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Introduction

A major task of physical oceanographers is to determine the ocean circulation from real data. Due to the cost and time-consuming factors of current meter measurement, physical oceanographers usually have a relatively frequent hydrographic observation. For example, the US Navy's Master Oceanographic Observational Data Set (MOODS) contains more than six million temperature profiles and nearly one million salinity profiles for the global ocean. How to use the hydrographic data becomes important in inferring the state of the ocean circulation, understanding it dynamically, and even perhaps forecasting it, through a quantitative combination of theory and observations (Wunsch 1996).

1.1 Basic Physics of the Inverse Problem

1.1.1 Basic Equations

Let (x, y, z) be the coordinates with x-axis in the zonal direction (eastward positive), y-axis in the latitudinal direction (northward positive), and z-axis in the vertical direction (upward positive). The unit vectors along the three axes are represented by ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$). The linear steady state is reached with the hydrostatic balance in vertical; and the balance among the Coriolis force, pressure gradient force, and gradient of Reynolds stress in horizontal. With the Boussinesq approximation, the basic equations are given by

$$-f(\tilde{v} - v) = A_z \frac{\partial^2 \tilde{u}}{\partial z^2} + A_h \nabla_h^2 \tilde{u}, \tag{1.1a}$$

$$f(\tilde{u} - u) = A_z \frac{\partial^2 \tilde{v}}{\partial z^2} + A_h \nabla_h^2 \tilde{v}, \tag{1.1b}$$

$$\frac{\partial p}{\partial z} = -\rho g, \tag{1.1c}$$

$$\nabla_h \bullet \tilde{\mathbf{V}}_2 + \frac{\partial \tilde{w}}{\partial z} = 0, \tag{1.1d}$$

where ρ is the in situ density; $f = 2\Omega \sin\varphi$, is the Coriolis parameter, Ω the Earth rotation rate, and φ the latitude. $\tilde{\mathbf{V}}_2 = (\tilde{u}, \tilde{v})$, is the horizontal velocity; $\tilde{w}$ is the vertical velocity; $\nabla_h = \mathbf{i}\partial/\partial x + \mathbf{j}\partial/\partial y$ is the horizontal gradient operator; $\mathbf{V}_2 = (u, v)$ is the geostrophic velocity representing the balance between the Coriolis force and the horizontal pressure (p) gradient force,

$$u = -\frac{1}{f\rho_0}\frac{\partial p}{\partial y}, \quad v = \frac{1}{f\rho_0}\frac{\partial p}{\partial x}, \tag{1.2}$$

where ρ_0 is the characteristic value ($1,025\,\mathrm{kg\,m^{-3}}$) of the sea water density. The two coefficients (A_z, A_h) are the vertical and horizontal eddy diffusivities. The horizontal diffusivity A_h is usually estimated by Smargrinsky parameterization,

$$A_h = \frac{D}{2}\Delta x \Delta y \left[\left(\frac{\partial \tilde{u}}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y}\right)^2 + \left(\frac{\partial \tilde{v}}{\partial y}\right)^2\right]^{1/2}.$$

Here, the nondimensional parameter D varies from 0.1 to 0.2 (Mellor 2003). For horizontal grid of $1° \times 1°$ as in many climatological temperature and salinity datasets, and for spatial variability of the velocity of $0.1\,\mathrm{m\,s^{-1}}$, the horizontal diffusivity is estimated as

$$A_h = 1.5 \times 10^3\,\mathrm{m^2\,s^{-1}}.$$

1.1.2 Ekman Number

The Ekman number can identify the relative importance of the horizontal gradient of the Reynolds stress $(A_h \nabla^2 \mathbf{V}_h)$ versus the Coriolis force $(f\mathbf{V}_h)$,

$$E = \frac{O(|A_h \nabla_2^2 \tilde{V}_2|)}{O(|f\tilde{V}_2|)} = \frac{A_h}{|f|L^2}.$$

