4\end{matrix}=\frac{60\cdot 0.01}{6}\begin{array}{c}\begin{matrix}1 & 2 & 5 & 4\end{matrix}\\ \begin{bmatrix}2 & 1 & 0 & 0\\ 1 & 2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\end{array}\begin{matrix}1\\2\\5\\4\end{matrix}= \quad (12)
$$

$$
=\begin{array}{c}\begin{matrix}1 & 2 & 5 & 4\end{matrix}\\ \begin{bmatrix}0.2 & 0.1 & 0 & 0\\ 0.1 & 0.2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\end{array}\begin{matrix}1\\2\\5\\4\end{matrix}.
$$

Matrix $[K_\alpha^2]$ will be calculated in a similar way.

$$
\left[K_\alpha^2\right]=\frac{\alpha L_{12}}{6}\begin{array}{c}\begin{matrix}2 & 3 & 6 & 5\end{matrix}\\ \begin{bmatrix}2 & 1 & 0 & 0\\ 1 & 2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\end{array}\begin{matrix}2\\3\\6\\5\end{matrix}=\begin{array}{c}\begin{matrix}2 & 3 & 6 & 5\end{matrix}\\ \begin{bmatrix}0.2 & 0.1 & 0 & 0\\ 0.1 & 0.2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\end{array}\begin{matrix}2\\3\\6\\5\end{matrix}. \quad (13)
$$

Once matrix coefficients $[K_\alpha^1]$ and $[K_\alpha^2]$, with the same global indexes, are added together, the following is obtained (index α in $K_{\alpha,ij}$ was omitted in order to simplify the notation)

$$K_{11} = K_{11}^1 = 0.2; K_{12} = K_{12}^1 = 0.1;\; K_{21} = K_{21}^1 = 0.1;$$
$$K_{22} = K_{22}^1 + K_{22}^2 = 0.2 + 0.2 = 0.4; K_{23} = K_{23}^2 = 0.1;$$
$$K_{32} = K_{32}^2 = 0.1; K_{33} = K_{33}^2 = 0.2.$$

Matrix $[K_\alpha]$, which results from boundary conditions on the boundary 1--2-3, has the form

$$[K_\alpha] = \begin{bmatrix} 0.2 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1 & 0.4 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{W/(m \cdot K)}. \qquad (14)$$

Matrix $[K]$ is obtained as a result of adding matrix $[K_c]$ formulated in (10) and matrix $[K_\alpha]$ formulated in (14)

$$[K] = \begin{bmatrix} 28.2 & -6{,}9 & 0 & -7 & -14 & 0 & 0 & 0 & 0 \\ -6.9 & 56.4 & -6.9 & -14 & -14 & -14 & 0 & 0 & 0 \\ 0 & -6.9 & 28.2 & 0 & -14 & -7 & 0 & 0 & 0 \\ -7 & -14 & 0 & 56 & -14 & 0 & -7 & -14 & 0 \\ -14 & -14 & -14 & -14 & 112 & -14 & -14 & -14 & -14 \\ 0 & -14 & -7 & 0 & -14 & 56 & 0 & -14 & -7 \\ 0 & 0 & 0 & -7 & -14 & 0 & 28 & -7 & 0 \\ 0 & 0 & 0 & -14 & -14 & -14 & -7 & 56 & -7 \\ 0 & 0 & 0 & 0 & -14 & -7 & 0 & -7 & 28 \end{bmatrix} \frac{\mathrm{W}}{\mathrm{mK}}. \qquad (15)$$

Next, one calculates the vectors on the right-hand-side of (31) from Ex. 11.10. Vector $\{f_Q^e\}$ is formulated in (3) from Ex. 11.13; it assumes the following form for the individual elements:

$$\left\{f_Q^1\right\}=\frac{\dot{q}_v a^2}{4}\begin{Bmatrix}1\\1\\1\\1\end{Bmatrix}\begin{matrix}1\\2\\5\\4\end{matrix}\ \text{W/m},\qquad \left\{f_Q^2\right\}=\frac{\dot{q}_v a^2}{4}\begin{Bmatrix}1\\1\\1\\1\end{Bmatrix}\begin{matrix}2\\3\\6\\5\end{matrix}\ \text{W/m},$$

$$\left\{f_Q^3\right\}=\frac{\dot{q}_v a^2}{4}\begin{Bmatrix}1\\1\\1\\1\end{Bmatrix}\begin{matrix}4\\5\\8\\7\end{matrix}\ \text{W/m},\qquad \left\{f_Q^4\right\}=\frac{\dot{q}_v a^2}{4}\begin{Bmatrix}1\\1\\1\\1\end{Bmatrix}\begin{matrix}5\\6\\9\\8\end{matrix}\ \text{W/m}. \tag{16}$$

Global vector $\{f_Q\}$ is obtained as a result of summing up the elements of vectors with the same global numbers.

$$\left\{f_Q\right\}=\frac{\dot{q}_v a^2}{4}\begin{Bmatrix}1\\2\\1\\2\\4\\2\\1\\2\\1\end{Bmatrix}=\begin{Bmatrix}250\\500\\250\\500\\1000\\500\\250\\500\\250\end{Bmatrix}\begin{matrix}1\\2\\3\\4\\5\\6\\7\\8\\9\end{matrix}\ \text{W/m}. \tag{17}$$

Vector $\{f_\alpha\}$ will be calculated in compliance with (2) from Ex. 11.14. Allowing for the fact that convection heat transfer is assigned on the surface 1-2 of element ① and on the surface 2-3 of element ②, vectors $\{f_\alpha^1\}$ and $\{f_\alpha^2\}$ have the form

$$\left\{f_\alpha^1\right\}=\frac{\alpha T_{cz} a}{2}\begin{Bmatrix}1\\1\\0\\0\end{Bmatrix}\begin{matrix}1\\2\\ \\ \end{matrix}=\begin{Bmatrix}6\\6\\0\\0\end{Bmatrix}\begin{matrix}1\\2\\ \\ \end{matrix}\ \text{W/m}, \tag{18}$$

$$\left\{f_\alpha^2\right\}=\frac{\alpha T_{cz} a}{2}\begin{Bmatrix}1\\1\\0\\0\end{Bmatrix}\begin{matrix}2\\3\\ \\ \end{matrix}=\begin{Bmatrix}6\\6\\0\\0\end{Bmatrix}\begin{matrix}2\\3\\ \\ \end{matrix}\ \text{W/m}. \tag{19}$$

As result of summing up the elements of vectors (18) and (19), with the same global numbers, one obtains

$$\{f_\alpha\} = \begin{Bmatrix} 6 \\ 12 \\ 6 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{matrix} \ \text{W/m} \ . \tag{20}$$

Heat flux $\dot{q}_B$ is assigned on surfaces 7-4 (⟨4⟩-⟨1⟩) and 4-1 (⟨4⟩-⟨1⟩), of the element ③ and ①, respectively. Vectors $\{f_q^1\}$ and $\{f_q^2\}$ calculated according to (10) from Ex. 11.14 are

$$\{f_q^1\} = \frac{\dot{q}_B a}{2} \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1000 \\ 0 \\ 0 \\ 1000 \end{Bmatrix} \begin{matrix} 1 \\ \\ \\ 4 \end{matrix} \ \text{W/m} \,, \tag{21}$$

$$\{f_q^3\} = \frac{\dot{q}_B a}{2} \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1000 \\ 0 \\ 0 \\ 1000 \end{Bmatrix} \begin{matrix} 4 \\ \\ \\ 7 \end{matrix} \ \text{W/m} \,. \tag{22}$$

Once these vectors are summed up (one should pay attention to the fact that the elements with the same global numbers should be summed up), one has

$$\{f_q\} = \begin{Bmatrix} 1000 \\ 0 \\ 0 \\ 2000 \\ 0 \\ 0 \\ 1000 \\ 0 \\ 0 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{matrix} \ \text{W/m} \ . \tag{23}$$

The sum of vectors

$$\{f\} = \{f_Q\} + \{f_\alpha\} + \{f_q\} \tag{24}$$

iso

$$\{f\} = \begin{Bmatrix} 250 \\ 500 \\ 250 \\ 500 \\ 1000 \\ 500 \\ 250 \\ 500 \\ 250 \end{Bmatrix} + \begin{Bmatrix} 6 \\ 12 \\ 6 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 1000 \\ 0 \\ 0 \\ 2000 \\ 0 \\ 0 \\ 1000 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1256 \\ 512 \\ 256 \\ 2500 \\ 1000 \\ 500 \\ 1250 \\ 500 \\ 250 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{matrix} \text{ W/m} . \tag{25}$$

