3

Intuitionistic and Type-2 Fuzzy Logic

We describe in this chapter two new areas in fuzzy logic, type-2 fuzzy logic systems and intuitionistic fuzzy logic. Basically, a type-2 fuzzy set is a set in which we also have uncertainty about the membership function. Of course, type-2 fuzzy systems consist of fuzzy if-then rules, which contain type-2 fuzzy sets. We can say that type-2 fuzzy logic is a generalization of conventional fuzzy logic (type-1) in the sense that uncertainty is not only limited to the linguistic variables but also is present in the definition of the membership functions. On the other hand, intuitionistic fuzzy sets can also be considered an extension of type-1 fuzzy sets in the sense that intuitionistic fuzzy sets not only use the membership function, but also a non-membership function to represent the uncertainty of belonging to a fuzzy set.

Fuzzy Logic Systems are comprised of rules. Quite often, the knowledge that is used to build these rules is uncertain. Such uncertainty leads to rules whose antecedents or consequents are uncertain, which translates into uncertain antecedent or consequent membership functions (Karnik & Mendel 1998). Type-1 fuzzy systems (like the ones seen in the previous chapter), whose membership functions are type-1 fuzzy sets, are unable to directly handle such uncertainties. We describe in this chapter, type-2 fuzzy systems, in which the antecedent or consequent membership functions are type-2 fuzzy sets. Such sets are fuzzy sets whose membership grades themselves are type-1 fuzzy sets; they are very useful in circumstances where it is difficult to determine an exact membership function for a fuzzy set.

The original fuzzy logic, founded by Lotfi Zadeh, has been around for more than 30 years, and yet it is unable to handle uncertainties (Mendel, 2001). That the original fuzzy logic (type-1 fuzzy logic) cannot do this sounds paradoxical because the word “fuzzy” has the connotation of uncertainty. The expanded fuzzy logic (type-2 fuzzy logic) is able to handle uncertainties because it can model and minimize their effects.

In what follows, we shall first introduce the basic concepts of type-2 fuzzy sets, and type-2 fuzzy reasoning. Then we will introduce and compare the different types of fuzzy inference systems that have been employed in various

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applications. We will also consider briefly type-2 fuzzy logic systems and the comparison to type-1 fuzzy systems. Then we will describe the concept of an intuitionistic fuzzy set and its applications. We will also describe intuitionistic fuzzy inference systems. Finally, we will address briefly the features and problems of fuzzy modeling with intuitionistic and type-2 fuzzy logic, which is concerned with the construction of fuzzy inference systems for modeling a given target system.

3.1 Type-2 Fuzzy Sets

The concept of a type-2 fuzzy set, was introduced by Zadeh (1975) as an extension of the concept of an ordinary fuzzy set (henceforth called a "type-1 fuzzy set"). A type-2 fuzzy set is characterized by a fuzzy membership function, i.e., the membership grade for each element of this set is a fuzzy set in [0, 1], unlike a type-1 set where the membership grade is a crisp number in [0, 1]. Such sets can be used in situations where there is uncertainty about the membership grades themselves, e.g., an uncertainty in the shape of the membership function or in some of its parameters. Consider the transition from ordinary sets to fuzzy sets. When we cannot determine the membership of an element in a set as 0 or 1, we use fuzzy sets of type-1. Similarly, when the situation is so fuzzy that we have trouble determining the membership grade even as a crisp number in [0, 1], we use fuzzy sets of type-2.

This does not mean that we need to have extremely fuzzy situations to use type-2 fuzzy sets. There are many real-world problems where we cannot determine the exact form of the membership functions, e.g., in time series prediction because of noise in the data. Another way of viewing this is to consider type-1 fuzzy sets as a first order approximation to the uncertainty in the real-world. Then type-2 fuzzy sets can be considered as a second order approximation. Of course, it is possible to consider fuzzy sets of higher types but the complexity of the fuzzy system increases very rapidly. For this reason, we will only consider very briefly type-2 fuzzy sets. Lets consider some simple examples of type-2 fuzzy sets.

Example 3.1. Consider the case of a fuzzy set characterized by a Gaussian membership function with mean m and a standard deviation that can take values in $[\sigma_1, \sigma_2]$, i.e.,

$$\mu(x) = \exp\{-1/2[(x-m)/\sigma]^2\}\ ; \quad \sigma \in [\sigma_1, \sigma_2] \tag{3.1}$$

Corresponding to each value of σ, we will get a different membership curve (see Fig. 3.1). So, the membership grade of any particular x (except $x = m$) can take any of a number of possible values depending upon the value of σ, i.e., the membership grade is not a crisp number, it is a fuzzy set. Figure 3.1 shows the domain of the fuzzy set associated with $x = 0.7$; however, the membership function associated with this fuzzy set is not shown in the figure.

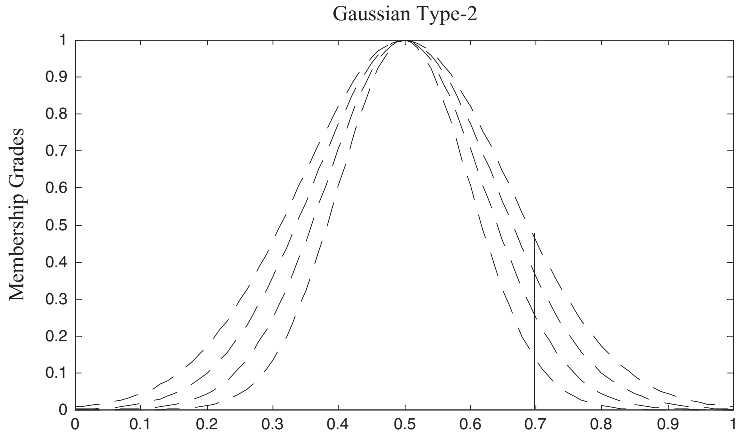

Fig. 3.1. A type-2 fuzzy set representing a type-1 fuzzy set with uncertain standard deviation

Example 3.2. Consider the case of a fuzzy set with a Gaussian membership function having a fixed standard deviation σ, but an uncertain mean, taking values in $[m_1,\ m_2]$, i.e.,

$$\mu(x) = \exp\left\{-1/2[(x-m)/\sigma]^2\right\} ; \quad m \in [m_1, m_2] \tag{3.2}$$

Again, $\mu(x)$ is a fuzzy set. Figure 3.2 shows an example of such a set.

