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Applications to Data Assimilation

Ocean data assimilation is an important component in ocean modeling. For practical application, near-real-time, global ocean data assimilation that provides, regular, complete descriptions of the temperature, salinity and velocity structures of the ocean are important in support of operational oceanography, seasonal-to-decadal climate forecasts and analyses, and oceanographic research. Usually, observational (T, S) data are assimilated into models using various techniques such as nudging, optimal interpolation, variational (VAR) methods (3D VAR, 4D VAR), and Kalman filter. The velocity data are not used due to lack of observations. Since ocean models usually have $(T, S, \mathbf{V})$ as the dependent variables, assimilation with (T, S) data only may be called unbalanced data assimilation. Ideal approach is to assimilate the (T, S) along with the corresponding balanced $\mathbf{V}$ data. This is called the balanced data assimilation.

In this chapter, a simple assimilation method (nudging) is used as an example showing the weakness of the unbalanced data assimilation and the strength of the balanced data assimilation. With the (T, S) data, the absolute velocity field $(\mathbf{V})$ can be calculated using the P-vector inverse method and therefore, using $(T, S, \mathbf{V})$ is the balanced data assimilation.

15.1 Data Nudging

Ocean data assimilation is a procedure that combines observational data from satellite or from ships and buoys (more direct measurements) with information from dynamical models to give the best possible estimate or analysis of the ocean state at a given time. This estimate can be used to initialize climate prediction models or to study ocean phenomena. The forecast skill of climate prediction models is sensitive to their initialization. Ideally, improvements in ocean data assimilation are reflected in improved forecasts.

In ocean data assimilation, observations are combined with information from predictive models in a manner that depends on statistical representations of the observational and model errors. By including more dynamical information in the model error representation, observations are used primarily to correct large-scale errors.

Nudging is a simple and popular data assimilation scheme. For example, the GFDL/NOAA global data assimilation system uses the Newtonian nudging to assimilate the observational temperature data (Rosati et al. 1996),

$$\frac{\partial T}{\partial t} + \ldots = -\gamma_{\mathrm{N}}(T - T_{\mathrm{obs}})\mathbb{S}, \tag{15.1}$$

where γ_{N} is the Newtonian damping coefficient and T_{obs} is the observed temperature. The nudging term [in right-hand of (17.1)] is to force T toward T_{obs}. For sufficiently long time period,

$$\mathop{\mathrm{Lim}}_{t\to\infty}(T) = T_{\mathrm{obs}}. \tag{15.2}$$

Burgers et al. (2002) used a shallow water model with data nudging to investigate the difference between balanced and unbalanced data assimilation for seasonal forecast of equatorial oceans. They found that the unbalanced data assimilation (updating the density field only) leads to distortion of the zonal velocity field around the equator.

15.2 Linear Shallow Water Model

Similar to Burgers (2002), a shallow-water model is used to illustrate the weakness of the unbalanced data assimilation. A linearized shallow water model on an f-plane is given by

$$\frac{\partial u}{\partial t} - f_0 v + \frac{\partial \Phi}{\partial x} = 0, \tag{15.3}$$

$$\frac{\partial v}{\partial t} + f_0 u + \frac{\partial \Phi}{\partial y} = 0, \tag{15.4}$$

$$\frac{\partial \Phi}{\partial t} + \bar{\Phi}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0, \tag{15.5}$$

where the Coriolis parameter is taken as a constant f_0 (the midlatitude f-plane assumption); $\bar{\Phi} = g\bar{h}$ and $\Phi = gh'$ is the deviation from $\bar{\Phi}$; and $h = \bar{h} + h'$ is the height of the free surface. Using Helmholtz's theorem, the velocity can be represented by a stream function ψ and a velocity potential (χ),

$$u = -\frac{\partial \psi}{\partial y} + \frac{\partial \chi}{\partial x}, \quad v = -\frac{\partial \psi}{\partial x} + \frac{\partial \chi}{\partial y}. \tag{15.6}$$

Because of the f-plane assumption, the Coriolis parameter passes through any differentiation. Thus, differentiating (15.3) with respect to y and (15.4) with respect to x and subtracting the first equation from the second equation yields the vorticity equation,

$$\frac{\partial}{\partial t}\nabla_h^2\psi + f_0\nabla_h^2\chi = 0. \tag{15.7}$$

