5

P-Vector

A perfect unit vector (i.e., P-vector) can be defined by the density only. This vector represents the orientation of the absolute velocity. Existence of the P-vector and its vertical turning guarantees the validity of the velocity inversion.

5.1 z-Coordinate System

5.1.1 Definition

When the constant ρ and q surfaces intersect (Fig. 5.1), it is true that

$$\nabla\rho \times \nabla q \neq 0. \tag{5.1}$$

A unit vector, called the perfect vector (or P-vector) by (Chu 1995a,b), can be defined by

$$\mathbf{P} = \frac{\nabla\rho \times \nabla q}{|\nabla\rho \times \nabla q|}. \tag{5.2}$$

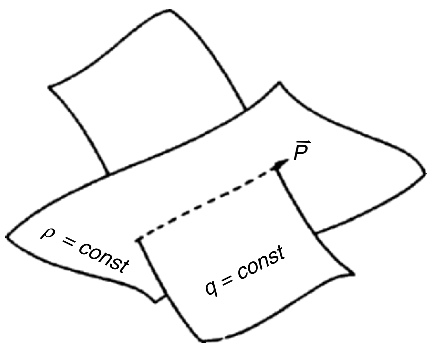

Fig. 5.1. Intersection of surfaces of ρ and q (from Chu 1995a, Marine Technology Society Journal)

The Needler's formula (1.19) becomes

$$\mathbf{V} = \gamma \mathbf{P}, \tag{5.3}$$

where γ is a scalar and its absolute value $|\gamma|$ is the speed. Thus, γ is called the speed parameter. The benefits of using the vector $\mathbf{P}$ are in two-folds: two-step determination of $\mathbf{V}$ and easy inclusion of three necessary conditions. A two-step method was proposed by Chu (1995a) (i.e., the P-vector inverse method): (a) determination of the unit vector $\mathbf{P}$, and (b) determination of the scalar γ from the thermal wind relation.

5.1.2 Horizontal P-Vector Field

Climatological $\mathbf{P}$ field in the North Atlantic Ocean was constructed from the climatological mean potential density field (ρ) which was computed from NOAA WOA annual mean temperature and salinity fields (Fig. 5.2). The data has 33 vertical levels (Table 3.1) with $1° \times 1°$ horizontal resolution,

$$\Delta x = \frac{2\pi R}{360} \cos \phi, \quad \Delta y = \frac{2\pi R}{360}, \tag{5.4}$$

where ϕ is the latitude, R the earth radius.

The vertical grid is the difference between the two vertical levels as illustrated in Table 3.1,

$$\Delta z_k = z_k - z_{k+1}. \tag{5.5}$$

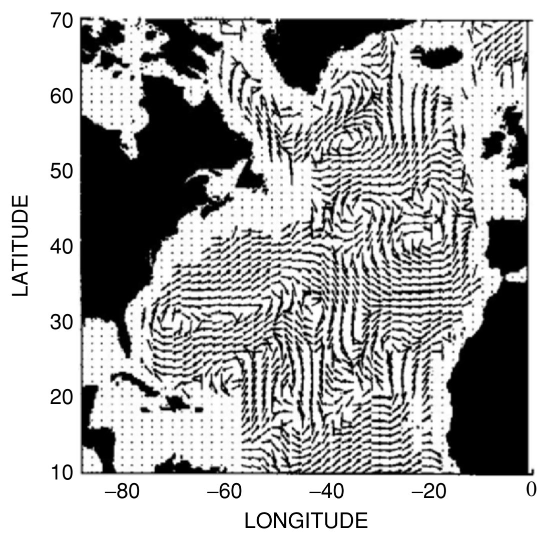

Fig. 5.2. North Atlantic Ocean $\mathbf{P}_h$ field at 150 m depth computed from WOA climatological annual mean T, S data (from Chu 1995a, Marine Technology Society Journal)

The potential vorticity q is calculated from ρ using (1.9), and the vector $\mathbf{P}$ is computed from (ρ, q) using (5.2). Although the unit vector, $\mathbf{P}$, only carries partial information about the velocity field [see (5.3)], it is rather surprising that its horizontal components, $\mathbf{P}_h = (P_x, P_y)$, catches the most important features of the North Atlantic general circulation. For example, at 150 m depth $\mathbf{P}_h$ resembles the classical views (Schmitz and McCartney 1993) quite well. The important features are: (1) anticyclonic subtropical gyre in the western part of the ocean between 20°N and 45°N, (2) recirculation cell on the western side (west of 40°W) of the subtropical gyre, (3) cyclonic–anticyclonic dipole in the areas (30°W − 10°W, about 30°N), and (4) high latitude cyclonic gyre (50°N − 60°N, 20°W − 50°W).

