4

Coordinate Systems

The hydrographic data are usually represented in the z-coordinate (or using pressure to represent depth) system. However, many inverse methods use the isopycnal coordinate system such as the box model (Wunsch 1978), original β-spiral method (Stommel and Scott 1977), and the P-vector model (Chu and Li 2000). This chapter describes isopycnal coordinate system, semi-isopycnal coordinate system, and the transformation of hydrographic data from the z to isopycnal and semi-isopycnal coordinate systems.

4.1 Isopycnal Coordinate System

As pointed out by Wunsch and Grant (1982), in determining large-scale circulation from hydrographic data, we can be reasonably confident on the assumptions of geostrophic balance, mass conservation, and no major crossisopycnal mixing (except where water masses are in contact with the atmosphere). The potential density ρ of each fluid element would be conserved.

The isopycnal coordinate system is represented by (x, y, ρ) with x- and y-axes in the horizontal plane, ρ-axis in vertical with unit vectors (**i, j, k**) in the three directions, repetitively. The benefit of using the isopycnal coordinate system is that the density ρ can be treated as constant in horizontal differentiation. The geostrophic balanced motion on an isopycnal surface (ρ) with a pressure p is given by Bleck and Smith (1990),

$$\mathbf{V} = \frac{1}{f}\mathbf{k} \times \nabla M, \tag{4.1}$$

where $\mathbf{V} = (u, v, 0)$, is the geostrophic velocity; $M = p/\rho + gz$, is the Montgomery potential. The hydrostatic balance is written by

$$\frac{p}{\rho^2} + \frac{\partial M}{\partial \rho} = 0. \tag{4.2}$$

The adiabatic density conservation and incompressible continuity equations are combined to give an equation for the thickness, $\partial p/\partial\rho$. The continuity equation is given by

$$\nabla \cdot \left(\frac{\partial p}{\partial \rho} \mathbf{V} \right) = 0. \tag{4.3}$$

Note that the differentiations with respect to x and y are on the isopycnal surface. Differentiation of (4.1) with respect to ρ and use of (4.2) lead to the thermal wind relation

$$\frac{\partial \mathbf{V}}{\partial \rho} = -\frac{1}{f\rho^2} \mathbf{k} \times \nabla p. \tag{4.4}$$

The continuity (4.3) can be rewritten by

$$\mathbf{V} \cdot \nabla \left(\frac{\partial p}{\partial \rho} \right) + \frac{\partial p}{\partial z} \nabla \cdot \mathbf{V} = 0, \tag{4.5}$$

Using (4.1) yields

$$\nabla \cdot \mathbf{V} = \nabla \left(\frac{1}{f} \right) \cdot (\mathbf{k} \times \nabla M) = -\frac{1}{f} \nabla f \cdot \mathbf{V}. \tag{4.6}$$

Substituting (4.6) in (4.5) yields the conservation of potential vorticity (q) on the isopycnal surface,

$$\mathbf{V} \cdot \nabla q = 0, \tag{4.7}$$

where

$$q = \frac{fg}{\partial p/\partial \rho}. \tag{4.8}$$

The derivative $\partial p/\partial\rho$ represents the thicknesses between two adjacent isopycnal levels. The (p, q) fields are computed on the isopycnal surface can be computed numerically after the hydrographic data are processed in the isopycnal surface. The climatological (p, q) fields on $\sigma^2 = 35.47(\mathrm{kg\,m^{-3}})$ isopycnal surface are calculated using the NODC annual mean climatological T, S data (Fig. 4.1).