Here, L is the characteristic horizontal length scale. For extra-equatorial regions (north of 8°N and south of 8°S), $|f| > 0.2 \times 10^{-4}\,\mathrm{s^{-1}}$ and for length scale of motion L being larger than 200 km, the Ekman number is estimated by

$$E < \frac{1.5 \times 10^3\,\mathrm{m^2 s^{-1}}}{(0.2 \times 10^{-4}\mathrm{s^{-1}}) \times (2 \times 10^5\mathrm{m})^2} = 1.875 \times 10^{-3}.$$

The horizontal gradient of the Reynolds stress can be neglected against the Coriolis force. For the equatorial regions especially near the equator, $|f|$ is very small. The Ekman number is not a small parameter. The horizontal gradient of the Reynolds stress, $(A_h \nabla_h^2 \tilde{u}, A_h \nabla_h^2 \tilde{v})$, cannot be neglected against the Coriolis force in the equatorial region.

1.1.3 Thermal Wind Relation

For large-scale motion in extra-equatorial regions, the geostrophic (1.2) and hydrostatic (1.1c) balances are usually satisfied. Differentiating the two equations in (1.2) with respect to z and utilizing (1.1c) yield the thermal wind relation,

$$\frac{\partial u}{\partial z} = \frac{g}{f\rho_0}\frac{\partial \rho}{\partial y}, \quad \frac{\partial v}{\partial z} = -\frac{g}{f\rho_0}\frac{\partial \rho}{\partial x}. \tag{1.3}$$

Vertical integration of the two equations in (1.3) with respect to z leads to the thermal wind relation,

$$u = u_0 + \frac{g}{f\rho_0}\int_{z_0}^{z} \frac{\partial \rho}{\partial y}\mathrm{d}z', \tag{1.4}$$

$$v = v_0 - \frac{g}{f\rho_0}\int_{z_0}^{z} \frac{\partial \rho}{\partial x}\mathrm{d}z', \tag{1.5}$$

which is the linkage between the geostrophic velocity and the in situ density. Here, (u, v), (u_0, v_0) are the geostrophic velocities at any depth z and at a reference depth z_0, respectively. It is noticed that only the density data determine the geostrophic shear. The reference-level velocity (u_0, v_0) needs to be determined. The continuity equation is given by

$$\nabla \bullet \mathbf{V} = 0. \tag{1.6}$$

where $\mathbf{V} = (u, v, w)$ is the three-dimensional velocity vector and w is the vertical velocity; and

$$\nabla \equiv \mathbf{i}\partial/\partial x + \mathbf{j}\partial/\partial y + \mathbf{k}\partial/\partial z,$$

is the three-dimensional gradient operator. The density equation (or thermodynamic equation) is generally written by

$$\frac{\partial \rho}{\partial t} + \mathbf{V} \bullet \nabla \rho = \text{Source} - \text{Sink},$$

which changes into

$$\mathbf{V} \bullet \nabla \rho = 0, \tag{1.7}$$

for steady-state and no source/sink terms (mass conservation).

The conservation of potential vorticity equation (Pedlosky 1986) can be obtained by differentiating (1.7) with respect to z, using geostrophic and hydrostatic balances (1.1) and (1.2), and including the latitudinal variation of the Coriolis parameter,

$$\mathbf{V} \bullet \nabla q = 0, \tag{1.8}$$

where q is the potential vorticity

$$q = f\frac{\partial \rho}{\partial z}. \quad (1.9)$$

It is noted that neglect of relative vorticity may induce a small but systematic error into the estimation of potential vorticity.

1.2 Reference Velocity

Determination of the reference-level velocity (u_0, v_0) needs the density conservation (1.7) and potential vorticity conservation (1.8). If the two conservation laws are not used, determination of the reference-level velocity becomes quite subjective. The simplest technique is the level-of-no-motion assumption.