5.1.3 P-Spiral

The three components of the vector $\mathbf{P}$ can be written by [see (5.2)],

$$P_x = \frac{J^{(x)}(\rho, q)}{J_2(\rho, q)}, \quad P_y = \frac{J^{(y)}(\rho, q)}{J_2(\rho, q)}, \quad P_z = \frac{J^{(z)}(\rho, q)}{J_2(\rho, q)}, \tag{5.6}$$

where $J^{(x)}(A, B), J^{(y)}(A, B)$, and $J^{(z)}(A, B)$ are two-dimensional Jacobians,

$$\begin{aligned} J^{(x)}(A, B) &\equiv \begin{vmatrix} \partial A/\partial y & \partial A/\partial z \\ \partial B/\partial y & \partial B/\partial z \end{vmatrix}, \quad J^{(y)}(A, B) \equiv \begin{vmatrix} \partial A/\partial z & \partial A/\partial x \\ \partial B/\partial z & \partial B/\partial x \end{vmatrix}, \\ J^{(z)}(A, B) &\equiv \begin{vmatrix} \partial A/\partial x & \partial A/\partial y \\ \partial B/\partial x & \partial B/\partial y \end{vmatrix}, \quad J_2 = \sqrt{\left[J^{(x)}\right]^2 + \left[J^{(y)}\right]^2 + \left[J^{(z)}\right]^2}. \end{aligned}$$

Let α be the angle between $\mathbf{P}_h$ and the x-axis,

$$P_x = \sin\,\alpha, \quad P_y = \cos\,\alpha. \tag{5.7}$$

Change of α with depth from z_m to z_k,

$$\Delta\;\alpha_{km} = \alpha_k - \alpha_m, \tag{5.8a}$$

is called the $\mathbf{P}$-spiral and

$$\sin(\Delta\;\alpha_{km}) = \begin{vmatrix} P_x^{(k)} & P_x^{(m)} \\ P_y^{(k)} & P_y^{(m)} \end{vmatrix} \tag{5.8b}$$

can be used to identify the $\mathbf{P}$-spiral between two levels. If $\sin(\Delta\;\alpha_{km}) \neq 0$, the $\mathbf{P}$-spiral exists; otherwise the $\mathbf{P}$-spiral does not exist (Fig. 5.3).

The P-vector expression (5.6) is used to explore the existence of the $\mathbf{P}$-spiral. Substituting (5.6) into (5.7) yields,

$$\begin{aligned} \sin\,\alpha &= \frac{J^{(y)}(\rho,q)}{\sqrt{\left[J^{(x)}(\rho,q)\right]^2 + \left[J^{(y)}(\rho,q)\right]^2}}, \\ \cos\,\alpha &= \frac{J^{(x)}(\rho,q)}{\sqrt{\left[J^{(x)}(\rho,q)\right]^2 + \left[J^{(y)}(\rho,q)\right]^2}}. \end{aligned} \tag{5.9}$$

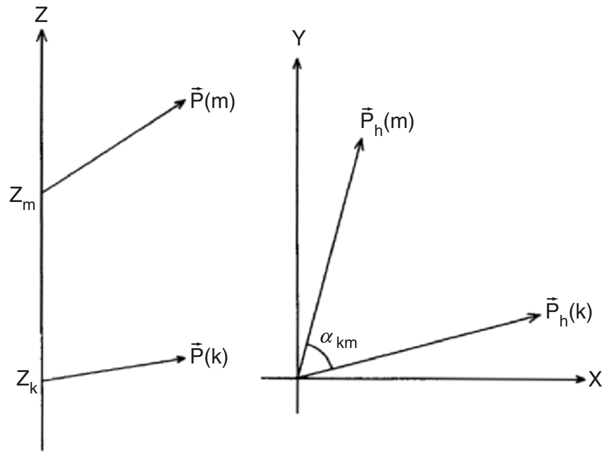

Fig. 5.3. (**a**) **P**-spiral and (**b**) turning angle α_{km} between two levels (from Chu 1995a, Marine Technology Society Journal)