Global equation system $[K]\{T\} = \{f\}$ assumes the form

$$\begin{bmatrix} 28.2 & -6.9 & 0 & -7 & -14 & 0 & 0 & 0 & 0 \\ -6.9 & 56.4 & -6.9 & -14 & -14 & -14 & 0 & 0 & 0 \\ 0 & -6.9 & 28.2 & 0 & -14 & -7 & 0 & 0 & 0 \\ -7 & -14 & 0 & 56 & -14 & 0 & -7 & -14 & 0 \\ -14 & -14 & -14 & -14 & 112 & -14 & -14 & -14 & -14 \\ 0 & -14 & -7 & 0 & -14 & 56 & 0 & -14 & -7 \\ 0 & 0 & 0 & -7 & -14 & 0 & 28 & -7 & 0 \\ 0 & 0 & 0 & -14 & -14 & -14 & -7 & 56 & -7 \\ 0 & 0 & 0 & 0 & -14 & -7 & 0 & -7 & 28 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{Bmatrix} = \begin{Bmatrix} 1256 \\ 512 \\ 256 \\ 2500 \\ 1000 \\ 500 \\ 1250 \\ 500 \\ 250 \end{Bmatrix} . \tag{26}$$

The equation system (26) will be transformed with the boundary condition (2), from which it follows that $T_3 = T_6 = T_9 = 100°C$

$$\begin{bmatrix} 28.2 & -6.9 & 0 & -7 & -14 & 0 & 0 & 0 & 0 \\ -6.9 & 56.4 & 0 & -14 & -14 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7 & -14 & 0 & 56 & -14 & 0 & -7 & -14 & 0 \\ -14 & -14 & 0 & -14 & 112 & 0 & -14 & -14 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 & -14 & 0 & 28 & -7 & 0 \\ 0 & 0 & 0 & -14 & -14 & 0 & -7 & 56 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{Bmatrix} = \begin{Bmatrix} 1256 \\ 2602 \\ 100 \\ 2500 \\ 5200 \\ 100 \\ 1250 \\ 2600 \\ 100 \end{Bmatrix} . \tag{27}$$

Global equation system (27) was solved using the Gauss elimination method and the following was obtained:

$$\begin{aligned} &T_1 = 238.63°\text{C}, && T_2 = 180.20°\text{C}, \\ &T_3 = 100°\text{C}, && T_4 = 240.68°\text{C}, \\ &T_5 = 181.80°\text{C}, && T_6 = 100°\text{C}, \\ &T_7 = 241.27°\text{C}, && T_8 = 182.21°\text{C}, \\ &T_9 = 100°\text{C}. \end{aligned} \quad (28)$$

Temperature distribution was also calculated by means of ANSYS program, while the region was divided into 2500 elements. The following temperatures were obtained for nodes, which correspond to nodes 1-9:

$$\begin{aligned} &T_1 = 238.51°\text{C}, && T_2 = 180.15°\text{C}, \\ &T_3 = 100°\text{C}, && T_4 = 240.53°\text{C}, \\ &T_5 = 181.67°\text{C}, && T_6 = 100°\text{C}, \\ &T_7 = 241.09°\text{C}, && T_8 = 182.08°\text{C}, \\ &T_9 = 100°\text{C}. \end{aligned} \quad (29)$$

The isotherm map is shown in Fig. 11.26.

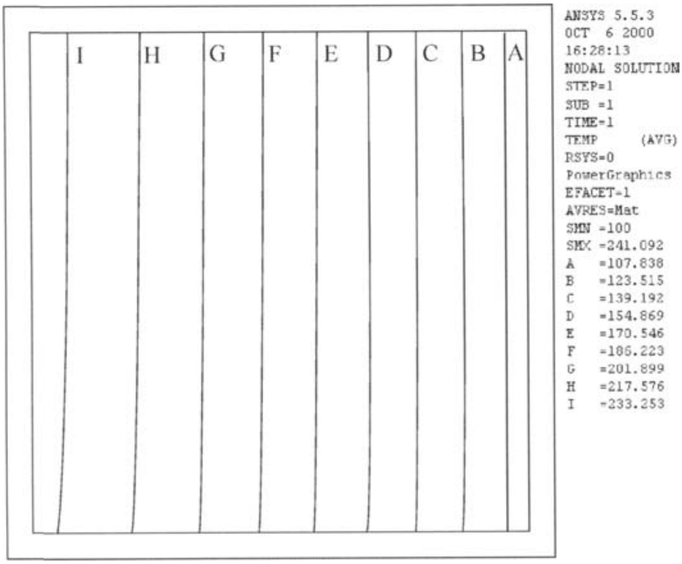


Fig. 11.26. Calculation results; calculations conducted by means of the ANSYS program

Exercise 11.19 Determining Two-Dimensional Temperature Distribution in a Straight Fin with Constant Thickness by Means of FEM

Determine temperature distribution in a fin by means of FEM. Assume the following values from Ex. 7.3 for the calculation: a = 0.003 m, λ = 50 W/(m·K), α = 100 W/(m^2·K), T_b = 95°C, T_{cz} = 20°C. Determine heat flow at the fin base and fin efficiency.

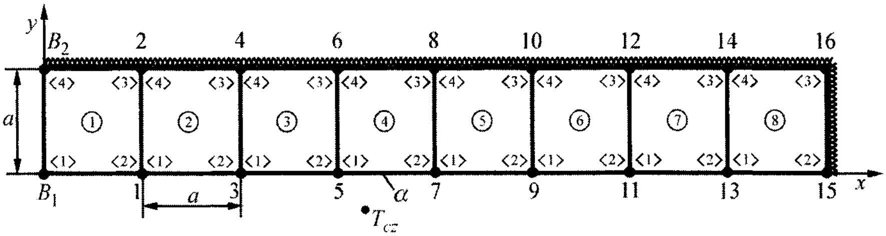


Fig. 11.27. A fin division into finite elements

Solution

Temperature distribution will be determined at the mid-point of the fin due to the symmetry of temperature field with respect to axis x. Table 11.4 contains local and global node numeration. Method II, discussed in Ex. 11.15, will be employed in the construction of the global equation system; the method is based on the summation of weighted residuals for elements with a common node. Due to the fact that the equations for individual nodes have similar structure, the equation for node three will be created and, subsequently, applied to nodes 1, 5, 7, 9, 11 and 13, while the equation for node 4 will be applied to nodes 2, 6, 8, 10, 12 and 14. Separate equations will be created for nodes 15 and 16.

• Node 3

Node 3 is shared by elements ② and ③ (Fig. 11.27). Thermal conduction matrixes $[K_c^2]$ and $[K_c^3]$ will be calculated according to (6) from Ex. 11.11, while matrixes $[K_\alpha^2]$ and $[K_\alpha^3]$ according to (7) from Ex. 11.12. Matrixes $[K_c^2]$ and $[K_\alpha^2]$ are expressed as follow:

$$\begin{matrix} & \langle 1\rangle & \langle 2\rangle & \langle 3\rangle & \langle 4\rangle & \\ \left[K_c^2\right]=\dfrac{\lambda}{6} & \left[\begin{matrix} 4 \\ -1 \\ -2 \\ -1 \end{matrix}\right. & \begin{matrix} -1 \\ 4 \\ -1 \\ -2 \end{matrix} & \begin{matrix} -2 \\ -1 \\ 4 \\ -1 \end{matrix} & \left.\begin{matrix} -1 \\ -2 \\ -1 \\ 4 \end{matrix}\right] & \begin{matrix} \langle 1\rangle \\ \langle 2\rangle \\ \langle 3\rangle \\ \langle 4\rangle \end{matrix} \end{matrix}, \quad (1)$$

Table 11.4. Local and global node numeration

Element No.	Local Node No.	Global Node No.
①	⟨1⟩	B_1
	⟨2⟩	1
	⟨3⟩	2
	⟨4⟩	B_2
②	⟨1⟩	1
	⟨2⟩	3
	⟨3⟩	4
	⟨4⟩	2
③	⟨1⟩	3
	⟨2⟩	5
	⟨3⟩	6
	⟨4⟩	4
④	⟨1⟩	5
	⟨2⟩	7
	⟨3⟩	8
	⟨4⟩	6
⑤	⟨1⟩	7
	⟨2⟩	9
	⟨3⟩	10
	⟨4⟩	8
⑥	⟨1⟩	9
	⟨2⟩	11
	⟨3⟩	12
	⟨4⟩	10
⑦	⟨1⟩	11
	⟨2⟩	13
	⟨3⟩	14
	⟨4⟩	12
⑧	⟨1⟩	13
	⟨2⟩	15
	⟨3⟩	16
	⟨4⟩	14

$$\begin{array}{c} \quad\quad\quad\quad \langle 1\rangle \ \langle 2\rangle \ \langle 3\rangle \ \langle 4\rangle \\ \left[K_\alpha^2\right]=\dfrac{\alpha a}{6}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{matrix}\langle 1\rangle \\ \langle 2\rangle \\ \langle 3\rangle \\ \langle 4\rangle\end{matrix}. \end{array} \tag{2}$$