Example 3.3. Consider a type-1 fuzzy set characterized by a Gaussian membership function (mean M and standard deviation σ_x), which gives one crisp membership $m(x)$ for each input $x \in X$, where

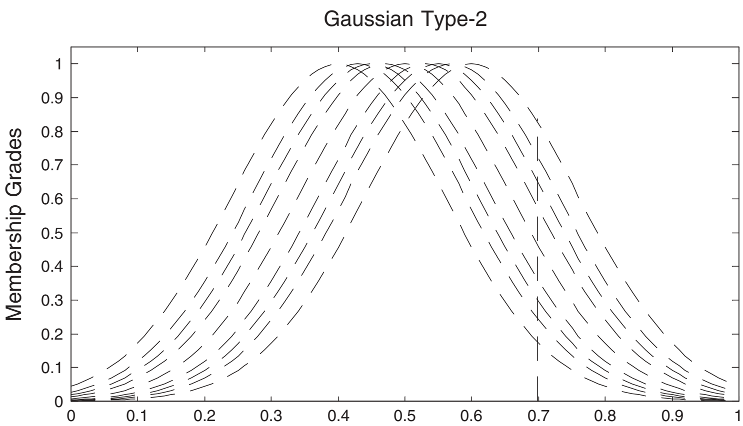

Fig. 3.2. A type-2 fuzzy set representing a type-1 fuzzy set with uncertain mean. The mean is uncertain in the interval [0.4, 0.6]

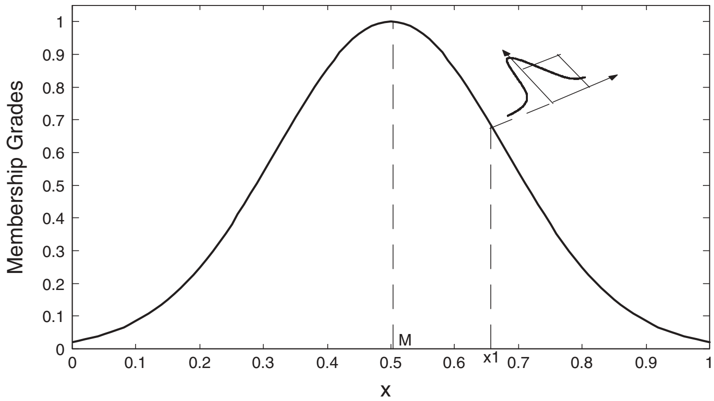

Fig. 3.3. A type-2 fuzzy set in which the membership grade of every domain point is a Gaussian type-1 set

$$m(x) = \exp\{-1/2[(x - M)/\sigma_x]^2\} \tag{3.3}$$

This is shown in Fig. 3.3. Now, imagine that this membership of x is a fuzzy set. Let us call the domain elements of this set "primary memberships" of x (denoted by μ_1) and membership grades of these primary memberships "secondary memberships" of x [denoted by $\mu_2(x, \mu_1)$]. So, for a fixed x, we get a type-1 fuzzy set whose domain elements are primary memberships of x and whose corresponding membership grades are secondary memberships of x. If we assume that the secondary memberships follow a Gaussian with mean $m(x)$ and standard deviation σ_m, as in Fig. 3.3, we can describe the secondary membership function for each x as

$$\mu_2(x, \mu_1) = e - 1/2\,[(\mu_1 - m(x))/\sigma_m]^2 \tag{3.4}$$

where $\mu_1 \in [0,\ 1]$ and m is as in (3.3).

We can formally define these two kinds of type-2 sets as follows.

Definition 3.1. ***Gaussian type-2***
A Gaussian type-2 fuzzy set is one in which the membership grade of every domain point is a Gaussian type-1 set contained in $[0,\ 1]$.

Example 3.3 shows an example of a Gaussian type-2 fuzzy set. Another way of viewing type-2 membership functions is in a three-dimensional fashion, in which we can better appreciate the idea of type-2 fuzziness. In Fig. 3.4 we have a three-dimensional view of a type-2 Gaussian membership function.

Definition 3.2. ***Interval type-2***
An interval type-2 fuzzy set is one in which the membership grade of every domain point is a crisp set whose domain is some interval contained in $[0, 1]$.

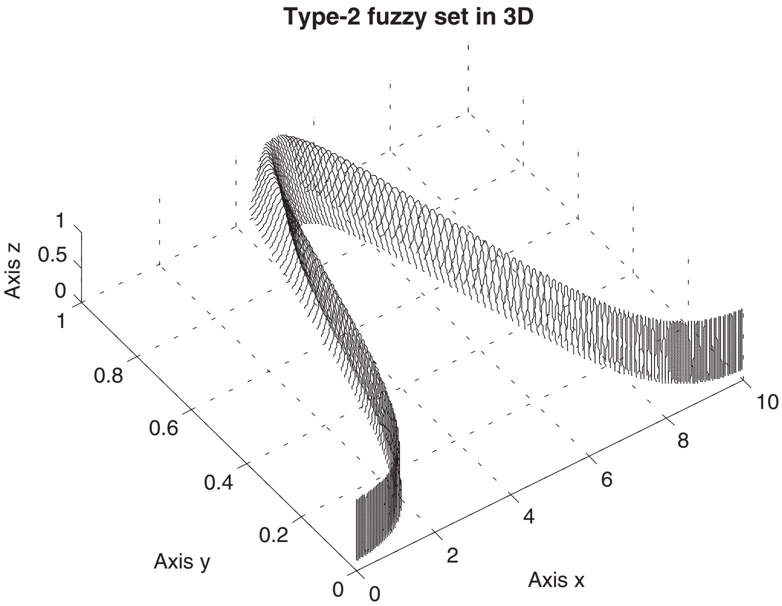

Fig. 3.4. Three-dimensional view of a type-2 membership function

Example 3.1 shows an example of an interval type-2 fuzzy set.

We will give some useful definitions on type-2 fuzzy sets in the following lines.

Definition 3.3. ***Footprint of uncertainty***
Uncertainty in the primary memberships of a type-2 fuzzy set, $\tilde{A}$, consists of a bounded region that we call the "footprint of uncertainty" (FOU). Mathematically, it is the union of all primary membership functions (Mendel, 2001).

We show as an illustration in Fig. 3.5 the footprint of uncertainty for a type-2 Gaussian membership function. This footprint of uncertainty can be obtained by projecting in two dimensions the three-dimensional view of the type-2 Gaussian membership function.

Definition 3.4. ***Upper and lower membership functions***
An "upper membership function" and a "lower membership functions" are two type-1 membership functions that are bounds for the FOU of a type-2 fuzzy set $\tilde{A}$. The upper membership function is associated with the upper bound of $FOU(\tilde{A})$. The lower membership function is associated with the lower bound of $FOU(\tilde{A})$.

We illustrate the concept of upper and lower membership functions as well as the footprint of uncertainty in the following example.