Differentiating (15.3) with respect to x and (15.4) with respect to y and adding the two equations yields the divergence equation,

$$\frac{\partial}{\partial t}\nabla_h^2\chi - f_0\nabla_h^2\psi + \nabla_h^2\Phi = 0. \tag{15.8}$$

Substituting (15.6) in (15.5) yields

$$\frac{\partial\Phi}{\partial t} + \bar{\Phi}\nabla_h^2\chi = 0. \tag{15.9}$$

Consider a periodic domain with dimensions $2\pi a$ in each of the x and y directions. The solutions (ψ, χ, Φ) have the form,

$$\begin{bmatrix}\psi(x,y,t)\\ \chi(x,y,t)\\ \Phi(x,y,t)\end{bmatrix} = \sum_n\sum_m \begin{bmatrix}\hat{\psi}^{mn}(t)\\ \mathrm{i}\hat{\chi}^{mn}(t)\\ f_0\sqrt{K}\hat{\Phi}^{mn}(t)\end{bmatrix}\exp\left[\frac{\mathrm{i}(mx+ny)}{a}\right], \tag{15.10}$$

where

$$K = \frac{(m^2+n^2)\bar{\Phi}}{a^2 f_0^2}. \tag{15.11}$$

Substituting (15.10) in (15.7)–(15.9) yields

$$\frac{\mathrm{d}\hat{\psi}^{mn}}{\mathrm{d}t} + \mathrm{i}f_0\hat{\chi}^{mn} = 0, \tag{15.12}$$

$$\mathrm{i}\frac{\mathrm{d}\hat{\chi}^{mn}}{\mathrm{d}t} - f_0\hat{\psi}^{mn} + f_0\sqrt{K}\hat{\Phi}^{mn} = 0, \tag{15.13}$$

$$\frac{\mathrm{d}\hat{\Phi}^{mn}}{\mathrm{d}t} - \mathrm{i}f_0\sqrt{K}\hat{\chi}^{mn} = 0. \tag{15.14}$$

The solutions of (15.12)–(15.14) are given by

$$\begin{bmatrix}\hat{\psi}(t)\\ \hat{\chi}(t)\\ \hat{\Phi}(t)\end{bmatrix} = \begin{bmatrix}A_\psi^{mn}\\ A_\chi^{mn}\\ A_\Phi^{mn}\end{bmatrix}\exp(-\mathrm{i}f_0\sigma t), \tag{15.15}$$

where $(A_\psi^{mn}, A_\chi^{mn}, A_\Phi^{mn})$ are the amplitudes for component (m, n), σ satisfies the following algebraic equation,

$$\sigma^3 - \sigma(K+1) = 0, \tag{15.16}$$

which has three roots

$$\sigma^{(1)} = 0, \quad \sigma^{(2)} = \sqrt{K+1}, \quad \sigma^{(3)} = -\sqrt{K+1}. \tag{15.17}$$

The three roots represent two dynamical modes: $\sigma^{(1)}$ for the geostrophic mode (or the Rossby mode), and ($\sigma^{(2)}$, $\sigma^{(3)}$) for the inertial-gravity mode. For each mode, the linear shallow water model has the solutions,

$$\begin{bmatrix} \psi(x,y,t) \\ \chi(x,y,t) \\ \Phi(x,y,t) \end{bmatrix} = \sum_n \sum_m \begin{bmatrix} A_\psi^{mn} \\ \mathrm{i}A_\chi^{mn} \\ f_0\sqrt{K}A_\Phi^{mn} \end{bmatrix} \exp\left[\frac{\mathrm{i}(mx+ny)}{a}\right] \exp[(-\mathrm{i}f_0\sigma(t-t_0)], \tag{15.18}$$

with the initial conditions (without data assimilation),

$$\begin{bmatrix} \psi(x,y,t_0) \\ \chi(x,y,t_0) \\ \Phi(x,y,t_0) \end{bmatrix} = \sum_n \sum_m \begin{bmatrix} A_\psi^{mn} \\ \mathrm{i}A_\chi^{mn} \\ f_0\sqrt{K}A_\varPhi^{mn} \end{bmatrix} \exp\left[\frac{\mathrm{i}(mx+ny)}{a}\right]. \tag{15.19}$$