Stiffness matrix $[K]$ is obtained by way of summing up matrices (1) and (2) (coefficients with the same indexes are added up)

$$\left[K^2\right]=\left[K_c^2\right]+\left[K_\alpha^2\right]=\begin{matrix} \langle 1\rangle & \langle 2\rangle & \langle 3\rangle & \langle 4\rangle & \\ \end{matrix}$$

$$\left[K^2\right]=\left[K_c^2\right]+\left[K_\alpha^2\right]=\begin{bmatrix} \frac{2\lambda}{3}+\frac{\alpha a}{3} & -\frac{\lambda}{6}+\frac{\alpha a}{6} & -\frac{\lambda}{3} & -\frac{\lambda}{6} \\ -\frac{\lambda}{6}+\frac{\alpha a}{6} & \frac{2\lambda}{3}+\frac{\alpha a}{3} & -\frac{\lambda}{6} & -\frac{\lambda}{3} \\ -\frac{\lambda}{3} & -\frac{\lambda}{6} & \frac{2\lambda}{3} & -\frac{\lambda}{6} \\ -\frac{\lambda}{6} & -\frac{\lambda}{3} & -\frac{\lambda}{6} & \frac{2\lambda}{3} \end{bmatrix}\begin{matrix}\langle 1\rangle \\ \langle 2\rangle \\ \langle 3\rangle \\ \langle 4\rangle\end{matrix}. \quad (3)$$

Matrixes $[K_c^3]$ and $[K_\alpha^3]$ are calculated in a similar way.

$$\begin{matrix} \langle 1\rangle & \langle 2\rangle & \langle 3\rangle & \langle 4\rangle \end{matrix}$$

$$\left[K_c^3\right]=\frac{\lambda}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix}\begin{matrix}\langle 1\rangle \\ \langle 2\rangle \\ \langle 3\rangle \\ \langle 4\rangle\end{matrix}, \quad (4)$$

$$\begin{matrix} \langle 1\rangle & \langle 2\rangle & \langle 3\rangle & \langle 4\rangle \end{matrix}$$

$$\left[K_\alpha^3\right]=\frac{\alpha a}{6}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{matrix}\langle 1\rangle \\ \langle 2\rangle \\ \langle 3\rangle \\ \langle 4\rangle\end{matrix}. \quad (5)$$

Rigidity matrix for element ③ is as follows:

$$\begin{matrix} \langle 1\rangle & \langle 2\rangle & \langle 3\rangle & \langle 4\rangle \end{matrix}$$

$$\left[K^3\right]=\left[K_c^3\right]+\left[K_\alpha^3\right]=\begin{bmatrix} \frac{2\lambda}{3}+\frac{\alpha a}{3} & -\frac{\lambda}{6}+\frac{\alpha a}{6} & -\frac{\lambda}{3} & -\frac{\lambda}{6} \\ -\frac{\lambda}{6}+\frac{\alpha a}{6} & \frac{2\lambda}{3}+\frac{\alpha a}{3} & -\frac{\lambda}{6} & -\frac{\lambda}{3} \\ -\frac{\lambda}{3} & -\frac{\lambda}{6} & \frac{2\lambda}{3} & -\frac{\lambda}{6} \\ -\frac{\lambda}{6} & -\frac{\lambda}{3} & -\frac{\lambda}{6} & \frac{2\lambda}{3} \end{bmatrix}\begin{matrix}\langle 1\rangle \\ \langle 2\rangle \\ \langle 3\rangle \\ \langle 4\rangle\end{matrix}. \quad (6)$$

Next, vectors $\{f_\alpha^2\}$ and $\{f_\alpha^3\}$ will be determined. According to (7) from Ex. 11.14, one has

$$\left\{f_\alpha^2\right\} = \frac{\alpha T_{cz} a}{2}\begin{Bmatrix}1\\1\\0\\0\end{Bmatrix}\begin{matrix}\langle 1\rangle\\\langle 2\rangle\\\langle 3\rangle\\\langle 4\rangle\end{matrix} = \begin{Bmatrix}\alpha T_{cz} a/2\\\alpha T_{cz} a/2\\0\\0\end{Bmatrix}\begin{matrix}\langle 1\rangle\\\langle 2\rangle\\\langle 3\rangle\\\langle 4\rangle\end{matrix}, \tag{7}$$

$$\left\{f_\alpha^3\right\} = \begin{Bmatrix}\alpha T_{cz} a/2\\\alpha T_{cz} a/2\\0\\0\end{Bmatrix}\begin{matrix}\langle 1\rangle\\\langle 2\rangle\\\langle 3\rangle\\\langle 4\rangle\end{matrix}. \tag{8}$$

Node 3, i.e. the node with global number 3, corresponds to local node ⟨2⟩ in element ② and to node ⟨1⟩ in element ③. Therefore, in the equation system for element ②

$$\left[K^2\right]\begin{Bmatrix}T_1\\T_3\\T_4\\T_2\end{Bmatrix} = \begin{Bmatrix}\alpha T_{cz} a/2\\\alpha T_{cz} a/2\\0\\0\end{Bmatrix} \tag{9}$$

only the second equation is taken into consideration; it has the form

$$\left[-\frac{\lambda}{6}+\frac{\alpha a}{6},\ \frac{2\lambda}{3}+\frac{\alpha a}{3},\ -\frac{\lambda}{6},\ -\frac{\lambda}{3}\right]\begin{Bmatrix}T_1\\T_3\\T_4\\T_2\end{Bmatrix} = \frac{\alpha T_{cz} a}{2}, \tag{10}$$

$$\left(-\frac{\lambda}{6}+\frac{\alpha a}{6}\right)T_1 + \left(\frac{2\lambda}{3}+\frac{\alpha a}{3}\right)T_3 - \frac{\lambda}{6}T_4 - \frac{\lambda}{3}T_2 = \frac{\alpha T_{cz} a}{2}. \tag{11}$$

In the equation system for element ③

$$\left[K^3\right]\begin{Bmatrix}T_3\\T_5\\T_6\\T_4\end{Bmatrix} = \begin{Bmatrix}\alpha T_{cz} a/2\\\alpha T_{cz} a/2\\0\\0\end{Bmatrix} \tag{12}$$

only the first equation is taken into consideration; it has the form

$$\left[\frac{2\lambda}{3}+\frac{\alpha a}{3},\ -\frac{\lambda}{6}+\frac{\alpha a}{6},\ -\frac{\lambda}{3},\ -\frac{\lambda}{6}\right]\begin{Bmatrix}T_3\\T_5\\T_6\\T_4\end{Bmatrix} = \frac{\alpha T_{cz} a}{2}, \tag{13}$$

$$\left(\frac{2\lambda}{3}+\frac{\alpha a}{3}\right)T_3+\left(-\frac{\lambda}{6}+\frac{\alpha a}{6}\right)T_5-\frac{\lambda}{3}T_6-\frac{\lambda}{6}T_4=\frac{\alpha T_{cz}a}{2}. \tag{14}$$

Once (11) and (14) are added, the equation for node 3 is obtained

$$\left(-\frac{\lambda}{6}+\frac{\alpha a}{6}\right)T_1-\frac{\lambda}{3}T_2+2\left(\frac{2\lambda}{3}+\frac{\alpha a}{3}\right)T_3-\frac{\lambda}{3}T_4+\left(-\frac{\lambda}{6}+\frac{\alpha a}{6}\right)T_5-\frac{\lambda}{3}T_6= \tag{15}$$
$$=\alpha T_{cz}a.$$

Analogically, one can write an equation for node 1

$$\left(-\frac{\lambda}{6}+\frac{\alpha a}{6}\right)T_b-\frac{\lambda}{3}T_b+2\left(\frac{2\lambda}{3}+\frac{\alpha a}{3}\right)T_1-\frac{\lambda}{3}T_2+\left(-\frac{\lambda}{6}+\frac{\alpha a}{6}\right)T_3-\frac{\lambda}{3}T_4= \tag{16}$$
$$=\alpha T_{cz}a,$$

which results in

$$2\left(\frac{2\lambda}{3}+\frac{\alpha a}{3}\right)T_1-\frac{\lambda}{3}T_2+\left(-\frac{\lambda}{6}+\frac{\alpha a}{6}\right)T_3-\frac{\lambda}{3}T_4=\left(\frac{\lambda}{2}-\frac{\alpha a}{6}\right)T_b+\alpha T_{cz}a. \tag{17}$$

- Nodes 5, 7, 9, 11, 13

Equation (15) can be applied to nodes that lie on the surface, which remains in contact with the medium; that excludes, however, node 15

$$\left(-\frac{\lambda}{6}+\frac{\alpha a}{6}\right)T_{i-2}-\frac{\lambda}{3}T_{i-1}+2\left(\frac{2\lambda}{3}+\frac{\alpha a}{3}\right)T_i-\frac{\lambda}{3}T_{i+1}+\left(-\frac{\lambda}{6}+\frac{\alpha a}{6}\right)T_{i+2}- \tag{18}$$
$$-\frac{\lambda}{3}T_{i+3}=\alpha a T_{cz};\qquad i=3,\ 5,\ 7,\ 9,\ 11,\ 13.$$