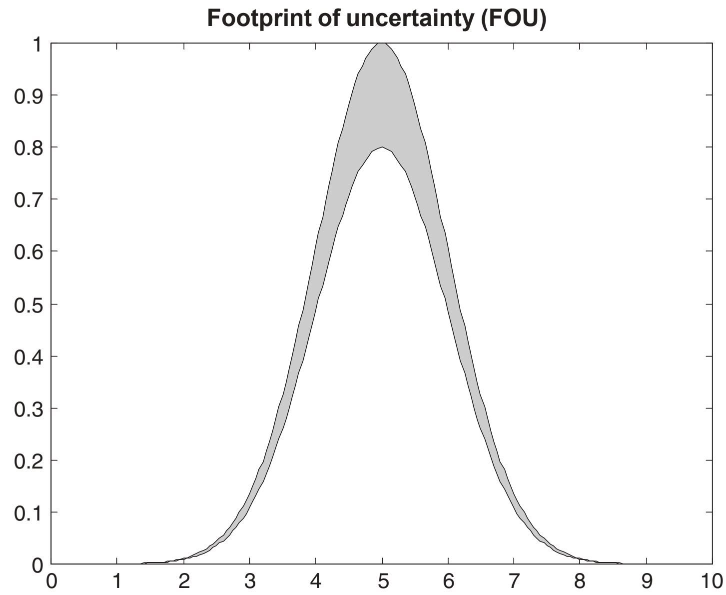

Fig. 3.5. Footprint of uncertainty of a sample type-2 Gaussian membership function

Example 3.4. Gaussian primary MF with uncertain standard deviation

For the Gaussian primary membership function with uncertain standard deviation (Fig. 3.1), the upper membership function is

$$\text{upper}(\text{FOU}(\tilde{A})) = N(m, \sigma_2; x) \tag{3.5}$$

And the lower membership function is

$$\text{lower}(\text{FOU}(\tilde{A})) = N(m, \sigma_1; x)\ . \tag{3.6}$$

We will describe the operations and properties of type-2 fuzzy sets in the following section.

3.2 Operations of Type-2 Fuzzy Sets

In this section we describe the set theoretic operations of type-2 fuzzy sets. We are interested in the case of type-2 fuzzy sets, $\tilde{A}_i$ $(i = 1, \ldots, r)$, whose secondary membership functions are type-1 fuzzy sets. To compute the union, intersection, and complement of type-2 fuzzy sets, we need to extend the binary operations of minimum (or product) and maximum, and the unary operation of negation, from crisp numbers to type-1 fuzzy sets, because at each x, $\mu_{\tilde{A}i}(x,\ u)$ is a function (unlike the type-1 case, where $\mu_{\tilde{A}i}(x)$ is a crisp number). The tool for computing the union, intersection, and complement of type-2 fuzzy sets is Zadeh's extension principle (Zadeh, 1975).

Consider two type-2 fuzzy sets $\tilde{A}_1$ and $\tilde{A}_2$, i.e.,

$$\tilde{A}_1 = \int_x \mu_{\tilde{A}1}(x)/x \tag{3.7}$$

and

$$\tilde{A}_2 = \int_x \mu_{\tilde{A}2}(x)/x \tag{3.8}$$

In this section, we focus our attention on set theoretic operations for such general type-2 fuzzy sets.

Definition 3.5. ***Union of type-2 fuzzy sets***
The union of $\tilde{A}_1$ and $\tilde{A}_2$ is another type-2 fuzzy set, just as the union of type-1 fuzzy sets A_1 and A_2 is another type-1 fuzzy set. More formally, we have the following expression

$$\tilde{A}_1 \cup \tilde{A}_2 = \int_{x \in X} \mu_{\tilde{A}1 \cup \tilde{A}2}(x)/x \tag{3.9}$$

We can explain (3.9) by the "join" operation (Mendel, 2001). Basically, the join between two secondary membership functions must be performed between every possible pair of primary memberships. If more than one combination of pairs gives the same point, then in the join we keep the one with maximum membership grade. We will consider a simple example to illustrate the union operation. In Fig. 3.6 we plot two type-2 Gaussian membership functions, and the union is shown in Fig. 3.7.

Definition 3.6. ***Intersection of type-2 fuzzy sets***
The intersection of $\tilde{A}_1$ and $\tilde{A}_2$ is another type-2 fuzzy set, just as the intersection of type-1 fuzzy sets A_1 and A_2 is another type-1 fuzzy set. More formally, we have the following expression

$$\tilde{A}_1 \cap \tilde{A}_2 = \int_{x \in X} \mu_{\tilde{A}1 \cap \tilde{A}2}(x)/x \tag{3.10}$$

We illustrate the intersection of two type-2 Gaussian membership functions in Fig. 3.8.

We can explain (3.10) by the "meet" operation (Mendel, 2001). Basically, the meet between two secondary membership functions must be performed between every possible pair of primary memberships. If more than one combination of pairs gives the same point, then in the meet we keep the one with maximum membership grade.