Differentiation of sin α with respect to z gives

$$\frac{\partial(\sin\,\alpha)}{\partial z} = \cos\,\alpha\frac{\partial\alpha}{\partial z}, \tag{5.10}$$

Substituting (5.9) into (5.10) yields

$$\Lambda \equiv \frac{\partial\alpha}{\partial z} = \frac{1}{\left[J^{(x)}\right]^2 + \left[J^{(y)}\right]^2}\left[J^{(x)}\frac{\partial J^{(y)}}{\partial z} - J^{(y)}\frac{\partial J^{(x)}}{\partial z}\right], \tag{5.11}$$

which represents the vertical turning of $\mathbf{P}_h$ when $\partial\alpha/\partial z \neq 0$, the **P**-spiral exists. The parameter Λ is called the P-vector turning parameter.

Since the potential vorticity q is calculated from ρ with $q = f\partial\rho/\partial z$, substitution of this relationship into (5.6) yields

$$J^{(x)}(\rho, q) = fJ^{(x)}(\rho, \rho_z) - \beta\rho_z^2, \quad J^{(y)}(\rho, q) = fJ^{(y)}(\rho, \rho_z), \tag{5.12}$$

$$\frac{\partial}{\partial z}J^{(x)}(\rho, q) = fJ^{(x)}(\rho, \rho_{zz}) - 2\beta\rho_x\rho_{zz}, \tag{5.13a}$$

$$\frac{\partial}{\partial z}J^{(y)}(\rho, q) = fJ^{(y)}(\rho, \rho_{zz}), \tag{5.13b}$$

where the subscript "z" denotes the vertical derivative. Substituting (5.12) and (5.13) into (5.11) yields

$$\Lambda = \frac{\rho_z J_3(\rho, \rho_z, \rho_{zz}) + \beta\left[2\rho_x\rho_{zz}J^{(y)}(\rho, \rho_z) - \rho_z^2 J^{(y)}(\rho, \rho_{zz})\right]/f}{\left[J^{(x)}(\rho, \rho_z) - \beta\rho_z^2/f\right]^2 + \left[J^{(y)}(\rho, \rho_z)\right]^2}, \tag{5.14}$$

where

$$J_3(A,B,C) \equiv \begin{vmatrix} \partial A/\partial x & \partial A/\partial y & \partial A/\partial z \\ \partial B/\partial x & \partial B/\partial x & \partial B/\partial x \\ \partial C/\partial x & \partial C/\partial x & \partial C/\partial x \end{vmatrix}, \tag{5.15}$$

is the three-dimensional Jacobian for (A, B, C).

What are the necessary conditions for the existence of the **P**-spiral? Equation (5.14) shows that

$$\Lambda = 0, \quad \text{for} \quad \rho_z = 0, \tag{5.16}$$

which is the first necessary condition – the **P**-spiral exists when the density is stratified (i.e., $\rho_z \neq 0$). For the f-plane approximation ($\beta = 0$), (5.14) becomes

$$\Lambda = \frac{\rho_z J_3(\rho, \rho_z, \rho_{zz})}{[J^{(x)}(\rho, \rho_z)]^2 + [J^{(y)}(\rho, \rho_z)]^2}, \tag{5.17}$$

which leads to the second necessary condition, the **P**-spiral exists, when ρ, ρ_z, and ρ_{zz} are functionally independent, i.e.,

$$J_3(\rho, \rho_z, \rho_{zz}) \neq 0. \tag{5.18}$$

This also indicates that the β-effect does not play any roles in the existence of **P**-spiral.