4.2 Semi-Isopycnal Coordinate System

Consider a series of potential density (σ_θ) surfaces, each marked by a constant value of σ with the depth,

$$z^{(\sigma)} = R(x, y, \sigma), \tag{4.9}$$

where R is the decreasing function with σ. The vertical distance between two closely spaced σ surfaces with increment of $\Delta\sigma$ is given by

$$h^{(\sigma)} = \frac{\partial z^{(\sigma)}}{\partial \sigma} \Delta\sigma. \tag{4.10}$$

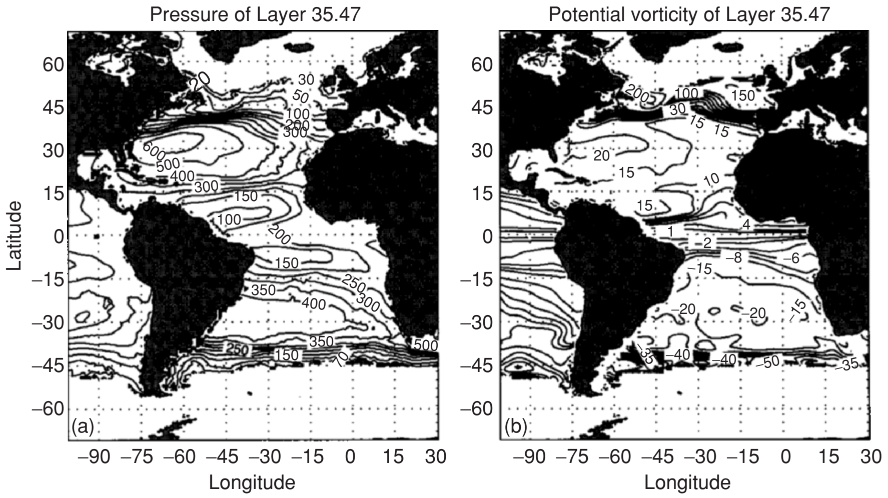

Fig. 4.1. Climatological (p, q) data on $\sigma_2 = 35.47(\mathrm{kg\,m^{-3}})$ isopycnal surface: (**a**) pressure (unit: db) and (**b**) potential vorticity (unit: $10^{-11}\,\mathrm{m^{-1}\,s^{-1}}$) (from Chu and Li 2000, Journal of Physical Oceanography)

The semi-isopycnal coordinate system is represented by $(x, y, z^{(\sigma)})$ with x- and y-axes in the horizontal plane, $z^{(\sigma)}$-axis in vertical with unit vectors $(\mathbf{i},\ \mathbf{j},\ \mathbf{k})$ in the three directions, repetitively (McDougall 1988).

The geostrophic balanced motion in the $(x, y, z^{(\sigma)})$ is given by

$$\mathbf{V}^{(\sigma)} = \frac{1}{f\rho}\mathbf{k} \times \nabla_\sigma p, \tag{4.11}$$

which is similar to (1.1) for the z-coordinate system. Here, the gradient operator at the $z^{(\sigma)}$-level is defined by

$$\nabla_\sigma \equiv \mathbf{i}\ \frac{\partial}{\partial x}\bigg|_\sigma + \mathbf{j}\ \frac{\partial}{\partial y}\bigg|_\sigma . \tag{4.12}$$

The continuity equation can be expressed by considering the flow between two isopycnal surfaces separated by an increment, $h^{(\sigma)}$, together with the diapycnal velocities $w_{\mathrm{u}}^{(\sigma)}$ and $w_{\mathrm{l}}^{(\sigma)}$ across the upper and lower isopycnal surfaces (Fig. 4.2). The continuity of an ocean is represented by

$$\frac{1}{h}\ \frac{\partial(h^{(\sigma)}u)}{\partial x}\bigg|_\sigma + \frac{\partial(h^{(\sigma)}v)}{\partial y}\bigg|_\sigma + \frac{w_u^{(\sigma)} - w_l^{(\sigma)}}{h} = 0. \tag{4.13}$$

For infinitesimally small h, (4.13) becomes

$$\mathbf{V}^{(\sigma)} \cdot \nabla_\sigma[\ln(h^{(\sigma)})] + \nabla_\sigma \cdot \mathbf{V}^{(\sigma)} + \frac{\partial w^{(\sigma)}}{\partial z} = 0. \tag{4.14}$$