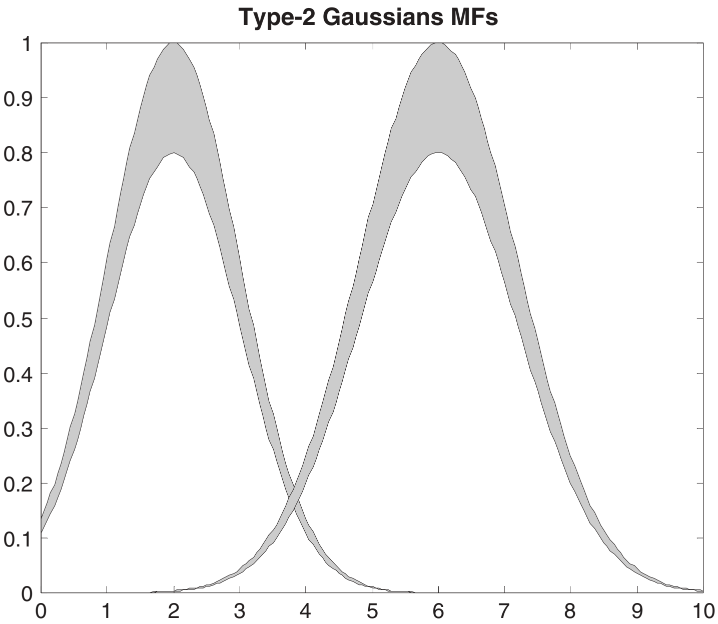

Fig. 3.6. Two sample type-2 Gaussian membership functions

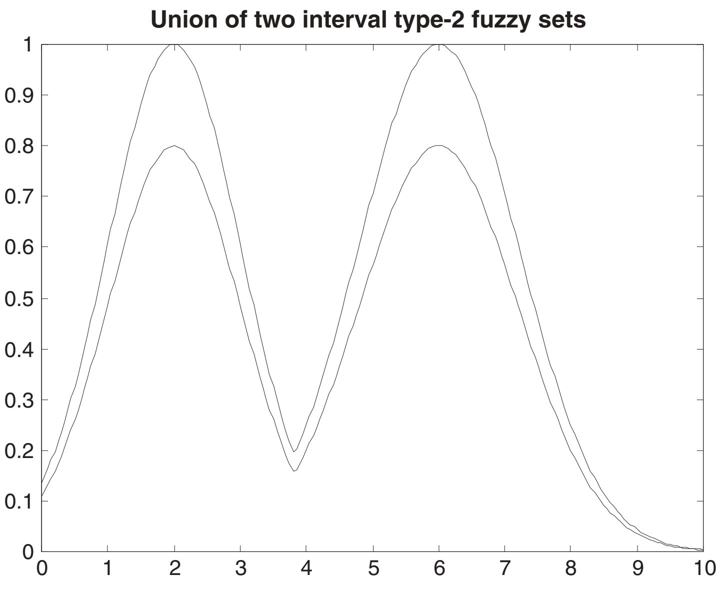

Fig. 3.7. Union of the two Gaussian membership functions

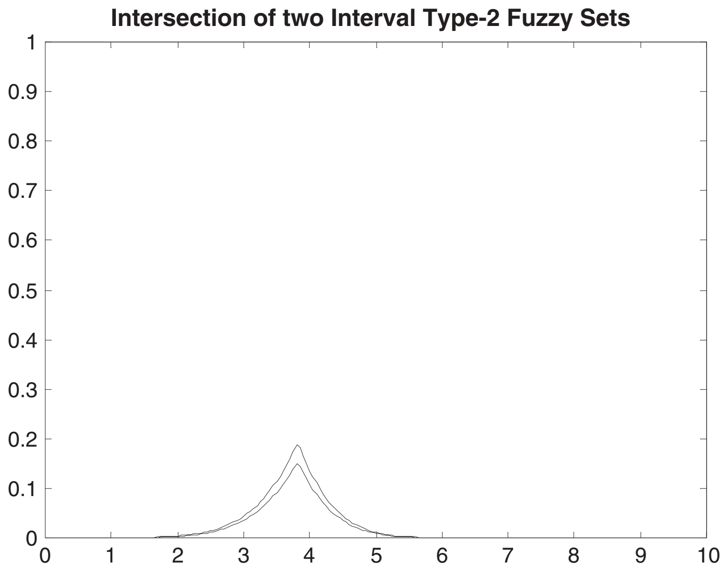

Fig. 3.8. Intersection of two type-2 Gaussian membership functions

Definition 3.7. ***Complement of a type-2 fuzzy set***
The complement of set $\tilde{A}$ is another type-2 fuzzy set, just as the complement of type-1 fuzzy set A is another type-1 fuzzy set. More formally we have

$$\tilde{A}' = \int_x \mu_{\tilde{A}'1}(x)/x \tag{3.11}$$

where the prime denotes complement in the above equation. In this equation $\mu_{\tilde{A}'1}$ is a secondary membership function, i.e., at each value of x $\mu_{\tilde{A}'1}$ is a function (unlike the type-1 case where, at each value of $x, \mu_{\tilde{A}'1}$ is a point value).

Example 3.5. Type-2 fuzzy set operations
In this example we illustrate the union, intersection and complement operations for two type-2 fuzzy sets $\tilde{A}_1$ and $\tilde{A}_2$, and for a particular element x for which the secondary membership functions in these two sets are $\mu_{\tilde{A}1}(x) = 0.5/0.1{+}0.8/0.2$ and $\mu_{\tilde{A}2}(x) = 0.4/0.5 + 0.9/0.9$. Using in the operations the minimum t-norm and the maximum t-conorm, we have the following results:

$$\begin{aligned}
\mu_{\tilde{A}1\cup\tilde{A}2}(x) &= \mu_{\tilde{A}1}(x) \cup \mu_{\tilde{A}2}(x) = (0.5/0.1 + 0.8/0.2) \cup (0.4/0.5 + 0.9/0.9) \\
&= (0.5 \wedge 0.4)/(0.1 \vee 0.5) + (0.5 \wedge 0.9)/(0.1 \vee 0.9) \\
&\quad +(0.8 \wedge 0.4)/(0.2 \vee 0.5) + (0.8 \wedge 0.9)/(0.2 \vee 0.9) \\
&= 0.4/0.5 + 0.5/0.9 + 0.4/0.5 + 0.8/0.9
\end{aligned}$$

$$
\begin{aligned}
&= \max\{0.4, 0.4\}/0.5 + \max\{0.5, 0.8\}/0.9 \\
&= 0.4/0.5 + 0.8/0.9
\end{aligned}
$$

$$
\begin{aligned}
\mu_{\tilde{A}1\cap\tilde{A}2}(x) &= \mu_{\tilde{A}1}(x) \cap \mu_{\tilde{A}2}(x) = (0.5/0.1 + 0.8/0.2) \cap (0.4/0.5 + 0.9/0.9) \\
&= (0.5 \wedge 0.4)/(0.1 \wedge 0.5) + (0.5 \wedge 0.9)/(0.1 \wedge 0.9) \\
&\quad +(0.8 \wedge 0.4)/(0.2 \wedge 0.5) + (0.8 \wedge 0.9)/(0.2 \wedge 0.9) \\
&= 0.4/0.1 + 0.5/0.1 + 0.4/0.2 + 0.8/0.2 \\
&= \max\{0.4, 0.5\}/0.1 + \max\{0.4, 0.8\}/0.2 \\
&= 0.5/0.1 + 0.8/0.2
\end{aligned}
$$

$$
\mu_{\tilde{A}'1}(x) = 0.5/(1 - 0.1) + 0.8/(1 - 0.2) = 0.5/0.9 + 0.8/0.8\ .
$$

3.3 Type-2 Fuzzy Systems

The basics of fuzzy logic do not change from type-1 to type-2 fuzzy sets, and in general, will not change for any type-n (Karnik & Mendel 1998). A higher-type number just indicates a higher "degree of fuzziness". Since a higher type changes the nature of the membership functions, the operations that depend on the membership functions change; however, the basic principles of fuzzy logic are independent of the nature of membership functions and hence, do not change. Rules of inference like Generalized Modus Ponens or Generalized Modus Tollens continue to apply.

The structure of the type-2 fuzzy rules is the same as for the type-1 case because the distinction between type-2 and type-1 is associated with the nature of the membership functions. Hence, the only difference is that now some or all the sets involved in the rules are of type-2. In a type-1 fuzzy system, where the output sets are type-1 fuzzy sets, we perform defuzzification in order to get a number, which is in some sense a crisp (type-0) representative of the combined output sets. In the type-2 case, the output sets are type-2; so we have to use extended versions of type-1 defuzzification methods. Since type-1 defuzzification gives a crisp number at the output of the fuzzy system, the extended defuzzification operation in the type-2 case gives a type-1 fuzzy set at the output. Since this operation takes us from the type-2 output sets of the fuzzy system to a type-1 set, we can call this operation "type reduction" and call the type-1 fuzzy set so obtained a "type-reduced set". The type-reduced fuzzy set may then be defuzzified to obtain a single crisp number; however, in many applications, the type-reduced set may be more important than a single crisp number.

Type-2 sets can be used to convey the uncertainties in membership functions of type-1 fuzzy sets, due to the dependence of the membership functions on available linguistic and numerical information. Linguistic information (e.g. rules from experts), in general, does not give any information about the shapes of the membership functions. When membership functions are determined or tuned based on numerical data, the uncertainty in the numerical data, e.g., noise, translates into uncertainty in the membership functions. In all such cases, any available information about the linguistic/numerical uncertainty can be incorporated in the type-2 framework. However, even with all of the advantages that fuzzy type-2 systems have, the literature on the applications of type-2 sets is scarce. Some examples are Yager (1980) for decision making, and Wagenknecht & Hartmann (1998) for solving fuzzy relational equations. We think that more applications of type-2 fuzzy systems will come in the near future as the area matures and the theoretical results become more understandable for the general public in the fuzzy arena.