5.2 Isopycnal Coordinate System

5.2.1 P-vector

The potential vorticity conservation requires that any water particle should move along q-isoline on the isopycnal surface, i.e., any q-isoline is a trajectory of water particles (Fig. 5.4). For each trajectory, the P-vector is defined as the unit tangential vector,

$$\mathbf{P} = \frac{1}{|\nabla q|}(\mathbf{k} \times \nabla q) = \frac{1}{|\nabla q|}(-\frac{\partial q}{\partial y}\mathbf{i} + \frac{\partial q}{\partial x}\mathbf{j}). \tag{5.19}$$

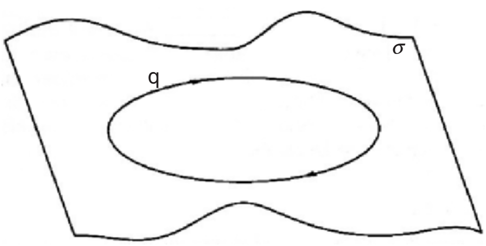

Fig. 5.4. Isoline of potential vorticity (q) is the trajectory on the isopycnal surface (from Chu and Li 2000, Journal of Physical Oceanography)

Equation (5.19) shows that the existence of **P** requires

$$\nabla q \neq 0, \tag{5.20}$$

showing the heterogeneity of potential vorticity on the isopycnal surface. At any point on the isopycnal surface, the vector **P** indicates the tangential direction of the trajectory and therefore, is parallel to the absolute velocity vector,

$$\mathbf{V} = \gamma(x, y, \rho)\mathbf{P}, \tag{5.21}$$

where the absolute value of γ is the speed of the currents,

$$|\gamma| = |\mathbf{V}|\,. \tag{5.22}$$

Now, the determination of the velocity on the isopycnal surface is also divided into two steps: (a) determination of the unit vector **P**, and (b) determination of γ. The positive (negative) values of γ indicate the same (opposite) direction of **V** with **P**.

5.2.2 Absolute Velocity Formula

Substituting (5.21) into the thermal wind relation (4.4) leads to

$$\frac{\partial \gamma}{\partial \rho}\mathbf{P} + \gamma \frac{\partial \mathbf{P}}{\partial \rho} = -\frac{1}{f\rho^2}\mathbf{k} \times \nabla p. \tag{5.23}$$

Vector-product of both sides of (5.23) by the vector **P** gives

$$\gamma(\mathbf{P} \times \frac{\partial \mathbf{P}}{\partial \rho}) = -\frac{1}{f\rho^2}\mathbf{P} \times (\mathbf{k} \times \nabla p). \tag{5.24}$$

Scalar-product of both sides of (5.24) by the vector **P** yields

$$\gamma = \frac{\mathbf{P} \bullet \nabla p}{f\rho^2 \mathbf{P} \bullet (\mathbf{k} \times \partial \mathbf{P}/\partial \rho)}. \tag{5.25}$$

Substituting (5.25) into (5.21) yields the absolute velocity,

$$\mathbf{V} = \frac{\mathbf{P} \bullet \nabla p}{f\rho^2 \mathbf{P} \bullet (\mathbf{k} \times \partial \mathbf{P}/\partial \rho)}\mathbf{P}, \tag{5.26}$$

which is similar to Needler's formula (1.19).

5.2.3 P-Vector Spiral

The angle between **P** and the x-axis on the isopycnal surface that changes with the isopycnal level is called the **P**-spiral. Figure 5.5 shows the turning between the two levels k and m with the turning angle (Chu 2000),

$$\Delta\ \alpha_{km} = \alpha_k - \alpha_m.$$

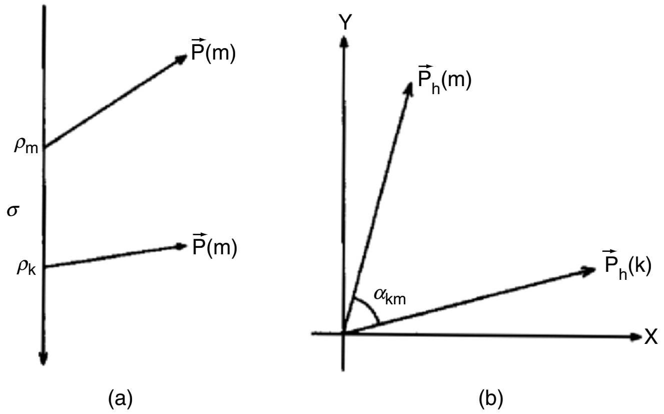

Fig. 5.5. Vertical turning of the P-vector: (**a**) P-vector at two isopycnal levels, and (**b**) turning angle between two isopycnal levels (from Chu 2000, Journal of Oceanography)