- Node 15

The equation for node 15 is obtained in a similar way as the equation for node 3 in element ②. Analogically to (10), one can obtain the equation for node 15

$$\left[-\frac{\lambda}{6}+\frac{\alpha a}{6},\ \frac{2\lambda}{3}+\frac{\alpha a}{3},\ -\frac{\lambda}{6},\ -\frac{\lambda}{3}\right]\begin{Bmatrix}T_{13}\\T_{15}\\T_{16}\\T_{14}\end{Bmatrix}=\frac{\alpha T_{cz}a}{2}, \tag{19}$$

from which, one obtains

$$\left(-\frac{\lambda}{6}+\frac{\alpha a}{6}\right)T_{13}+\left(\frac{2\lambda}{3}+\frac{\alpha a}{3}\right)T_{15}-\frac{\lambda}{6}T_{16}-\frac{\lambda}{3}T_{14}=\frac{\alpha T_{cz}a}{2}. \tag{20}$$

• Node 4

Convection heat transfer does not occure on the surface of elements ② and ③. The algebraic equation system for element ② has the form

$$\left[K_c^2\right]\begin{Bmatrix} T_1 \\ T_3 \\ T_4 \\ T_2 \end{Bmatrix}=0, \tag{21}$$

where $[K_c^2]$ is formulated in (1).

The equation for node 4 (the third equation in the system (21)) has the form

$$\frac{\lambda}{6}\left[-2, \quad -1, \quad 4, \quad -1\right]\begin{Bmatrix} T_1 \\ T_3 \\ T_4 \\ T_2 \end{Bmatrix}=0, \tag{22}$$

$$-\frac{\lambda}{3}T_1-\frac{\lambda}{6}T_3+\frac{2\lambda}{3}T_4-\frac{\lambda}{6}T_2=0. \tag{23}$$

The equation system for element ③ has the form

$$\left[K_c^3\right]\begin{Bmatrix} T_3 \\ T_5 \\ T_6 \\ T_4 \end{Bmatrix}=0, \tag{24}$$

where $[K_c^2]$ is formulated in (4).

The equation for node 4 (the fourth equation in the system (24)) has the form

$$\frac{\lambda}{6}\left[-1, \quad -2, \quad -1, \quad 4\right]\begin{Bmatrix} T_3 \\ T_5 \\ T_6 \\ T_4 \end{Bmatrix}=0, \tag{25}$$

from which, one obtains

$$-\frac{\lambda}{6}T_3-\frac{\lambda}{3}T_5-\frac{\lambda}{6}T_6+\frac{2\lambda}{3}T_4=0. \tag{26}$$

Once (23) and (26) are added together, an algebraic equation (heat balance) for node 4 is obtained

$$-\frac{\lambda}{3}T_1-\frac{\lambda}{6}T_2-\frac{\lambda}{3}T_3+\frac{4\lambda}{3}T_4-\frac{\lambda}{3}T_5-\frac{\lambda}{6}T_6=0. \quad (27)$$

Equation for node 2 can be written analogically (indexes are reduced by 2 and the boundary conditions are accounted for)

$$-\frac{\lambda}{3}T_b-\frac{\lambda}{6}T_b-\frac{\lambda}{3}T_1+\frac{4\lambda}{3}T_2-\frac{\lambda}{3}T_3-\frac{\lambda}{6}T_4=0; \quad (28)$$

hence, one gets

$$-\frac{\lambda}{3}T_1+\frac{4\lambda}{3}T_2-\frac{\lambda}{3}T_3-\frac{\lambda}{6}T_4=\frac{\lambda}{2}T_b. \quad (29)$$

• Nodes 6, 8, 10, 12, 14

One can write a general equation for nodes 4, 6, 8, 10, 12 and 14 on the basis of (27)

$$-\frac{\lambda}{3}T_{i-3}-\frac{\lambda}{6}T_{i-2}-\frac{\lambda}{3}T_{i-1}+\frac{4\lambda}{3}T_i-\frac{\lambda}{3}T_{i+1}-\frac{\lambda}{6}T_{i+2}=0, \quad (30)$$
$$i=4,\ 6,\ 8,\ 10,\ 12,\ 14.$$

The equation system for element ⑧ has the form

$$\left[K_c^2\right]\begin{Bmatrix}T_{13}\\T_{15}\\T_{16}\\T_{14}\end{Bmatrix}=0, \quad (31)$$

since $[K_c^8] = [K_c^2]$. $[K_c^2]$ is formulated in (1).

• Node 16

The equation for node 16 (the third equation in the system (31)) has the form

$$\frac{\lambda}{6}[-2,\ -1,\ 4,\ -1]\begin{Bmatrix}T_{13}\\T_{15}\\T_{16}\\T_{14}\end{Bmatrix}=0, \quad (32)$$

hence, one has

$$-\frac{\lambda}{3}T_{13}-\frac{\lambda}{6}T_{15}+\frac{2\lambda}{3}T_{16}-\frac{\lambda}{6}T_{14}=0. \quad (33)$$

The equation system made of (17), (18), (20), (29), (30) and (33) defines node temperature distribution. Such system has the form

$$2\left(\frac{2\lambda}{3}+\frac{\alpha a}{3}\right)T_1-\frac{\lambda}{3}T_2+\left(-\frac{\lambda}{6}+\frac{\alpha a}{6}\right)T_3-\frac{\lambda}{3}T_4=\left(\frac{\lambda}{2}-\frac{\alpha a}{6}\right)T_b+\alpha T_{cz}a,$$
$$\left(-\frac{\lambda}{6}+\frac{\alpha a}{6}\right)T_{i-2}-\frac{\lambda}{3}T_{i-1}+2\left(\frac{2\lambda}{3}+\frac{\alpha a}{3}\right)T_i-\frac{\lambda}{3}T_{i+1}+\left(-\frac{\lambda}{6}+\frac{\alpha a}{6}\right)T_{i+2}-$$
$$-\frac{\lambda}{3}T_{i+3}=\alpha a T_{cz},\quad i=3,5,7,9,11,13,$$
$$\left(-\frac{\lambda}{6}+\frac{\alpha a}{6}\right)T_{13}+\left(\frac{2\lambda}{3}+\frac{\alpha a}{3}\right)T_{15}-\frac{\lambda}{6}T_{16}-\frac{\lambda}{3}T_{14}=\frac{\alpha T_{cz}a}{2} \tag{34}$$
$$-\frac{\lambda}{3}T_1+\frac{4\lambda}{3}T_2-\frac{\lambda}{3}T_3-\frac{\lambda}{6}T_4=\frac{\lambda}{2}T_b,$$
$$-\frac{\lambda}{3}T_{i-3}-\frac{\lambda}{6}T_{i-2}-\frac{\lambda}{3}T_{i-1}+\frac{4\lambda}{3}T_i-\frac{\lambda}{3}T_{i+1}-\frac{\lambda}{6}T_{i+2}=0,$$
$$i=4,6,8,10,12,14$$
$$-\frac{\lambda}{3}T_{13}-\frac{\lambda}{6}T_{15}+\frac{2\lambda}{3}T_{16}-\frac{\lambda}{6}T_{14}=0.$$

The equation system (34) will be solved using the Gauss-Seidel method; due to this reason, it will be written in the form

$$T_1=\frac{1}{8+4Bi}\left[2T_2+(1-Bi)T_3+2T_4+(3-Bi)T_b+6BiT_{cz}\right]$$
$$T_i=\frac{1}{8+4Bi}\left[(1-Bi)T_{i-2}+2T_{i-1}+2T_{i+1}+(3-Bi)T_{i+2}+2T_{i+3}+6BiT_{cz}\right], \tag{35}$$
$$i=3,5,7,9,11,13$$

$$T_{15}=\frac{1}{4+2Bi}\left[(1-Bi)T_{13}+2T_{14}+T_{16}+3BiT_{cz}\right]$$
$$T_2=\frac{1}{8}\left(2T_1+2T_3+T_4+3T_b\right)$$
$$T_i=\frac{1}{8}\left(2T_{i-3}+T_{i-2}+2T_{i-1}+2T_{i+1}+T_{i+2}\right),\qquad i=4,6,8,10,12,14 \tag{35}$$
$$T_{16}=\frac{1}{4}\left(2T_{13}+T_{14}+T_{15}\right),$$

where $Bi=\alpha a/\lambda$ is the Biot number.