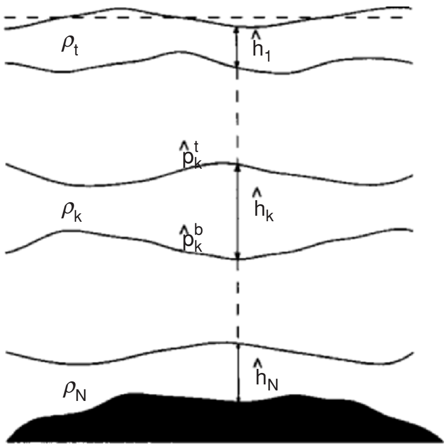

Fig. 4.2. Vertical discretization of the multilayer ocean with the kth layer having potential density ρ_k, layer thickness $\hat{h}_k$, top and bottom pressures $(\hat{p}_k^{\mathrm{t}}, \hat{p}_k^{\mathrm{b}})$, respectively (Chu and Li 2000, Journal of Physical Oceanography)

Horizontal divergence can be calculated from (4.11),

$$\nabla_\sigma \cdot \mathbf{V}^{(\sigma)} = -\frac{\nabla_\sigma f}{f^2 \rho} \cdot (\mathbf{k} \times \nabla_\sigma p) = -\frac{\nabla_\sigma f}{f} \cdot \mathbf{V}^{(\sigma)}. \tag{4.15}$$

Substituting (4.15) in (4.14) yields the conservation of potential vorticity,

$$\mathbf{V}^{(\sigma)} \cdot \nabla_\sigma [q^{(\sigma)}] = \frac{\partial w^{(\sigma)}}{\partial z}, \quad q^{(\sigma)} \equiv \ln[Q^{(\sigma)}], \quad Q^{(\sigma)} \equiv \frac{f}{h^{(\sigma)}}, \tag{4.16}$$

where $Q^{(\sigma)}$ is the potential vorticity; and $q^{(\sigma)}$ is a conservative quantity representing the potential vorticity (we may call it the pseudo potential vorticity). The diapycnal velocity $w^{(\sigma)}$ is not only from the vertical diffusivity, thermobaricity, and cabbeling, but also from lateral mixing along the neutral tangent plane (McDougall 1988).

4.3 Isopycnal Surfaces Determined from Data

Three reference levels: (surface, 2,000 decibar (db), 4,000 db) are used for potential density computation (Keffer 1985; Talley 1988): σ_0 (or sometime called σ_θ) using the surface, σ_2 using 2,000 db, and σ_4 using 4,000 db. The potential densities σ_2 and σ_4 provide better representations for levels near 2,000 and 4,000 db. Here,

$$\sigma_m = \rho_m - 1{,}000\,\mathrm{kg\,m^{-3}}, \quad m = 0, 2, 4. \tag{4.17}$$

The following ranges for the σ_m-values are considered,

$$22.200 \le \sigma_0 \le 27.725, \quad 31.625 \le \sigma_2 \le 37.150, \quad 41.30 \le \sigma_4 \le 45.90. \tag{4.18}$$

Note that 27.725, 37.15, and 45.90 are maximum values forσ_0, σ_2, and $\sigma_4$computed from the GDEM T, S data set. The potential densities (σ_0, σ_2, σ_4) are discretized with the increment,

$$\Delta\sigma = 0.025\,\mathrm{kg\,m^{-3}}, \tag{4.19}$$

for σ_0, σ_2 and

$$\Delta\sigma = 0.02\,\mathrm{kg\,m^{-3}}, \tag{4.20}$$

for σ_4. Thus, there are 222 σ_0 -layers, 222 σ_2-layers, and 280 σ_4 -layers. Within each layer, the density is vertically uniform.

The cubic spline is used to interpolate T, S data into 246 z-levels with three different increments in order to well resolve isopycnal surfaces: 5 m from 0- to 100-m depth, 10 m from 100- to 1,000-m depth, 20 m from 1,000- to 2,500-m depth, and 50 m below 2,500-m depth. Thus, a high-resolution z-coordinate data set $[\hat{T}(z), \hat{S}(z), \hat{\sigma}_m(z)]$ has been built. The symbol "hat" indicates the data either directly from observations or computed from observational data. For simplicity and no loss of generality, we will use $\sigma_\theta(\sigma_0)$ for illustration.