From the P-vector definition (5.19), we have

$$\sin\,\alpha = P_x = -\frac{1}{|\nabla q|}\frac{\partial q}{\partial y}, \quad \cos\,\alpha = P_y = \frac{1}{|\nabla q|}\frac{\partial q}{\partial x}. \tag{5.27}$$

Differentiating sin α with respect to ρ and using (5.27) yield the **P** turning parameter

$$\Lambda \equiv \frac{\partial \alpha}{\partial \rho} = -\mathbf{P} \bullet (\mathbf{k} \times \frac{\partial \mathbf{P}}{\partial \rho}). \tag{5.28}$$

Substituting (5.19) into (5.28) yields

$$\Lambda = \frac{J(q, q_\rho)}{|\nabla q|^2}, \tag{5.29}$$

where

$$J(A, B) \equiv \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial B}{\partial x}\frac{\partial A}{\partial y}.$$

Two necessary conditions should be satisfied such that the **P**-spiral exists (i.e., $\partial\alpha/\partial\rho \neq 0$): (a) heterogeneity of potential vorticity on the isopycnal surface (i.e., $\nabla q \neq 0$), and (b) no functional dependence between q and q_ρ.

5.3 Semi-Isopycnal Coordinate

5.3.1 P-Vector

The conservation of potential vorticity is given by [see (4.16)]

$$\mathbf{V}^{(\sigma)} \bullet \nabla_\sigma \left[q^{(\sigma)}\right] = \frac{\partial w^{(\sigma)}}{\partial z}, \quad q^{(\sigma)} \equiv \ln\left[Q^{(\sigma)}\right], \; Q^{(\sigma)} \equiv \frac{f}{h^{(\sigma)}}.$$

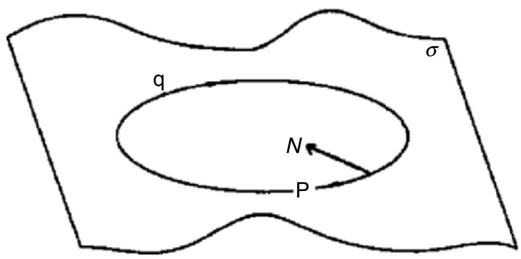

Fig. 5.6. Isoline of potential vorticity (q) is the trajectory of the water particle on the isopycnal surface. Here, (P, N) are the tangential and normal unit vectors of the trajectory (from Chu 2000, Journal of Oceanography)

The two vectors $\mathbf{V}^{(\sigma)}$ and $\nabla_\sigma[q^{(\sigma)}]$ are usually not perpendicular if the vertical convergence of the diapycnal velocity exists ($\partial w^{(\sigma)}/\partial z \neq 0$). The velocity does not parallel the tangential direction of the $q^{(\sigma)}$-isoline. However, the tangential and normal unit vectors of the $q^{(\sigma)}$-isoline (Fig. 5.6) on the potential density surface can be defined by (Fig. 5.6),

$$\mathbf{P} = \frac{1}{\left|\nabla_\sigma q^{(\sigma)}\right|}\left[\frac{\partial q^{(\sigma)}}{\partial y}\mathbf{i} - \frac{\partial q^{(\sigma)}}{\partial x}\mathbf{j}\right], \quad \mathbf{N} = \frac{\nabla_\sigma q^{(\sigma)}}{\left|\nabla_\sigma q^{(\sigma)}\right|}. \tag{5.30}$$

The horizontal velocity at the isopycnal surface can be represented by (Chu 2000)

$$\mathbf{V}^{(\sigma)} = \gamma\mathbf{P} + \frac{\partial w^{(\sigma)}/\partial z}{\left|\nabla_\sigma q^{(\sigma)}\right|}\mathbf{N}. \tag{5.31}$$

Here, γ is the parameter to be determined by the thermal wind relation. Nonzero vertical convergence of diapycnal velocity ($\partial w^{(\sigma)}/\partial z \neq 0$) causes the crossing $q^{(\sigma)}$-isoline motion.