This system will be solved using the same data that was given in Ex.7.3: a = 0.003m, λ = 50 W/(m·K), α = 100 W/(m^2·K), T_b = 95°C, T_{cz} = 20°C, $Bi=\alpha a/\lambda$ = (100·0.003)/50 = 0.006. If we assume that calculation tolerance δ = 0.00001 K in Gauss-Seidel method is the solution to system (35), we will obtain the results, which are shown in Table 11.5.

Table 11.5. Temperature in control volume nodes shown in Fig. 11.27

Node No.	Temperature		Node No.	Temperature	
	FEM	Analytical Method		FEM	Analytical Method
1	91.90	91.88	9	84.20	84.18
2	92.14	92.09	10	84.39	84.37
3	89.38	89.34	11	83.26	83.23
4	89.58	89.55	12	83.45	83.42
5	87.25	87.22	13	82.70	82.67
6	87.45	87.42	14	82.88	82.86
7	85.53	85.50	15	82.51	82.48
8	85.73	85.70	16	82.70	82.67

Next, we will calculate heat flow

$$\dot{Q} = 2\int_0^a \left(-\lambda \frac{\partial T^1}{\partial x} \right)\Bigg|_{x=0} dy = -2\lambda \int_0^a \frac{\partial T^1}{\partial x}\Bigg|_{x=0} dy, \tag{36}$$

where $T^1(x, y)$ stands for the temperature distribution in element ①, formulated as

$$T^1(x,y) = N_1^1 T_b + N_2^1 T_1 + N_3^1 T_2 + N_4^1 T_b, \tag{37}$$

where,

$$N_1^1 = \left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{a}\right),$$
$$N_2^1 = \frac{x}{a}\left(1 - \frac{y}{a}\right), \tag{38}$$
$$N_3^1 = \frac{xy}{a^2}, \quad N_4^1 = \frac{y}{a}\left(1 - \frac{x}{a}\right).$$

Derivative $\partial T^1/\partial x$ is

$$\frac{\partial T^1}{\partial x} = \left(\frac{\partial N_1^1}{\partial x} + \frac{\partial N_4^1}{\partial x} \right) T_b + \frac{\partial N_2^1}{\partial x} T_1 + \frac{\partial N_3^1}{\partial x} T_2, \tag{39}$$

where

$$\frac{\partial N_1^1}{\partial x} = -\frac{1}{a}\left(1 - \frac{y}{a}\right), \qquad \frac{\partial N_4^1}{\partial x} = -\frac{y}{a^2},$$
$$\frac{\partial N_2^1}{\partial x} = \frac{1}{a}\left(1 - \frac{y}{a}\right), \qquad \frac{\partial N_3^1}{\partial x} = \frac{y}{a^2}. \tag{40}$$

By substituting (40) into (39), one obtains

$$\frac{\partial T^1}{\partial x}=\frac{T_1-T_b}{a}+\frac{T_2-T_1}{a^2}y, \tag{41}$$

while after substitution of (41) into (36) and integration, one has

$$\dot{Q}=-2\lambda\left[T_1-T_b+\frac{1}{2}(T_2-T_1)\right]=2\lambda\left(T_b-\frac{T_1+T_2}{2}\right)\text{W/m}. \tag{42}$$

Maximum heat flow given off by an isothermal fin with the base temperature T_b is

$$\dot{Q}_{\max}=2\cdot 8\alpha a\left(T_b-T_{cz}\right)=16\alpha a\left(T_b-T_{cz}\right). \tag{43}$$

Fin efficiency is defined as

$$\eta=\frac{\dot{Q}}{\dot{Q}_{\max}}=\frac{2\lambda\left(T_b-\frac{T_1+T_2}{2}\right)}{16\alpha a\left(T_b-T_{cz}\right)}=\frac{1}{8Bi}\frac{T_b-\frac{T_1+T_2}{2}}{T_b-T_{cz}}, \tag{44}$$

where $Bi=\alpha a/\lambda$.

After substitution of the numerical values, one obtains

$$\eta=\frac{1}{8\cdot 0.006}\cdot\frac{95-\frac{91.90+92.14}{2}}{95-20}=0.828. \tag{45}$$

As one can see, the determined fin efficiency differs from the value $\eta_e=0.887$ obtained by means of the analytical formula (Ex. 7.3). Relative error is at

$$\Delta\eta=\frac{\eta-\eta_e}{\eta_e}\cdot 100\%=\frac{0.828-0.887}{0.887}\cdot 100\%=-6.6\%. \tag{46}$$

Rather large error $\Delta\eta$ arises from the approximation of medium temperature gradient within the width of fin thickness by means of difference quotient

$$\left.\frac{\partial \bar{T}}{\partial x}\right|_{x=0}=\frac{1}{a}\int_0^a\left.\frac{\partial T}{\partial x}\right|_{x=0}dy=\frac{1}{a}\left(T_b-\frac{T_1+T_2}{2}\right)$$

with an accuracy of 1st order.

One should add that heat flow $\dot{Q}$ at the fin base can be also calculated from formula

$$\dot{Q}=2a\dot{q}_b, \tag{47}$$

where $\dot{q}_b$ is the heat flux at the fin base. Aside from the given fin base temperature, assigned at points B_1 and B_2 (Fig. 11.27), temperatures in nodes 1 and 2 (global numeration) are known from the FEM calculations. The equation system for element ① has the form

$$\left[K_c^1\right]\begin{Bmatrix} T_b \\ T_1 \\ T_2 \\ T_b \end{Bmatrix} = \left\{f_q\right\}, \tag{48}$$

where $\{f_q\}$ follows from the heat flux $\dot{q}_b$ assigned at the fin-base. Because $\dot{q}_b$ is assigned on the side ⟨4⟩-⟨1⟩ of element ①, vector $\{f_q\}$ has the form then

$$\left\{f_q\right\} = \begin{Bmatrix} \dot{q}_b a/2 \\ 0 \\ 0 \\ \dot{q}_b a/2 \end{Bmatrix}. \tag{49}$$

The first equation in the system (48) has the form

$$\frac{\lambda}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \end{bmatrix}\begin{Bmatrix} T_b \\ T_1 \\ T_2 \\ T_b \end{Bmatrix} = \frac{\dot{q}_b a}{2}, \tag{50}$$

hence, we have

$$\frac{4\lambda}{6}T_b - \frac{\lambda}{6}T_1 - \frac{2\lambda}{6}T_2 - \frac{\lambda}{6}T_b = \frac{\dot{q}_b a}{2}. \tag{51}$$

Heat flux $\dot{q}_b$ at point B_1 determined from (51) is

$$\dot{q}_b = \frac{\lambda}{a}\left(T_b - \frac{T_1 + 2T_2}{3}\right). \tag{52}$$

The fourth equation in the system (49) has the form

$$\frac{\lambda}{6}\begin{bmatrix} -1 & -2 & -1 & 4 \end{bmatrix}\begin{Bmatrix} T_b \\ T_1 \\ T_2 \\ T_b \end{Bmatrix} = \frac{\dot{q}_b a}{2}, \tag{53}$$

hence, we obtain

$$-\frac{\lambda}{6}T_b-\frac{2\lambda}{6}T_1-\frac{\lambda}{6}T_2+\frac{4\lambda}{6}T_b=\frac{\dot{q}_b a}{2}. \tag{54}$$

Heat flux $\dot{q}_b$ at point B_2 determined from (54) is

$$\dot{q}_b=\frac{\lambda}{a}\left(T_b-\frac{2T_1+T_2}{3}\right). \tag{55}$$

Arithmetic average of the heat flux in nodes B_1 and B_2 given by (52) and (55) is

$$\dot{q}_b=\frac{\lambda}{a}\left(T_b-\frac{T_1+T_2}{2}\right). \tag{56}$$

By substituting (56) into (47), (42) is obtained. Both methods for calculating heat flux give identical results.

In order to improve accuracy, fin efficiency will be calculated using a different method.

Heat flow given off by the fin can be expressed in the following way:

$$\dot{Q}=2\alpha a\left[\frac{1}{2}(T_b-T_{cz})+(T_1-T_{cz})+(T_3-T_{cz})+(T_5-T_{cz})+(T_7-T_{cz})+\right.$$
$$\left.+(T_9-T_{cz})+(T_{11}-T_{cz})+(T_{13}-T_{cz})+\frac{1}{2}(T_{15}-T_{cz})\right], \tag{57}$$

$$\dot{Q}=2\alpha\Delta x\left[\frac{1}{2}T_b+T_1+T_3+T_5+T_7+T_9+T_{11}+T_{13}+\frac{1}{2}T_{15}-8T_{cz}\right], \tag{58}$$

$$\dot{Q}=2\cdot 100\cdot 0.003\left(\frac{95}{2}+91.90+89.38+87.25+85.53+\right.$$
$$\left.+84.20+83.26+82.70+\frac{82.51}{2}-8\cdot 20\right)=319.785\ \text{W/m}.$$

Fin efficiency determined by means of FEM is

$$\eta=\frac{\dot{Q}}{\dot{Q}_{\max}}=\frac{319.785}{360}=0.888. \tag{59}$$

Relative error from the determination of efficiency is

$$\Delta\eta=\frac{\eta-\eta_e}{\eta_e}\cdot 100\%=\frac{0.888-0.887}{0.887}\cdot 100\%=0.112\%,$$

This method, therefore, is much more accurate than the earlier presented method in which $\dot{Q}$ is determined from (42).