1.2.1 Level-of-no-Motion

Inability to determine the reference-level velocity distorted and stymied the study of oceanography for many decades. Therefore, to avoid complete paralysis, oceanographers made the assumption that if one went deeply enough into the sea, the fluid movement would become as weak as to be negligible (Wunsch 1996). The "level-of-no-motion" assumption,

$$u_0 = 0, \quad v_0 = 0,$$

has been used widely by the oceanographic community with various levels (z_0) ranging from 500 to 4,000 decibars. Defant (1941) proposed to use the level of minimum geostrophic shear representing the level-of-no-motion. Thus, the level was permitted to slope across the ocean. Obviously, the two levels are not the same. However, Defant's method is a rational one in the absence of any other criterion – the depth of minimum geostrophic shear is the depth for which the resulting velocities and transports are least sensitive to perturbation in the level-of-no-motion (Wunsch 1996).

1.2.2 Determination of Reference-Level Velocity

Several inverse methods have been developed to determine the reference-level velocity (u_0, v_0) using the conservation of mass and density (1.6), (1.7) and to avoid the ambiguity caused by the level-of-no-motion assumption. Those inverse methods are classified into two major categories: area determination, such as the box model (Hidaka 1940a,b; Wunsch 1978), and local determination, such as the β-spiral method (Stommel and Scott 1977; Schott and Stommel 1978; Behringer and Stommel 1980) and the Bernoulli method (Killworth 1986).

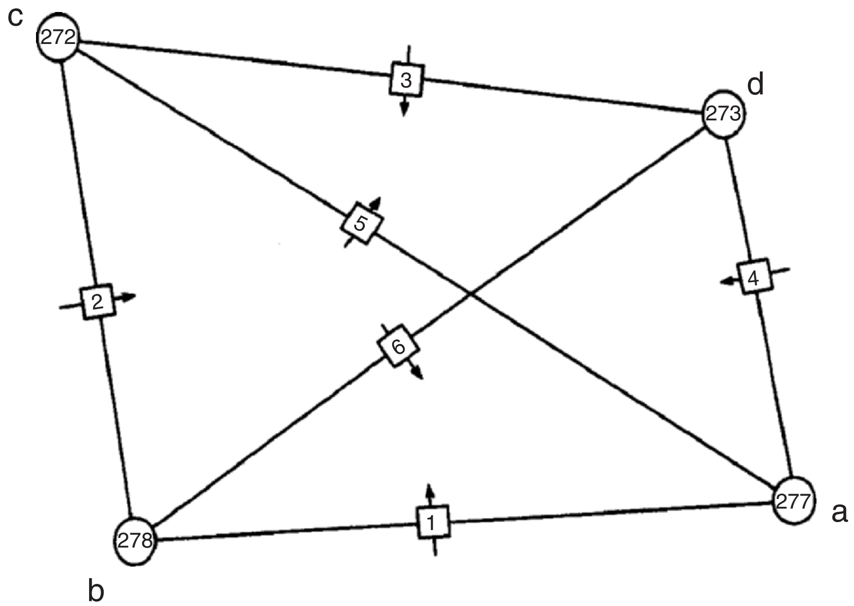

Fig. 1.1. Mass and salt conservation in defined triangular volumes used by Hidaka (1940b) in estimating reference-level velocities $v_0^{(i)}(i = 1, 2, \ldots, 6)$. Here, the *circled numbers* are station identifiers, and *boxed integers* are interface labels used to identify the flows between volumes

(a) Hidaka's Attempt. Hidaka (1940a,b) made the first attempt to use the mass and salt conservation for determining the reference-level velocity from hydrographic data. He constructed triangles from the hydrographic stations with the side connecting pair of stations. For four hydrographic stations, there are six straight lines (Fig. 1.1). Let the reference-level velocity in each station pair along each line be denoted $v_0^{(i)}$ $(i = 1, 2, \ldots, 6)$, which are unknowns. The conservation requires that the product of net mass and salt flux with volumes (three triangles) must be zero. Thus, Hidaka obtained six algebraic equations with six unknowns. Hidaka (1940b) solved this system and obtained numerical values of the reference-level velocities. However, Defant (1961) demonstrated that such a system of equations was ill-conditioned and numerical values of the reference-level velocities produced by Hidaka (1940b) were meaningless.