3.3.1 Singleton Type-2 Fuzzy Logic Systems

This section discusses the structure of a singleton type-2 Fuzzy Logic Systems (FLS), which is a system that accounts for uncertainties about the antecedents or consequents in rules, but does not explicitly account for input measurement uncertainties. More complicated (but, more versatile) non-singleton type-2 FLSs, which account for both types of uncertainties, are discussed later.

The basics of fuzzy logic do not change from type-1 to type-2 fuzzy sets, and in general will not change for type-n. A higher type number just indicates a higher degree of fuzziness. Since a higher type changes the nature of the membership functions, the operations that depend on the membership functions change, however, the basic principles of fuzzy logic are independent of the nature of membership functions and hence do not change. Rules of inference, like Generalized Modus Ponens, continue to apply.

A general type-2 FLS is shown in Fig. 3.9. As discussed before a type-2 FLS is very similar to type-1 FLS, the major structural difference being that the defuzzifier block of a type-1 FLS is replaced by the output processing block in type-2 FLS. That block consists of type-reduction followed by defuzzification.

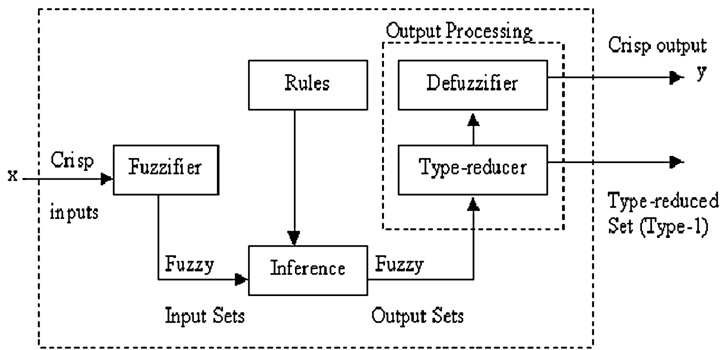

Fig. 3.9. Type-2 Fuzzy Logic System

During our development of a type-2 FLS, we assume that all the antecedent and consequent sets in rules are type-2, however, this need not necessarily be the case in practice. All results remain valid as long as long as just one set is type-2. This means that a FLS is type-2 as long as any one of its antecedent or consequent sets is type-2.

In the type-1 case, we generally have fuzzy if-then rules of the form

$$R^l: \text{ IF } x_1 \text{ is } A_1^l \text{ and } \ldots \; x_p \text{ is } A_p^l, \text{ THEN } y \text{ is } Y^l \quad l = 1, \ldots, M \tag{3.12}$$

As mentioned earlier, the distinction between type-1 and type-2 is associated with the nature of the membership functions, which is not important when forming the rules. The structure of the rules remains exactly the same in the type-2 case, but now some or all of the sets involved are type-2.

Consider a type-2 FLS having r inputs $x_1 \in X_1, \ldots, x_r \in X_r$ and one output $y \in Y$. As in the type-1 case, we can assume that there are M rules; but, in the type-2 case the lth rule has the form

$$R^l: \text{ IF } x_1 \text{ is } \tilde{A}_1^l \text{ and } \ldots \; x_p \text{ is } \tilde{A}_p^l, \text{ THEN } y \text{ is } \hat{Y}^l \quad 1 = 1, \ldots, M \tag{3.13}$$

This rule represents a type-2 fuzzy relation between the input space $X_1 \times \ldots \times X_r$, and the output space, Y, of the type-2 fuzzy system.

In a type-1 FLS the inference engine combines rules and gives a mapping from input type-1 fuzzy sets to output type-1 fuzzy sets. Multiple antecedents in rules are combined by the t-norm. The membership grades in the input sets are combined with those in the output sets using composition. Multiple rules may be combined using the t-conorm or during defuzzification by weighted summation. In the type-2 case the inference process is very similar. The inference engine combines rules and gives a mapping from input type-2 fuzzy sets to output type-2 fuzzy sets. To do this one needs to compute unions and intersections of type-2 fuzzy sets, as well as compositions of type-2 relations.

In the type-2 fuzzy system (Fig. 3.9), as in the type-1 fuzzy system, crisp inputs are first fuzzified into fuzzy input sets that then activate the inference block, which in the present case is associated with type-2 fuzzy sets. In this section, we describe singleton fuzzification and the effect of such fuzzification on the inference engine. The "fuzzifier" maps a crisp point $\mathbf{x} = (x_1, \ldots, x_r)^T \in X_1 \times X_2 \ldots \times X_r \equiv \mathbf{X}$ into a type-2 fuzzy set $\tilde{A}_x$ in $\mathbf{X}$.

The type-2 output of the inference engine shown in Fig. 3.9 must be processed next by the output processor, the first operation of which is type-reduction. Type-reduction methods include (Mendel, 2001): centroid, center-of-sums, height, modified height, and center-of-sets. Lets assume that we perform centroid type-reduction. Then each element of the type-reduced set is the centroid of some embedded type-1 set for the output type-2 set of the FLS. Each of these embedded sets can be thought of as an output set of an associated type-1 FLS, and, correspondingly, the type-2 FLS can be viewed of as a collection of many different type-1 FLSs. Each type-1 FLS is embedded in the type-2 FLS; hence, the type-reduced set is a collection of the outputs

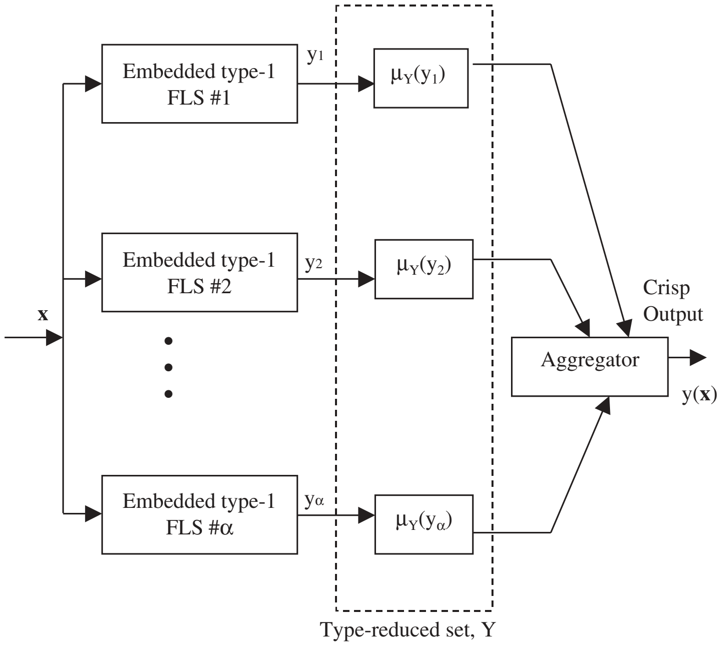

Fig. 3.10. A type-2 FLS viewed as a collection of embedded type-1 FLSs

of all of the embedded type-1 FLSs (see Fig. 3.10). The type-reduced set lets us represent the output of the type-2 FLS as a fuzzy set rather than as a crisp number, which is something that cannot be done with a type-1 fuzzy system.