Exercise 11.20 Determining Two-Dimensional Temperature Distribution by Means of FEM in a Straight Fin with Constant Thickness (ANSYS Program)

Determine temperature distribution and efficiency of a fin presented in Fig. 11.28. For the calculation, adopt the values from Ex. 7.3: $w = 0.003$ m, l = 0.024m, α = 100 W/(m·K)^2, T_b = 95°C, T_{cz} = 20°C, λ = 50 W/(m·K). Calculate fin efficiency for cases a) and b) presented in Fig. 11.28, i.e. when the fin tip is thermally insulated and heat transfer occurs at the tip. Furthermore, for the case a) determine fin efficiency by means of the analytical formula; make use of the results obtained in Ex. 7.3.

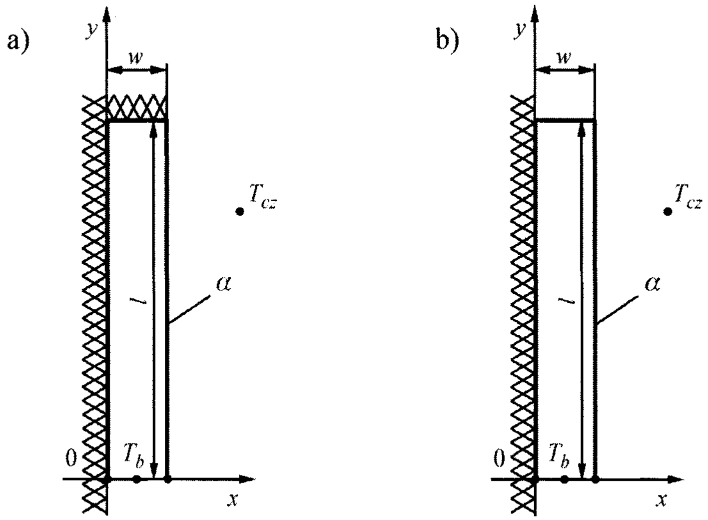


Fig. 11.28. Diagram of a fin with constant thickness: (**a**) fin tip thermally insulated, (**b**) heat transfer occurs between the fin and surrounding at the tip

Solution

Calculations were carried out by means of the ANSYS program, used for calculating by FEM. Half of the fin cross-section was divided into 288 elements. Temperature was calculated in 343 nodes (Fig. 11.29). In a case when the fin tip is thermally insulated (Fig. 11.28a), temperatures of the tip calculated by means of the analytical formulas are (Ex. 7.3)

$$T(0, l) = 82.67°\text{C} \qquad T(w, l) = 82.48°\text{C}.$$

Corresponding approximate temperatures obtained by means of FEM are

$$T(0, l) = 82.67°\text{C} \qquad T(w, l) = 82.49°\text{C}.$$

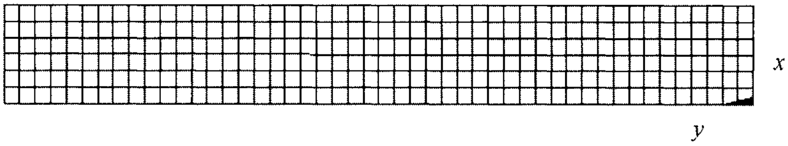


Fig. 11.29. Division of the half of fin cross-section into finite elements

The results obtained by means of the analytical solution and FEM are in a very good agreement.

Fin efficiency will be calculated from formula

$$\eta = \frac{\dot{Q}}{\dot{Q}_{\max}}, \tag{1}$$

where $\dot{Q}$ is the fin-to-surroundings transferred heat flow, which is formulated as

$$\dot{Q} = 2 \cdot \bar{\dot{q}}_x \Big|_{x=0} w\,. \tag{2}$$

In the case of the analytical solution, the mean heat flux at the fin base $\bar{\dot{q}}_x \Big|_{x=0}$ calculated by means of (7) from Ex. 7.3 is at $\bar{\dot{q}}_x \Big|_{x=0}$ = 53253 W/m^2. Therefore,

$$\dot{Q} = 2 \cdot 53253 \cdot 0.003 = 319.52\,\text{W/m}\,. \tag{3}$$

Maximum heat flow $\dot{Q}_{\max}$, i.e. heat flow transferred by an isothermal fin with fin base temperature T_b is formulated as

$$\dot{Q}_{\max} = 2\alpha l\left(T_b - T_{cz}\right) = 2 \cdot 100 \cdot 0.024(95-20) = 360\ \text{W/m}\,. \tag{4}$$

Fin efficiency calculated by means of the analytical solution is

$$\eta = \frac{\dot{Q}}{\dot{Q}_{\max}} = \frac{319.52}{360} = 0.8876\,. \tag{5}$$

Heat flow at the fin base determined by means of FEM for the fin shown in Fig. 11.28a is

$$\dot{Q} = \dot{Q}_b = 2 \cdot 157.88 = 315.76\ \text{W/m}\,. \tag{6}$$

Fin efficiency calculated by means of FEM is

$$\eta = \frac{\dot{Q}}{\dot{Q}_{\max}} = \frac{315.76}{360} = 0.8771\,. \tag{7}$$

and is very close to the value of $\eta = 0.8876$ obtained by means of the analytical solution.

One can specify the heat flow given off by the fin, if the heat flow transferred by the lateral fin surfaces is determined first.

$$\dot{Q} = 2\int_0^l \alpha\left[T(w,y) - T_{cz}\right]dy = 319.5\ \text{W/m}\ ;$$

Next, efficiency $\eta = 0.8875$ is obtained from (7) and is almost identical to the one obtained from the analytical formula. It is evident, therefore, that node temperatures are more accurately calculated in FEM than the boundary heat flux.

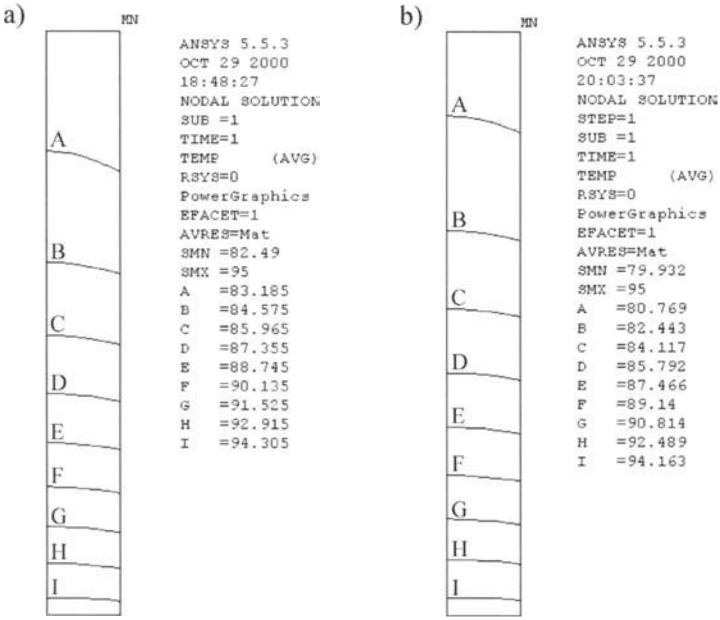


Fig. 11.30. Layout of isotherms in the fin cross-section: (**a**) thermally insulated fin tip; $T(0, l) = 82.67$°C, $T(w, l) = 82.49$°C; (**b**) heat transfer at the fin tip; $T(0, l) = 80.11$°C, $T(w, l) = 79.93$°C

Next, temperature distribution and fin efficiency were calculated, while a consideration was given to a tip heat transfer. The layout of isotherms in the fin cross-section is presented in Fig.11.30b. As one can see, temperatures $T(0, l) = 80.11$°C, $T(w, l) = 79.93$°C are slightly lower than they are in the case when the fin tip is thermally insulated. Also the maximum flow $\dot{Q}_{\max}$ calculated from formula

$$\dot{Q}_{max} = 2\alpha(l + w)(T_b - T_{cz}) = 2 \cdot 100(0.024 + 0.003)(95 - 20) = 405 \text{ W/m}$$

is larger due to the fin tip heat transfer. Heat flow at the fin base calculated by means of FEM is at $\dot{Q} = \dot{Q}_b = 345.84$ W/m. Fin efficiency, therefore, is

$$\eta = \frac{\dot{Q}}{\dot{Q}_{max}} = \frac{345.84}{405} = 0.8539\,.$$

The above value approximates the obtained value, while the heat transfer at the fin tip is neglected (7). In spite of the fact that fin efficiency is low, the fin-diffused heat flow is large, since the heat transfer takes place at the fin tip.