4.4 Data Transformation

The transformation is fulfilled by comparing the z-coordinate potential density data $\hat{\sigma}_\theta(z_j)$ with the discrete σ_θ-values at the bottom of the kth isopycnal layer, $\sigma_\theta^{\mathrm{b}}(k)$,

$$\sigma_\theta^{\mathrm{b}}(k) = 22.1875 + (k-1)\Delta\sigma_\theta, \quad \sigma_\theta(k) = \frac{1}{2}[\sigma_\theta^{\mathrm{b}}(k) + \sigma_\theta^{\mathrm{b}}(k+1)], \tag{4.21}$$

where the superscript b indicates the bottom of the kth isopycnal layer. The geometric depth for the bottom of the $\sigma_\theta(k)$-layer is obtained by

$$\hat{D}_k^{\mathrm{b}} = -z_j, \quad \text{if}\, \hat{\sigma}_\theta(z_j) = \sigma_\theta^{\mathrm{b}}(k), \tag{4.22}$$

and

$$\hat{D}_k^{\mathrm{b}} = -z_j - \frac{\sigma_\theta(k) - \hat{\sigma}_\theta(z_j)}{\hat{\sigma}_\theta(z_{j+1}) - \hat{\sigma}_\theta(z_j)}, \quad \text{if}\, \hat{\sigma}_\theta(z_j) < \sigma_\theta^{\mathrm{b}}(k) < \hat{\sigma}_\theta(z_{j+1}). \tag{4.23}$$

The thickness of the kth isopycnal layer is calculated by

$$\hat{h}_k = \hat{D}_k^{\mathrm{b}} - \hat{D}_{k-1}^{\mathrm{b}}. \tag{4.24}$$

After $\hat{h}_k$ is obtained, we may compute the hydrostatic pressure field. Starting from the surface ($k = 1$), the kth layer has density ρ_k and thickness $\hat{h}_k$, as shown in Fig. 4.2. Pressure is not uniform within the layer with

$$\hat{p}_k^t = g\sum_{i=1}^{k-1} \rho_i \hat{h}_i, \tag{4.25}$$

at the top of the kth layer, and

$$\hat{p}_k^{\mathrm{b}} = g \sum_{i=1}^{k} \rho_i \hat{h}_i, \tag{4.26}$$

at the bottom of the kth layer. The mean value

$$\hat{p}_k \equiv \frac{\hat{p}_k^{\mathrm{t}} + \hat{p}_k^{\mathrm{b}}}{2} = g \left(\sum_{i=1}^{k-1} \rho_i \hat{h}_i + \frac{1}{2} \rho_k \hat{h}_k \right), \tag{4.27}$$

can be used to represent the pressure at the middle of the kth layer (Fig. 4.2). The potential vorticity (4.18) is discretized by

$$\hat{q}_k = \frac{f g \delta \sigma_\theta}{\hat{p}_k^b - \hat{p}_k^t}. \tag{4.28}$$

Earlier work of McCartney (1982), Keffer (1985), and Talley (1988) also shows the benefit of using potential vorticity in ocean circulation studies.

Questions and Exercises

(1) Derive the conservation of the potential vorticity equation for the isopycnal coordinate system from the geostrophic balance,

$$\mathbf{V} = \frac{1}{f} \mathbf{k} \times \nabla M, \tag{E4.1}$$

the hydrostatic balance,

$$\frac{p}{\rho^2} + \frac{\partial M}{\partial \rho} = 0, \tag{E4.2}$$

and the continuity equation,

$$\nabla \cdot \left(\frac{\partial p}{\partial \rho} \mathbf{V} \right) = 0. \tag{E4.3}$$

(2) The potential vorticity is defined by

$$q = \frac{f g}{\partial p / \partial \rho}, \tag{E4.4}$$

in the isopycnal surface coordinate and by

$$q^{(\sigma)} \equiv \ln \left[\frac{f}{h^{(\sigma)}} \right], \tag{E4.5}$$

in the semi-isopycnal surface coordinate, discuss the difference between the two.

(3) What is the similarity and difference between the isopycnal and semi-isopycnal coordinate systems? When you analyze the (T, S) profile data, what coordinate system will you choose among z-coordinate, isopycnal-coordinate, and semi-isopycnal coordinate? Why?