When Newtonian nudging (15.1) is used to assimilate the observational data, (15.7)–(15.9) are changed into

$$\frac{\partial}{\partial t}\nabla_h^2\tilde{\psi} + f_0\nabla_h^2\tilde{\chi} = -\gamma_{\mathrm{N}}\nabla_h^2(\tilde{\psi} - \psi_{\mathrm{obs}}), \tag{15.20}$$

$$\frac{\partial}{\partial t}\nabla_h^2\tilde{\chi} - f_0\nabla_h^2\tilde{\psi} + \nabla_h^2\tilde{\Phi} = -\gamma_{\mathrm{N}}\nabla_h^2(\tilde{\chi} - \chi_{\mathrm{obs}}), \tag{15.21}$$

$$\frac{\partial\tilde{\Phi}}{\partial t} + \bar{\Phi}\nabla_h^2\tilde{\chi} = -\gamma_{\mathrm{N}}(\tilde{\Phi} - \Phi_{\mathrm{obs}}). \tag{15.22}$$

The ocean model is integrated between the two consecutive time instances, say from $t = t_0$ to $t = t_1$, the observational data ($\Phi_{\mathrm{obs}}, \psi_{\mathrm{obs}}, \chi_{\mathrm{obs}}$) are treated as time-independent during that period ($t_0 < t < t_1$). Equations (15.20)–(15.22) constitute a set of coupled linear partial differential equations with time-independent forcing terms: $\gamma_{\mathrm{N}}\nabla_h^2\psi_{\mathrm{obs}}, \gamma_{\mathrm{N}}\nabla_h^2\chi_{\mathrm{obs}}, \gamma_{\mathrm{N}}\varPhi_{\mathrm{obs}}$.

15.3 Balanced Data Assimilation

The balanced data assimilation is to use the observational data ($\Phi_{\mathrm{obs}}, \psi_{\mathrm{obs}}, \chi_{\mathrm{obs}}$) which are in geostrophic balance,

$$f_0\psi_{\mathrm{obs}} = \Phi_{\mathrm{obs}}, \tag{15.23}$$

$$\nabla_h^2\chi_{\mathrm{obs}} = 0. \tag{15.24}$$

Equations (15.20)–(15.22) can be transformed into homogeneous equations,

$$\frac{\partial}{\partial t}\nabla_h^2\psi + f_0\nabla_h^2\chi + \gamma_{\mathrm{N}}\nabla_h^2\psi = 0, \tag{15.25}$$

$$\frac{\partial}{\partial t}\nabla_h^2\chi - f_0\nabla_h^2\psi + \nabla_h^2\Phi + \gamma_{\mathrm{N}}\nabla_h^2\chi = 0, \tag{15.26}$$

$$\frac{\partial \Phi}{\partial t} + \bar{\Phi}\nabla_h^2\chi + \gamma_{\mathrm{N}}\Phi = 0, \tag{15.27}$$

where

$$\begin{aligned}\tilde{\psi}(x,y,t) &= \psi(x,y,t) + \psi_{\mathrm{obs}}(x,y,t),\\ \tilde{\chi}(x,y,t) &= \chi(x,y,t) + \chi_{\mathrm{obs}}(x,y,t),\\ \tilde{\Phi}(x,y,t) &= \Phi(x,y,t) + \Phi_{\mathrm{obs}}(x,y,t).\end{aligned} \tag{15.28}$$

The solutions of homogeneous linear equations (15.25)–(15.27) are given by

$$\begin{bmatrix}\psi(x,y,t)\\ \chi(x,y,t)\\ \Phi(x,y,t)\end{bmatrix} = \sum_n\sum_m \begin{bmatrix}A_\psi^{mn}\\ \mathrm{i}A_\chi^{mn}\\ f_0\sqrt{K}A_\Phi^{mn}\end{bmatrix} \exp\left[\frac{\mathrm{i}(mx+ny)}{a}\right]\exp[(-\mathrm{i}f_0\hat{\sigma}(t-t_0)], \tag{15.29}$$

where $(\pi A_\psi^{mn}, A_\chi^{mn}, A_\Phi^{mn})$ are the initial $(t = t_0)$ amplitudes for the component (m, n) and $\hat{\sigma}$ satisfies the algebraic equation,

$$\left(\hat{\sigma} + \frac{\mathrm{i}\gamma_{\mathrm{N}}}{f_0}\right)^3 - \left(\hat{\sigma} + \frac{\mathrm{i}\gamma_{\mathrm{N}}}{f_0}\right)(K+1) = 0, \tag{15.30}$$

where

$$\hat{\sigma} = -\frac{\mathrm{i}\gamma_{\mathrm{N}}}{f_0} + \sigma. \tag{15.31}$$