Referring to Fig. 3.10, when the antecedent and consequent membership functions of the type-2 FLS have continuous domains, the number of embedded sets is uncountable. Figure 3.10 shows a situation in which we have assumed that the membership functions have discrete (or discretized) domains. The memberships in the type-reduced set, $\mu_Y(y_i)$, represent the level of uncertainty associated with each embedded type-1 FLS. A crisp output can be obtained by aggregating the outputs of all embedded type-1 FLSs by, e.g., finding the centroid of the type-reduced set.

If all of the type-2 uncertainties were to disappear, the secondary membership functions for all antecedents and consequents would each collapse to a single point, which shows that the type-2 FLS collapses to a type-1 FLS.

If we think of a type-2 FLS as a "perturbation" of a type-1 FLS, due to uncertainties in their membership functions, then the type-reduced set of the type-2 FLS can be thought of as representing the uncertainty in the crisp output due to the perturbation. Some measure of the spread of the type-reduced set may then be taken to indicate the possible variation in the crisp output due to the perturbation. This is analogous to using confidence intervals in a stochastic-uncertainty situation.

We defuzzify the type-reduced set to get a crisp output from the type-2 FLS. The most natural way to do this seems to be finding the centroid of the type-reduced set. Finding the centroid is equivalent to finding the weighted average of the outputs of all the type-1 FLSs that are embedded in the type-2 FLS, where the weights correspond to the memberships in the type-reduced set (see Fig. 3.10). If the type-reduced set Y for an input $\mathbf{x}$ is discretized or is discrete and consists of α points, then the expression for its centroid is

$$y(\mathbf{x}) = \left[\sum_{k=1}^{\alpha} y_k \mu_y(y_k)\right] \Bigg/ \left[\sum_{k=1}^{\alpha} \mu_y(y_k)\right] \tag{3.14}$$

If α is large then data storage may be a problem for the computation of (3.14). This equation can, however, be evaluated using parallel processing, in this case data storage will not be problem. Currently, however, most researchers still depend on software for simulations and cannot make use of parallel processing. We can, however, use a recursive method to vastly reduce the memory required for storing the data that are needed to compute the defuzzification output. From (3.14), we can calculate

$$A(i) = A(i-1) + y_i \mu_y(y_i) A(0) = 0 \tag{3.15}$$

and

$$B(i) = B(i-1) + y_i \mu_y(y_i) B(0) = 0 \tag{3.16}$$

for $i = 1, \ldots, \alpha$. With these formulas we just need to store A and B during each iteration.

From our previous discussions about the five elements that comprise the Fig. 3.9 type-2 FLS, we see that there are many possibilities to choose from, even more than for a type-1 FLS. To begin, we must decide on the kind of defuzzification (singleton or non-singleton). We must also choose a FOU for each type-2 membership function, decide on the functional forms for both the primary and secondary membership functions, and choose the parameters of the membership functions (fixed a-priori or tuned during a training procedure). Then we need to choose the composition (max-min, max-product), implication (minimum, product), type-reduction method (centroid, center-of-sums, height, modified height, center-of-sets), and defuzzifier. Clearly, there is an even greater richness among type-2 FLSs than there is among type-1 FLSs. In other words, there are more design degrees of freedom associated with a type-2 FLS than with a type-1 FLS; hence, a type-2 FLS has the potential to outperform a type-1 FLS because of the extra degrees of freedom.

3.3.2 Non-Singleton Fuzzy Logic Systems

A non-singleton FLS is one whose inputs are modeled as fuzzy numbers. A type-2 FLS whose inputs are modeled as type-1 fuzzy numbers is referred to as "type-1 non-singleton type-2 FLS". This kind of a fuzzy system not only accounts for uncertainties about either the antecedents or consequents in rules, but also accounts for input measurement uncertainties.

A type-1 non-singleton type-2 FLS is described by the same diagram as in singleton type-2 FLS, see Fig. 3.9. The rules of a type-1 non-singleton type-2 FLS are the same as those for the singleton type-2 FLS. What are different is the fuzzifier, which treats the inputs as type-1 fuzzy sets, and the effect of this on the inference block. The output of the inference block will again be a type-2 fuzzy set; so, the type-reducers and defuzzifier that we described for a singleton type-2 FLS apply as well to a type-1 non-singleton type-2 FLS.

We can also have a situation in which the input are modeled as type-2 fuzzy numbers. This situation can occur, e.g., in time series forecasting when the additive measurement noise is non-stationary. A type-2 FLS whose inputs are modeled as type-2 fuzzy numbers is referred to as "type-2 non-singleton type-2 FLS".

A type-2 non-singleton type-2 FLS is described by the same diagram as in singleton type-2 FLS, see Fig. 3.9. The rules of a type-2 non-singleton type-2 FLS are the same as those for a type-1 non-singleton type-2 FLS, which are the same as those for a singleton type-2 FLS. What is different is the fuzzifier, which treats the inputs as type-2 fuzzy sets, and the effect of this on the inference block. The output of the inference block will again be a type-2 fuzzy set; so, the type-reducers and defuzzifier that we described for a type-1 non-singleton type-2 FLS apply as well to a type-2 non-singleton type-2 FLS.

3.3.3 Sugeno Type-2 Fuzzy Systems

All of our previous FLSs were of the Mamdani type, even though we did not refer to them as such. In this section, we will need to distinguish between the two kinds of FLSs, we refer to our previous FLSs as "Mamdani" FLSs. Both kinds of FLS are characterized by if-then rules and have the same antecedent structures. They differ in the structures of their consequents. The consequent of a Mamdani rule is a fuzzy set, while the consequent of a Sugeno rule is a function.

A type-1 Sugeno FLS was proposed by Takagi & Sugeno (1985), and Sugeno & Kang (1988), in an effort to develop a systematic approach to generating fuzzy rules from a given input-output data set. We will consider in this section the extension of first-order type-1 Sugeno FLS to its type-2 counterpart, with emphasis on interval sets.

Consider a type-2 Sugeno FLS having r inputs $x_1 \in X_1, \ldots, x_r \in X_r$ and one output $y \in Y$. A type-2 Sugeno FLS is also described by fuzzy if-then rules that represent input-output relations of a system. In a general first-order type-2 Sugeno model with a rule base of M rules, each having r antecedents, the ith rule can be expressed as

$$R^l : \text{ IF } x_1 \text{ is } \tilde{A}_1^l \text{ and } \ldots\ x_p \text{ is } \tilde{A}_p^l, \text{ THEN } Y^i = C_0^i + C_1^i x_1 + \cdots + C_r^i x_r \tag{3.17}$$

where $i = 1, \ldots, M; C_j^i (j = 1, \ldots, r)$ are consequent type-1 fuzzy sets; Y^i, the output of the ith rule, is also a type-1 fuzzy set (because it is a linear combination of type-1 fuzzy sets); and $\tilde{A}_k^i$ $(k = 1, \ldots, r)$ are type-2 antecedent fuzzy sets. These rules let us simultaneously account for uncertainty about antecedent membership functions and consequent parameter values. For a type-2 Sugeno FLS there is no need for type-reduction, just as there is no need for defuzzification in a type-1 Sugeno FLS.