(b) Box model. Following Hidaka's (1940a,b) lead on the conservation of mass and salt with the flow into and out of a volume of ocean (Fig. 1.2), Wunsch (1978) constructed an inverse method (or called the box model) to determine the reference-level velocity. In the recent book, Wunsch (1996) described his method as follows. Consider a closed volume depicted in Fig. 1.2, the flow into and out of the volume should conserve the mass. Make the convention that velocities and transports are positive to the north and/or east and that the sign of the unit normal for a closed volume is positive inward.

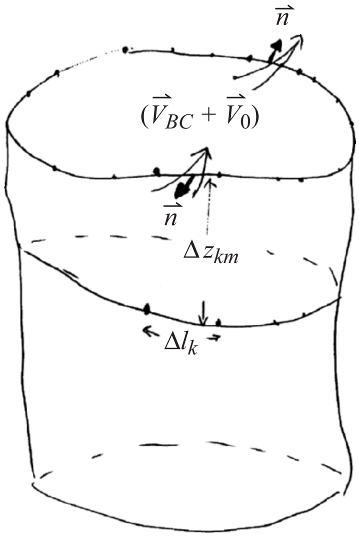

Fig. 1.2. Mass conservation for each layer of the volume

Choose a reference depth $z_0(s_j)$ where s is an arc length along the volume periphery and j denotes the station pair number, and compute the thermal wind relation to this reference-level for each station-pair. If the Ekman flow is assumed negligible, consider first the total amount of fluid moving geostrophically into and out of the closed volume shown in Fig. 1.2,

$$\sum_{j}^{J} \sum_{k}^{K} \rho_j(k) \left[v_{Rj}(k) + v_{0j} \right] \delta_j \Delta a_j(k) \simeq 0, \tag{1.10}$$

where $v_{Rj}(k)$ and $\Delta a_j(k)$ are the thermal wind (relative velocity) and the differential area for the station pair j at the depth interval k; v_{0j} is the reference-level velocity, and δ_j is the unit normal (± 1) for the volume with the pair j. Equation (1.10) is a discrete approximation to the area integrals over the boundary section and can be carried out in a variety of different approximations. Everything is known in (1.10) except for the reference-level velocities. Equation (1.10) has been written as approximately equal to zero, rather than precisely so, in anticipation of the need to grapple with errors in the various terms of the sum.

Equation (1.10) is one equation with J unknowns, $v_{0j}(j = 1, 2, \ldots, J)$, and addition of some further constraints would be helpful. It is assumed that the volume of water in the ocean lying in fixed density intervals does not change significantly. Define the depth of fluid of density ρ_i as

$$z(\rho_i, x, y) = k_i(x, y).$$

Then mass conservation in the density interval $\rho_i < \rho < \rho_{i+1}$ is

$$\sum_{j}^{J} \sum_{k_i(j)}^{k_{i+1}(j)} \rho_j(k)\,[v_{Rj}(k) + v_{0j}]\delta_j \Delta a_j(k) \simeq 0. \tag{1.11}$$

Since $v_{Rj}(k)$is assumed as known, (1.11) can be written as

$$\sum_{j}^{J} \sum_{k_i(j)}^{k_{i+1}(j)} \rho_j(k) v_{0j} \delta_j \Delta a_j(k) \simeq -\sum_{j}^{J} \sum_{k_i(j)}^{k_{i+1}(j)} \rho_j(k) v_{Rj}(k) \delta_j \Delta a_j(k). \tag{1.12}$$

Usually, the number of unknowns $(v_{01}, v_{02}, \ldots, v_{0J})$ is larger than the number of the constraints in the box model (under-determined system). Obviously, if the velocity does not have vertical turning, $v_{Rj}(k) = 0$, the reference-level velocities cannot be obtained by the box model.