Substitution of $q^{(\sigma)} \equiv \ln\left[f/h^{(\sigma)}\right]$ into (5.30) yields

$$P_x = \left[\frac{\beta}{f} - \frac{\partial \ln(h^{(\sigma)})}{\partial y}\right] / \left[\left(\frac{\beta}{f} - \frac{\partial \ln(h^{(\sigma)})}{\partial y}\right)^2 + \left(\frac{\partial \ln(h^{(\sigma)})}{\partial x}\right)^2\right]^{1/2}, \tag{5.32}$$

$$P_y = \frac{\partial \ln(h^{(\sigma)})}{\partial x} / \left[\left(\frac{\beta}{f} - \frac{\partial \ln(h^{(\sigma)})}{\partial y}\right)^2 + \left(\frac{\partial \ln(h^{(\sigma)})}{\partial x}\right)^2\right]^{1/2}, \tag{5.33}$$

where P_x and P_y are the two components of the unit vector $\mathbf{P}$. Without vertical change of the diapycnal velocity equation, (5.31) becomes,

$$\mathbf{V}^{(\sigma)} = \gamma\mathbf{P}, \tag{5.34}$$

and any water particle that moves along $q^{(\sigma)}$-isoline on the potential density, i.e., any $q^{(\sigma)}$-isoline, is a trajectory of water particles (Fig. 5.6). We call this case ($\partial w^{(\sigma)}/\partial z = 0$) as the no diapycnal exchange case.

5.3.2 P-Spiral

From the P-vector definition (5.30), the angle (α) from the x-axis to the vector $\mathbf{P}$ counterclockwise is given by

$$P_x = \sin\,\alpha, \quad P_y = \cos\,\alpha. \tag{5.35}$$

Existence of the **P**-spiral between potential-density levels becomes the nonzero ($\partial\alpha/\partial z \neq 0$). A nondimensional **P**-spiral parameter defined by

$$\Lambda = h^{(\sigma)} \frac{\partial \alpha}{\partial z^{(\sigma)}} \tag{5.36}$$

is used to represent the **P**-spiral. Substituting (5.35) into (5.36) yields

$$\Lambda = -h^{(\sigma)} \frac{\partial P_x/\partial z^{(\sigma)}}{P_y} = h^{(\sigma)} \frac{\partial P_y/\partial z^{(\sigma)}}{P_x}, \tag{5.37}$$

and substituting (5.32) and (5.33) into (5.37) yields

$$\Lambda = h^{(\sigma)} \frac{J(q^{(\sigma)}, \partial q^{(\sigma)}/\partial z^{(\sigma)})}{\left|\nabla_\sigma q^{(\sigma)}\right|^2}. \tag{5.38}$$

Thus, the necessary condition for the existence of the **P**-spiral is the nonzero Jacobian of $q^{(\sigma)}$and $\partial q^{(\sigma)}/\partial z^{(\sigma)}$, that is, the nonzero $q^{(\sigma)}$ and $\partial q^{(\sigma)}/\partial z^{(\sigma)}$, and the independence between $q^{(\sigma)}$ and $\partial q^{(\sigma)}/\partial z^{(\sigma)}$.

Substituting (4.16) into (5.38) yields

$$\Lambda = h^{(\sigma)} \frac{-\frac{\beta}{f}\frac{\partial^2\left[\ln\, h^{(\sigma)}\right]}{\partial x \partial z^{(\sigma)}} + J\left[\ln\, h^{(\sigma)}, \frac{\partial\left(\ln\, h^{(\sigma)}\right)}{\partial z^{(\sigma)}}\right]}{\left[\frac{\partial\left(\ln\, h^{(\sigma)}\right)}{\partial x}\right]^2 + \left[\frac{\beta}{f} - \frac{\partial\left(\ln\, h^{(\sigma)}\right)}{\partial y}\right]^2}, \tag{5.39}$$