3.4 Introduction to Intuitionistic Fuzzy Logic

The intuitionistic fuzzy sets where defined as an extension of the ordinary fuzzy sets (Atanassov, 1999). As opposed to a fuzzy set in X (Zadeh, 1971), given by

$$B = \{(x, \mu_B(x)) \mid x \in X\} \tag{3.18}$$

where $\mu_B : X \to [0, 1]$ is the membership function of the fuzzy set B, an intuitionistic fuzzy set A is given by

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\} \tag{3.19}$$

where $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ are such that

$$0 \leq \mu_A + \nu_A \leq 1 \tag{3.20}$$

and $\mu_A(x); \nu_A(x) \in [0, 1]$ denote a degree of membership and a degree of non-membership of $x \in A$, respectively.

For each intuitionistic fuzzy set in X, we will call

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{3.21}$$

a "hesitation margin" (or an "intuitionistic fuzzy index") of $x \in A$ and, it expresses a hesitation degree of whether x belongs to A or not. It is obvious that $0 \leq \pi_A(x) \leq 1$, for each $x \in X$.

On the other hand, for each fuzzy set B in X, we evidently have that

$$\pi_B(x) = 1 - \mu_B(x) - [1 - \mu_B(x)] = 0 \text{ for each } x \in X\ . \tag{3.22}$$

Therefore, if we want to fully describe an intuitionistic fuzzy set, we must use any two functions from the triplet (Szmidt & Kacprzyk, 2002):

- Membership function,
- Non-membership function,
- Hesitation margin.

In other words, the application of intuitionistic fuzzy sets instead of fuzzy sets means the introduction of another degree of freedom into a set description (i.e. in addition to μ_A we also have ν_A or π_A).

Since the intuitionistic fuzzy sets being a generalization of fuzzy sets give us an additional possibility to represent imperfect knowledge, they can make it possible to describe many real problems in a more adequate way.

Basically, intuitionistic fuzzy sets based models maybe adequate in situations when we face human testimonies, opinions, etc. involving two (or more) answers of the type (Szmidt & Kacprzyk, 2002):

- Yes,
- No,
- I do not know, I am not sure, etc.

Voting can be a good example of such a situation, as human voters may be divided into three groups of those who:

- Vote for,
- Vote against,
- Abstain or give invalid votes.

This third group is of great interest from the point of view of, say, customer behavior analysis, voter behavior analysis, etc., because people from this third undecided group after proper enhancement (eg., different marketing activities) can finally become sure, i.e. become persons voting for (or against), customers wishing to buy products advertised, etc.

3.5 Intuitionistic Fuzzy Inference Systems

Fuzzy inference in intuitionistic systems has to consider the fact that we have the membership μ functions as well as the non-membership ν functions. In this case, we propose that the conclusion of an intuitionistic fuzzy system can be a linear combination of the results of two classical fuzzy systems, one for the membership functions and the other for the non-membership functions.

Assume that IFS is the output of an intuitionistic fuzzy system, then with the following equation we can calculate the total output as a linear combination:

$$\mathrm{IFS} = (1-\pi)\mathrm{FS}_{\mu} + \pi\mathrm{FS}_{\nu} \tag{3.23}$$

where FS_{μ} is the traditional output of a fuzzy system using the membership function μ, and FS_{ν} is the output of a fuzzy system using the non-membership function ν. Of course (3.23) for $\pi = 0$ will reduce to the output of a traditional fuzzy system, but for other values of π the result of IFS will be different as we are now taking into account the hesitation margin π.

The advantage of this method for computing the output IFS of an intuitionistic fuzzy system is that we can use our previous machinery of traditional fuzzy systems for computing FS_{μ} and FS_{ν}. Then, we only perform a weighted average of both results to obtain the final output IFS of the intuitionistic fuzzy inference system. We consider below a simple example to illustrate these ideas.

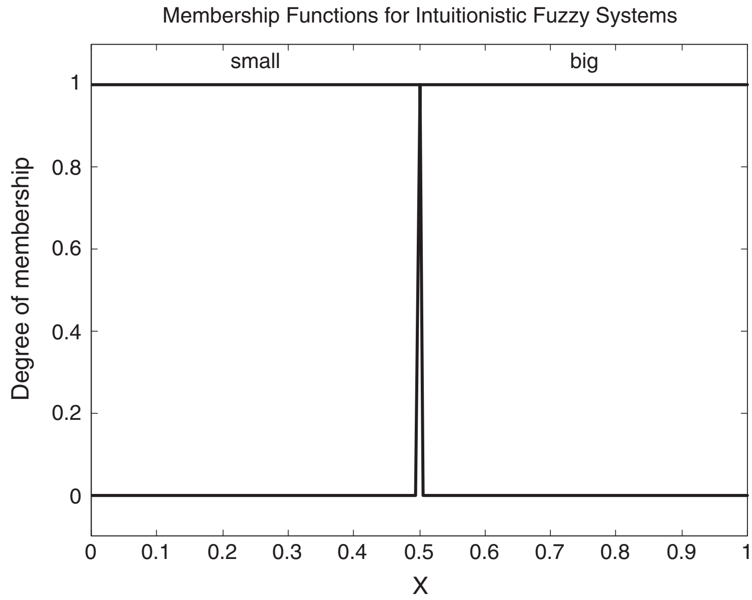

Fig. 3.11. Membership functions for the "small" and "big" linguistic values of the input variable

Example 3.6. : Let us assume that we have a simple intuitionistic fuzzy system of only two rules:

$$\begin{array}{ll} \text{IF } x \text{ is small} & \text{THEN } y \text{ is big} \\ \text{IF } x \text{ is big} & \text{THEN } y \text{ is small} \end{array}$$

We will consider for simplicity uniform rectangular membership functions for both linguistic variables. We show in Fig. 3.11 the membership functions for the linguistic values "small" and "big" of the input linguistic variable. We also show in Fig. 3.12 the non-membership functions for the linguistic values of the output variable. It is clear from Fig. 3.12 that in this case the membership and non-membership functions are not complementary, which is due to the fact that we have an intuitionistic fuzzy system.