Exercise 11.21 Determining Two-Dimensional Temperature Distribution by Means of FEM in a Hexagonal Fin with Constant Thickness (ANSYS Program)

Determine temperature distribution and fin efficiency by means of FEM. Fin diagram, which results from plate fin division, is presented in Fig. 11.31. For the calculation, assume the values from Ex. 6.20: thickness of the plate-fin t = 0.000115 m, d = 0.00759 m, λ = 165 W/(m·K), α = 40 W/(m^2·K); T_b = 100°C; T_{cz} = 0°C. Compare calculated fin efficiency value with the values obtained in Ex. 6.20.

Solution

Calculations were carried out by means of FEM and ANSYS programs. The analyzed region was divided into 1377 elements (Fig. 11.32). Temperature was determined in 2934 nodes. The layout of isotherms on the fin surface is shown in Fig. 11.33. Note that temperature on the outer fin boundary is non-uniform due to the irregular shape of the fin. Maximum temperature on the outer boundary is T_{max} = 93.162°C, while minimum T_{min}= 90.379°C. Temperatures at characteristic points marked in Fig. 11.31 are, correspondingly: T_1 = 91.56°C, T_2 = 90.47°C, T_3 = 90.38°C, T_4= 100°C, T_5= 100°C .

Fin efficiency will be calculated from formula

$$\eta = \frac{\dot{Q}}{\dot{Q}_{max}}, \tag{1}$$

where $\dot{Q}_{max}$ is the heat flow transferred by the isothermal fin whose base temperature is $T_b = 100°C$; the heat flow is formulated as

$$\dot{Q}_{max} = \alpha A_{\dot{z}} \left(T_b - T_{cz} \right), \tag{2}$$

while (Fig. 6.24, Ex. 6.20)

$$\begin{gathered} A_{\dot{z}} = 2\left(2A_{OAB} + 4A_{OBC} - \frac{\pi d^2}{4} \right) = 2\left(2 \cdot 2.1089 \cdot 10^{-5} + \right. \\ \left. + 4 \cdot 5.6156 \cdot 10^{-5} - 4.52452 \cdot 10^{-5}\right) = 4.43 \cdot 10^{-4}\ \mathrm{m}^2. \end{gathered} \tag{3}$$

Maximum heat flow $\dot{Q}_{max}$ is at

$$\dot{Q}_{max} = 40 \cdot 4.43 \cdot 10^{-4} \cdot (100 - 0) = 1.772\ \mathrm{W}.$$

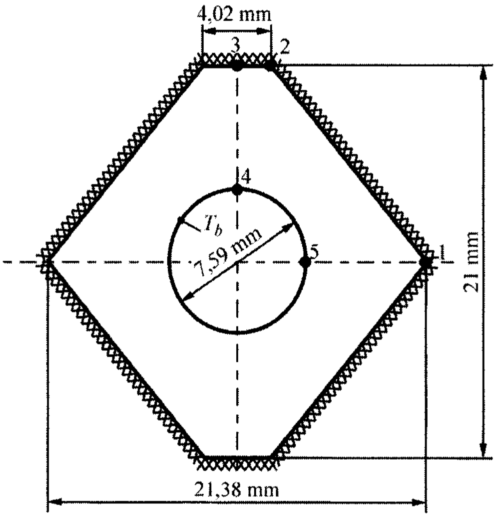


Fig. 11.31. Diagram of a conventional fin after plate-fin division

Fin-transferred heat flow $\dot{Q}$ can be determined from formula

$$\dot{Q} = \int_{A_{\dot{z}}} \alpha \left(T_s - T_{cz} \right) dA = \alpha \left(T_{sr} - T_{cz} \right) A_{\dot{z}}, \tag{4}$$

where $A_{\dot{z}}$ is the lateral area of the fin surface that exchanges heat with surroundings, while T_{sr} is the average temperature of the fin surface formulated as

$$T_{sr} = \frac{\int_{A_{\dot{z}}} T_s dA}{A_{\dot{z}}} \approx \frac{\sum_{e=1}^{N_e} T_{sr,e} \cdot A_e}{A_{\dot{z}}}. \tag{5}$$

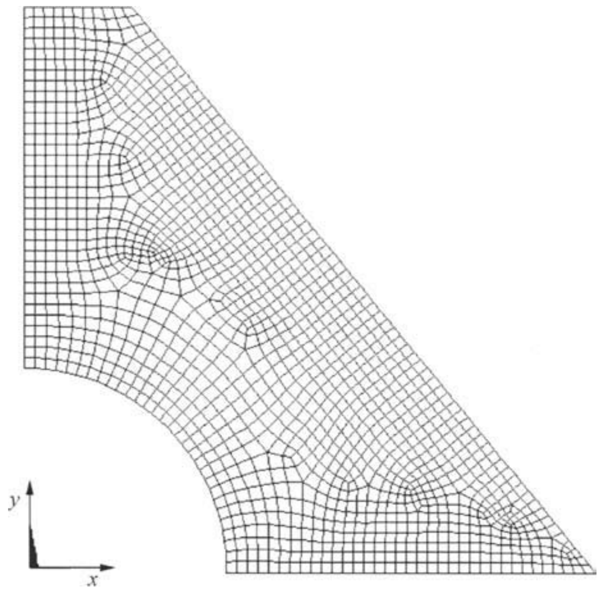


Fig. 11.32. Division of 1/8 of a fin cross-section into finite elements

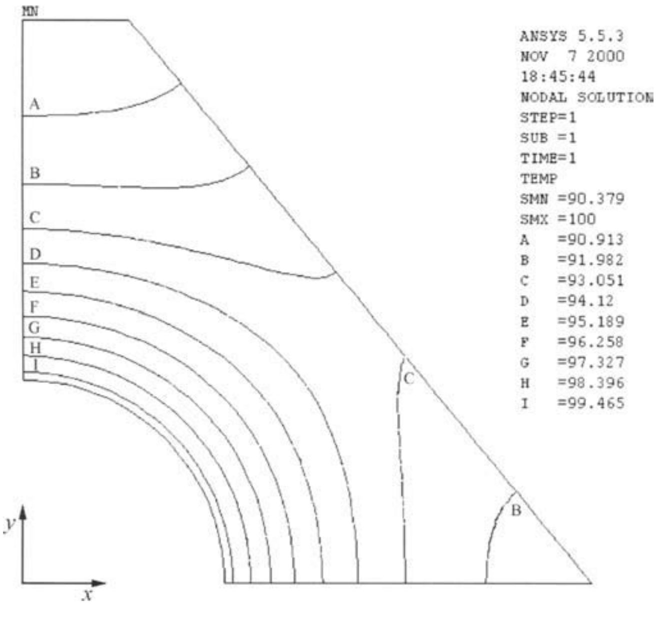


Fig. 11.33. The layout of isotherms on the fin surface

Temperature T_s is the temperature of the fin outer surface, which is in contact with surroundings. Temperature $T_{sr,e}$ is the average temperature of the element's surface exposed to surroundings, while A_e is the element's surface area from the side exposed to surroundings. Symbol N_e stands for the number of elements, which the analyzed region was divided to.

Average temperature determined from (5) by means of the ANSYS program measures T_{sr} = 93.80°C. Such method of determining fin-transferred heat flow $\dot{Q}$ is more accurate than the method that uses formula

$$\dot{Q} = \dot{Q}_b = \pi dt \left(-\lambda \frac{dT}{dr} \right) \Bigg|_{r=d/2}, \tag{6}$$

since an accurate determination of $\lambda(dT/dr)$ in FEM enforces the need to divide the region into a very large number of elements.

Fin-transferred heat flow $\dot{Q}$ determined by means of (4) with the help of the ANSYS program comes to

$$\dot{Q} = 1.6622 \text{ W}.$$

It is a heat flow transferred by lateral fin surfaces. Fin efficiency is at

$$\eta = \frac{\dot{Q}}{\dot{Q}_{\max}} = \frac{1.6622}{1.772} = 0.9380.$$

Calculated efficiency differs insignificantly from the efficiency of the equivalent circular fin η_e = 0.9394 and from the fin efficiency determined by means of the segment method, equal to η_e = 0.9373.