(c) β-Spiral method. On the basis of geostrophic balance (1.1), hydrostatic balance (1.2), conservation of mass (1.6), and conservation of potential vorticity (1.8), the β-spiral method was developed (Stommel and Scott 1977; Olbers et al. 1985) to determine the reference-level velocity locally. Solving the density conservation (1.7) for w leads to

$$\begin{aligned}(\frac{\partial \rho}{\partial z})^2 \frac{\partial w}{\partial z} = -\left(\frac{\partial \rho}{\partial z}\right)\left(\frac{\partial u}{\partial z}\frac{\partial \rho}{\partial x} + u\frac{\partial^2 \rho}{\partial x \partial z} + \frac{\partial v}{\partial z}\frac{\partial \rho}{\partial y} + v\frac{\partial^2 \rho}{\partial y \partial z}\right) \\ + \left(u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y}\right)\frac{\partial^2 \rho}{\partial z^2}.\end{aligned} \tag{1.13}$$

Substitution the equation of the geostrophic balance (1.1) into the mass conservation equation (1.6) leads to the linear vorticity balance,

$$\beta v = f\frac{\partial w}{\partial z}, \quad \beta = \frac{\mathrm{d}f}{\mathrm{d}y}, \tag{1.14}$$

which eliminates w. The thermal wind relations (1.4)–(1.5) are used

$$u_R = \frac{g}{f\rho_0}\int_{z_0}^{z} \frac{\partial \rho}{\partial y}\mathrm{d}z', \quad v_R = -\frac{g}{f\rho_0}\int_{z_0}^{z} \frac{\partial \rho}{\partial x}\mathrm{d}z', \tag{1.15}$$

Substituting (1.4)–(1.5) in (1.13) and using (1.15) yields

$$\begin{aligned}(u_R + u_0)\left[\frac{\partial \rho}{\partial x} - \frac{\partial \rho}{\partial z}\frac{\partial^2 \rho}{\partial x \partial z}\right] \\ + (v_R + v_0)\left[\frac{\partial \rho}{\partial y} - \frac{\partial \rho}{\partial z}\frac{\partial^2 \rho}{\partial y \partial z} - \frac{\beta}{f}\left(\frac{\partial^2 \rho}{\partial z^2}\right)^2\right] = 0,\end{aligned} \tag{1.16}$$

which is the algebraic equation for determining the reference-level velocity (u_0, v_0). If the vertical axis is discretized into a series of depths z_i, then $i = 1, 2, \ldots, I$. There are I equations in (1.16) and two unknowns u_0, v_0. Usually I is much larger than two. Thus, the β-spiral method is an over-determined system.

(d) Equivalence between the β-spiral and box models. Davis (1978) pointed out that the β-spiral method (Stommel and Scott 1977) and the box method (Wunsch 1978), no matter how different in appearance, are based on the same order of dynamical sophistication and differ from implicit assumptions about the scales of oceanic variability and different definitions of the smooth field. The physical principle for both methods are the existence of a conservative tracer which allows determination of a family of material (usually the potential vorticity) surfaces $z = h(x, y)$ such as

$$\nabla_h \bullet \int_h^{h+\delta h} [\mathbf{V}_R + \mathbf{V}_0] \mathrm{d}z = 0, \tag{1.17}$$

which comes from the box model. Taking the variation of (1.17) with respect to h, Davis (1978) obtained

$$\mathbf{V}_0 \bullet \nabla_h \frac{\partial h}{\partial z} - \frac{\beta}{f} v_0 = -\frac{\partial}{\partial z}(\mathbf{V}_R \bullet \nabla_h h) + \frac{\beta}{f} v_R, \tag{1.18}$$

which differs from the original β-spiral method (Stommel and Scott 1977) by the term, $-(\partial \mathbf{V}_R/\partial z) \bullet \nabla_h h$. Coats (1981) used (1.18) to calculate the absolute velocity in the northeastern Pacific Ocean.