From Fig. 3.12 we can clearly see that the hesitation margin π is 0.05 for both cases. As a consequence (3.22) can be written for our example as follows:

$$\mathrm{IFS} = 0.95\mathrm{FS}_{\mu} + 0.05\mathrm{FS}_{\nu} \tag{3.24}$$

Now, let us assume that we want to calculate the output of the fuzzy system for a given input value of $x = 0.45$. In this case, we have that x is small with $\mu = 1$ and x is not small with $\nu = 0$, and x is big with $\mu = 0$ and x is not big with $\nu = 0.5$. As a consequence of these facts we have that,

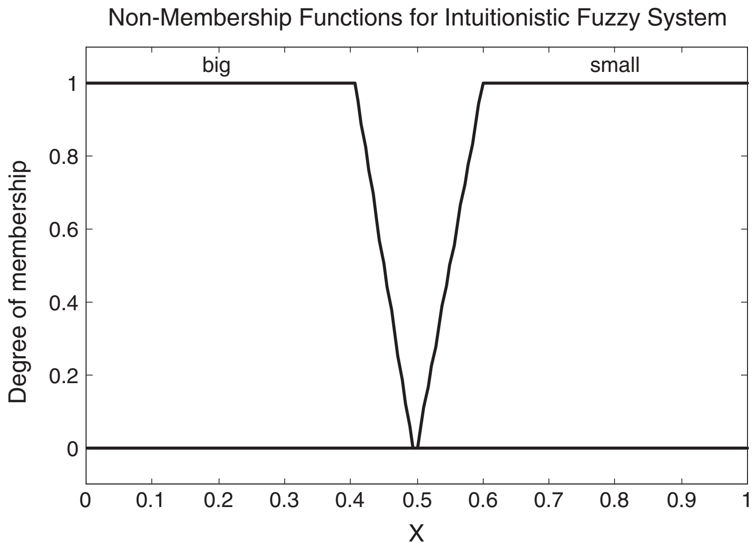

Fig. 3.12. Non-membership functions for the "small" and "big" linguistic values of the output variable

$$\mathrm{IFS} = \mathrm{IFS}_{\mathrm{small}} + \mathrm{IFS}_{\mathrm{big}}$$
$$\mathrm{IFS} = 0.95\mathrm{FS}_{\mu\,\mathrm{small}} + 0.05\mathrm{FS}_{\nu\,\mathrm{small}} + 0.95\mathrm{FS}_{\mu\,\mathrm{big}} + 0.05\mathrm{FS}_{\nu\,\mathrm{big}}$$
$$\mathrm{IFS} = 0.95\mathrm{FS}_{\mu\,\mathrm{small}} + 0.05\mathrm{FS}_{\nu\,\mathrm{big}}$$
$$\mathrm{IFS} = 0.95(0.75) + 0.05(0.765)$$
$$\mathrm{IFS} = 0.74075$$

Of course, we can compare this intuitionistic fuzzy output with the traditional one (of 0.75), the difference between these two output values is due to the hesitation margin. We have to mention that in this example the difference is small because the hesitation margin is also small. We show in Table 3.1 the results of the intuitionistic fuzzy system for several input values.

We can appreciate from Table 3.1 the difference between the outputs of the intuitionistic fuzzy system and the output of the classical one.

Table 3.1. Sample results of the intuitionistic fuzzy system for several input values

Input Values, x	Membership Result	Non-Membership Result	Intuitionistic Result
0.2500	0.7500	0.7766	0.741330
0.3500	0.7500	0.7766	0.741330
0.4500	0.7500	0.7650	0.740750
0.5500	0.2500	0.2359	0.249295
0.6500	0.2500	0.2250	0.248750
0.7500	0.2500	0.2250	0.248750

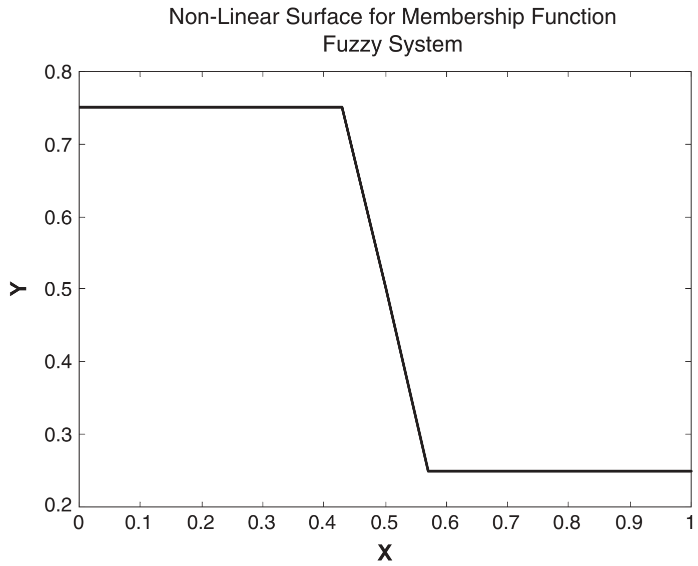

Fig. 3.13. Non-linear surface of the membership function fuzzy system

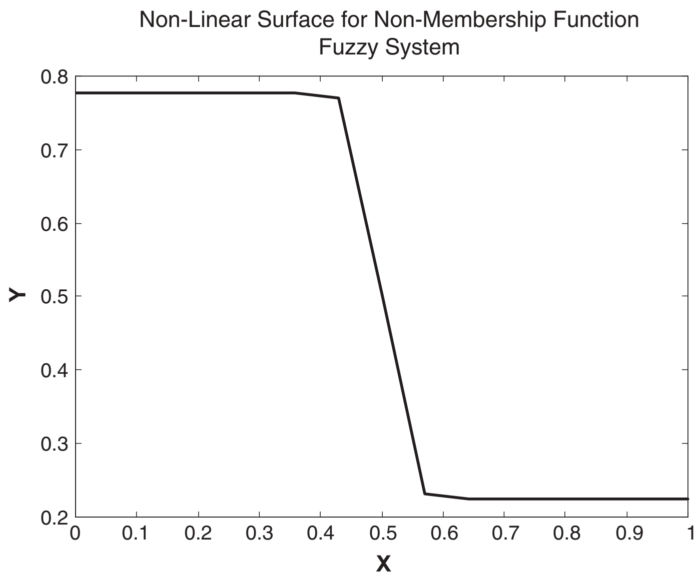

Fig. 3.14. Non-linear surface of non-membership function fuzzy system

Finally, we show in Figs. 3.13 and 3.14 the non-linear surfaces for the fuzzy systems of the membership and non-membership functions, respectively. We can appreciate from these figures that the surfaces are similar, but they differ slightly because of the hesitation margin.

3.6 Summary

In this chapter, we have presented the main ideas underlying intuitionistic and type-2 fuzzy logic and we have only started to point out the many possible applications of these powerful computational theories. We have discussed in some detail type-2 fuzzy set theory, fuzzy reasoning and fuzzy inference systems. At the end, we also gave some remarks about type-2 fuzzy modeling with the Mamdani and Sugeno approaches. We have also discussed in some detail intuitionistic fuzzy sets and intuitionistic fuzzy inference systems. In the following chapters, we will show how intuitionistic and type-2 fuzzy logic (in some cases, in conjunction with other methodologies) can be applied to solve real world complex problems. This chapter will serve as a basis for the new hybrid intelligent methods, for modeling, simulation, and pattern recognition that will be described later this book.