Exercise 11.22 Determining Axisymmetrical Temperature Distribution in a Cylindrical and Conical Pin by Means of FEM (ANSYS Program)

Determine temperature distribution in cylindrical and conical pins, shown in Fig. 11.34, by means of FEM and with the use of the ANSYS program. Pins of this kind are used in gas-fired cast-iron heating boilers with an aim to increase the heat flow transferred from combustion gases to water. Pin dimensions are given in Fig.11.34. Both pins are almost identical in volume. Assume the following values for the calculation: water temperature T_w = 75°C, temperature of combustion gases T_{sp} = 400°C. Thermal conductivity of the material from which the wall and pins are made of is λ =

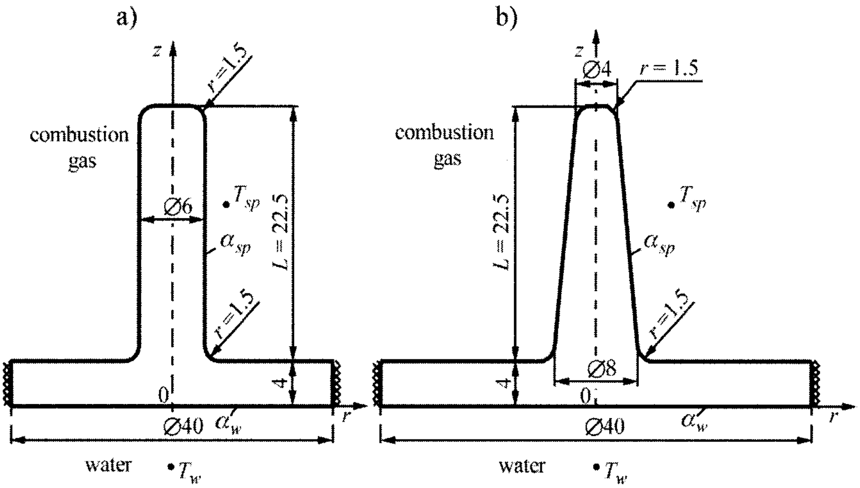


Fig. 11.34. Pinned heating surfaces: (**a**) cylindrical pin, (**b**) conical pin

48 W/(m·K). Heat transfer coefficients on the water and combustion gases side are, respectively $\alpha_w = 1000$ W/(m^2·K) and $\alpha_{sp} = 80$ W/(m^2·K).

Draw the layout of isotherms in the longitudinal cross-section of the pins and determine maximum temperatures. Also calculate pin-transferred heat flows from combustion gases to a boiler wall by determining heat flow at the base of the pins $\dot{Q}_b$ for the coordinate $z = 0.004$ m. Which of the pins ensures a larger flow of transferred heat when maximum temperature is decreased? Calculate temperature distribution and heat flux at the base of the cylindrical pin by means of the formulas obtained when a radial temperature drop is neglected.

Solution

Temperature in the cylindrical pin was determined in 3401 nodes when longitudinal cross-section was divided into 3195 elements (Fig. 11.35a), while in the conical pin in 3647 nodes when longitudinal cross-section was divided into 3439 elements (Fig. 11.35b).

Maximum temperature of the cylindrical pin at $T_{w,\max} = 186.65$°C is larger than the maximum temperature of the conical pin at $T_{s,\max} = 150.747$°C (Fig. 11.36). A lower maximum temperature of the conical pin is due to the fact that the pin has a more advantageous shape, since the surface area of the cross-section becomes larger as the heat flow, which is conducted through the pin's cross-section increases too.

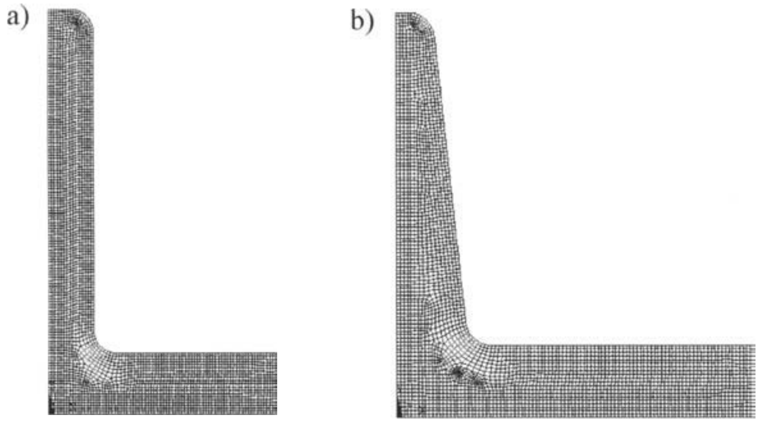


Fig. 11.35. Half of the pin's longitudinal cross-section divided into finite elements: (**a**) cylindrical pin, (**b**) conical pin

In the case of the cylindrical peg, constituent heat flux $\dot{q}_z = -\lambda \partial T/\partial z$ is much larger near the base than anywhere else and that contributes to a large increase in pin temperature within this region. Following that, heat flow at the pin base $z = 0.004$ m were calculated from formula

$$\dot{Q}_b = -2\pi \int_0^{r_b + r} \lambda \frac{\partial T}{\partial z} r dr,$$

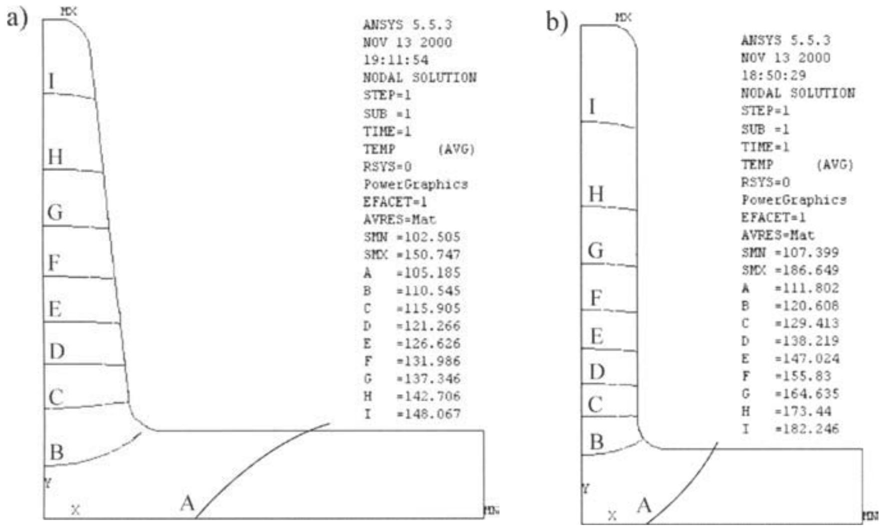


Fig. 11.36. Layout of isotherms on the pin surface: (**a**) cylindrical pin, (**b**) conical pin

where $r_b = 0.003$ m for cylindrical pin, $r_b = 0.004$ m for conical pin and $r = 0.0015$ m.

For cylindrical pin $\dot{Q}_{b,w} = 8.892$ W, while for conical pin $\dot{Q}_{b,s} = 8.456$ W. As one can see, the shape of the conical pin is a very advantageous, since in spite of the fact that the flow of transferred heat is almost the same, the temperature at the tip is much lower than it is at the tip of the cylindrical pin. Temperature of the cylindrical pin can be approximately calculated from the formula below (while disregarding radius at the curves and radial temperature drop):

$$T(z) = T_{sp} + \left(T_b - T_{sp}\right)\frac{\cosh\left[m\left(z - L - 0.004\right)\right]}{\cosh mL}, \tag{1}$$

where T_b is an average temperature at the peg base determined by means of FEM. This temperature measures approximately $T_b \approx 118.469$°C. Parameter m formulated as

$$m = \sqrt{\frac{4\alpha_{sp}}{\lambda d}}, \tag{2}$$

where d is the peg's diameter, measures

$$m = \sqrt{\frac{4 \cdot 80}{48 \cdot 0.006}} = 33.33(3)\ 1/\text{m}.$$

Therefore, the tip temperature of the cylindrical pin is

$$T(L + 0.004) = T_{w,\max} = 400 + (118.469 - 400)\frac{1}{\cosh(33.3333 \cdot 0.0225)} =$$
$$= 182.548°\text{C}.$$

As one can see, this temperature is close to the temperature $T_{w,\max}$ = 186.65°C obtained by means of FEM. Heat flow at the pin base can be calculated from formula

$$\dot{Q} = -\frac{\pi d^2}{4}\lambda\left.\frac{\partial T}{\partial z}\right|_{z=0.004} = -\lambda\frac{\pi d^2}{4}m\left(T_b - T_{sp}\right)\text{tgh}\, mL = -48\frac{\pi 0.006^2}{4}\times$$
$$\times 33.3333(118.469 - 400)\,\text{tgh}(33.3333 \cdot 0.0225) = 8.0893\ \text{W}. \tag{3}$$

The obtained value approximates the value determined by means of FEM, which is equal to $\dot{Q}_{b,w} = 8.892$ W. However, from the calculations carried out with the use of FEM, it is clear that pin-base-temperature is higher than the pin-free wall temperature from the combustion-gases-side, i.e. wall temperature for $z = 0.004$ m at a significant distance from the pin axis for, e.g. $r > 2d$. Due to the application of FEM, one can use the actual dimensions of the pins shape in the calculation, e.g. the curved edges or the two-dimensional character of the temperature field in the pin and the wall, to which the pin is attached.

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