1.3 Necessary Conditions for Inversion

After mathematical manipulation on (1.1), (1.2), (1.6), and (1.7) Needler (1982) obtained the following equation

$$\rho \mathbf{V}(x, y, z) = \frac{g\mathbf{k} \bullet (\nabla \rho \times \nabla q)}{\nabla (f \partial q/\partial z) \bullet (\nabla \rho \times \nabla q)} (\nabla \rho \times \nabla q), \tag{1.19}$$

to determine geostrophic velocity ($\mathbf{V}$) from density (ρ). Here, $\mathbf{k}$ is the unit vector along the z-axis. As pointed by Needler (1982), direct use of (1.19) for calculating $\mathbf{V}$ is almost impossible. The validity of the Needler's formula (1.19) requires

$$\nabla \rho \times \nabla q \neq 0, \tag{1.20a}$$

$$\nabla (f \partial q/\partial z) \neq 0, \tag{1.20b}$$

$$\nabla (f \partial q/\partial z) \bullet (\nabla \rho \times \nabla q) \neq 0. \tag{1.20c}$$

Vertical derivative of the left-hand side of (1.20a) gives,

$$f\frac{\partial}{\partial z}(\nabla\rho \times \nabla q) = \mathbf{k}\beta\frac{\partial(\rho, q)}{\partial(z, x)} + \nabla\rho \times \nabla\left(f\frac{\partial q}{\partial z}\right). \tag{1.21}$$

The condition of

$$\nabla(f\partial q/\partial z) = 0,$$

is equivalent to the disappearance of vertical turning of the horizontal component of the vector $\nabla\rho \times \nabla q$.

Three necessary conditions can be derived for determination of the absolute velocity $\mathbf{V}$ from the density field ρ using the Needler's formula. Equation (1.20a) implies noncoincidence of theρ-surface with the q surface (first necessary condition). Equation (1.20b) shows the existence of vertical turning of the horizontal component of the vector $\nabla\rho \times \nabla q$ (second necessary condition). Equation (1.20c) requires that the vector $\nabla\rho \times \nabla q$ should not exist on the iso-surface of $f\partial q/\partial z$ (third necessary condition). If the three necessary conditions are satisfied, the absolute velocity can be determined exclusively from the density field.

Questions and Exercises

(1) Starting from the mass conservation,

$$\mathbf{V} \bullet \nabla\rho = 0, \tag{E1.1}$$

and using the geostrophic and hydrostatic balances, derive the conservation of potential vorticity,

$$\mathbf{V} \bullet \nabla q = 0, \quad q = f\frac{\partial\rho}{\partial z}, \tag{E1.2}$$

Discuss the physical significance of the potential vorticity conservation.

(2) Derive the Needler's formula from (1.1), (1.2), (1.6), and (1.7),

$$\rho\mathbf{V}(x, y, z) = \frac{g\mathbf{k} \bullet (\nabla\rho \times \nabla q)}{\nabla(f\partial q/\partial z) \bullet (\nabla\rho \times \nabla q)}(\nabla\rho \times \nabla q), \tag{E1.3}$$

and list all the vectors in the Needler's formula, and find the relationships among these vectors.

(3) Does the level-of-no-motion exist? Why?

(4) Derive the formula for the β-spiral method,

$$\mathbf{V}_0 \bullet \nabla_2\frac{\partial h}{\partial z} - \frac{\beta}{f}v_0 = -\frac{\partial}{\partial z}(\mathbf{V}_R \bullet \nabla_2 h) + \frac{\beta}{f}v_R, \tag{E1.4}$$

from mass conservation of the box model (1.17). Discuss the difference between the β-spiral and box methods.

(5) What conditions can be drawn from the Needler's formula (E1.3)?

(6) Since the Needler's formula (E1.3) directly relates the absolute velocity to the density field, can it be used to compute the absolute velocity from density? If not, explain why?