which indicates that the **P**-spiral can be generated by laterally and vertically inhomogeneous thickness between two closely-spaced potential-density surfaces (Chu 2000). The spatial variability of the parameter Λ represents the spatial variability of the **P**-spirals. For illustration, we chose six locations (Fig. 5.7), A (54°W, 20°N), B (36°W, 28°N), C (20°W, 55°N), D (57°W, 40°N), E (18°W, 6°S), and F (30°W, 45°N), where the first three locations were studied extensively in the past (e.g., Stommel and Schott 1997; Schott and Stommel 1978). These six points represent the Antilles Current, North Atlantic Gyre, Norwegian Current, Gulf Stream, South Equatorial Current, Antarctic Circumpolar Current, respectively. The P-vector turning angles between the two consecutive isopycnal layers ($\Delta\ \alpha_{km}$) are computed for the six locations (Fig. 5.8). For $\Delta\ \alpha_{km} = 0.2$, the vertical turning angle is around 11°. For $\beta = 0$, (5.39) becomes

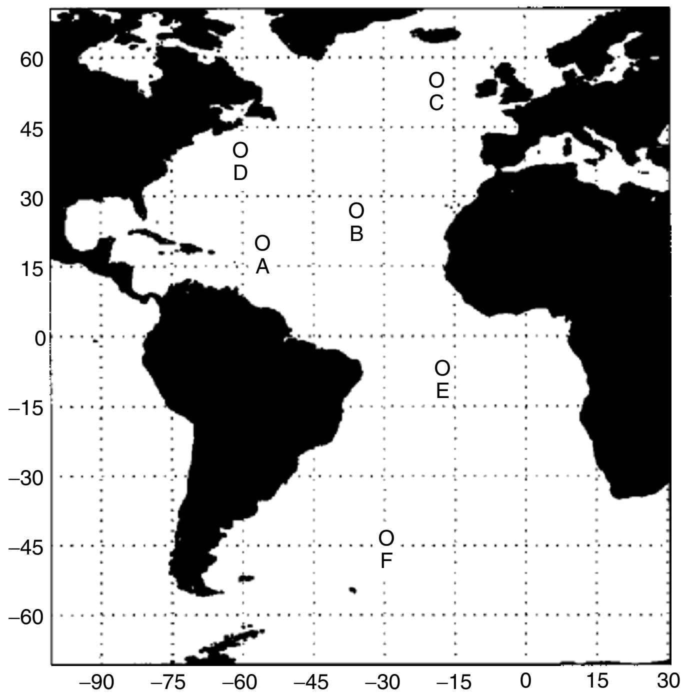

Fig. 5.7. Six locations: Antilles Current (A), North Atlantic Gyre (B), Norwegian Current (C), Gulf Stream (D), South Equatorial Current (E), Antarctic Circumpolar Current (F) (from Chu 2000, Journal of Oceanography)

$$\Lambda = h^{(\sigma)} \frac{J\left[\ln h^{(\sigma)}, \frac{\partial\left(\ln h^{(\sigma)}\right)}{\partial z^{(\sigma)}}\right]}{\left[\frac{\partial\left(\ln h^{(\sigma)}\right)}{\partial x}\right]^2 + \left[\frac{\partial\left(\ln h^{(\sigma)}\right)}{\partial y}\right]^2}, \tag{5.40}$$

showing the existence of the **P**-spiral without the β-effect.

Questions and Exercises

(1) What are the most important features of the P-vector defined by (5.3) for the z-coordinate system?
(2) Why the P-vector lie on the interface of isopycnal surface and iso-potential vorticity surface?
(3) What is the **P**-spiral? Why is the **P**-spiral so important in determining absolute velocity from hydrographic data?
(4) Is the β-effect important in determining absolute velocity from hydrographic data? If the answer is no, please explain why.
(5) What are the three necessary conditions for the determination of absolute velocity from hydrographic data using the P-vector?