The solutions of (15.20)–(15.22) are given by

$$\begin{bmatrix}\tilde{\psi}(x,y,t)\\ \tilde{\chi}(x,y,t)\\ \tilde{\Phi}(x,y,t)\end{bmatrix} = \begin{bmatrix}\psi_{\mathrm{obs}}(x,y,t)\\ \chi_{\mathrm{obs}}(x,y,t)\\ \Phi_{\mathrm{obs}}(x,y,t)\end{bmatrix}$$

$$\begin{bmatrix}\tilde{\psi}(x,y,t)\\ \tilde{\chi}(x,y,t)\\ \tilde{\Phi}(x,y,t)\end{bmatrix} = \begin{bmatrix}\psi_{\mathrm{obs}}(x,y,t)\\ \chi_{\mathrm{obs}}(x,y,t)\\ \Phi_{\mathrm{obs}}(x,y,t)\end{bmatrix} + \sum_n\sum_m \begin{bmatrix}A_\psi^{mn}\\ \mathrm{i}A_\chi^{mn}\\ f_0\sqrt{K}A_\Phi^{mn}\end{bmatrix}$$

$$\exp\left[\frac{\mathrm{i}(mx+ny)}{a}\right]\exp[(-\mathrm{i}f_0\sigma(t-t_0)]\exp(-\gamma_{\mathrm{N}}t). \tag{15.32}$$

Since σ has three real values $(0,\ \sqrt{K+1}, -\sqrt{K+1})$ [see (15.17)], the first term in the right-hand side of (15.32) tends to 0 as $t \to \infty$. This means that the solutions are nudging to the observational values [second term in the right-hand side of (15.32)] as $t \to \infty$,

$$\begin{bmatrix} \tilde{\psi}(x,y,t) \\ \tilde{\chi}(x,y,t) \\ \tilde{\Phi}(x,y,t) \end{bmatrix} \to \begin{bmatrix} \psi_{\text{obs}}(x,y,t) \\ \chi_{\text{obs}}(x,y,t) \\ \Phi_{\text{obs}}(x,y,t) \end{bmatrix}, \tag{15.33}$$

which satisfies the nudging condition (15.2). If we continue to integrate the model, there are no spurious solutions to be generated since the observational data are geostrophically balanced.

15.4 Unbalanced Data Assimilation

The unbalanced data assimilation is to use the observational data $(\Phi_{\text{obs}}, \psi_{\text{obs}}, \chi_{\text{obs}})$ which are not in geostrophic balance. For example, ocean data assimilation is usually using (T, S), but not (u, v) data. For the shallow water model, it is equivalent to assimilate $\mathit{\Phi}_{\text{obs}}$, but not $(\psi_{\text{obs}}, \chi_{\text{obs}})$. The shallow water equations with the Newtonian nudging are given by

$$\frac{\partial}{\partial t}\nabla_h^2\tilde{\psi} + f_0\nabla_h^2\tilde{\chi} = 0, \tag{15.34}$$

$$\frac{\partial}{\partial t}\nabla_h^2\tilde{\chi} - f_0\nabla_h^2\tilde{\psi} + \nabla_h^2\tilde{\Phi} = 0, \tag{15.35}$$

$$\frac{\partial\tilde{\Phi}}{\partial t} + \bar{\Phi}\nabla_h^2\tilde{\chi} = -\gamma_{\text{N}}(\tilde{\Phi} - \Phi_{\text{obs}}). \tag{15.36}$$

Since the variable $\tilde{\Phi}$ occurs in (15.30) and (15.31), it is hard to transform (15.29)–(15.31) into homogeneous equations directly [unlike (15.20–15.22)]. The dependent variables are decomposed into Fourier components

$$\begin{bmatrix} \tilde{\psi}(x,y,t) \\ \tilde{\chi}(x,y,t) \\ \tilde{\Phi}(x,y,t) \end{bmatrix} = \sum_n \sum_m \begin{bmatrix} \hat{\psi}^{mn}(t) \\ \mathrm{i}\hat{\chi}^{mn}(t) \\ f_0\sqrt{K}\hat{\mathit{\Phi}}^{mn}(t) \end{bmatrix} \exp\left[\frac{\mathrm{i}(mx+ny)}{a}\right]. \tag{15.37}$$