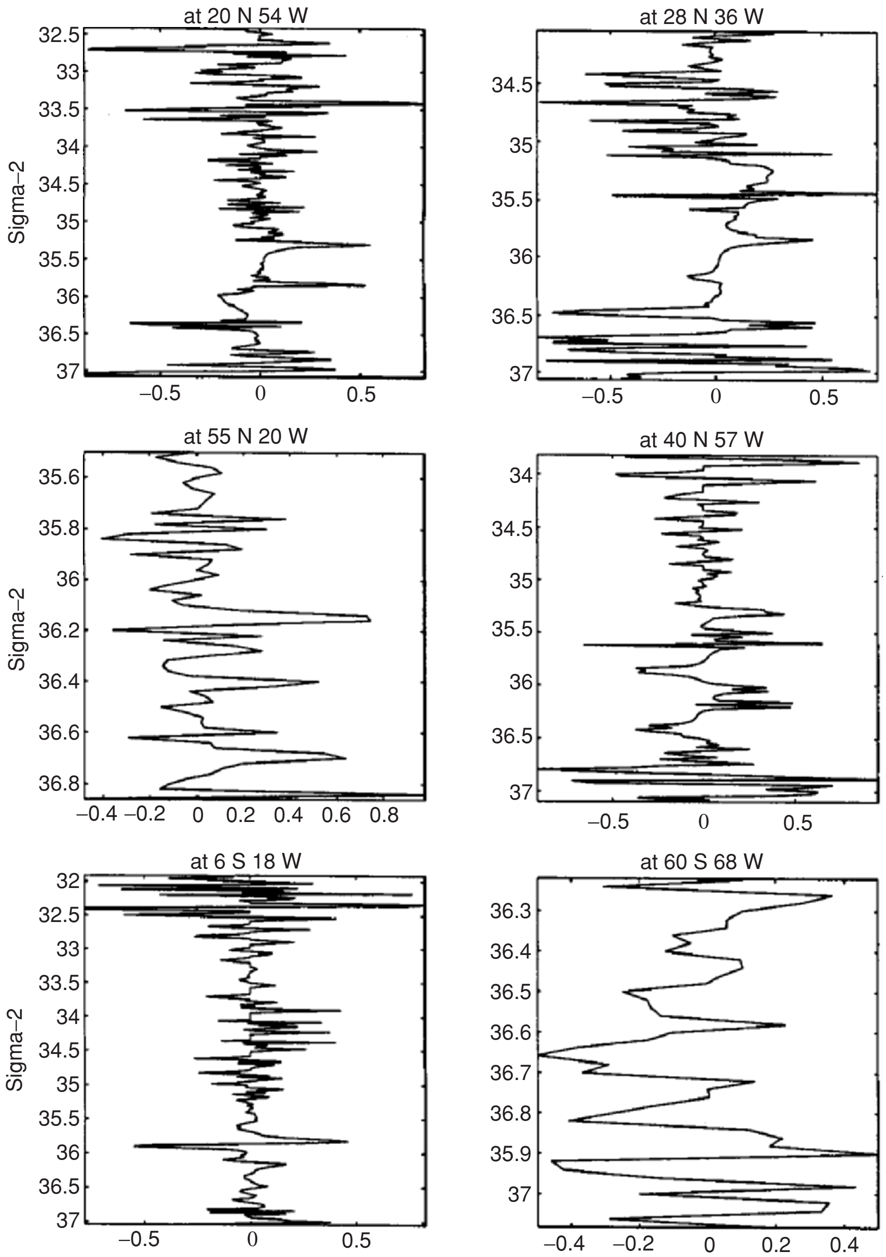

Fig. 5.8. The values of $\Delta\alpha$ for two consecutive isopycnal levels are usually small in the upper isopycnal layers, and become large in deeper isopycnal layers (from Chu 2000, Journal of Oceanography)

(6) Form the geostrophic balance

$$\mathbf{V} = \frac{1}{f}\mathbf{k} \times \nabla M, \tag{E5.1}$$

and the hydrostatic balance,

$$\frac{p}{\rho^2} + \frac{\partial M}{\partial \rho} = 0, \tag{E5.2}$$

for the isopycnal coordinate system. Derive the absolute velocity equation,

$$\mathbf{V} = \frac{\mathbf{P} \bullet \nabla p}{f\rho^2 \mathbf{P} \bullet (\mathbf{k} \times \partial \mathbf{P}/\partial \rho)}\mathbf{P}. \tag{E5.3}$$

Discuss the physical significance of (E5.3).

(7) Compare the P-vector turning parameters in z-coordinate, isopycnal-coordinate, and semi-isopycnal coordinate. Find in which coordinate system the P-vector turning parameter provides more concise information.

(8) What is the physical interpretation of Fig. 5.8? Does it imply that the isopycnal- or semi-isopycnal coordinate is a good choice in determining absolute velocity from hydrographic data? Why?