Substituting (15.37) in (15.34)–(15.36) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \hat{\psi}^{mn} \\ \hat{\chi}^{mn} \\ \hat{\Phi}^{mn} \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{i}f_0 & 0 \\ -\mathrm{i}f_0 & 0 & \mathrm{i}f_0\sqrt{K} \\ 0 & \mathrm{i}f_0\sqrt{K} & -\gamma_{\text{N}} \end{bmatrix} \begin{bmatrix} \hat{\psi}^{mn} \\ \hat{\chi}^{mn} \\ \hat{\Phi}^{mn} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \gamma_{\text{N}}\Phi_{\text{obs}}^{mn}(t) \end{bmatrix}, \tag{15.38}$$

which can be written in the matrix form,

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \mathbf{A}\mathbf{X} + \mathbf{F}, \tag{15.39}$$

with

$$\mathbf{X}(t) = \begin{bmatrix} \hat{\psi}^{mn}(t) \\ \hat{\chi}^{mn}(t) \\ \hat{\Phi}^{mn}(t) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & -\mathrm{i}f_0 & 0 \\ -\mathrm{i}f_0 & 0 & \mathrm{i}f_0\sqrt{K} \\ 0 & \mathrm{i}f_0\sqrt{K} & -\gamma_\mathrm{N} \end{bmatrix}, \quad \mathbf{F}(t) = \begin{bmatrix} 0 \\ 0 \\ \gamma_\mathrm{N}\Phi_\mathrm{obs}^{mn}(t) \end{bmatrix}. \tag{15.40}$$

Integrating (15.39) from t_0 to t gives

$$\mathbf{X}(t) = \mathbf{T}(t, t_0)\mathbf{X}(t_0) + \int_{t_0}^{t} \mathbf{T}(t, s)\mathbf{F}(s)\mathrm{d}s, \tag{15.41}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{T}(t, s) = \mathbf{A}\mathbf{T}(t, s), \tag{15.42}$$

and $\mathbf{T}(T, S)$ is the transition matrix. Since the solution $\mathbf{X}(t)$ depends on $\mathbf{F}$ not only at time instance t but during the time period $[t_0, t]$, the solutions of (15.34)–(15.36) are not likely nudging to the observational data at certain time instance. Therefore, unbalanced data assimilation should be avoided. Weakness of unbalanced data assimilation needs further investigation theoretically and numerically.

MODAS is the US Navy's ocean operational system to assimilate a wide range of ocean observations into twice daily global three-dimensional (T, S) fields with various horizontal resolutions (see Sect. 3.2.1). MODAS (T, S) fields are regarded as pseudo-"observational" data set and used for ocean model assimilation and acoustic calculation.

If only the MODAS (T, S) data are assimilated into ocean models, it is the unbalanced data assimilation. To avoid this, three-dimensional absolute velocity field $(\mathbf{V})$ should be calculated from the MODAS gridded (T, S) fields using the P-vector inverse method.

Questions and Exercises

(1) Discuss the major differences between balanced and unbalanced data assimilations.

(2) Mathematical difference between balanced and unbalanced data assimilations is in the eigenvalue equation. The eigenvalue equation for the balanced data assimilation is written by

$$\left(\hat{\sigma} + \frac{\mathrm{i}\gamma_\mathrm{N}}{f_0}\right)^3 - \left(\hat{\sigma} + \frac{\mathrm{i}\gamma_\mathrm{N}}{f_0}\right)(K+1) = 0, \tag{E15.1}$$

which is similar to the dynamical system without data assimilation,

$$\sigma^3 - \sigma(K+1) = 0. \tag{E15.2}$$

The eigenvalue equation for the unbalanced data assimilation does not have such a form. Why the balanced data assimilation will provide more realistic solutions?

(3) Select a region and a numerical ocean model of your interest. The monthly WOA (T, S) data are used. The corresponding absolute velocity $(\mathbf{V})$ data are downloaded from the enclosed DVD-ROM. The balanced data assimilation is to use $(T, S, \mathbf{V})$ data. The unbalanced data assimilation is to use (T, S) data. Run the numerical model with the two types of data assimilation for several years. Analyze the difference between the two model runs, and discuss the results.